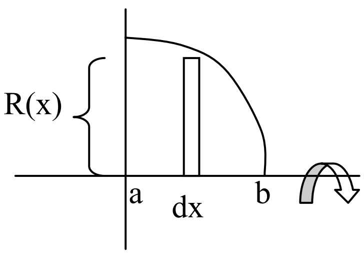
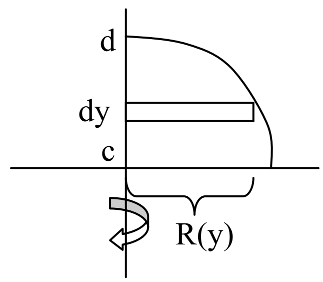
Solids of Revolution

Solids of revolution are created by taking an area and revolving it around an *axis of rotation*. There are two methods to determine the volume of the solid of revolution: the **disk method** and the **shell method**.

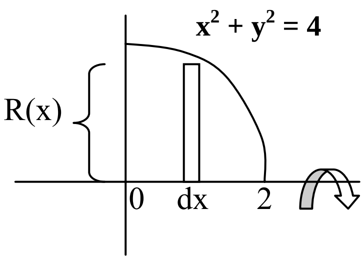
Disk method: In the disk method, small rectangles that are perpendicular to the axis of rotation are rotated around, building up a series of disks that stack on each other in sequence. The width of a rectangle is expressed as either *dx* or *dy*. If **the area abuts** the axis of rotation, then the height (or length) of a rectangle (R for "radius") is the distance from the axis of rotation to the boundary curve, which will be the *function value* R(x) or R(y). If rotating around the x-axis, R is a function of x; if rotating around the y-axis, R is a function of y.

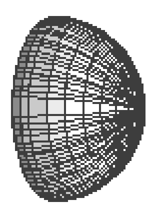




$$V = \pi\int_a^b \left[R(x)\right]^2 dx \qquad\qquad V = \pi\int_c^d \left[R(y)\right]^2 dy$$

1) Find the volume of the solid formed by rotating the region bounded by $x^2 + y^2 = 4$, $y = 0$, and $x = 0$ around the x-axis.



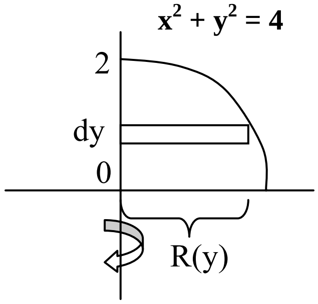


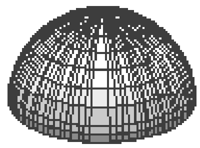
$$V = \pi\int_a^b \left[R(x)\right]^2 dx$$

R(x) = y but it must be in terms of x: $x^2 + y^2 = 4 \;\rightarrow\; y = \sqrt{4 - x^2} = R(x)$

$$V = \pi\int_0^2 \left(\sqrt{4-x^2}\right)^2 dx \;\rightarrow\; = \pi\int_0^2 4 - x^2\, dx = \pi\left[4x - \frac{x^3}{3}\right]_0^2 = \frac{16\pi}{3}$$

Find the volume of the solid formed by rotating the region bounded by $x^2 + y^2 = 4$, $y = 0$, and $x = 0$ around the y-axis.





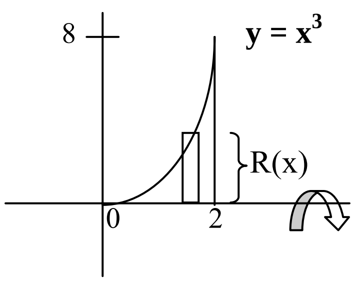
$$V = \pi\int_c^d [R(y)]^2\,dy$$

$R(y) = x$ but it <u>must be in terms of y</u>: $x^2 + y^2 = 4 \;\rightarrow\; x = \sqrt{4-y^2} = R(y)$

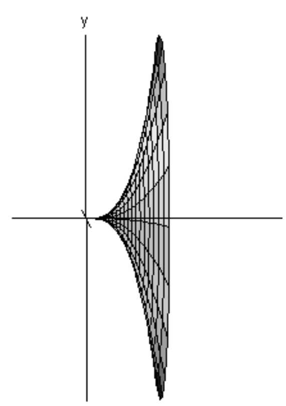
$$V = \pi\int_0^2 \left(\sqrt{4-y^2}\right)^2 dy = \frac{16\pi}{3}$$

(Note that the volume can be verified by using the volume formula for a sphere: $V = 4\pi\, r^3/3$.)

2) Find the volume of the solid formed by rotating the region bounded by $y = x^3$, $y = 0$, $x = 0$, and $x = 2$ around the x-axis.



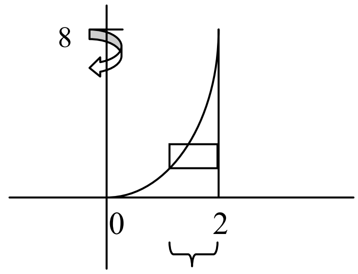
$$V = \pi\int_a^b [R(x)]^2\,dx$$



$R(x) = y = x^3.$

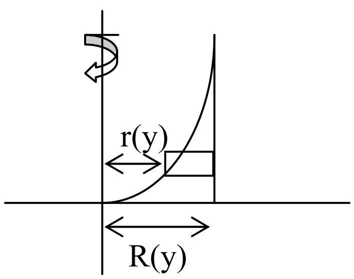
$$V = \pi\int_0^2 \left(x^3\right)^2 dx = \pi\int_0^2 x^6 dx = \pi\left[\frac{x^7}{7}\right]_0^2 = \frac{128\pi}{7}$$

Find the volume of the solid formed by rotating the region bounded by $y = x^3$, $y = 0$, $x = 0$, and $x = 2$ around the y-axis.



L(y) = length of rectangle

What happens if we rotate this area around the y-axis? We still need to use the *length of the rectangle* because that's what is forming the solid, but now it's no longer just the distance to the axis of rotation because there's a gap – the region is not right up against the axis of rotation. So we switch to a modified disk method, called the **washer method** (disk + hole). In the washer method, we use the distance from the axis of rotation to the "outer" curve and the distance from the axis to the "inner" curve.



Washer: $V = \pi \int_c^d [R(y)]^2 - [r(y)]^2 \, dy$

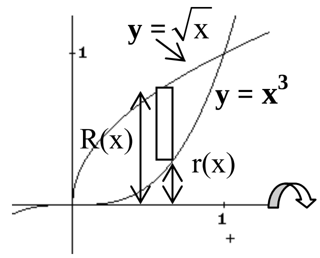
R(y) is constant for this area – it's just 2.
r(y) = x, but it must be in terms of y $\rightarrow$ $x = y^{1/3} = r(y)$
Don't forget to use the limits of integration for y, not x!

$$V = \pi \int_0^8 [2]^2 - \left[y^{1/3}\right]^2 dy \;=\; \pi \int_0^8 4 - y^{2/3} dy$$

$$= \pi \left[4y - \frac{3y^{5/3}}{5} \right|_0^8 = \frac{64\pi}{5}$$

3) Let's look at the area formed by the intersections of the curves $y = \sqrt{x}$ and $y = x^3$.

If we rotate this area around the x-axis, then the 'upper' curve (farthest from the axis) is $y = \sqrt{x}$ and the 'lower' curve (nearest the axis) is $y = x^3$.

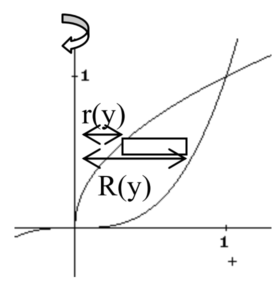


$R(x) = \sqrt{x}$ and $r(x) = x^3$.

$$V = \pi\int_a^b [R(x)]^2 - [r(x)]^2\,dx$$

$$= \pi\int_0^1 \left[\sqrt{x}\right]^2 - \left[x^3\right]^2 dx = \pi\int_0^1 x - x^6\,dx = \pi\left[\frac{x^2}{2} - \frac{x^7}{7}\right]_0^1 = \frac{5\pi}{14} \approx 1.122$$

If we rotate this area around the y-axis, the 'upper' curve (farthest from the axis) is $y = x^3$ and the 'lower' curve (nearest the axis) is $y = \sqrt{x}$. The distances from the axis to these curves are x-values, but they must be written as functions of y.



$R(y) = x = \sqrt[3]{y}$ and $r(y) = x = y^2$.

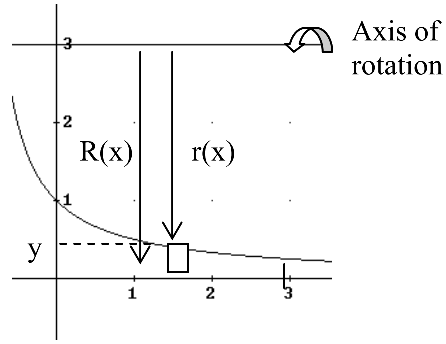
$$V = \pi\int_c^d [R(y)]^2 - [r(y)]^2\,dy$$

$$= \pi\int_0^1 \left[y^{1/3}\right]^2 - \left[y^2\right]^2 dy = \pi\int_0^1 y^{2/3} - y^4\,dy = \pi\left[\frac{3y^{5/3}}{5} - \frac{y^5}{5}\right]_0^1 = \frac{2\pi}{5} \approx 1.257$$

4) One more: calculate the volume formed by revolving the region bounded by $y = \dfrac{1}{1+x}$, y = 0, x = 0, and x = 3 <u>about the line y = 3.</u>

The distance R(x) *from the axis* of y = 3 is just 3.

The distance r(x) *from the axis* is (3 – y) for the function $y = \dfrac{1}{1+x} \quad \rightarrow \quad r(x) = 3 - \dfrac{1}{1+x}$



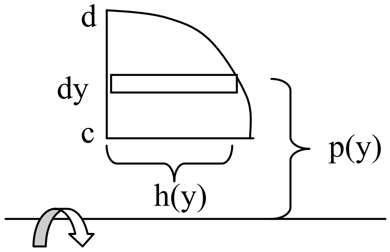
$$V = \pi\int_a^b [R(x)]^2 - [r(x)]^2\,dx \quad \rightarrow \quad V = \pi\int_0^3 [3]^2 - \left[3 - \frac{1}{1+x}\right]^2 dx \quad = \quad \pi\int_0^3 \frac{6}{1+x} - \frac{1}{(1+x)^2}\,dx$$

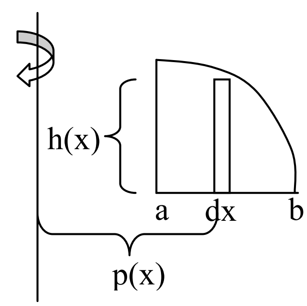
$$= \pi\left[6\ln(1+x) + \frac{1}{1+x}\right|_0^3 \quad = \quad \pi\,[12\ln 2 - \tfrac{3}{4}] \quad \approx \quad 23.77$$

Shell method: In the shell method, small rectangles that are <u>parallel</u> to the axis of rotation are rotated around, building up a series of shells that build the solid from the inside out. The width of a rectangle is again expressed as either dx or dy. P(x) or p(y) is the distance from the rectangle to the axis of rotation; h(x) or h(y) is the length (or height) of the rectangle. Gaps between the region and the axis are accounted for in the limits of integration, so do not require extra calculation as in the washer method.

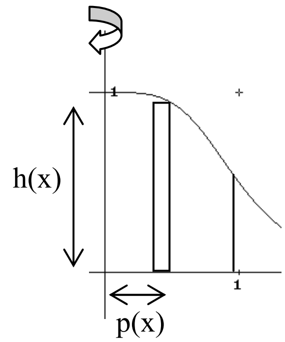
The shell method added to the disk method gives you a choice of *whichever integration would be easier* – integrating with respect to x or to y. Sometimes you can't (easily) solve for x in terms of y or vice versa, and are forced into a specific method.

$$V = 2\pi\int_c^d p(y)\,h(y)\,dy \qquad\qquad V = 2\pi\int_a^b p(x)\,h(x)\,dx$$





1) Find the volume formed by revolving the region bounded by $y = \dfrac{1}{x^4+1}$, $y = 0$, $x = 0$, and $x = 1$ about the y axis.

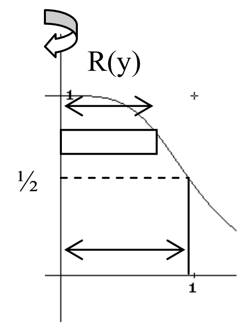


$$p(x) = x \qquad h(x) = y = \frac{1}{x^4+1}$$

$$V = 2\pi \int_a^b p(x)\,h(x)\,dx = 2\pi \int_0^1 \frac{x}{x^4+1}\,dx$$

$$= \pi \int_0^1 \frac{(2x)}{(x^2)^2+1}\,dx = \pi \left[\arctan x^2\right|_0^1 = \frac{\pi^2}{4}$$

If you tried to do this using the disk method, you would first have to split the area into two parts: a rectangle up to y = ½ and the function curve from there to y = 1.



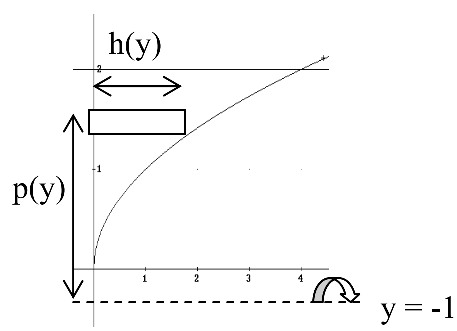
For the bottom part, R(y) = 1 $\qquad V = \pi \int_c^d \left[R(y)\right]^2 dy$

For the top part, $R(y) = x = \sqrt[4]{\dfrac{1-y}{y}}$

$$V = \pi \int_0^{1/2} 1^2\,dy + \pi \int_{1/2}^1 \sqrt{\frac{1-y}{y}}\,dy \quad \rightarrow \quad \text{try using some calculus software!}$$

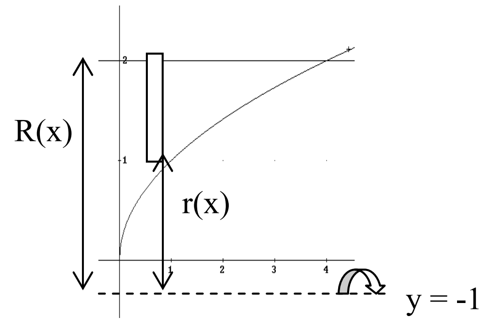
2) Find the volume of the solid produced by rotating the region bounded by $y = \sqrt{x}$, y = 2, and x = 0 around the line y = -1.

<u>Shell method:</u> $\qquad p(y) = y + 1 \qquad h(y) = x = y^2$



$$V = 2\pi \int_c^d p(y)\,h(y)\,dy = 2\pi \int_0^2 (y+1)(y^2)\,dy$$

$$= 2\pi \int_0^2 (y^3+y^2)\,dy = 2\pi \left[\frac{y^4}{4} + \frac{y^3}{3}\right|_0^2 = \frac{40\pi}{3}$$



Disk method: $R(x) = 3 \qquad r(x) = y + 1 = \sqrt{x} + 1$

$$V = \pi\int_a^b [R(x)]^2 - [r(x)]^2\,dx$$

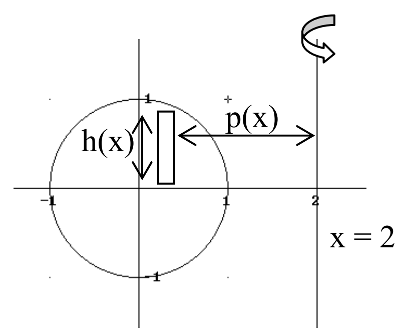
$$= \pi\int_0^4 [3]^2 - \left[\sqrt{x}+1\right]^2 dx \;=\; \pi\int_0^4 8 - x - 2\sqrt{x}\,dx$$

$$= \pi\left[8x - \frac{x^2}{2} - \frac{4x^{3/2}}{3}\right|_0^4 \;=\; \frac{40\pi}{3}$$

Neither of these integrations is very difficult, but the shell method yields somewhat simpler calculations.

3) Find the volume of the torus generated by revolving the circle given by $x^2 + y^2 = 1$ around the line $x = 2$.

$$V = 2\pi\int_a^b p(x)\,h(x)\,dx$$



We can work with just the top half of the circle and then double the volume.

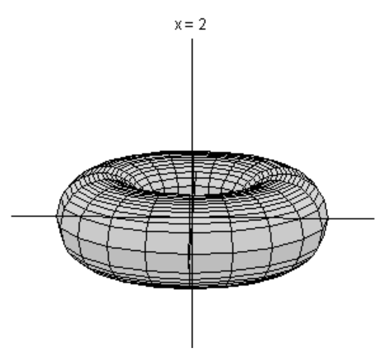
Shell method: $p(x) \;=\; 2 - x \qquad h(x) \;=\; y \;=\; \sqrt{1-x^2}$

$$V = (2)2\pi\int_{-1}^{1}(2-x)\sqrt{1-x^2}\,dx \;=\; 4\pi\left[2\int_{-1}^{1}\sqrt{1-x^2}\,dx - \int_{-1}^{1}x\sqrt{1-x^2}\,dx\right]$$

The first integral can be obtained from a table or by recognizing it is simply the area of the circle $\pi r^2 \;=\; \pi$.

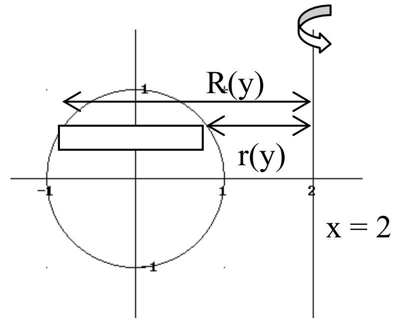
The second integral can be solved using a *u* substitution: $= +\left.\frac{\left(1-x^2\right)^{3/2}}{3}\right|_{-1}^{1} \;=\; 0$

$V = 4\pi(\pi) \;=\; 4\pi^2$



Disk method: This is also doable using the disk method and based on the symmetry of the circle.

$$V = \pi\int_c^d [R(y)]^2 - [r(y)]^2 \, dy$$



$R(y) \; = \; 2 + x \; = \; 2 + \sqrt{1-y^2}$
$r(y) \; = \; 2 - x \; = \; 2 - \sqrt{1-y^2}$

$$V \; = \; (2)\pi\int_0^1 \left[2+\sqrt{1-y^2}\right]^2 - \left[2-\sqrt{1-y^2}\right]^2 dy$$

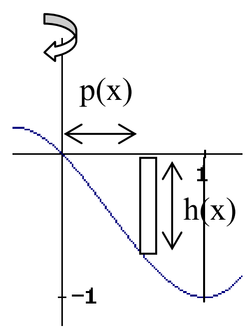
$$= \; 2\pi\int_0^1 8\sqrt{1-y^2}\, dy \; = \; 16\pi\int_0^1 \sqrt{1-y^2}\, dy$$

The integral can again be obtained from a table or by recognizing it is the area of a quarter circle $\pi r^2/4 \; = \; \pi/4$

$$V \; = \; (16\pi)\left(\frac{\pi}{4}\right) \; = \; 4\pi^2$$

4) Find the volume of the solid produced by rotating the region bounded by $y = x^3 - x^2 - x$, $y = 0$, $x = 0$, and $x = 1$ around the y-axis.

For this function, x cannot be easily solved for in terms of y, so the shell method is required.



$p(x) \; = \; x$
since y is negative in this region, $h(x) = - y \; = \; - x^3 + x^2 + x$

$$V = 2\pi\int_a^b p(x)\,h(x)\,dx \; = \; 2\pi\int_0^1 x\,(-x^3+x^2+x)\,dx$$

$$= \; 2\pi\int_0^1 (-x^4+x^3+x^2)\,dx \; = \; 2\pi\left[-\frac{x^5}{5}+\frac{x^4}{4}+\frac{x^3}{3}\right|_0^1 \; = \; \frac{23\pi}{30} \; \approx \; 2.41$$