1 **a** $\int y^{-3}\ dy = \int x\ dx$

$-\frac{1}{2}y^{-2} = \frac{1}{2}x^2 + c$

$y^{-2} = k - x^2$

$y^2 = \dfrac{1}{k - x^2}$

b $y = \frac{1}{2}$ when $x = 1$

$\therefore\ \frac{1}{4} = \dfrac{1}{k - 1}$

$k = 5$

$\therefore\ y^2 = \dfrac{1}{5 - x^2}$

2 $\dfrac{dy}{dx} = \dfrac{x}{\cos y \sin^3 y}$

$\int \cos y \sin^3 y\ dy = \int x\ dx$

$\frac{1}{4}\sin^4 y = \frac{1}{2}x^2 + c$

$\sin^4 y = 2x^2 + k$

$y = \frac{\pi}{4}$ when $x = 1$

$\therefore\ (\frac{1}{\sqrt{2}})^4 = 2 + k$

$\frac{1}{4} = 2 + k$

$k = -\frac{7}{4}$

$\therefore\ \sin^4 y = 2x^2 - \frac{7}{4}$

3 **a** $\dfrac{dN}{dt} = kN$

$\int \dfrac{1}{N}\ dN = \int k\ dt$

$\ln |N| = kt + c$

$N = e^{kt + c} = e^c \times e^{kt}$

$\therefore\ N = Ae^{kt}$

b $t = 0, N = 200 \quad \therefore\ A = 200$

$t = 2, N = 3000 \quad \therefore\ 3000 = 200e^{2k}$

$\therefore\ k = \frac{1}{2}\ln 15 = 1.354$

$\therefore\ N = 200e^{1.354t}$

$\therefore\ 10\,000 = 200e^{1.354t}$

$t = \frac{1}{1.354}\ln 50 = 2.889$ hours

= 2 hours 53 minutes

c 5 per second = 18 000 per hour

$\dfrac{dN}{dt} = 200 \times 0.1354e^{1.354t}$

$\therefore\ 18\,000 = 270.8e^{1.354t}$

$t = \frac{1}{1.354}\ln\frac{18000}{270.8} = 3.099$ hours

= 3 hours 6 minutes

4 **a** $u = \ln x,\ \dfrac{du}{dx} = \dfrac{1}{x};\ \dfrac{dv}{dx} = x,\ v = \frac{1}{2}x^2$

$\int x \ln x\ dx = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x\ dx$

$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + c$

$= \frac{1}{4}x^2(2 \ln x - 1) + c$

b $\int \dfrac{1}{y}\ dy = \int x \ln x\ dx$

$\ln |y| = \frac{1}{4}x^2(2 \ln x - 1) + c$

$y > 0 \quad \therefore\ \ln y = \frac{1}{4}x^2(2 \ln x - 1) + c$

$y = 4$ when $x = 2$

$\therefore\ \ln 4 = 2 \ln 2 - 1 + c$

$c = 1$

$\therefore\ \ln y = \frac{1}{4}x^2(2 \ln x - 1) + 1$

when $x = 1,\ \ln y = \frac{1}{4}(0 - 1) + 1 = \frac{3}{4}$

$\therefore\ y = e^{\frac{3}{4}}$

5 $\frac{dy}{dx} = e^y \times e^x \cos x$

$\int e^{-y}\ dy = \int e^x \cos x\ dx$

$u = e^x,\ \frac{du}{dx} = e^x;\ \frac{dv}{dx} = \cos x,\ v = \sin x$

$I = \int e^x \cos x\ dx = e^x \sin x - \int e^x \sin x\ dx$

$u = e^x,\ \frac{du}{dx} = e^x;\ \frac{dv}{dx} = \sin x,\ v = -\cos x$

$I = e^x \sin x - (-e^x \cos x + \int e^x \cos x\ dx)$

$2I = e^x \sin x + e^x \cos x + C$

$-e^{-y} = \frac{1}{2}e^x(\sin x + \cos x) + c$

$e^{-y} = k - \frac{1}{2}e^x(\sin x + \cos x)$

$y = 0$ when $x = 0$

$\therefore\ 1 = k - \frac{1}{2}$

$k = \frac{3}{2}$

$\therefore\ e^{-y} = \frac{3}{2} - \frac{1}{2}e^x(\sin x + \cos x)$

$[\ 2e^{-y} = 3 - e^x(\sin x + \cos x)\ ]$

6 **a** $-\frac{d\theta}{dt} = k\theta$

$\int \frac{1}{\theta}\ d\theta = \int -k\ dt$

$\ln|\theta| = -kt + c$

$\theta = e^{-kt + c} = e^c \times e^{-kt}$

$\therefore\ \theta = Ae^{-kt}$

$t = 0,\ \theta = 25 - 10 = 15$

$\therefore\ A = 15$

$\therefore\ \theta = 15e^{-kt}$

b $t = 30,\ \theta = 20 - 10 = 10$

$\therefore\ 10 = 15e^{-30k}$

$k = -\frac{1}{30}\ln\frac{2}{3} = 0.0135$ (3sf)

c $\theta = 15 - 10 = 5$

$\therefore\ 5 = 15e^{-0.01352t}$

$t = -\frac{1}{0.01352}\ln\frac{1}{3}$

$= 81.3$ minutes (3sf)

7 **a** $A = \pi r^2\ \therefore\ r = \sqrt{\frac{A}{\pi}}$

$P = 2\pi r = 2\pi\sqrt{\frac{A}{\pi}} = 2\sqrt{\pi A}$

$\frac{dA}{dt} = cP = c \times 2\sqrt{\pi A} = 2c\sqrt{\pi} \times \sqrt{A}$

$\therefore\ \frac{dA}{dt} = k\sqrt{A}$

b $\int A^{-\frac{1}{2}}\ dA = \int k\ dt$

$2A^{\frac{1}{2}} = kt + C$

$\sqrt{A} = \frac{1}{2}kt + \frac{1}{2}C = pt + q$

$\therefore\ A = (pt + q)^2$

c $t = 0, A = 25\quad \therefore\ \sqrt{25} = 0 + q$

$q = 5$

$t = 20, A = 40\quad \therefore\ \sqrt{40} = 20p + 5$

$p = \frac{2\sqrt{10} - 5}{20}$

$A = 50 \quad\Rightarrow\quad 50 = (\frac{2\sqrt{10} - 5}{20}t + 5)^2$

$t = (5\sqrt{2} - 5) \div \frac{2\sqrt{10} - 5}{20}$

$= 31.3$ minutes (3sf)

8 **a** $\frac{(x+4)}{(1+x)(2-x)} \equiv \frac{A}{1+x} + \frac{B}{2-x}$

$x + 4 \equiv A(2 - x) + B(1 + x)$

$x = -1 \Rightarrow A = 1,\ x = 2 \Rightarrow B = 2$

$\therefore\ \frac{(x+4)}{(1+x)(2-x)} \equiv \frac{1}{1+x} + \frac{2}{2-x}$

b $\int \frac{1}{y}\ dy = \int (\frac{1}{1+x} + \frac{2}{2-x})\ dx$

$\ln|y| = \ln|1 + x| - 2\ln|2 - x| + c$

$y = 2$ when $x = 3$

$\therefore\ \ln 2 = \ln 4 - 0 + c$

$c = -\ln 2$

$\therefore\ \ln|y| = \ln|1 + x| - 2\ln|2 - x| - \ln 2$

$[\ y = \frac{1+x}{2(2-x)^2}\ ]$

9 **a** $-\frac{dV}{dt} = aV$ where a is a positive constant

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

if angle at vertex = 2θ, $\tan\theta = \frac{r}{h}$

$\therefore r = bh$ where b is a positive constant

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi b^2 h^3$

$\therefore \frac{dV}{dh} = \pi b^2 h^2$

$\therefore -\pi b^2 h^2 \times \frac{dh}{dt} = a \times \frac{1}{3}\pi b^2 h^3$

$\frac{dh}{dt} = -kh$ where k is a positive constant

b $\int \frac{1}{h}\, dh = \int -k\, dt$

$\ln|h| = -kt + c$

$h = e^{-kt+c} = e^c \times e^{-kt} = Ae^{-kt}$

$t = 0, h = 12 \quad \therefore A = 12$

$t = 20, h = 10 \quad \therefore 10 = 12e^{-20k}$

$\therefore k = -\frac{1}{20}\ln\frac{5}{6}$

$\therefore h = 12e^{-kt}$, $k = 0.00912$ (3sf)

c $6 = 12e^{-0.009116t}$

$t = -\frac{1}{0.009116}\ln\frac{1}{2}$

$= 76.0$ (3sf)