

$$1 \quad \mathbf{a} \quad y = \int (x+2)^3 dx$$

$$y = \frac{1}{4}(x+2)^4 + c$$

$$\mathbf{c} \quad x = \int 3e^{2t} + 2 dt$$

$$x = \frac{3}{2}e^{2t} + 2t + c$$

$$\mathbf{e} \quad N = \int t\sqrt{t^2+1} dt$$

$$N = \frac{1}{2} \int 2t(t^2+1)^{\frac{1}{2}} dt$$

$$N = \frac{1}{2} \times \frac{2}{3}(t^2+1)^{\frac{3}{2}} + c$$

$$N = \frac{1}{3}(t^2+1)^{\frac{3}{2}} + c$$

$$2 \quad \mathbf{a} \quad y = \int e^{-x} dx$$

$$y = -e^{-x} + c$$

$$y = 3 \text{ when } x = 0$$

$$\therefore 3 = -1 + c$$

$$c = 4$$

$$\therefore y = 4 - e^{-x}$$

$$\mathbf{c} \quad \frac{du}{dx} = \frac{4x}{x^2-3}$$

$$u = \int \frac{4x}{x^2-3} dx = 2 \int \frac{2x}{x^2-3} dx$$

$$u = 2 \ln|x^2-3| + c$$

$$u = 5 \text{ when } x = 2$$

$$\therefore 5 = 0 + c$$

$$c = 5$$

$$\therefore u = 2 \ln|x^2-3| + 5$$

$$3 \quad \mathbf{a} \quad \frac{x-8}{x^2-x-6} \equiv \frac{A}{x-3} + \frac{B}{x+2}$$

$$x-8 \equiv A(x+2) + B(x-3)$$

$$x=3 \Rightarrow -5 = 5A \Rightarrow A = -1$$

$$x=-2 \Rightarrow -10 = -5B \Rightarrow B = 2$$

$$\frac{x-8}{x^2-x-6} \equiv \frac{2}{x+2} - \frac{1}{x-3}$$

$$\mathbf{b} \quad y = \int 4 \cos 2x dx$$

$$y = 2 \sin 2x + c$$

$$\mathbf{d} \quad \frac{dy}{dx} = \frac{1}{2-x}$$

$$y = \int \frac{1}{2-x} dx$$

$$y = -\ln|2-x| + c$$

$$\mathbf{f} \quad y = \int xe^x dx$$

$$u = x, \frac{du}{dx} = 1; \frac{dv}{dx} = e^x, v = e^x$$

$$y = xe^x - \int e^x dx$$

$$y = xe^x - e^x + c \quad [y = e^x(x-1) + c]$$

$$\mathbf{b} \quad y = \int \tan^3 t \sec^2 t dt$$

$$y = \frac{1}{4} \tan^4 t + c$$

$$y = 1 \text{ when } t = \frac{\pi}{3}$$

$$\therefore 1 = \frac{1}{4}(\sqrt{3})^4 + c$$

$$c = 1 - \frac{9}{4} = -\frac{5}{4}$$

$$\therefore y = \frac{1}{4} \tan^4 t - \frac{5}{4} \quad [y = \frac{1}{4}(\tan^4 t - 5)]$$

$$\mathbf{d} \quad y = \int 3 \cos^2 x dx$$

$$y = \frac{3}{2} \int (1 + \cos 2x) dx$$

$$y = \frac{3}{2}(x + \frac{1}{2} \sin 2x) + c = \frac{3}{4}(2x + \sin 2x) + c$$

$$y = \pi \text{ when } x = \frac{\pi}{2}$$

$$\therefore \pi = \frac{3}{4}(\pi + 0) + c$$

$$c = \frac{\pi}{4}$$

$$\therefore y = \frac{3}{4}(2x + \sin 2x) + \frac{\pi}{4}$$

$$[y = \frac{1}{4}(6x + 3 \sin 2x + \pi)]$$

$$\mathbf{b} \quad \frac{dy}{dx} = \frac{x-8}{x^2-x-6}$$

$$y = \int \frac{x-8}{x^2-x-6} dx = \int \left( \frac{2}{x+2} - \frac{1}{x-3} \right) dx$$

$$y = 2 \ln|x+2| - \ln|x-3| + c$$

$$y = \ln 9 \text{ when } x = 1$$

$$\therefore \ln 9 = 2 \ln 3 - \ln 2 + c$$

$$c = \ln 2 \quad (\ln 9 = \ln 3^2 = 2 \ln 3)$$

$$\therefore y = 2 \ln|x+2| - \ln|x-3| + \ln 2$$

$$\text{when } x = 2, y = 2 \ln 4 - 0 + \ln 2 = \ln(4^2 \times 2)$$

$$= \ln 32$$

$$4 \quad \mathbf{a} \quad \int \frac{1}{2y+3} dy = \int dx$$

$$\frac{1}{2} \ln |2y+3| = x + c$$

$$[y = \frac{1}{2}(ke^{2x} - 3)]$$

$$\mathbf{c} \quad \int \frac{1}{y} dy = \int x dx$$

$$\ln |y| = \frac{1}{2}x^2 + c$$

$$[y = ke^{\frac{1}{2}x^2}]$$

$$\mathbf{e} \quad \int y dy = \int (x^2 - 2) dx$$

$$\frac{1}{2}y^2 = \frac{1}{3}x^3 - 2x + c$$

$$[y^2 = \frac{2}{3}x^3 - 4x + k]$$

$$\mathbf{g} \quad \int e^{3-y} dy = \int x^{-\frac{1}{2}} dx$$

$$-e^{3-y} = 2x^{\frac{1}{2}} + c$$

$$[y = 3 - \ln(k - 2\sqrt{x})]$$

$$\mathbf{i} \quad \int \frac{1}{y} dy = \int x \sin x dx$$

$$u = x, \frac{du}{dx} = 1; \frac{dv}{dx} = \sin x, v = -\cos x$$

$$\ln |y| = -x \cos x + \int \cos x dx$$

$$\ln |y| = \sin x - x \cos x + c$$

$$[y = ke^{\sin x - x \cos x}]$$

$$\mathbf{k} \quad \int \frac{y-3}{y(y-1)} dy = \int x dx$$

$$\frac{y-3}{y(y-1)} \equiv \frac{A}{y} + \frac{B}{y-1}$$

$$y-3 \equiv A(y-1) + By$$

$$y=0 \Rightarrow A=3, y=1 \Rightarrow B=-2$$

$$\int \left(\frac{3}{y} - \frac{2}{y-1}\right) dy = \int x dx$$

$$3 \ln |y| - 2 \ln |y-1| = \frac{1}{2}x^2 + c$$

$$\mathbf{b} \quad \int \operatorname{cosec}^2 2y dy = \int dx$$

$$-\frac{1}{2} \cot 2y = x + c$$

$$[\cot 2y = k - 2x]$$

$$\mathbf{d} \quad \int \frac{1}{y} dy = \int \frac{1}{x+1} dx$$

$$\ln |y| = \ln |x+1| + c$$

$$[y = k(x+1)]$$

$$\mathbf{f} \quad \int \sec^2 y dy = \int 2 \cos x dx$$

$$\tan y = 2 \sin x + c$$

$$\mathbf{h} \quad y \frac{dy}{dx} = x(y^2 + 3)$$

$$\int \frac{y}{y^2+3} dy = \int x dx$$

$$\frac{1}{2} \int \frac{2y}{y^2+3} dy = \int x dx$$

$$\frac{1}{2} \ln |y^2+3| = \frac{1}{2}x^2 + c$$

$$[y^2 = ke^{x^2} - 3]$$

$$\mathbf{j} \quad \frac{dy}{dx} = \frac{e^{2x}}{e^y}$$

$$\int e^y dy = \int e^{2x} dx$$

$$e^y = \frac{1}{2}e^{2x} + c$$

$$[y = \ln(\frac{1}{2}e^{2x} + c)]$$

$$\mathbf{l} \quad \int y^{-2} dy = \int \ln x dx$$

$$u = \ln x, \frac{du}{dx} = \frac{1}{x}; \frac{dv}{dx} = 1, v = x$$

$$-y^{-1} = x \ln x - \int dx$$

$$-y^{-1} = x \ln x - x + c$$

$$[y = \frac{1}{x - x \ln x + k}]$$

$$5 \quad \mathbf{a} \quad \int 2y \, dy = \int x \, dx$$

$$y^2 = \frac{1}{2}x^2 + c$$

$$y = 3 \text{ when } x = 4$$

$$\therefore 9 = 8 + c$$

$$c = 1$$

$$\therefore y^2 = \frac{1}{2}x^2 + 1$$

$$\mathbf{c} \quad \int \frac{1}{y} \, dy = \int \cot^2 x \, dx$$

$$\int \frac{1}{y} \, dy = \int (\operatorname{cosec}^2 x - 1) \, dx$$

$$\ln|y| = -\cot x - x + c$$

$$y = 1 \text{ when } x = \frac{\pi}{2}$$

$$\therefore 0 = 0 - \frac{\pi}{2} + c$$

$$c = \frac{\pi}{2}$$

$$\therefore \ln|y| = \frac{\pi}{2} - \cot x - x$$

$$\mathbf{e} \quad \int \cot y \, dy = \int x^2 \, dx$$

$$\int \frac{\cos y}{\sin y} \, dy = \int x^2 \, dx$$

$$\ln|\sin y| = \frac{1}{3}x^3 + c$$

$$y = \frac{\pi}{6} \text{ when } x = 0$$

$$\therefore \ln \frac{1}{2} = 0 + c$$

$$c = -\ln 2$$

$$\therefore \ln|\sin y| = \frac{1}{3}x^3 - \ln 2$$

$$[2 \sin y = e^{\frac{1}{3}x^3}]$$

$$\mathbf{g} \quad \int \sin y \, dy = \int xe^{-x} \, dx$$

$$u = x, \frac{du}{dx} = 1; \frac{dv}{dx} = e^{-x}, v = -e^{-x}$$

$$-\cos y = -xe^{-x} + \int e^{-x} \, dx$$

$$-\cos y = -xe^{-x} - e^{-x} + c$$

$$\cos y = (x+1)e^{-x} + k$$

$$y = \pi \text{ when } x = -1$$

$$\therefore -1 = 0 + k$$

$$k = -1$$

$$\therefore \cos y = (x+1)e^{-x} - 1$$

$$\mathbf{b} \quad \int (y+1)^{-3} \, dy = \int \, dx$$

$$-\frac{1}{2}(y+1)^{-2} = x + c$$

$$(y+1)^{-2} = k - 2x$$

$$y = 0 \text{ when } x = 2$$

$$\therefore 1 = k - 4$$

$$k = 5$$

$$\therefore (y+1)^{-2} = 5 - 2x$$

$$[(y+1)^2 = \frac{1}{5-2x}]$$

$$\mathbf{d} \quad \int \frac{1}{y+2} \, dy = \int \frac{1}{x-1} \, dx$$

$$\ln|y+2| = \ln|x-1| + c$$

$$y = 6 \text{ when } x = 3$$

$$\therefore \ln 8 = \ln 2 + c$$

$$c = \ln 4$$

$$\therefore \ln|y+2| = \ln|x-1| + \ln 4$$

$$[y+2 = 4(x-1) \Rightarrow y = 4x - 6]$$

$$\mathbf{f} \quad \frac{dy}{dx} = \frac{\sqrt{y}}{\sqrt{x+3}}$$

$$\int y^{\frac{1}{2}} \, dy = \int (x+3)^{-\frac{1}{2}} \, dx$$

$$2y^{\frac{3}{2}} = 2(x+3)^{\frac{1}{2}} + c$$

$$\sqrt{y} = \sqrt{x+3} + k$$

$$y = 16 \text{ when } x = 1$$

$$\therefore 4 = 2 + k$$

$$k = 2$$

$$\therefore \sqrt{y} = \sqrt{x+3} + 2$$

$$[y = (\sqrt{x+3} + 2)^2]$$

$$\mathbf{h} \quad \int \frac{\sin y}{1+\cos y} \, dy = \int \frac{1}{2}x^{-2} \, dx$$

$$-\int \frac{-\sin y}{1+\cos y} \, dy = \int \frac{1}{2}x^{-2} \, dx$$

$$-\ln|1+\cos y| = -\frac{1}{2}x^{-1} + c$$

$$\ln|1+\cos y| = \frac{1}{2}x^{-1} + k$$

$$y = \frac{\pi}{3} \text{ when } x = 1$$

$$\therefore \ln \frac{3}{2} = \frac{1}{2} + k$$

$$k = \ln \frac{3}{2} - \frac{1}{2}$$

$$\therefore \ln|1+\cos y| = \frac{1}{2}x^{-1} + \ln \frac{3}{2} - \frac{1}{2}$$

$$[(1+\cos y)^2 = \frac{9}{4}e^{\frac{1-x}{x}}]$$

6 a  $\frac{dN}{dt} = kN$

b  $\int \frac{1}{N} dN = \int k dt$

$$\ln |N| = kt + c$$

$$N = e^{kt+c} = e^c \times e^{kt}$$

$$N = Ae^{kt}$$

c  $t = 0, N = 40 \therefore A = 40$

$$t = 5, N = 60 \therefore 60 = 40e^{5k}$$

$$\therefore k = \frac{1}{5} \ln \frac{3}{2} = 0.0811 \text{ (3sf)}$$

d  $t = 12 \therefore N = 40e^{0.08109 \times 12}$

$$= 106 \text{ (3sf)}$$

8 a  $-\frac{dP}{dt} = k\sqrt{P}$

b  $\int P^{-\frac{1}{2}} dP = \int -k dt$

$$2P^{\frac{1}{2}} = -kt + c$$

$$\sqrt{P} = \frac{1}{2}c - \frac{1}{2}kt = a - bt$$

$$\therefore P = (a - bt)^2$$

c  $t = 0, P = 400 \therefore \sqrt{400} = a - 0$

$$a = 20$$

d  $t = 30, P = 100 \therefore \sqrt{100} = 20 - 30b$

$$b = \frac{1}{3}$$

$$\therefore P = (20 - \frac{1}{3}t)^2$$

$$t = 50 \therefore P = (20 - \frac{50}{3})^2$$

$$= 11\frac{1}{9}$$

7 a  $-\frac{dm}{dt} = km$

$$\int \frac{1}{m} dm = \int -k dt$$

$$\ln |m| = -kt + c$$

$$m = e^{-kt+c} = e^c \times e^{-kt} = Ae^{-kt}$$

$$t = 0, m = 24 \therefore A = 24$$

$$m = 24e^{-kt}$$

b  $t = 20, m = 22.6 \therefore 22.6 = 24e^{-20k}$

$$\therefore k = -\frac{1}{20} \ln \frac{22.6}{24} = 0.00301 \text{ (3sf)}$$

c  $\frac{dm}{dt} = -km = -0.003005 \times 22.6$

$\therefore$  decreasing at 0.0679 grams per day

d  $m = 12 \therefore 12 = 24e^{-0.003005t}$

$$t = -\frac{1}{0.003005} \ln \frac{1}{2} = 231 \text{ days (nearest day)}$$

9 a  $\frac{1}{(1+x)(1-x)} \equiv \frac{A}{1+x} + \frac{B}{1-x}$

$$1 \equiv A(1-x) + B(1+x)$$

$$x = -1 \Rightarrow A = \frac{1}{2}, x = 1 \Rightarrow B = \frac{1}{2}$$

$$\frac{1}{(1+x)(1-x)} \equiv \frac{1}{2(1+x)} + \frac{1}{2(1-x)}$$

b  $t = 0, m = 0, \frac{dm}{dt} = 0.5$

$$\therefore 0.5 = k \times 1, k = 0.5$$

c  $\int \frac{1}{(1+m)(1-m)} dm = \int \frac{1}{2} e^{-t} dt$

$$\int \left( \frac{\frac{1}{2}}{1+m} + \frac{\frac{1}{2}}{1-m} \right) dm = \int \frac{1}{2} e^{-t} dt$$

$$\frac{1}{2} \ln |1+m| - \frac{1}{2} \ln |1-m| = -\frac{1}{2} e^{-t} + c$$

$$\ln |1+m| - \ln |1-m| = C - e^{-t}$$

$$t = 0, m = 0 \therefore 0 - 0 = C - 1, C = 1$$

$$\ln |1+m| - \ln |1-m| = 1 - e^{-t}$$

for  $0 \leq m < 1$ ,  $1+m > 0$  and  $1-m > 0$

$$\therefore \ln \left( \frac{1+m}{1-m} \right) = 1 - e^{-t}$$

d  $m = 0.1 \therefore \ln \frac{1.1}{0.9} = 1 - e^{-t}$

$$t = -\ln \left( 1 - \ln \frac{1.1}{0.9} \right) = 0.2240 \text{ hrs} = 13.4 \text{ mins}$$

e  $t \rightarrow \infty, \ln \left( \frac{1+m}{1-m} \right) \rightarrow 1$

limiting value of  $m$  is given by  $\frac{1+m}{1-m} = e$

$$1+m = e(1-m)$$

$$m(1+e) = e-1$$

$$m = \frac{e-1}{1+e} = 0.4621$$

$\therefore$  max. produced  $\approx 462$  g