

Volumes by Cylindrical Shells: the Shell Method

Another method of find the volumes of solids of revolution is the *shell method*. It can usually find volumes that are otherwise difficult to evaluate using the Disc / Washer method.

General formula: $V = \int 2\pi(\text{shell radius})(\text{shell height}) dx$

The Shell Method (about the y -axis)

The volume of the solid generated by revolving about the y -axis the region between the x -axis and the graph of a continuous function $y = f(x)$, $a \leq x \leq b$ is

$$V = \int_a^b 2\pi[\text{radius}] \cdot [\text{shell height}] dx = \int_a^b 2\pi x f(x) dx$$

Similarly,

The Shell Method (about the x -axis)

The volume of the solid generated by revolving about the x -axis the region between the y -axis and the graph of a continuous function $x = f(y)$, $c \leq y \leq d$ is

$$V = \int_c^d 2\pi[\text{radius}] \cdot [\text{shell height}] dy = \int_c^d 2\pi y f(y) dy$$

Comment: An easy way to remember which method to use to find the volume of a solid of revolution is to note that the Disc / Washer method is used if the independent variable of the function(s) and the axis of rotation is the same (e.g., the area under $y = f(x)$, revolved about the x -axis); while the Shell method should be used if the independent variable is different from the axis of rotation (e.g., the area under $x = f(y)$, revolved about the x -axis).

Ex. Find the volume of the solid generated by revolving about the y -axis the region bounded by the parabola $y = 2 - \frac{x^2}{4}$ and the x -axis, $0 \leq x \leq 2$.

$$\begin{aligned} V &= \int_0^2 2\pi x \left(2 - \frac{x^2}{4} \right) dx = 2\pi \int_0^2 \left(2x - \frac{x^3}{4} \right) dx = 2\pi \left(x^2 - \frac{x^4}{16} \right) \Big|_0^2 \\ &= 2\pi \left(\left(2^2 - \frac{2^4}{16} \right) - (0 - 0) \right) = 2\pi \left(4 - \frac{16}{16} \right) = 2\pi(3) = 6\pi \end{aligned}$$

Ex. Find the volume of the solid generated by revolving about the x -axis the region bounded by the parabola $x = y^2$ and the x -axis, $0 \leq y \leq \sqrt{2}$.

The height of the shell (= right curve - left curve) is $2 - y^2$.

$$V = \int_0^{\sqrt{2}} 2\pi y(2 - y^2) dy = 2\pi \int_0^{\sqrt{2}} (2y - y^3) dy = 2\pi \left(y^2 - \frac{y^4}{4} \right) \Big|_0^{\sqrt{2}} = 2\pi \left(2 - \frac{4}{4} - 0 + 0 \right) = 2\pi$$

Bonus example 1: Find the above volume using the Disc method instead.

We will first need to rewrite the function in terms of x : $y = \sqrt{x}$; the limits of integration are now from $x = 0$ to $x = 2$ (since $x = y^2$, as y goes from 0 to $\sqrt{2}$, x goes from 0 to 2). Hence,

$$V = \int_0^2 \pi(\sqrt{x})^2 dx = \pi \int_0^2 x dx = \frac{\pi}{2} x^2 \Big|_0^2 = \frac{\pi}{2} (2^2 - 0^2) = \frac{4\pi}{2} = 2\pi$$

Revolving about an axis the region between 2 curves

The version of Shell method, analogous to the Washer method, to find the volume of a solid generated by revolving the area between 2 curves about an axis of rotation is:

(About the y -axis)

The volume of the solid generated by revolving about the y -axis the region between the graphs of continuous functions $y = F(x)$ and $y = f(x)$, $F(x) \geq f(x)$, $a \leq x \leq b$ is

$$V = \int_a^b 2\pi[\text{radius}] \cdot [\text{shell height}] dx = \int_a^b 2\pi x[F(x) - f(x)] dx$$

Notice the shell height is now just the difference of the heights of the 2 curves.

Similarly, the volume of the solid generated by revolving about the x -axis the region between the graphs of continuous functions $x = F(y)$ and $x = f(y)$, $F(y) \geq f(y)$, $c \leq y \leq d$ is

(About the x -axis)

$$V = \int_c^d 2\pi[\text{radius}] \cdot [\text{shell height}] dy = \int_c^d 2\pi y[F(y) - f(y)] dy$$

Ex. Find the volume of the solid generated by revolving the first quadrant region bounded by $y = 8 - x^2$, $y = 2x$, and the y -axis, about the y -axis.

The upper curve is the parabola $y = 8 - x^2$, and $y = 2x$ is the lower curve. Their first quadrant intersection is at $(2, 4)$, therefore,

$$\begin{aligned} V &= \int_0^2 2\pi x((8 - x^2) - 2x) dx = 2\pi \int_0^2 x(8 - 2x - x^2) dx = 2\pi \int_0^2 (8x - 2x^2 - x^3) dx \\ &= 2\pi \left(4x^2 - \frac{2}{3}x^3 - \frac{x^4}{4} \right) \Big|_0^2 = 2\pi \left(16 - \frac{16}{3} - \frac{16}{4} \right) - 0 = 2\pi \left(\frac{20}{3} \right) = \frac{40\pi}{3} \end{aligned}$$

Revolving about a line other than the x - or y -axis

If the axis of rotation is not one of the two coordinate-axes, but rather an arbitrary horizontal or vertical line, the volume can be similarly calculated, with some slight adjustments.

Ex. Find the volume of the solid generated by revolving the region bounded by $y = x^3$, $y = 0$, and $x = 2$, about the line $x = 3$.

The axis of rotation, $x = 3$, is a line parallel to the y -axis, therefore, the shell method is to be used. The height of the shell is $f(x) = x^3$, $0 \leq x \leq 2$; and the radius is $3 - x$ (as measured from the axis of rotation: when $x = 0$, $r = 3$, and when $x = 2$, $r = 1$). Hence,

$$\begin{aligned} V &= \int_0^2 2\pi[\text{radius}] \cdot [\text{shell height}] dx = \int_0^2 2\pi(3-x)x^3 dx = 2\pi \int_0^2 (3x^3 - x^4) dx \\ &= 2\pi \left(\frac{3}{4}x^4 - \frac{x^5}{5} \right) \Big|_0^2 = 2\pi \left(12 - \frac{32}{5} \right) - 0 = 2\pi \left(\frac{28}{5} \right) = \frac{56\pi}{5} \end{aligned}$$

We could also find this volume using the washer method. First the function needs to be rewritten in terms of y : $x = f(y) = y^{1/3}$, $0 \leq y \leq 8$. The washer has an outer radius of $R = 3 - y^{1/3}$ (as measured from the axis of rotation $x = 3$ to the *farther away* of two curves; which is the curve on the left); and the inner radius is $r = 1$ (the distance between the line $x = 2$ to the axis of rotation). Hence,

$$\begin{aligned} V &= \int_0^8 \pi([R(y)]^2 - [r(y)]^2) dy = \pi \int_0^8 ([3 - y^{1/3}]^2 - 1^2) dy = \pi \int_0^8 (9 - 6y^{1/3} + y^{2/3} - 1) dy \\ &= \pi \int_0^8 (8 - 6y^{1/3} + y^{2/3}) dy = \pi \left(8y - \frac{9}{2}y^{4/3} - \frac{3}{5}y^{5/3} \right) \Big|_0^8 = \pi \left(64 - 72 + \frac{96}{5} \right) - 0 = \pi \left(\frac{-40 + 96}{5} \right) = \frac{56\pi}{5} \end{aligned}$$

Ex. Find the volume of the solid generated by revolving the region bounded by $y = x^2$, $y = 0$, $x = -1$, and $x = 1$, about the line $x = 2$.

The axis of rotation, $x = 2$, is a line parallel to the y -axis, therefore, the shell method is to be used. The height of the shell is $f(x) = x^2$, $-1 \leq x \leq 1$; and the radius is $2 - x$ (as measured from the axis of rotation: when $x = -1$, $r = 3$, and when $x = 1$, $r = 1$). Hence,

$$V = \int_{-1}^1 2\pi(2-x)x^2 dx = 2\pi \int_{-1}^1 (2x^2 - x^3) dx = 2\pi \left(\frac{2}{3}x^3 - \frac{x^4}{4} \right) \Big|_{-1}^1 = 2\pi \left(\left(\frac{2}{3} - \frac{1}{4} \right) - \left(-\frac{2}{3} - \frac{1}{4} \right) \right) = \frac{8\pi}{3}$$

Since the distance between any point x to the line $x = k$ along the x -axis is $|x - k|$, we have

(Shell Method, about the line $x = k$, i.e., a line parallel to the y -axis)

The volume of the solid generated by revolving about the line $x = k$ the region between the graphs of continuous functions $y = F(x)$ and $y = f(x)$, $F(x) \geq f(x)$, $a \leq x \leq b$, k not between a and b , is

$$V = \int_a^b 2\pi[\text{radius}] \cdot [\text{shell height}] dx = \int_a^b 2\pi|x - k|[F(x) - f(x)] dx$$

Note that we are integrating with respect to x , since the axis of rotation is parallel to the y -axis.

Similarly,

(About the line $y = k$, k not between c and d)

$$V = \int_c^d 2\pi[\text{radius}] \cdot [\text{shell height}] dy = \int_c^d 2\pi|y - k|[F(y) - f(y)] dy$$

The axis of rotation is parallel to the x -axis, so the integration is done with respect to y .