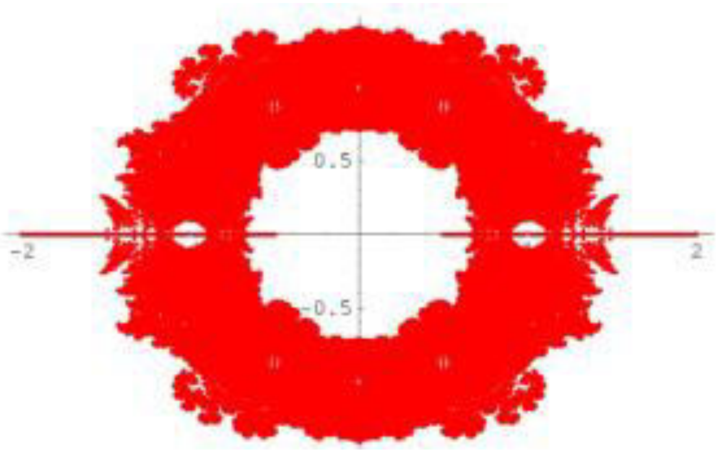
Maths Investigation Ideas for A-level, IB and Gifted GCSE Students



All this content taken from my site at www.ibmathsresources.com – you might find it easier to follow hyperlinks from there. I thought I'd put a selection of the posts I've made over the past year into a word document – these are all related to maths investigations or enrichment ideas. Some are suitable for top set students across the year groups, others may only be suitable for sixth form students. However, whatever level you teach there'll definitely be something of use!

Index:

Maths Exploration Topics: 100+ ideas for investigations.

Algebra and number

1) Modular arithmetic - This technique is used throughout Number Theory. For example, Mod 3 means the remainder when dividing by 3.
2) Goldbach's conjecture: "Every even number greater than 2 can be expressed as the sum of two primes." One of the great unsolved problems in mathematics.
3) Probabilistic number theory
4) Applications of complex numbers: The stunning graphics of Mandelbrot and Julia Sets are generated by complex numbers.
5) Diophantine equations: These are polynomials which have integer solutions. Fermat's Last Theorem is one of the most famous such equations.
6) Continued fractions: These are fractions which continue to infinity. The great Indian mathematician Ramanujan discovered some amazing examples of these.
7) Patterns in Pascal's triangle: There are a large number of patterns to discover - including the Fibonacci sequence.
8) Finding prime numbers: The search for prime numbers and the twin prime conjecture are some of the most important problems in mathematics. There is a $1 million prize for solving the Riemann Hypothesis and $250,000 available for anyone who discovers a new, really big prime number.
9) Random numbers
10) Pythagorean triples: A great introduction into number theory - investigating the solutions of Pythagoras' Theorem which are integers (eg. 3,4,5 triangle).
11) Mersenne primes: These are primes that can be written as 2^n -1.
12) Magic squares and cubes: Investigate magic tricks that use mathematics. Why do magic squares work?
13) Loci and complex numbers
14) Egyptian fractions: Egyptian fractions can only have a numerator of 1 - which leads to some interesting patterns. 2/3 could be written as 1/6 + 1/2. Can all fractions with a numerator of 2 be written as 2 Egyptian fractions?
15) Complex numbers and transformations
16) Euler's identity: An equation that has been voted the most beautiful equation of all time, Euler's identity links together 5 of the most important numbers in mathematics.
17) Chinese remainder theorem. This is a puzzle that was posed over 1500 years ago by a Chinese mathematician. It involves understanding the modulo operation.
18) Fermat's last theorem: A problem that puzzled mathematicians for centuries - and one that has only recently been solved.
19) Natural logarithms of complex numbers
20) Twin primes problem: The question as to whether there are patterns in the primes has fascinated mathematicians for centuries. The twin prime conjecture states that there are infinitely many consecutive primes (eg. 5 and 7 are consecutive primes). There has been a recent breakthrough in this problem.
21) Hypercomplex numbers

22) Diophantine application: Cole numbers
23) Odd perfect numbers: Perfect numbers are the sum of their factors (apart from the last factor). ie 6 is a perfect number because 1 + 2 + 3 = 6.
24) Euclidean algorithm for GCF
25) Palindrome numbers: Palindrome numbers are the same backwards as forwards.
26) Fermat's little theorem: If p is a prime number then a^p - a is a multiple of p.
27) Prime number sieves
28) Recurrence expressions for phi (golden ratio): Phi appears with remarkable consistency in nature and appears to shape our understanding of beauty and symmetry.
29) The Riemann Hypothesis - one of the greatest unsolved problems in mathematics - worth $1million to anyone who solves it (not for the faint hearted!)
30) Time travel to the future: Investigate how traveling close to the speed of light allows people to travel "forward" in time relative to someone on Earth. Why does the twin paradox work?
31) Graham's Number - a number so big that thinking about it could literally collapse your brain into a black hole.
32) RSA code - the most important code in the world? How all our digital communications are kept safe through the properties of primes.
33) The Chinese Remainder Theorem: This is a method developed by a Chinese mathematician Sun Zi over 1500 years ago to solve a numerical puzzle. An interesting insight into the mathematical field of Number Theory.
34) Cesaro Summation: Does 1 – 1 + 1 – 1 … = 1/2?. A post which looks at the maths behind this particularly troublesome series.
35) Fermat's Theorem on the sum of 2 squares - An example of how to use mathematical proof to solve problems in number theory.
36) Can we prove that 1 + 2 + 3 + 4 .... = -1/12 ? How strange things happen when we start to manipulate divergent series.
37) Mathematical proof and paradox - a good opportunity to explore some methods of proof and to show how logical errors occur.

Geometry

1) Non-Euclidean geometries: This allows us to "break" the rules of conventional geometry - for example, angles in a triangle no longer add up to 180 degrees.
2) Hexaflexagons: These are origami style shapes that through folding can reveal extra faces.
3) Minimal surfaces and soap bubbles: Soap bubbles assume the minimum possible surface area to contain a given volume.
4) Tesseract – a 4D cube: How we can use maths to imagine higher dimensions.
5) Stacking cannon balls: An investigation into the patterns formed from stacking canon balls in different ways.
6) Mandelbrot set and fractal shapes: Explore the world of infinitely generated pictures and fractional dimensions.
7) Sierpinksi triangle: a fractal design that continues forever.
8) Squaring the circle: This is a puzzle from ancient times - which was to find out whether a square could be created that had the same area as a given circle. It is now used as a saying to represent something impossible.
9) Polyominoes: These are shapes made from squares. The challenge is to see how many different shapes can be made with a given number of squares - and how can they fit together?
10) Tangrams: Investigate how many different ways different size shapes can be fitted together.
11) Understanding the fourth dimension: How we can use mathematics to imagine (and test for) extra dimensions.
12) The Riemann Sphere - an exploration of some non-Euclidean geometry. Straight lines are not straight, parallel lines meet and angles in a triangle don't add up to 180 degrees.

Calculus/analysis and functions

1) The harmonic series: Investigate the relationship between fractions and music, or investigate whether this series converges.
2) Torus – solid of revolution: A torus is a donut shape which introduces some interesting topological ideas.
3) Projectile motion: Studying the motion of projectiles like cannon balls is an essential part of the mathematics of war. You can also model everything from Angry Birds to stunt bike jumping. A good use of your calculus skills.
4) Why e is base of natural logarithm function: A chance to investigate the amazing number e.

Statistics and modelling

1) Traffic flow: How maths can model traffic on the roads.
2) Logistic function and constrained growth
3) Benford's Law - using statistics to catch criminals by making use of a surprising distribution.
4) Bad maths in court - how a misuse of statistics in the courtroom can lead to devastating miscarriages of justice.
5) The mathematics of cons - how con artists use pyramid schemes to get rich quick.
6) Impact Earth - what would happen if an asteroid or meteorite hit the Earth?
7) Black Swan events - how usefully can mathematics predict small probability high impact events?
8) Modelling happiness - how understanding utility value can make you happier.
9) Does finger length predict mathematical ability? Investigate the surprising correlation between finger ratios and all sorts of abilities and traits.
10) Modelling epidemics/spread of a virus
11) The Monty Hall problem - this video will show why statistics often lead you to unintuitive results.
12) Monte Carlo simulations
13) Lotteries
14) Bayes' theorem: How understanding probability is essential to our legal system.
15) Birthday paradox: The birthday paradox shows how intuitive ideas on probability can often be wrong. How many people need to be in a room for it to be at least 50% likely that two people will share the same birthday? Find out!
16) Are we living in a computer simulation? Look at the Bayesian logic behind the argument that we are living in a computer simulation.
17) Does sacking a football manager affect results? A chance to look at some statistics with surprising results.
18) Which times tables do students find most difficult? A good example of how to conduct a statistical investigation in mathematics.

Games and game theory

1) The prisoner's dilemma: The use of game theory in psychology and economics.
2) Sudoku
3) Gambler's fallacy: A good chance to investigate misconceptions in probability and probabilities in gambling. Why does the house always win?
4) Bluffing in Poker: How probability and game theory can be used to explore the the best strategies for bluffing in poker.
5) Knight's tour in chess: This chess puzzle asks how many moves a knight must make to visit all squares on a chess board.
6) Billiards and snooker
7) Zero sum games
8) How to "Solve" Noughts and Crossess (Tic Tac Toe) - using game theory. This topics provides a fascinating introduction to both combinatorial Game Theory and Group Theory.

9) Maths and football - Do managerial sackings really lead to an improvement in results? We can analyse the data to find out. Also look at the finances behind Premier league teams

Topology and networks

1) Knots
2) Steiner problem
3) Chinese postman problem
4) Travelling salesman problem
5) Königsberg bridge problem: The use of networks to solve problems. This particular problem was solved by Euler.
6) Handshake problem: With n people in a room, how many handshakes are required so that everyone shakes hands with everyone else?
7) Möbius strip: An amazing shape which is a loop with only 1 side and 1 edge.
8) Klein bottle
9) Logic and sets
10) Codes and ciphers: ISBN codes and credit card codes are just some examples of how codes are essential to modern life. Maths can be used to both make these codes and break them.
11) Zeno's paradox of Achilles and the tortoise: How can a running Achilles ever catch the tortoise if in the time taken to halve the distance, the tortoise has moved yet further away?
12) Four colour map theorem - a puzzle that requires that a map can be coloured in so that every neighbouring country is in a different colour. What is the minimum number of colours needed for any map?

Further ideas:

1) Radiocarbon dating - understanding radioactive decay allows scientists and historians to accurately work out something's age - whether it be from thousands or even millions of years ago.
2) Gravity, orbits and escape velocity - Escape velocity is the speed required to break free from a body's gravitational pull. Essential knowledge for future astronauts.
3) Mathematical methods in economics - maths is essential in both business and economics - explore some economics based maths problems.
4) Genetics - Look at the mathematics behind genetic inheritance and natural selection.
5) Elliptical orbits - Planets and comets have elliptical orbits as they are influenced by the gravitational pull of other bodies in space. Investigate some rocket science!
6) Logarithmic scales – Decibel, Richter, etc. are examples of log scales - investigate how these scales are used and what they mean.
7) Fibonacci sequence and spirals in nature - There are lots of examples of the Fibonacci sequence in real life - from pine cones to petals to modelling populations and the stock market.
8) Change in a person's BMI over time - There are lots of examples of BMI stats investigations online - see if you can think of an interesting twist.
9) Designing bridges - Mathematics is essential for engineers such as bridge builders - investigate how to design structures that carry weight without collapse.
10) Mathematical card tricks - investigate some maths magic.
Voting systems
11) Flatland by Edwin Abbott - This famous book helps understand how to imagine extra dimension. You can watch a short video on it here
12) Towers of Hanoi puzzle - This famous puzzle requires logic and patience. Can you find the pattern behind it?
13) Different number systems - Learn how to add, subtract, multiply and divide in Binary. Investigate how binary is used - link to codes and computing.
14) Methods for solving differential equations - Differential equations are amazingly powerful at modelling real life - from population growth to to pendulum motion. Investigate how to solve them.
15) Modelling epidemics/spread of a virus - what is the mathematics behind understanding how epidemics

occur?
16) Hyperbolic functions - These are linked to the normal trigonometric functions but with notable differences. They are useful for modelling more complex shapes.

Statistics and Probability Investigations

Primary or Secondary data?

The main benefit of primary data is that you can really personalise your investigation. It allows you scope to investigate something that perhaps no-one else has ever done. It also allows you the ability to generate data that you might not be able to find online. The main drawback is that collecting good quality data in sufficient quantity to analyze can be time consuming. You should aim for an absolute minimum of 50 pieces of data – and ideally 60-100 to give yourself a good amount of data to look at.

The benefits of secondary data are that you can gain access to good quality raw data on topics that you wouldn't be able to collect data on personally – and it's also much quicker to get the data. Potential drawbacks are not being able to find the raw data that fits what you want to investigate – or sometimes having too much data to wade through.

Secondary data sources:

1) The Census at School website is a fantastic source of secondary data to use. If you go to the random data generator you can download up to 200 questionnaire results from school children around the world on a number of topics (each year's questionnaire has up to 20 different questions). Simply fill in your email address and the name of your school and then follow the instructions.

2) If you're interested in sports statistics then the Olympic Database is a great resource. It contains an enormous amount of data on winning times and distances in all events in all Olympics. Follow links at the top of the page to similar databases on basketball, golf, baseball and American football.

3) If you prefer football, the the Guardian stats centre has information on all European leagues – you can see when a particular team scores most of their goals, how many goals they score a game, how many red cards they average etc. You can also find a lot of football stats on the Who Scored website. This gives you data on things like individual players' shots per game, pass completion rate etc.

4) The Guardian Datablog has over 800 data files to view or download – everything from the Premier League football accounts of clubs to a list of every Dr Who villain, US gun crime, UK unemployment figures, UK GCSE results by gender, average pocket money and most popular baby names. You will need to sign into Google to download the files.

5) The World Bank has a huge data bank - which you can search by country or by specific topic. You can compare life-expectancy rates, GDP, access to secondary education, spending on military, social inequality, how many cars per 1000 people and much much more.

6) Gapminder is another great resource for comparing development indicators – you can plot 2 variables on a graph (for example urbanisation against unemployment, or murder rates against urbanisation) and then run them over a number of years. You can also download Excel speadsheets of the associated data.

7) Wolfram Alpha is one of the most powerful maths and statistics tools available – it has a staggering amount of information that you can use. If you go to the examples link above, then you can choose from data on everything from astronomy, the human body, geography, food nutrition, sports, socioeconomics, education and shopping.

Example Maths Studies IA Investigations:

Some of these ideas taken from the excellent Oxford IB Maths Studies textbook.

Correlations:

1) Is there a correlation between hours of sleep and exam grades?
Studies have shown that a good night's sleep raises academic attainment.
2) Is there a correlation between height and weight?
The NHS use a chart to decide what someone should weigh depending on their height. Does this mean that height is a good indicator of weight?
3) Is there a correlation between arm span and foot height?
This is also a potential opportunity to discuss the Golden Ratio in nature.
4) Is there a correlation between the digit ratio and maths ability?
Studies show there is a correlation between digit ratio and everything from academic ability, aggression and even sexuality.
5) Is there a correlation between smoking and lung capacity?
6) Is there a correlation between GDP and life expectancy?
Run the Gapminder graph to show the changing relationship between GDP and life expectancy over the past few decades.
7) Is there a correlation between numbers of yellow cards a game and league position?
Use the Guardian Stats data to find out if teams which commit the most fouls also do the best in the league.
8) Is there a correlation between Olympic 100m sprint times and Olympic 15000m times?
Use the Olympic database to find out if the 1500m times have go faster in the same way the 100m times have got quicker over the past few decades.
9) Is there a correlation between sacking a football manager and improved results?
A recent study suggests that sacking a manager has no benefit and the perceived improvement in results is just regression to the mean.
10) Is there a correlation between time taken getting to school and the distance a student lives from school?
11) Does eating breakfast affect your grades?
12) Is there a correlation between stock prices of different companies?
Use Google Finance to collect data on company share prices.
13) Does teenage drinking affect grades?
A recent study suggests that higher alcohol consumption amongst teenagers leads to greater social stress and poorer grades.
14) Is there a correlation between unemployment rates and crime?
If there are less work opportunities, do more people turn to crime?
15) Is there a correlation between female participation in politics and wider access to further education?
16) Is there a correlation between blood alcohol laws and traffic accidents?
17) Is there a correlation between height and basketball ability?
18) Is there a correlation between stress and blood pressure?

Normal distributions:

1) Are a sample of student heights normally distributed?
We know that adult population heights are normally distributed – what about student heights?
2) Are a sample of flower heights normally distributed?
3) Are a sample of student weights normally distributed?
4) Are a sample of student reaction times normally distributed?
Conduct this BBC reaction time test to find out.
5) Are a sample of student digit ratios normally distributed?
6) Are the IB maths test scores normally distributed?
IB test scores are designed to fit a bell curve. Investigate how the scores from different IB subjects compare.
7) Are the weights of "1kg" bags of sugar normally distributed?

Other statistical investigations

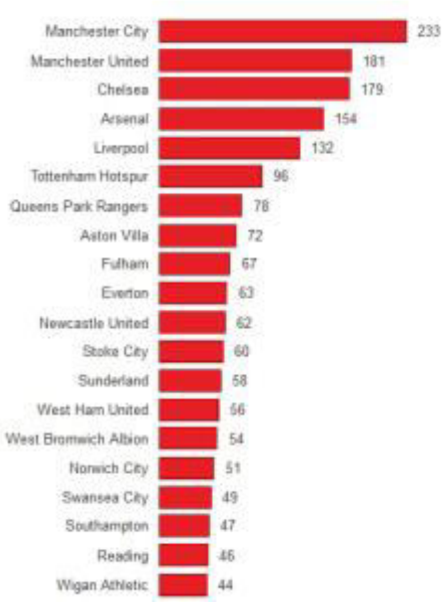
1) Does gender affect hours playing sport?
A UK study showed that primary school girls play much less sport than boys.
2) Investigation into the distribution of word lengths in different languages.
The English language has an average word length of 5.1 words. How does that compare with other languages?
3) Do bilingual students have a greater memory recall than non-bilingual students?
Studies have shown that bilingual students have better "working memory" – does this include memory recall?
4) Investigation about the distribution of sweets in packets of Smarties. A chance to buy lots of sweets! Also you could link this with some optimisation investigation.

Probability and statistics

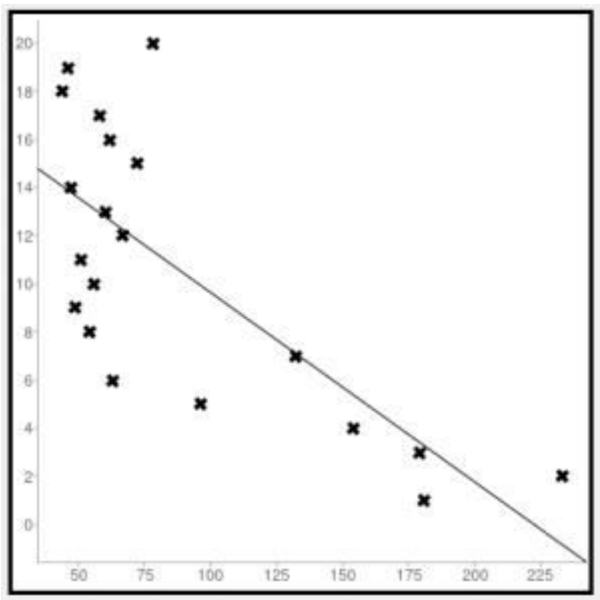
1) The probability behind poker games
2) Finding expected values for games of chance in a casino.
3) Birthday paradox:
The birthday paradox shows how intuitive ideas on probability can often be wrong. How many people need to be in a room for it to be at least 50% likely that two people will share the same birthday? Find out!
4) Which times tables do students find most difficult?
A good example of how to conduct a statistical investigation in mathematics.
5) Handshake problem
With n people in a room, how many handshakes are required so that everyone shakes hands with everyone else?

Statistics Topics

Is there a correlation between Premier League wages and league position?



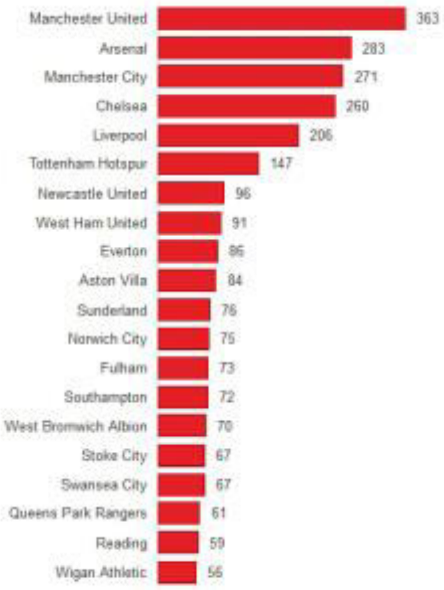
The Guardian has just released its 2012-13 Premier League season data analysis - which shows exactly how much each club in the Premier League spent on wages last year (see the bar chart above). This can be easily plotted on a scatter graph to test how strong the correlation is between spending and league position. (y axis is league position, x axis is wage bill in millions of pounds).



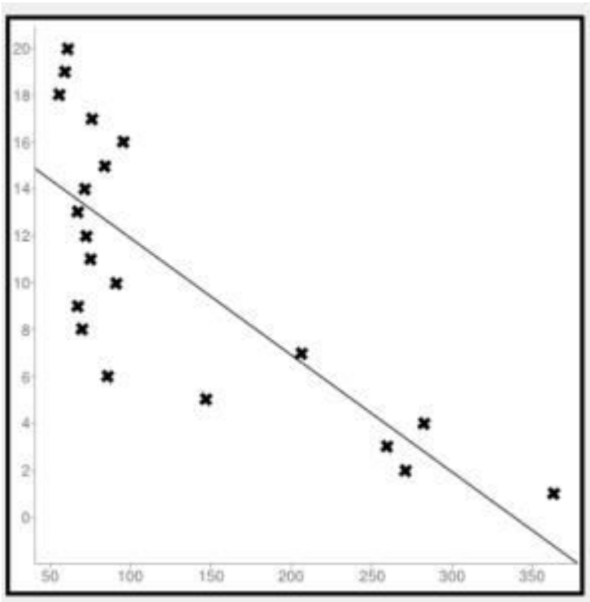
The mean spending on wages is 89 million pounds. Our regression line is y = -0.08x + 17.52. We can see some of the big outliers are QPR (with a big wage bill but low premier league position) and Everton (with a low wage bill relative to others who finished in a similar position).

The Pearson's product moment correlation coefficient (r) is -0.73. This is negative because in our case league position is numerically lower the higher up the league you are. This shows a pretty strong correlation between league spending and league position. An r value of -1 would be a perfect correlation in our case, whereas 0 would be no correlation.

Is there a correlation between turnover and league position?



We can also see what the correlation is between league position and overall club turnover (see the bar chart above). Here we can see there is a huge gulf between the top few clubs and everyone else in the league. There's only 40 million pounds difference between the bottom ranked club for revenue Wigan and Newcastle, with the 7th biggest revenue. But then a massive jump up to those with the top 6 revenues.



This time we have a mean turnover of 128 million pounds and a regression line of y = -0.05x + 16.89. The Pearson's r value this time is r = -0.79, so there is a slightly stronger correlation than from wages - and this is a strong correlation overall. So, both wage bills and turnover provide a pretty good predictor of where a team will finish - and also a decent yardstick to measure how well a team has done relative to their resources.

The Mathematics of Bluffing

This post is based on the fantastic [PlusMaths article on bluffing]- which is a great introduction to this topic. If you're interested then it's well worth a read. This topic shows the power of mathematics in solving real world problems - and combines a wide variety of ideas and methods - probability, Game Theory, calculus, psychology and graphical analysis.

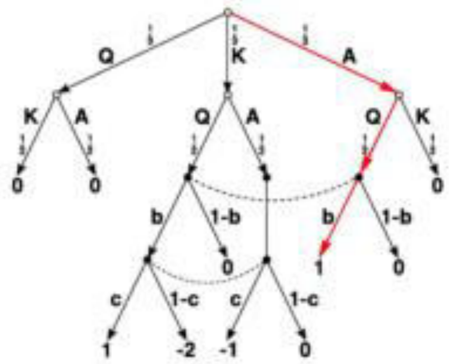
You would probably expect that there is no underlying mathematical strategy for good bluffing in poker - indeed that a good bluffing strategy would be completely random so that other players are unable to spot when a bluff occurs. However it turns out that this is not the case.

As explained by John Billingham in the PlusMaths article, when considering this topic it helps to really simplify things first. So rather than a full poker game we instead consider a game with only 2 players and only 3 cards in the deck (1 Ace, 1 King, 1 Queen).

The game then plays as follows:
1) Both players pay an initial £1 into the pot.
2) The cards are dealt - with each player receiving 1 card.
3) Player 1 looks at his card and can:
(a) check
(b) bet an additional £1
4) Player 2 then can respond:
a) If Player 1 has checked, Player 2 must also check. This means both cards are turned over and the highest card wins.
b) If Player 1 has bet £1 then Player 2 can either match (call) that £1 bet or fold. If the bets are matched then the cards are turned over and the highest card wins.

So, given this game what should the optimal strategy be for Player 1? An Ace will always win a showdown, and a Queen always lose - but if you have a Queen and bet, then your opponent who may only have a King might decide to fold thinking you actually have an Ace.

In fact the optimal strategy makes use of Game Theory - which can mathematically work out exactly how often you should bluff:



This tree diagram represents all the possible outcomes of the game. The first branch at the top represents the 3 possible cards that Player 2 can be dealt (A,K,Q) each of which have a probability of 1/3. The second branch represents the remaining 2 possible cards that Player 1 has - each with probability 1/2. The numbers at the bottom of the branches represent the potential gain or loss from betting strategies for Player 2 - this is calculated by comparing the profit/loss relative to if both players had simply shown their cards at the beginning of the game.

For example, Player 2 has no way of winning any money with a Queen - and this is represented by the left branch £0, £0. Player 2 will always win with an Ace. If Player 1 has a Queen and bluffs then Player 2 will call the bet and so will have gained an additional £1 of his opponents money relative to a an initial game showdown (represented by the red branch). Player 1 will always check with a King (as were he to bet then Player 2 would always call with an Ace and fold with a Queen) and so the AK branch also has a £0 outcome relative to an initial showdown.

So, the only decisions the game boils down to are:

1) Should Player 1 bluff with a Queen? (Represented with a probability of b on the tree diagram)
2) Should Player 2 call with a King? (Represented with a probability of c on the tree diagram).

Now it's simply a case of adding the separate branches of the tree diagram to find the expected value for Player 2.

The right hand branch (for AQ and AK) for example gives:

1/3 . 1/2 . b . 1
1/3 . 1/2 . (1-b) . 0
1/3 . 1/2 . 0

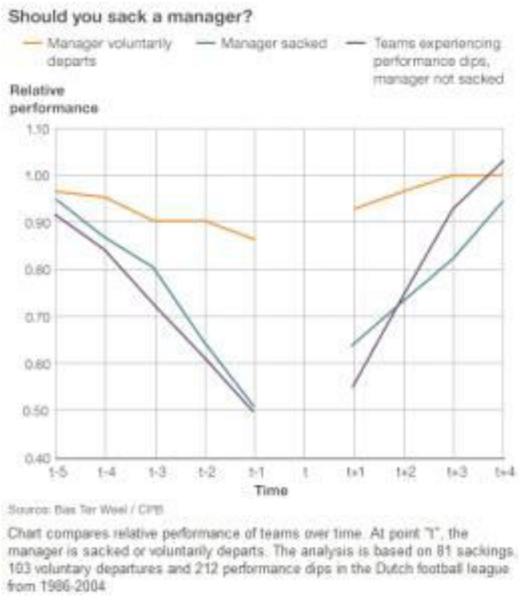
So, working out all branches gives:

Expected Value for Player 2 = 0.5b(c-1/3) - c/6
Expected Value for Player 1 = -0.5b(c-1/3) + c/6

(Player 1's Expected Value is simply the negative of Player 2's. This is because if Player 2 wins £1 then Player 1 must have lost £1). The question is what value of b (Player 1 bluff) should be chosen by Player 1 to maximise his earnings? Equally, what is the value of c (Player 2 call) that maximises Player 2's earnings?

It is possible to analyse these equations numerically to find the optimal values (this method is explained in the article), and also to employ both graphical means or calculus to arrive at the optimum strategy of c = 1/3 and b = 1/3.

Does Sacking a Manager Affect Results?



In sports leagues around the world, managers are often only a few bad results away from the sack - but is this all down to a misunderstanding of statistics?

According to the Guardian, in the 21 year history of the Premier League, approximately 140 managers have been sacked. In more recent years the job is getting ever more precarious - 12 managers lost their jobs in 2013, and 20 managers in the top flight have been shown the door in the last 2 years. Indeed, there are now only three Premier League managers who have held their position for more than 2 years (Arsene Wenger, Sam Allardyce and Alan Pardew).

Owners appear attracted to the idea that a new manager can bring a sudden improvement in results - and indeed most casual observers of football would agree that new managers often seem to pull out some good initial results. But according to Dutch economist Dr Bas ter Weel this is just a case of regression to the mean - if a team has been underperforming relative to their abilities then over the long run we would expect them to improve to get closer to the mean value.

As the BBC reported:

"Changing a manager during a crisis in the season does improve the results in the short term," Dr Bas ter Weel says. "But this is a misleading statistic because not changing the manager would have had the same result."

Ter Weel analysed managerial turnover across 18 seasons (1986-2004) of the Dutch premier division, the Eredivisie. As well as looking at what happened to teams who sacked their manager when the going got tough, he looked at those who had faced a similar slump in form but who stood by their boss to ride out the crisis.

He found that both groups faced a similar pattern of declines and improvements in form.

By looking at the graph at the top of the page it is clear to see that sacking a manager may have appeared to lead to an improvement in results - but that actually had the manager not been sacked results would have been even better!

We can understand regression to the mean better by considering coin tosses as a crude model for football games (ignoring draws). If we get a head the team wins, if we get a tail the team loses. So this is a distinctly average team - which over a season we would expect to finish around mid-table. However over that season they will have "good runs" and "bad runs."

Coin Flips= 38 Heads= 20 Tails= 18

This graphic above is the result of 38 coin tosses (the length of a Premier League season). Even though it's the result of random throws you can see a run of 6 wins in a row - a good run. There's also a run of 8 defeats and only 2 wins in 10 games - which would have more than a few Chairman thinking about getting a new manager.

Being aware of regression to the mean - i.e that over the long term results tend towards the mean would help owners to have greater confidence in riding out "bad runs" - and maybe would keep a few more managers in their jobs.

The Birthday Problem

http://www.youtube.com/watch?v=a2ey9a70yY0

One version of the birthday problem is as follows:

How many people need to be in a room such that there is a greater than 50% chance that 2 people share the same birthday.

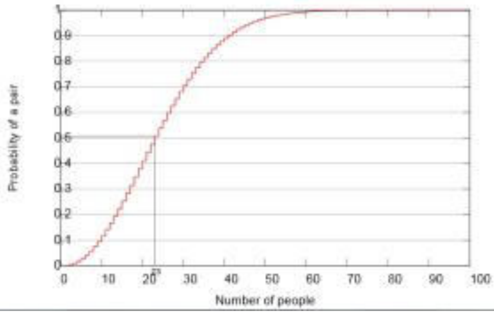
This is an interesting question as it shows that probabilities are often counter-intuitive. The answer is that you only need 23 people before you have a 50% chance that 2 of them share a birthday. So, why do you only need 23 people?

The key to understanding this question is realising that when comparing if any of the (n) people in the room share a birthday, you are not simply making n comparisons - but C(n,2) (n choose 2) comparisons. If there are people A, B, C and D in a room I don't just make 4 comparisons, I have to compare AB, AC, AD, BC, BD, CD. This is the same calculation as working out 4 choose 2 = 6 comparisons.

Therefore when there are 23 people in the room you actually need to make C(23,2) comparisons = 253. This goes someway to explaining why the number is much lower than you would expect - but it still doesn't tell us where the number 23 came from.

If we simplify things so that we don't have leap years then we can approximate the problem by working out the probability p(no shared birthday). When there are 2 people in the room the probability that person B does not share his birthday with person A is 364/365. When a third person enters the room the probability that C doesn't share his birthday with A or B is 363/365. Carrying on in this manner, when the 23rd person enters the room, the probability that he doesn't share a birthday with anyone already there is 343/365.

We then work out p(no shared birthday) = 364/365 x 363/365...x 343/365 = 0.4927
So p(shared birthday) = 1 - 0.49 = 0.51 (2 dp). Therefore when there are 23 people in the room the probability of a shared birthday is around 51%.



However, the method outlined above is a little unsatisfactory - as we start by using the fact that we know the answer is 23 and then work backwards. So how can we discover the result independently? One possibility is to use the Poisson approximation, P(λ):

$$\Pr(X = k) = \frac{\lambda^k e^{-\lambda}}{k!},$$

(where X is the number of shared birthdays). If we want the probability of at least 1 shared birthday then we can find 1 - P(X=0). When k = 0 the formula reduces to $e^{-\lambda}$. Therefore P(X>0) = 1 - $e^{-\lambda}$

So, if we want the probability to be greater than 0.5 we can then set up an inequality and solve:

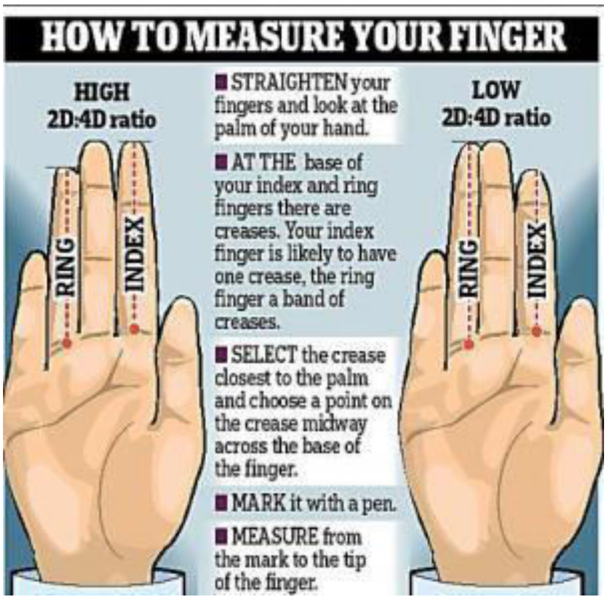
$1 - e^{-\lambda} > 0.5$
$0.5 > e^{-\lambda}$
$\ln(0.5) > -\lambda$

Next we use the fact that λ is the expected value - and so will be given by C(n,2)/365. This is because the number of potential birthday combinations divided by 365 will give us the mean number of shared birthdays. We also use the formula for C(n,r) which is n!/(k!(n-k)!). This gives C(n,2) = n(n-1)/2

$-\ln(0.5) < C(n,2)/365$
$253 < C(n,2)$
$253 < n(n-1)/2$
$0 < n^2 - n - 506$

Which we can solve using the quadratic formula to give n = 23 almost exactly.

Does Digit Ratio Predict Maths Ability?



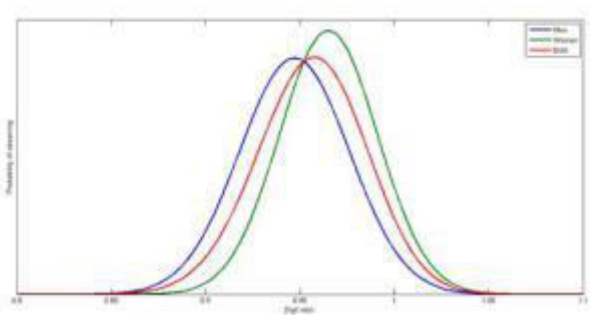
Some of the studies on the 2D: 4D finger ratios (as measured in the picture above) are interesting when considering what factors possibly affect mathematical ability. A 2007 study by Mark Brosnan from the University of Bath found that:

"Boys with the longest ring fingers relative to their index fingers tend to excel in math. The boys with the lowest ratios also were the ones whose abilities were most skewed in the direction of math rather than literacy.

With the girls, there was no correlation between finger ratio and numeracy, but those with higher ratios--presumably indicating low testosterone levels--had better scores on verbal abilities. The link, according to the researchers, is that testosterone levels in the womb influence both finger length and brain development.

In men, the ring (fourth) finger is usually longer than the index (second); their so-called 2D:4D ratio is lower than 1. In females, the two fingers are more likely to be the same length. Because of this sex difference, some scientists believe that a low ratio could be a marker for higher prenatal testosterone levels, although it's not clear how the hormone might influence finger development."

In the study, Brosnan photocopied the hands of 74 boys and girls aged 6 and 7. He worked out the 2D:4D finger ratio by dividing the length of the index finger (2D) with the length of the ring finger (4D). They then compared the finger ratios with standardised UK maths and English tests. The differences found were small, but significant.



Another study of 136 men and 137 women, looked at the link between finger ratio and aggression. The results are plotted in the graph above - which clearly show this data follows a normal distribution. The men are represented with the blue line, the women the green line and the overall cohort in red. You can see that the male distribution is shifted to the left as they have a lower mean ratio. (Males: mean 0.947, standard deviation 0.029, Females: mean 0.965, standard deviation 0.026).

The 95% confidence interval for average length is 0.889-1.005 for males and 0.913-1.017 for females. That means that 95% of the male and female populations will fall into these categories.

The correlation between digit ratio and everything from personality, sexuality, sporting ability and management has been studied. If a low 2D:4D ratio is indeed due to testosterone exposure in the womb (which is not confirmed), then that raises the question as to why testosterone exposure should affect mathematical ability. And if it is not connected to testosterone, then what is responsible for the correlation between digit ratios and mathematical talent?

I think this would make a really interesting Internal Assessment investigation at either Studies or Standard Level. Also it works well as a class investigation at KS3 and IGCSE into correlation and scatter diagrams. Does the relationship still hold for when you look at algebraic skills rather than numeracy? Or is algebraic talent distinct from numeracy talent?

A detailed academic discussion of the scientific literature on this topic is available here.

Amanda Knox and Bad Maths in Courts

This post is inspired by the recent BBC News article, "Amanda Knox and Bad Maths in Courts." The article highlights the importance of good mathematical understanding when handling probabilities - and how mistakes by judges and juries can sometimes lead to miscarriages of justice.

A scenario to give to students:

A murder scene is found with two types of blood - that of the victim and that of the murderer. As luck would have it, the unidentified blood has an incredibly rare blood disorder, only found in 1 in every million men. The capital and surrounding areas have a population of 20 million - and the police are sure the murderer is from the capital. The police have already started cataloging all citizens' blood types for their new super crime-database. They already have nearly 1 million male samples in there - and bingo - one man, Mr XY, is a match. He is promptly marched off to trial, there is no other evidence, but the jury are told that the odds are 1 in a million that he is innocent. He is duly convicted. The question is, how likely is it that he did not commit this crime?

Answer:

We can be around 90% confident that he did not commit this crime. Assuming that there are approximately 10 million men in the capital, then were everyone cataloged on the database we would have on average 10 positive matches. Given that there is no other evidence, it is therefore likely that he is only a 1 in 10 chance of being guilty. Even though P(Fail Test/Innocent) = 1/1,000,000, P(Innocent/Fail test) = 9/10.

Amanda Knox

Eighteen months ago, Amanda Knox and Raffaele Sollecito, who were previously convicted of the murder of British exchange student Meredith Kercher, were acquitted. The judge at the time ruled out re-testing a tiny DNA sample found at the scene, stating that, "The sum of the two results, both unreliable… cannot give a reliable result."

This logic however, whilst intuitive is not mathematically correct. As explained by mathematician Coralie Colmez in the BBC News article, by repeating relatively unreliable tests we can make them more reliable - the larger the pooled sample size, the more confident we can be in the result.

SALFORD UNI MAN SAYS SALLY CLARK CONVICTION MAY BE WRONG

Maths professor challenges double baby murder case

Sally Clark

One of the most (in)famous examples of bad maths in the court room is that of Sally Clark - who was convicted of the murder of her two sons in 1999. It has been described as, "one of the great miscarriages of justice in modern British legal history." Both of Sally Clark's children died from cot-death whilst still babies. Soon afterwards she was arrested for murder. The case was based on a seemingly incontrovertible statistic - that the chance of 2 children from the same family dying from cot-death was 1 in 73 million. Experts testified to this, the jury were suitably convinced and she was convicted.

The crux of the prosecutor's case was that it was so statistically unlikely that this had happened by chance, that she must have killed her children. However, this was bad maths - which led to an innocent woman being jailed for four years before her eventual acquittal.

Independent Events

The 1 in 73 million figure was arrived at by simply looking at the probability of a single cot-death (1 in 8500) and then squaring it - because it had happened twice. However, this method only works if both events are independent - and in this case they clearly weren't. Any biological or social factors which contribute to the death of a child due to cot-death will also mean that another sibling is also at elevated risk.

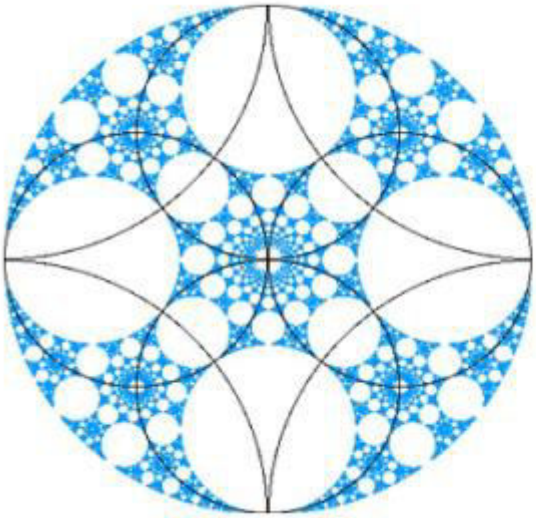
Prosecutor's Fallacy

Additionally this figure was presented in a way which is known as the "prosecutor's fallacy" - the 1 in 73 million figure (even if correct) didn't represent the probability of Sally Clark's innocence, because it should have been compared against the probability of guilt for a double homicide. In other words, the probability of a false positive is not the same as the probability of innocence. In mathematical language, P(Fail Test/Innocent) is not equal to P(Innocent/Fail test).

Subsequent analysis of the Sally Clark case by a mathematics professor concluded that rather than having a 1 in 73 million chance of being innocent, actually it was about 4-10 times more likely this was due to natural causes rather than murder. Quite a big turnaround - and evidence of why understanding statistics is so important in the courts.

Geometry Topics

Circular Inversion - Reflecting in a Circle

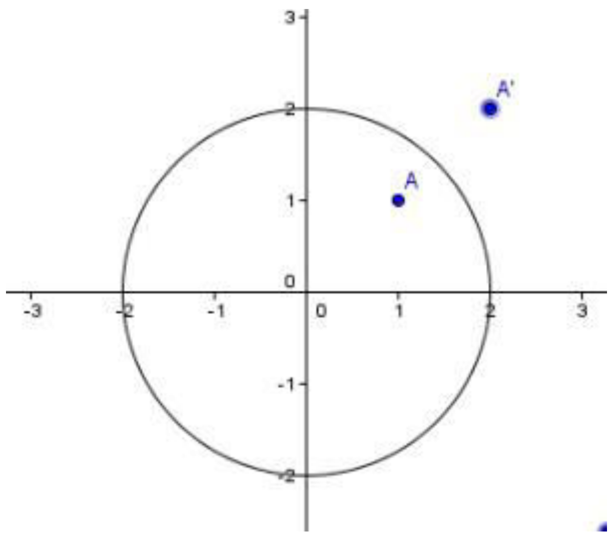


This topic is a great introduction to the idea of *mapping* - where one point is mapped to another. This is a really useful geometrical tool as it allows complex shapes to be transformed into isomorphic (equivalent) shapes which can sometimes be easier to understand and work with mathematically.

One example of a mapping is a circular inversion. The inversion rule maps a point P onto a point P' according to the rule:

$OP \times OP' = r^2$

To understand this, we start with a circle radius r centred on O. The inversion therefore means that the distance from O to P multiplied by the distance from O to P' will always give the constant value r^2



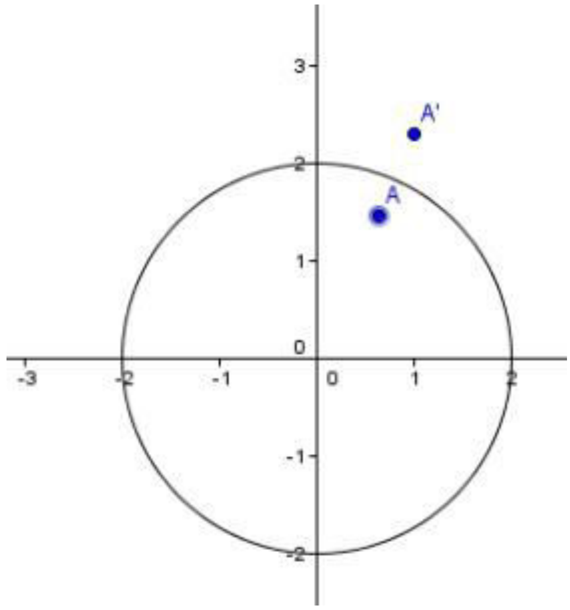
This is an example of the circular inversion of the point A to the point A'.

We have the distance of OA as √2 and the radius of the circle as 2. Therefore using the formula we can find OA' by:

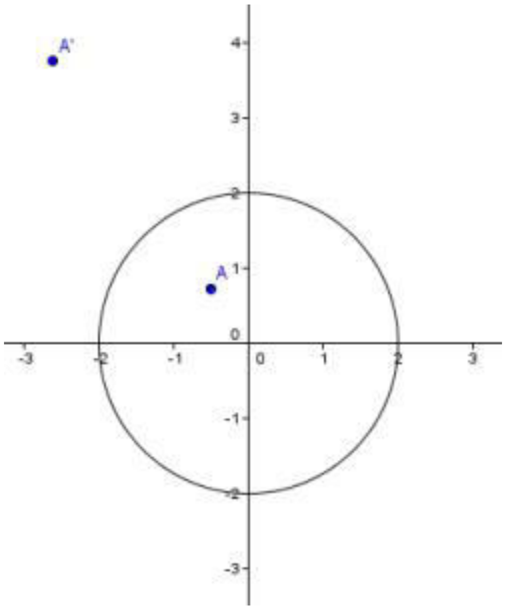
OA' = r^2 / OA = 4/√2

OA' = $2^{1.5}$. This means that the point A' is a distance of $2^{1.5}$ away from O on the same line as OA.

We can check that the Geogebra plot is correct - because this point A' is plotted at (2,2) - which is indeed (using Pythagoras) a distance of $2^{1.5}$ from O.

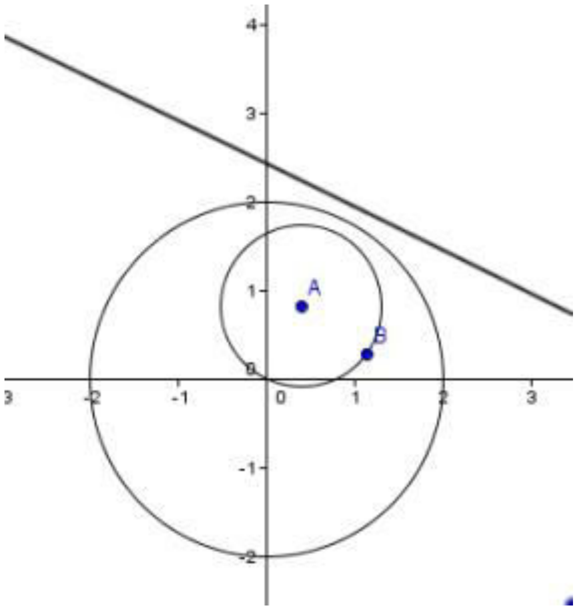


A point near to the edge of the circle will have an inversion also close to the circle

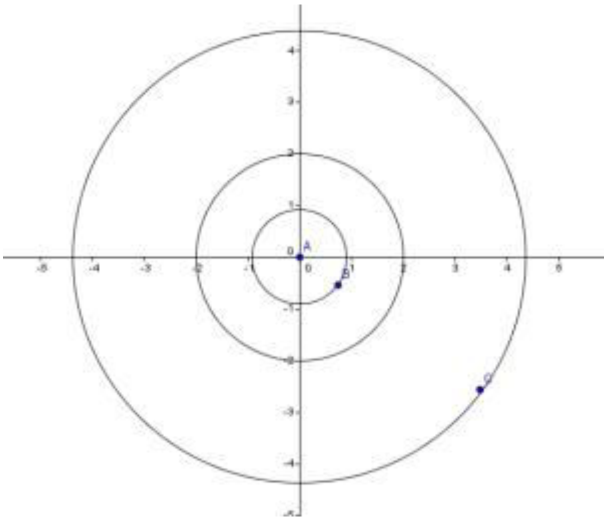


A point near to the centre will have an inversion a long way from the circle. The point (0,0) will be undefined as no point outside the circle will satisfy the inversion equation.

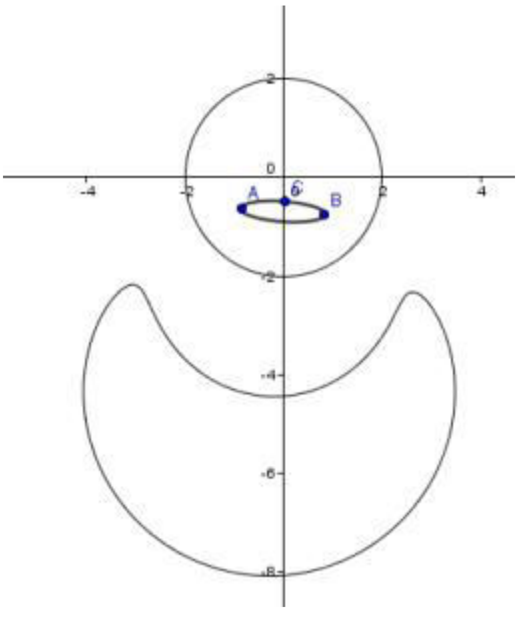
So, that is the basic idea behind circular inversion - though it gets a lot more interesting when we start inverting shapes rather than just points.



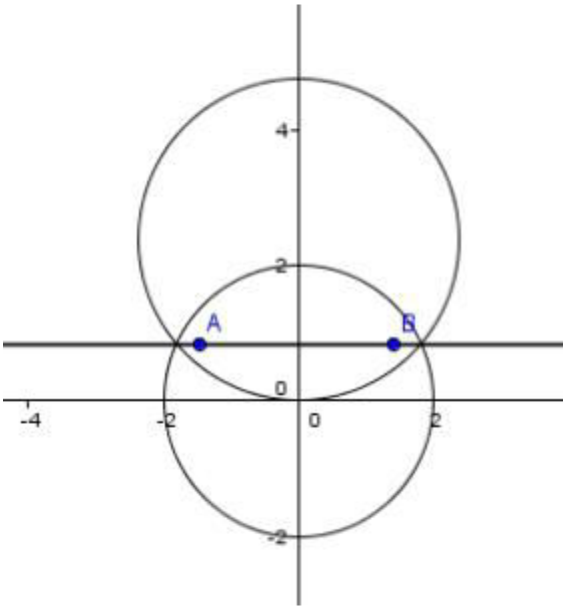
Circles through the origin map onto straight lines to infinity (see above).



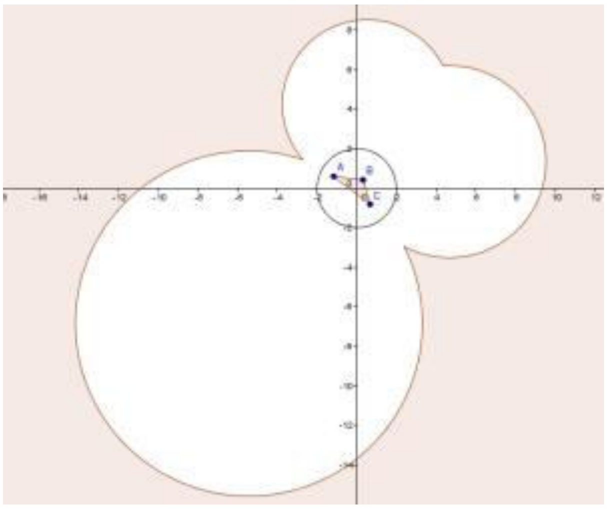
Circles centred on the origin map to other circles centred on the origin (above).



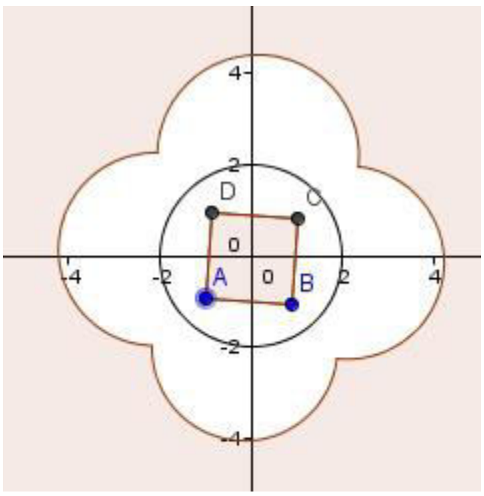
Ellipses create these shapes (above).



The straight line through A B maps to a circle through the origin (above).



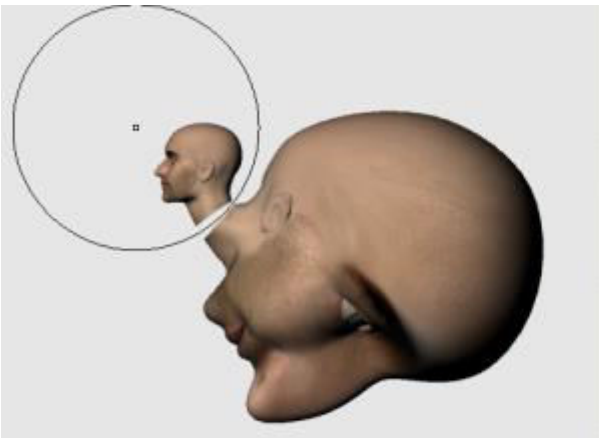
The solid triangle ABC maps to the pink region (above).



The solid square ABCD maps to the pink region (above).

These shapes can all be explored using the *reflect object in circle* button on Geogebra.

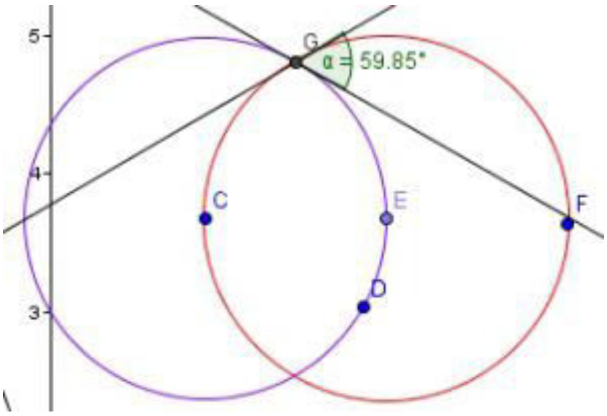
It is possible to extend the formula to 3 dimensions to give spherical inversion:



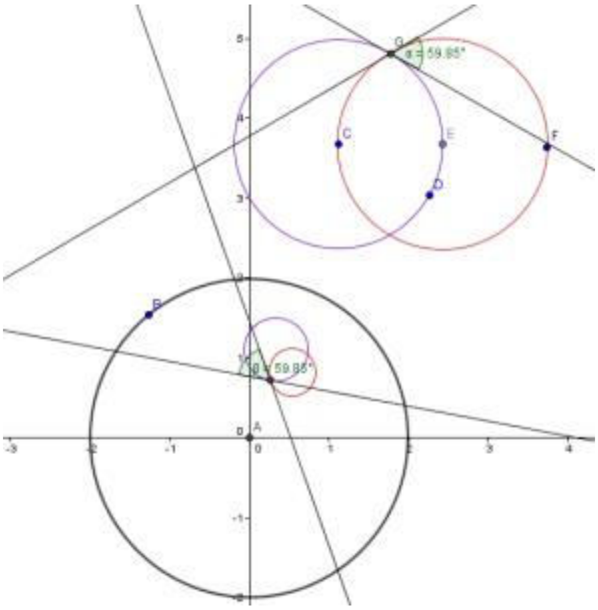
The above image is a 3D human head inverted in a sphere (from the Space Symmetry Structures website. There's lots to explore on this topic - it's a good example of how art can be mathematically generated, as well as introducing isomorphic structures.

Circular inversions II

There are some other interesting properties of circular inversions. One of which is that they preserve the "angle" between intersecting circles. Firstly, how can circles have an angle between them? Well, we draw 2 tangents to both the circles at the point of intersection, and then measure the angle between the 2 tangents:

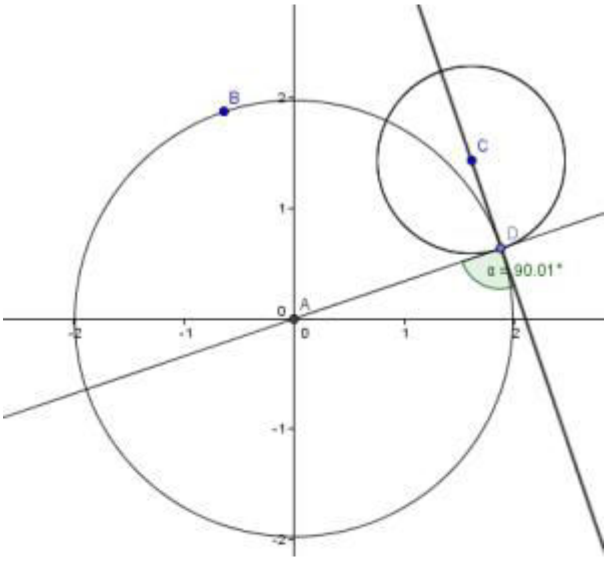


Therefore we can see that the "angle" between these 2 circles is 59.85 degrees. If we then carry out a circular inversion we see the following:



The inversion has been done with regards to the black circle centred around the origin. The red and blue circles are mapped from outside the the black circle onto circles inside the black circle. Now if we do the same as before - by finding the 2 tangents at the point of intersection, we find that the angle has remained the same - it is still 59.85 degrees.

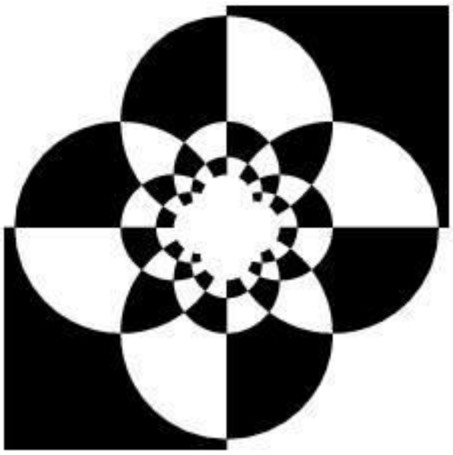
It is also possible to find circles which remain unchanged under the inversion. This happens when a circle is orthogonal (at a 90 degree angle) to the circle with which the inversion is being carried out.



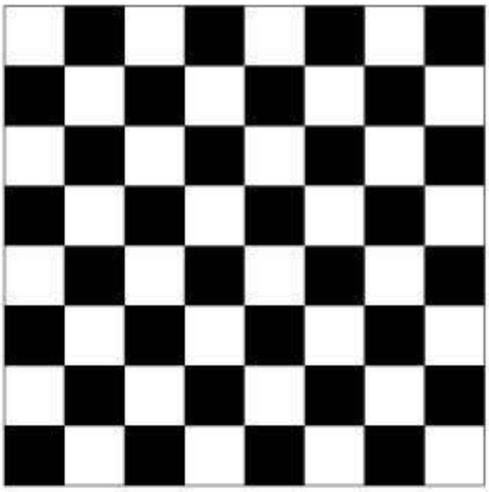
The small circle has an angle of 90 degrees with the large circle, and therefore when we invert with respect to the large circle, we map the small circle onto itself.

The question is, why is all this useful? Well, an entire branch of mathematics (non-Euclidean geometry) is concerned with being able to map points in our traditional Euclidean worldview (the geometry of high school triangles, parallel lines and circle theorems) to different geometrical systems entirely. Circular inversion is a good introduction to this concept.

Also, circular inversion can sometimes make studying mathematical shapes easier to understand and explain. For example, (from Wolfram):

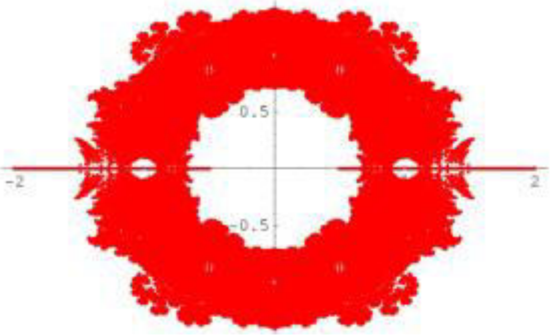


It would be very difficult to explain mathematically how the shape above is generated - whilst there are patterns, it is not obvious how to explain them. However, if we invert this shape through a circular inversion (with the circle at centre of the image) then we get the following:



This is the image inside the circle - and now we can clearly see the pattern behind the generated image. So, inversion has a lot of potential for simplifying geometrical problems.

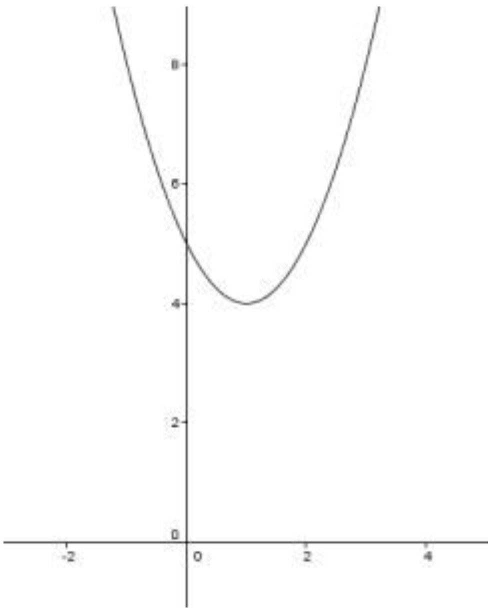
Graphically Understanding Complex Roots



If you have studied complex numbers then you'll be familiar with the idea that many polynomials have complex roots. For example $x^2 + 1 = 0$ has the solution x = i and -i. We know that the solution to $x^2 - 1 = 0$ (x = 1 and -1) gives the two x values at which the graph crosses the x axis, but what does a solution of x = i or -i represent graphically?

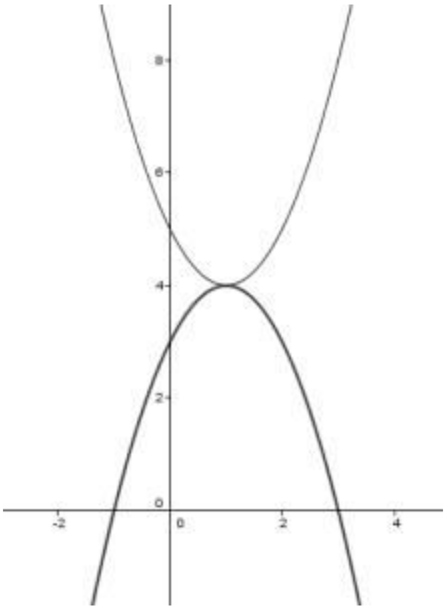
There's a great post on Maths Fun Facts which looks at basic idea behind this and I'll expand on this in a little more detail. This particular graphical method only works with quadratics:

Step 1



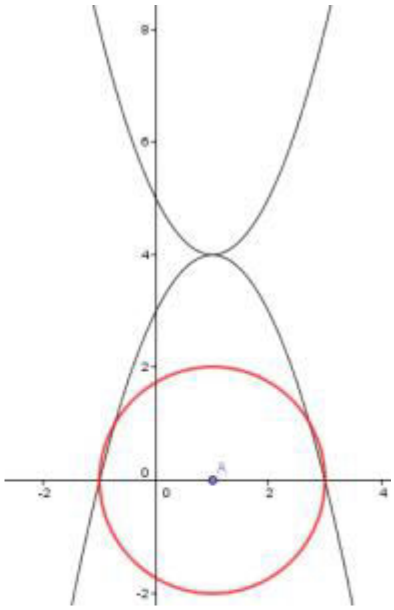
You have a quadratic graph with complex roots, say $y = (x - 1)^2 + 4$. Written in this form we can see the minimum point of the graph is at (1,4) so it doesn't cross the x axis.

Step 2



Reflect this graph downwards at the point of its vertex. We do this by transforming $y = (x - 1)^2 + 4$ into $y = -(x - 1)^2 + 4$

Step 3



We find the roots of this new equation using the quadratic formula or by rearranging - leaving the plus or minus sign in.

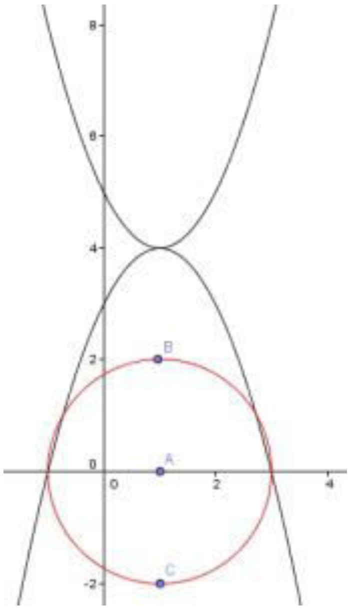
$-(x - 1)^2 + 4 = 0$

$(x-1) = \pm 2$

$x = 1 \pm 2$

Plot a circle with centre (1,0) and radius of 2. This will touch both roots.

Step 4



We can now represent the complex roots of the initial equation by rotating the 2 real roots we've just found 90 degrees anti-clockwise, with the centre of rotation the centre of the circle.

The points B and C on the diagram are a representation of the complex roots (if we view the graph as representing the complex plane). The complex roots of the initial equation are therefore given by $x = 1 \pm 2i$.

General case

It's relatively straightforward to show algebraically what is happening:

If we take the 2 general equations:

1) $y = (x-a)^2 + b$
2) $y = -(x-a)^2 + b$ (this is the reflection at the vertex of equation 1)

($b > 0$). Then the first equation will always have complex roots. The roots of both equations will be given by:

1) $a \pm i\sqrt{b}$

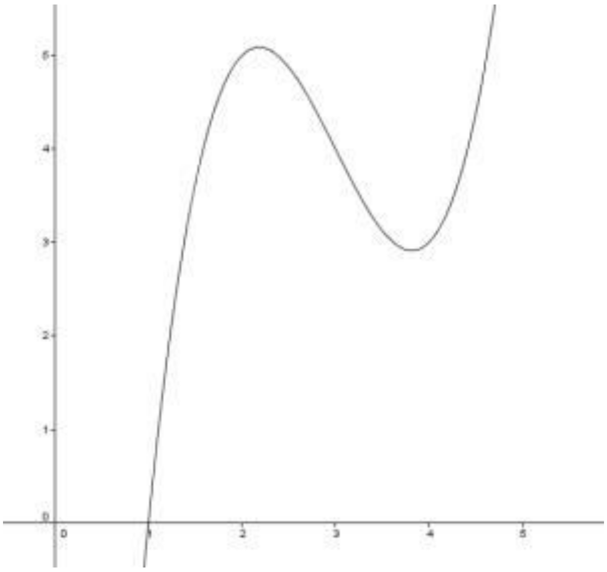
2) $a \pm \sqrt{b}$

So we can think of (2) as representing a circle of radius $\sqrt{b}$, centred at $(a,0)$. Therefore multiplying $\sqrt{b}$ by i has the effect of rotating the point $(\sqrt{b}, 0)$ 90 degrees anti-clockwise around the point $(a,0)$. Therefore the complex roots will be graphically represented by those points at the top and bottom of this circle. $(a, \sqrt{b})$ and $(a, -\sqrt{b})$

Graphically finding complex roots of a cubic

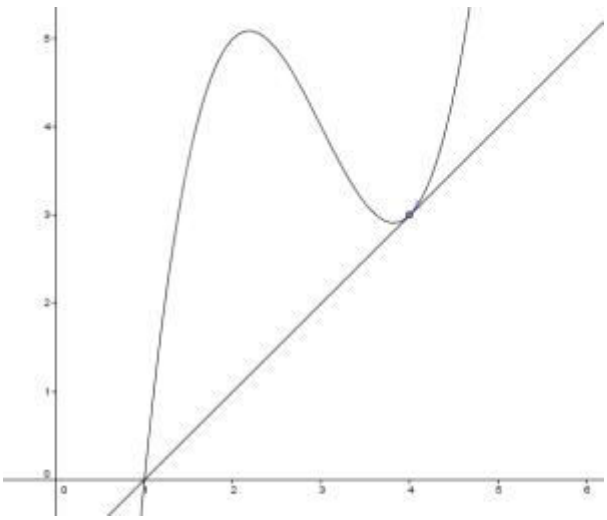
There is also a way of graphically calculating the complex roots of a cubic with 1 real and 2 complex roots. This method is outlined with an algebraic explanation [here]

Step 1



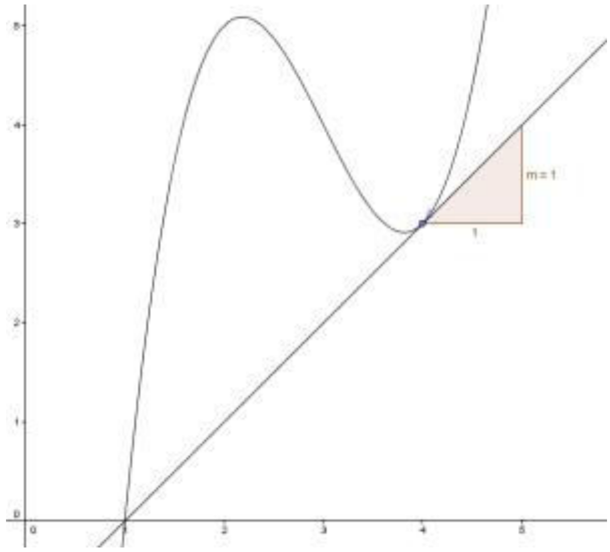
We plot a cubic with 1 real and 2 complex roots, in this case y = $x^3 - 9x^2 + 25x - 17$.

Step 2



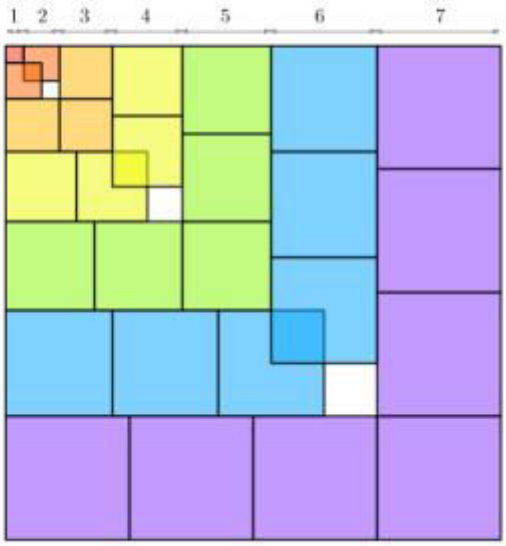
We find the line which goes through the real root (1,0) and which is also a tangent to the function.

Step 3



If the x co-ordinate of the tangent intersection with the cubic is a and the gradient of the tangent is m, then the complex roots are a ± mi. In this case the tangent x intersection is at 4 and the gradient of the tangent is 1, therefore the complex roots are 4 ± 1i.

Visualising Algebra through Geometry



The image above illustrates which of the following identities?

This picture above is a fantastic example of how we can use geometry to visualise an algebraic expression. It's taken from Brilliant - which is a fantastic new forum for sharing maths puzzles. This particular puzzle was created and uploaded by Arron Kau. The question is, which of the following mathematical identities does this image represent?

$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}\left[n(n+1)\right]^2$

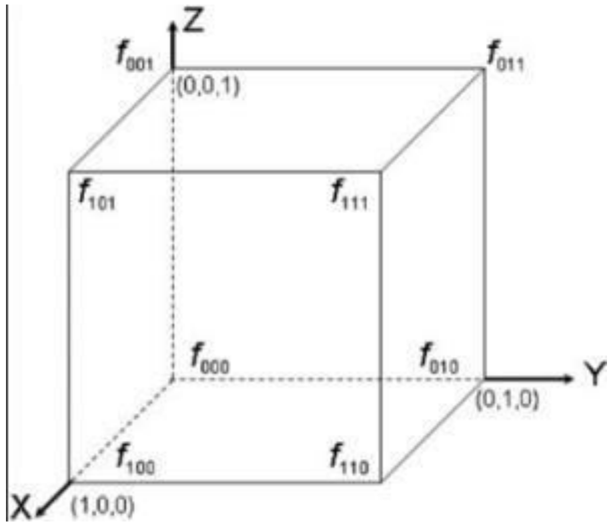
$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$

$1 + 3 + 5 + \cdots + (2n - 1) = n^2$

See if you can work it out!

Another example of the power of geometry in representing mathematical problems is provided by Ian Stewart's Cabinet of Mathematical Curiosities. The puzzle itself is pretty famous:

A farmer wants to cross a river and take with him a wolf, a goat, and a cabbage. There is a boat that can fit himself plus either the wolf, the goat, or the cabbage. If the wolf and the goat are alone on one shore, the wolf will eat the goat. If the goat and the cabbage are alone on the shore, the goat will eat the cabbage. How can the farmer bring the wolf, the goat, and the cabbage across the river?



The standard way of solving it is trial and error with some logic thrown in. However, as Ian Stewart points out, we can actually utilise 3 dimensional geometry to solve the puzzle. We start with a 3D wolf-goat-cabbage (w,g,c) space (shown in the diagram). All 3 start at (0,0,0). 0 represents this side of the bank, and 1 represents the far side of the bank. The target is to get therefore to (1,1,1). In (w,g,c) space , the x direction represents the wolf's movements, the y direction the goat and z the cabbage. Therefore the 8 possible triplet combinations are represented by the 8 vertices on a cube.

We can now cross out the 4 paths:

(0,0,0) to (1,00) as this leaves the goat with the cabbages

(0,0,0) to (0,0,1) as this leaves the wolf with the goat

(0,1,1) to (1,1,1) as the farmer would leave the goat and cabbage alone

(1,1,0) to (1,1,1) as the farmer would leave the wolf and goat alone.

which reduces the puzzle to a geometric problem – where we travel along the remaining edges – and the 2 solutions are immediately evident.

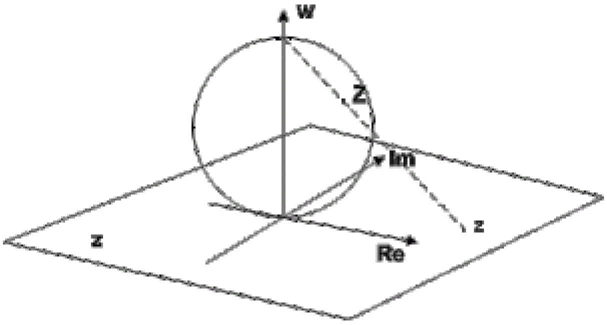
(eg. (0,0,0) – (0,1,0) – (1,1,0) – (1,0,0) – (1,0,1)- (1,1,1))

What's really nice about this solution is that it shows how problems seemingly unrelated to mathematics can be "translated" in mathematics – and also it shows how geometrical space can be used for problem solving.

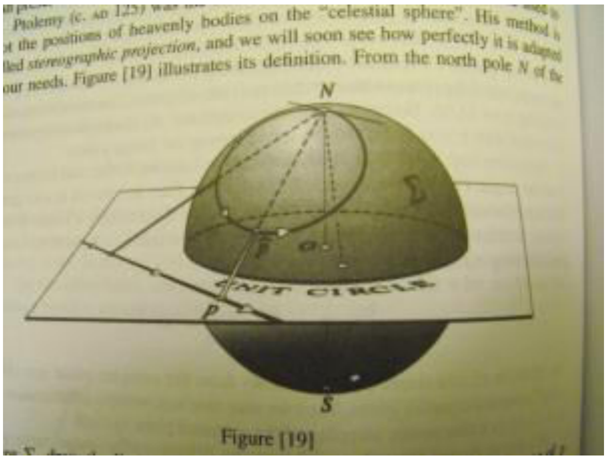
Solution to the initial puzzle:

This is quite a surprising identity. You can see it by seeing that there are (for example) 2 squares of length 2 - this gives you a total area of 2x2x2 = 2^3. Adding all the squares will give you the same area as a square of sides (1+2+3....)(1+2+3....) - hence the result.

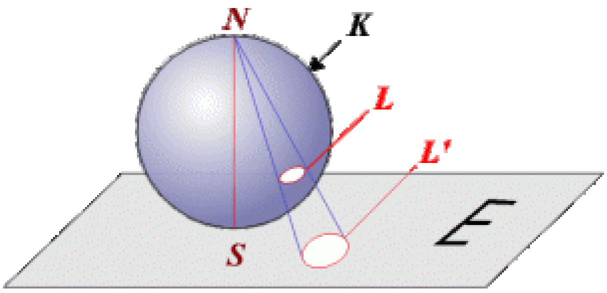
The Riemann Sphere



The Riemann Sphere is a fantastic glimpse of where geometry can take you when you escape from the constraints of Euclidean Geometry - the geometry of circles and lines taught at school. Riemann, the German 19th Century mathematician, devised a way of representing every point on a plane as a point on a sphere. He did this by first centering a sphere on the origin - as shown in the diagram above. Next he took a point on the complex plane ($z = x + iy$) and joined up this point to the North pole of the sphere (marked W). This created a straight line which intersected the sphere at a single point at the surface of the sphere (say at z'). Therefore every point on the complex plane (z) can be represented as a unique point on the sphere (z') - in mathematical language, there is a one-to-one mapping between the two. The only point on the sphere which does not equate to a point on the complex plane is that of the North pole itself (W). This is because no line touching W and another point on the sphere surface can ever reach the complex plane. Therefore Riemann assigned the value of infinity to the North pole, and therefore the the sphere is a 1-1 mapping of all the points in the complex plane (and infinity).



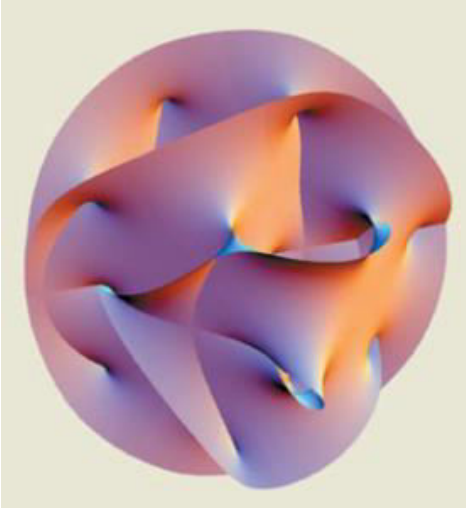
So what does this new way of representing the two dimensional (complex) plane actually allow us to see? Well, it turns on its head our conventional notions about "straight" lines. A straight line on the complex plane is projected to a circle going through North on the Riemann sphere (as illustrated above). Because North itself represents the point at infinity, this allows a line of infinite length to be represented on the sphere.



Equally, a circle drawn on the Riemann sphere not passing through North will project to a circle in the complex plane (as shown in the diagram above). So, on the Riemann sphere - which remember is isomorphic (mathematically identical) to the extended complex plane, straight lines and circles differ only in their position on the sphere surface. And this is where it starts to get really interesting - when we have two isometric spaces there is no way an inhabitant could actually know which one is his own reality. For a two dimensional being living on a Riemann sphere, travel in what he regarded as straight lines would in fact be geodesic (a curved line joining up A and B on the sphere with minimum distance).

By the same logic, our own 3 dimensional reality is isomorphic to the projection onto a 4 dimensional sphere (hypersphere) - and so our 3 dimensional universe is indistinguishable from a a curved 3D space which is the surface of a hypersphere. This is not just science fiction - indeed Albert Einstein was one to suggest this as a possible explanation for the structure of the universe. Indeed, such a scenario would allow there to be an infinite number of 3D universes floating in the 4th dimension - each bounded by the surface of their own personal hypersphere. Now that's a bit more interesting than the Euclidean world of straight lines and circle theorems.

Imagining the 4th Dimension



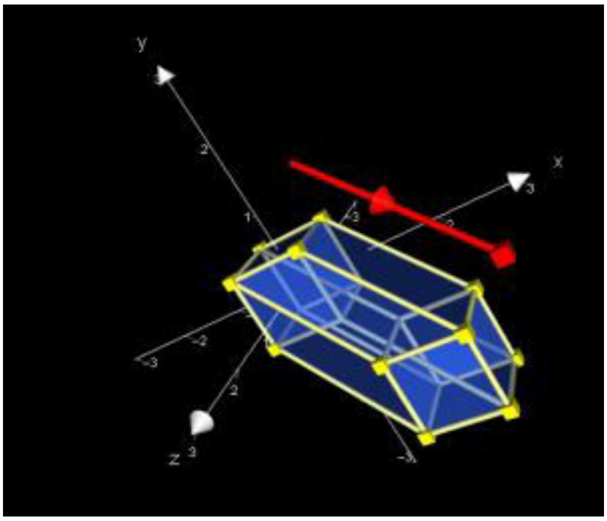
Imagining extra dimensions is a fantastic ToK topic - it is something which seems counter-intuitively false, something which we have no empirical evidence to support, and yet it is something which seems to fit the latest mathematical models on string theory (which requires 11 dimensions). Mathematical models have consistently been shown to be accurate in describing reality, but when they predict a reality that is outside our realm of experience then what should we believe? Our senses? Our intuition? Or the mathematical models?

Carl Sagan produced a great introduction to the idea of extra dimensions based on the Flatland novel. This imagines reality as experienced by two dimensional beings.

http://www.youtube.com/watch?v=-wv0vxVRGMY

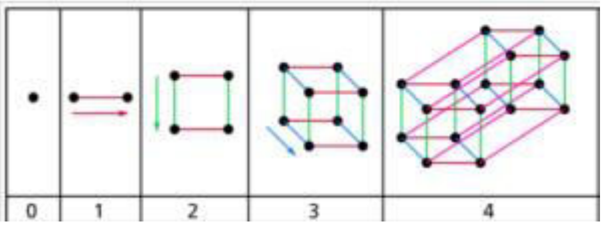
Mobius strips are a good gateway into the weird world of topology - as they are 2D shapes with only 1 side. There are some nice activities to do with Mobius strips - first take a pen and demonstrate that you can cover all of the strip without lifting the pen. Next, cut along the middle of the strip and see the resulting shape. Next start again with a new strip, but this time start cutting from nearer the edge (around 1/3 in). In both cases have students predict what they think will happen.

Next we can move onto the Hypercube (or Tesseract). We can see an Autograph demonstration of what the fourth dimensional cube looks like here.



The page allows you to model 1, then 2, then 3 dimensional traces - each time representing a higher dimensional cube.

It's also possible to create a 3 dimensional representation of a Tesseract using cocktail sticks - you simply need to make 2 cubes, and then connect one vertex in each cube to the other as in the diagram below:

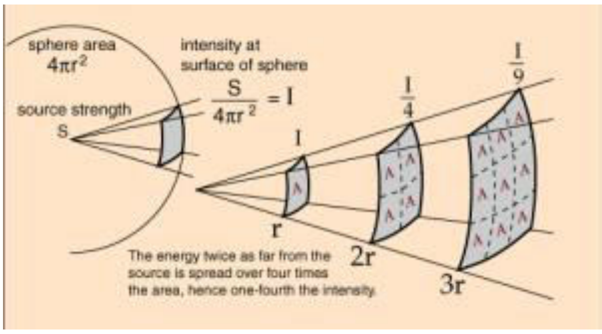


For a more involved discussion (it gets quite involved!) on imagining extra dimensions, this 10 minute cartoon takes us through how to imagine 10 dimensions.

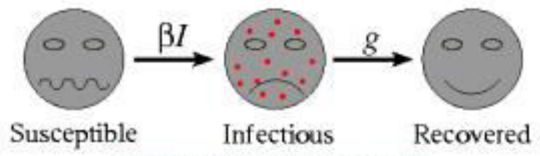
http://www.youtube.com/watch?v=mu5URbh-Lh0

It might also be worth touching on why mathematicians believe there might be 11 dimensions. Michio Kaku has a short video (with transcript) here and Brian Greene also has a number of good videos on the subject.

All of which brings us onto empirical testing - if a mathematical theory can not be empirically tested then does it differ from a belief? Well, interestingly this theory can be tested - by looking for potential violations to the gravitational inverse square law.



The current theory expects that the extra dimensions are themselves incredibly small - and as such we would only notice their effects on an incredibly small scale. The inverse square law which governs gravitational attraction between 2 objects would be violated on the microscopic level if there were extra dimensions - as the gravitational force would "leak out" into these other dimensions. Currently physicists are carrying out these tests - and as yet no violation of the inverse square law has been found, but such a discovery would be one of the greatest scientific discoveries in history.



Movement rates between classes of the SIR model

Modelling Infectious Diseases

Using mathematics to model the spread of diseases is an incredibly important part of preparing for potential new outbreaks. As well as providing information to health workers about the levels of vaccination needed to protect a population, it also helps govern first response actions when new diseases potentially emerge on a large scale (for example, Bird flu, SARS and Ebola have all merited much study over the past few years).

The basic model is based on the SIR model - this is represented by the picture above (from Plus Maths which has an excellent and more detailed introduction to this topic). The SIR model looks at how much of the population is susceptible to infection, how many of these go on to become infectious, and how many of these go on to recover (and in what timeframe).

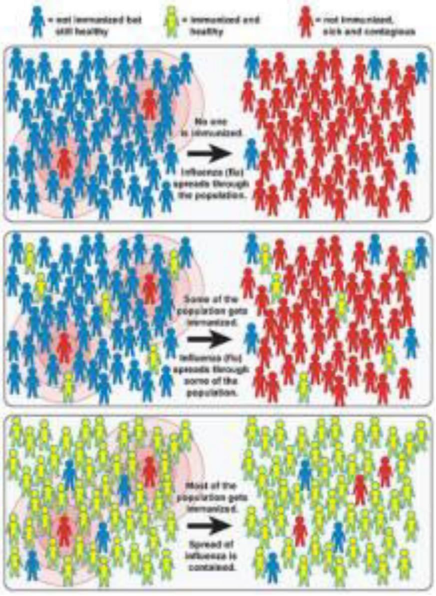
The value of R_0 for some well-known diseases	
Disease	R_0
AIDS	2 to 5
Smallpox	3 to 5
Measles	16 to 18
Malaria	> 100

Another important parameter is R_0, this is defined as how many people an infectious person will pass on their infection to in a totally susceptible population. Some of the R_0values for different diseases are shown above. This shows how an airbourne infection like measles is very infectious - and how malaria is exceptionally hard to eradicate because infected people act almost like a viral storage bank for mosquitoes.

One simple bit of maths can predict what proportion of the population needs to be vaccinated to prevent the spread of viruses. The formula is:

$V_T = 1 - 1/R_0$

Where V_T is the proportion of the population who require vaccinations. In the case of something like the HIV virus (with an R_0 value of between 2 and 5), you would only need to vaccinate a maximum of 80% of the population. Measles however requires around 95% vaccinations. This method of protecting the population is called herd immunity



This graphic above shows how herd immunity works. In the first scenario no members of the population are immunised, and that leads to nearly all the population becoming ill - but in the third scenario, enough members of the population are immunised to act as buffers against the spread of the infection to non-immunised people.

$$\frac{dS}{dt} = -\beta IS$$
$$\frac{dI}{dt} = \beta IS - \nu I$$
$$\frac{dR}{dt} = \nu I$$

The equations above represent the simplest SIR (susceptible, infectious, recovered) model - though it is still somewhat complicated!

dS/dt represents the rate of change of those who are susceptible to the illness with respect to time. dI/dt represents the rate of change of those who are infected with respect to time. dR/dt represents the rate of change of those who have recovered with respect to time.

For example, if dI/dt is high then the number of people becoming infected is rapidly increasing. When dI/dt is zero then there is no change in the numbers of people becoming infected (number of infections remain steady). When dI/dt is negative then the numbers of people becoming infected is decreasing.

The constants β and ν are chosen depending on the type of disease being modelled. β represents the contact rate - which is how likely someone will get the disease when in contact with someone who is ill. ν is the recovery rate which is how quickly people recover (and become immune.

ν can be calculated by the formula:

$D = 1/\nu$

where D is the duration of infection.

β can then be calculated if we know R_0 by the formula:

$R_0 = \beta/\nu$

Modelling measles

So, for example, with measles we have an average infection of about a week, (so if we want to work in days, $7 = 1/\nu$ and so $\nu = 1/7$). If we then take $R_0 = 15$ then:

$R_0 = \beta/\nu$
$15 = \beta/0.14$
$\beta = 2.14$

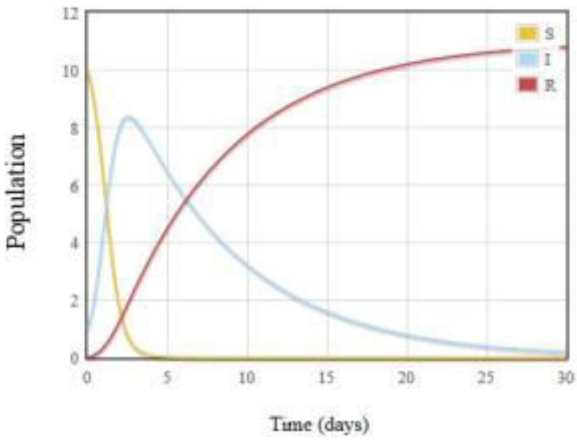
Therefore our 3 equations for rates of change become:

$dS/dt = -2.14\, I\, S$

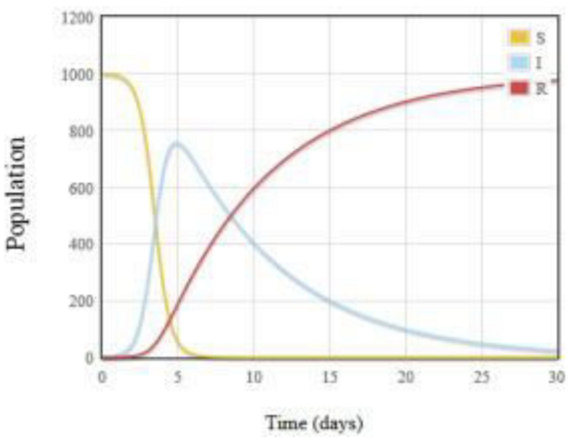
$dI/dt = 2.14\, I\, S - 0.14\, I$

$dR/dt = 0.14\, I$

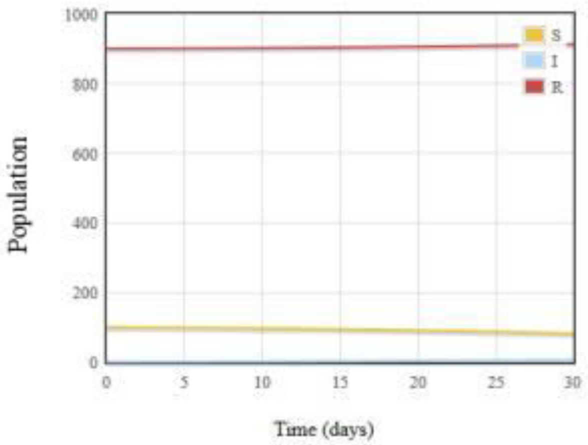
Unfortunately these equations are very difficult to solve - but luckily we can use a computer program to plot what happens. We need to assign starting values for S, I and R - the numbers of people susceptible, infectious, recovered (immune) from measles. Let's say we have a total population of 11 people - 10 who are susceptible, 1 who is infected and 0 who are immune. This gives the following outcome:



This shows that the infection spreads incredibly rapidly - by day 2, 8 people are infected. By day 10 most people are immune but the illness is still in the population, and by day 30 the entire population is immune and the infection has died out.

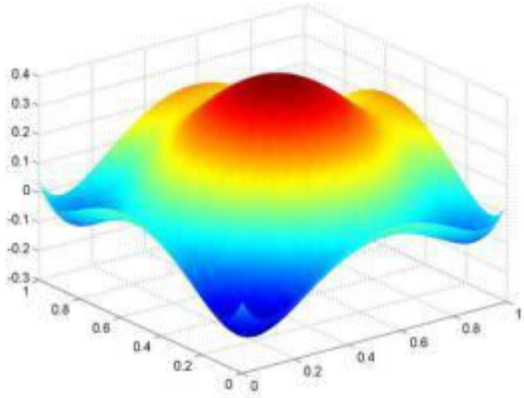


An illustration of just how rapidly measles can spread is provided by the graphic above. This time we start with a population of 1000 people and only 1 infected individual - but even now, within 5 days over 75% of the population are infected.



This last graph shows the power of herd immunity. This time there are 100 susceptible people, but 900 people are recovered (immune), and there is again one infectious person. This time the infection never takes off in the community - those who are already immune act as a buffer against infection.

Real life use of Differential Equations

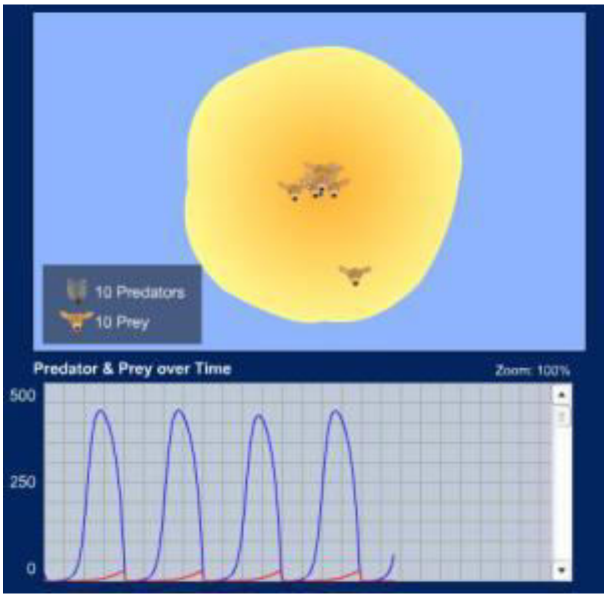


Differential equations have a remarkable ability to predict the world around us. They are used in a wide variety of disciplines, from biology, economics, physics, chemistry and engineering. They can describe exponential growth and decay, the population growth of species or the change in investment return over time. A differential equation is one which is written in the form dy/dx = ... Some of these can be solved (to get y = ...) simply by integrating, others require much more complex mathematics.

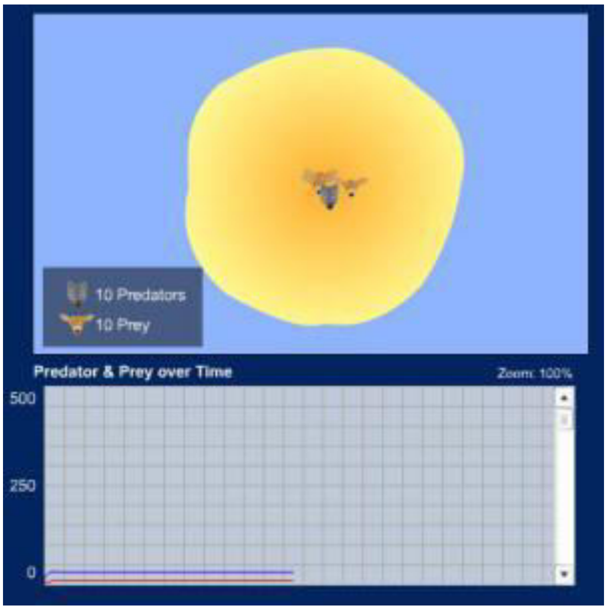
Population Models

One of the most basic examples of differential equations is the Malthusian Law of population growth dp/dt = rp shows how the population (p) changes with respect to time. The constant r will change depending on the species. Malthus used this law to predict how a species would grow over time.

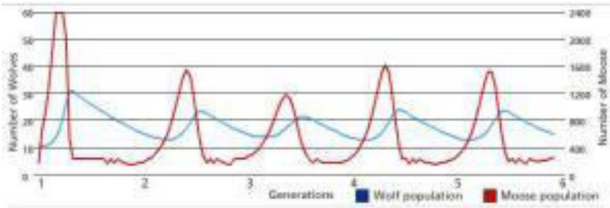
More complicated differential equations can be used to model the relationship between predators and prey. For example, as predators increase then prey decrease as more get eaten. But then the predators will have less to eat and start to die out, which allows more prey to survive. The interactions between the two populations are connected by differential equations.



The picture above is taken from an online predator-prey simulator . This allows you to change the parameters (such as predator birth rate, predator aggression and predator dependance on its prey). You can then model what happens to the 2 species over time. The graph above shows the predator population in blue and the prey population in red - and is generated when the predator is both very aggressive (it will attack the prey very often) and also is very dependent on the prey (it can't get food from other sources). As you can see this particular relationship generates a population boom and crash - the predator rapidly eats the prey population, growing rapidly - before it runs out of prey to eat and then it has no other food, thus dying off again.



This graph above shows what happens when you reach an equilibrium point - in this simulation the predators are much less aggressive and it leads to both populations have stable populations.



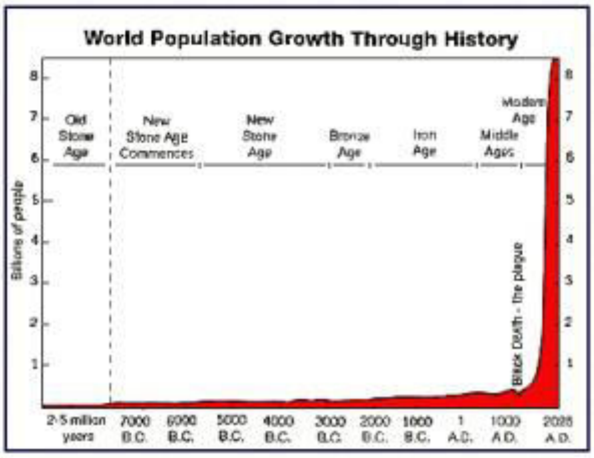
There are also more complex predator-prey models - like the one shown above for the interaction between moose and wolves. This has more parameters to control. The above graph shows almost-periodic behaviour in the moose population with a largely stable wolf population.

Some other uses of differential equations include:

1) In medicine for modelling cancer growth or the spread of disease
2) In engineering for describing the movement of electricity
3) In chemistry for modelling chemical reactions
4) In economics to find optimum investment strategies
5) In physics to describe the motion of waves, pendulums or chaotic systems.

With such ability to describe the real world, being able to solve differential equations is an important skill for mathematicians. If you want to learn more, you can read about how to solve them here.

Black Swans and Civilisation Collapse

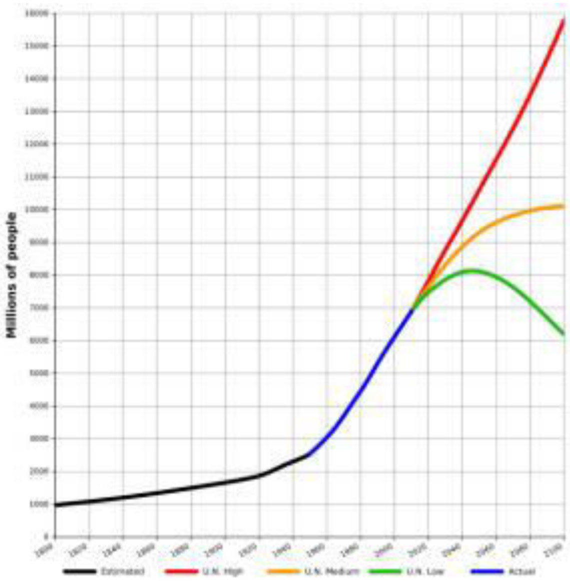


A really interesting branch of mathematics is involved in making future predictions about how civilisation will evolve in the future - and indeed looking at how robust our civilisation is to external shocks. This is one area in which mathematical models do not have a good record as it is incredibly difficult to accurately assign probabilities and form policy recommendations for events in the future.

Malthusian Catastrophe

One of the most famous uses of mathematical models in this context was by Thomas Malthus in 1798. He noted that the means of food production were a fundamental limiting factor on population growth - and that if population growth continued beyond the means of food production that there would be (what is now termed) a "Malthusian catastrophe" of a rapid population crash.

As it turns out, agrarian productivity has been able to keep pace with the rapid population growth of the past 200 years.



Looking at the graph we can see that whilst it took approximately 120 years for the population to double from 1 billion to 2 billion, it only took 55 years to double again. It would be a nice exercise to try and see what equation fits this graph - and also look at the rate of change of population (is it now slowing down?) The three lines at the end of the graph are the three different UN predictions - high end, medium and low end estimate. There's a pretty stark difference between high end and low end estimates by 2100 - between 16 billion and 6 billion! So what does that tell us about the accuracy of such predictions?

Complex Civilisations

More recently academics like Joseph Tainter and Jared Diamond have popularised the notion of civilisations as vulnerable to collapse due to ever increasing complexity. In terms of robustness of civilisation one can look at an agrarian subsistence example. Agrarian subsistence is pretty robust against civilisation collapse - small self sufficient units may themselves be rather vulnerable to famines and droughts on an individual level, but as a society they are able to ride out most catastrophes intact.

The next level up from agrarian subsistence is a more organised collective - around a central authority which is able to (say) provide irrigation technology through a system of waterways. Immediately the complexity of society has increased, but the benefits of irrigation allow much more crops to be grown and thus the society can support a larger population. However, this complexity comes at a cost - society now is reliant on those irrigation channels - and any damage to them could be catastrophic to society as a whole.

To fast forward to today, we have now an incredibly complex society, far far removed from our agrarian past - and whilst that means we have an unimaginably better quality of life, it also means society is more vulnerable to collapse than ever before. To take the example of a Coronal Mass Ejection - in which massive solar discharges hit the Earth. The last large one to hit the Earth was in 1859 but did negligible damage as this was prior to the electrical age. Were the same event to happen today, it would cause huge damage - as we are reliant on electricity for everything from lighting to communication to refrigeration to water supplies. A week without electricity for an urban centre would mean no food, no water, no lighting, no communication and pretty much the entire breakdown of society.

That's not to say that such an event will happen in our lifetimes - but it does raise an interesting question about intelligent life - if advanced civilisations continue to evolve and in the process grow more and more complex then is this a universal limiting factor on progress? Does ever increasing complexity leave civilisations so vulnerable to catastrophic events that their probabilities of surviving through them grow ever smaller?

Black Swan Events

One of the great challenges for mathematical modelling is therefore trying to assign probabilities for these "Black Swan" events. The term was coined by economist Nassim Taleb - and used to describe rare, low probability events which have very large consequences. If the probability of a very large scale asteroid impact is (say) estimated as 1-100,000 years - but were it to hit it is estimated to cause $35 trillion of damage (half the global GDP) then what is the rational response to such a threat? Dividing the numbers suggests that we should in such a scenario be spending $3.5billion every year on trying to address such an event - and yet which politician would justify such spending on an event that might not happen for another 100,000 years?

I suppose you would have to conclude therefore that our mathematical models are pretty poor at predicting future events, modelling population growth or dictating future and current policy. Which stands in stark contrast to their abilities in modelling the real world (minus the humans). Will this improve in the future, or are we destined to never really be able to predict the complex outcomes of a complex world?

Pure Mathematics

Fermat's Theorem on the sum of two squares

Not as famous as Fermat's Last Theorem (which baffled mathematicians for centuries), Fermat's Theorem on the sum of two squares is another of the French mathematician's theorems.

Fermat asserted that all odd prime numbers p of the form 4n + 1 can be expressed as:

$$p = x^2 + y^2$$

where x and y are both integers. No prime numbers of the form 4n+3 can be expressed this way.

This is quite a surprising theorem - why would we expect only some prime numbers to be expressed as the sum of 2 squares? To give some examples:

13 is a prime number of the form 4n+1 and can be written as $3^2 + 2^2$.
17 is also of the form 4n + 1 and can be written as $4^2 + 1^2$.
$29 = 5^2 + 2^2$.
$37 = 6^2 + 1^2$.

Prime numbers of the form 4n + 3 such as 7, 11, 19 can't be written in this way.

The proof of this theorem is a little difficult. It is however easier to prove a similar (though not logically equivalent!) theorem:

All sums of $x^2 + y^2$ (x and y integers) are either of the form 4n + 1 or even.

In other words, for some n:

$x^2 + y^2 = 4n + 1$ or

$x^2 + y^2 = 2n$

We can prove this by looking at the possible scenarios for the choices of x and y.

Case 1:

x and y are both even (i.e. $x = 2n$ and $y = 2m$ for some n and m). Then

$x^2 + y^2 = (2n)^2 + (2m)^2$
$x^2 + y^2 = 4n^2 + 4m^2$
$x^2 + y^2 = 2(2n^2 + 2m^2)$

which is even.

Case 2:

x and y are both odd (i.e. $x = 2n+1$ and $y = 2m+1$ for some n and m).
Then $x^2 + y^2 = (2n+1)^2 + (2m+1)^2$
$x^2 + y^2 = 4n^2 + 4n + 1 + 4m^2 + 4m + 1$
$x^2 + y^2 = 4n^2 + 4n + 4m^2 + 4m + 2$
$x^2 + y^2 = 2(2n^2 + 2m^2 + 2m + 2n + 1)$.
which is even.

Case 3:

One of x and y is odd, one is even. Let's say x is odd and y is even. (i.e. $x = 2n+1$ and $y = 2m$ for some n and m).
Then $x^2 + y^2 = (2n+1)^2 + (2m)^2$
$x^2 + y^2 = 4n^2 + 4n + 1 + 4m^2$
$x^2 + y^2 = 4(n^2+m^2+n) + 1$
which is in the form $4k+1$ (with $k = (n^2+m^2+n)$)

Therefore, the sum of any 2 integer squares will either be even or of the form $4n+1$. Unfortunately this does not necessarily imply the reverse: that all numbers of the form $4n+1$ are the sum of 2 squares (which would then prove Fermat's Theorem). This is because,

A implies B
Does not necessarily mean that
B implies A

For example,

If A is "cats" and B is "have 4 legs"
A implies B (All cats have 4 legs)
B implies A (All things with 4 legs are cats).

A is logically sound, whereas B is clearly false.

This is a nice example of some basic number theory - such investigations into expressing numbers as the composition of 2 other numbers have led to some of the most enduring and famous mathematical puzzles.

The Goldbach Conjecture suggests that every even number greater than 2 can be expressed as the sum of 2 primes and has remained unsolved for over 250 years. Fermat's Last Theorem lasted over 350 years before finally someone proved that $a^2 + b^2 = c^2$ has no positive integers a, b, and c which solve the equation for n greater than 2

e's are good - He's Leonard Euler.

Along with pi, e is one of the most important constants in mathematics. It is an irrational number which carries on forever. The first few digits are 2.71828182845945...

Leonard Euler

e is sometime named after Leonard Euler (Euler’s number). He wasn't the first mathematician to discover e - but he was the first mathematician to publish a paper using it. Euler is not especially well known outside of mathematics, yet he is undoubtedly one of the true great mathematicians. He published over 800 mathematical papers on everything from calculus to number theory to algebra and geometry.

Why is e so important?

Lots of functions in real life display exponential growth. Exponential growth is used to describe any function of the form a^x where a is a constant. One example of exponential growth is the chessboard and rice problem, (if I have one grain of rice on the first square, two on the second, how many will I have on the 64th square?) This famous puzzle demonstrates how rapidly numbers grow with exponential growth.

http://www.youtube.com/watch?v=g6ifomrjlsI

Sketch
$y = 2^x$
$y = e^x$
$y = 3^x$

for between x = 0 and 3. You can see that $y = e^x$ is between $y=2^x$ and $y = 3^x$ on the graph, so why is e so much more useful than these numbers? By graphical methods you can find the gradient when the graphs cross the y axis. For the function $y = e^x$ this gradient is 1. This is because the derivative of e^x is still e^x - which makes it really useful in calculus.

The beauty of e.

e appears in a host of different and unexpected mathematical contexts, from probability models like the normal distribution, to complex numbers and trigonometry.

Euler's Identity is frequently voted the most beautiful equation of all time by mathematicians, it links 5 of the most important constants in mathematics together into a single equation.

$$e^{i\pi} + 1 = 0$$

Infinite fraction: e can be represented as a continued infinite fraction can students you spot the pattern? - the LHS is given by 2 then 1,2,1 1,4,1 1,6,1 etc.

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{\mathbf{2} + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\mathbf{4} + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}}}} = 1 + \cfrac{1}{\mathbf{0} + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\mathbf{2} + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\mathbf{4} + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}}}}}}.$$

Infinite sum of factorials: e can also be represented as the infinite sum of factorials:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

A limit: e can also be derived as the limit to the following function. It was this limit that Jacob Bernoulli investigated - and he is in fact credited with the first discovery of the constant.

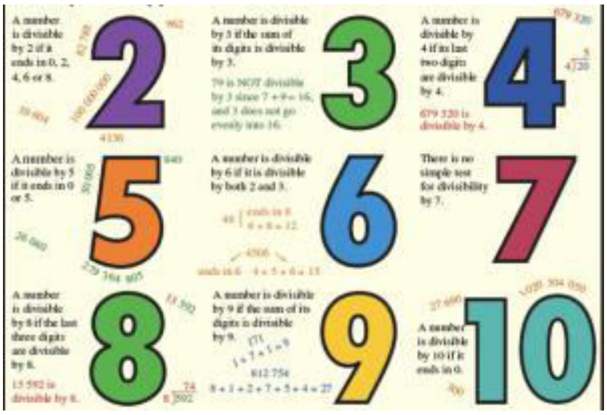
$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n,$$

Complex numbers and trigonometry : e can be used to link both trigonometric identities and complex numbers:

$$e^{ix} = \cos x + i \sin x,$$

You can explore more of the mathematics behind the number e here.

Divisibility Tests and Palindromic Numbers



Divisibility tests allow us to calculate whether a number can be divided by another number. For example, can 354 be divided by 3? Can 247,742 be divided by 11? So what are the rules behind divisibility tests, and more interestingly, how can we prove them?

Divisibility rule for 3

The most well known divisibility rule is that for dividing by 3. All you need to do is add the digits of the number and if you get a number that is itself a multiple of 3, then the original number is divisible by 3. For example, 354 is divisible by 3 because 3+5+4 = 12 and 12 can be divided by 3.

We can prove this using the modulo function. This allows us to calculate the remainder when any number is divided by another. For example, $21 \equiv 3 \pmod{6}$. This means that the remainder when 21 is divided by 6 is 3.

So we first start with our number that we want to divide by 3:

$n = a + 10b + 100c + 1000d + ...$

Now, dividing by 3 is the same as working in mod 3, so we can rewrite this n in terms of mod 3:

$n = a + 1b + 1c + 1d + ...\ (\text{mod } 3)$

(this is because $10 \equiv 1 \text{ mod } 3$, $100 \equiv 1 \text{ mod } 3$, $1000 \equiv 1 \text{ mod } 3$ etc)

Now, for a number to be divisible by 3 this sum needs to add to a multiple of 3. Therefore if $a + b + c + d + ... \equiv 0 \pmod{3}$ then the original number is also divisible by 3.

Divisibility rule for 11

This rule is much less well known, but it's quite a nice one. Basically you take the digits of any number and alternately subtract and add them. If the answer is a multiple of 11 (or 0) then the original number is divisible by 11. For example, 121 is a multiple of 11 because 1-2+1 = 0. 247,742 is because 2-4+7-7+4-2 = 0.

Once again we can prove this using the modulo operator.

$n = a + 10b +100c + 1000d + ...$

and this time work in mod 11:

$n = a + -1b + 1c + -1d + \ldots\ldots \pmod{11}$

this is because for ease of calculation we can write $10 \equiv -1 \pmod{11}$. This is because $-1 \equiv 10 \equiv 21 \equiv 32 \equiv 43 \pmod{11}$. All numbers 11 apart are the same in mod 11. Meanwhile $100 \equiv 1 \pmod{11}$. This alternating pattern will continue.

Therefore if we alternately subtract and add digits, then if the answer is divisible by 11, then the original number will be as well.

Palindromic Numbers

Palindromic numbers are numbers which can be read the same forwards as backwards. For example, 247,742 is a palindromic number, as is 123,321. Any palindromic number which is an even number of digits is also divisible by 11. We can see this by considering (for example) the number:

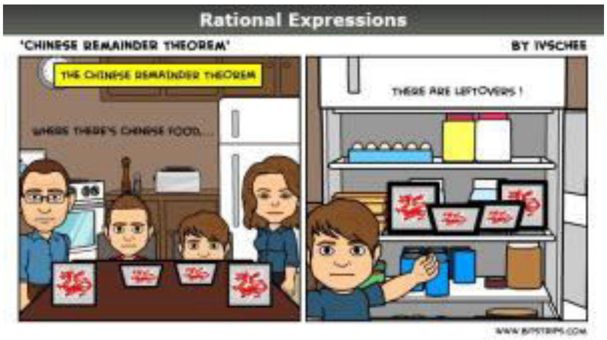
$n = a + 10b + 100c + 1000c + 10{,}000b + 100{,}000a$

Working in mod 11 we will then get the same pattern as previously:

$n = a - b + c - c + b - a \pmod{11}$

so $n = 0 \pmod{11}$. Therefore n is divisible by 11. This only works for even palindromic numbers as when the numbers are symmetric they cancel out.

The Chinese Remainder Theorem



The Chinese Remainder Theorem is a method to solve the following puzzle, posed by Sun Zi around the 4th Century AD.

What number has a remainder of 2 when divided by 3, a remainder of 3 when divided by 5 and a remainder of 2 when divided by 7?

There are a couple of methods to solve this. Firstly it helps to understand the concept of modulus - for example 21 mod 6 means the remainder when 21 is divided by 6. In this case the remainder is 3, so we can write 21 ≡ 3 (mod 6). The ≡ sign means "equivalent to" and is often used in modulus questions.

Method 1:

1) We try to solve the first part of the question, *What number has a remainder of 2 when divided by 3,*

to do this we list the values of x ≡ 2 (mod 3). x = 2,5,8,11,14,17......

2) We then look at the values in this list and see which ones also satisfy the second part of the question, *What number has a remainder of 3 when divided by 5*

From the list x = 2,5,8,11,14,17...... we can see that x = 8 has a remainder of 3 when divided by 5 (i.e 8 ≡ 3 (mod 5))

3) We now start from 8 and count up in multiples of 15 (3 x 5 because we have mod 3 and mod 5)

8, 23, 38, 53 .....

4) We look for a number on this list which satisfies the last part of the question, *What number has a remainder of 2 when divided by 7?*

With x = 8, 23, 38, 53 ..... we can see that x = 23 has a remainder of 2 when divided by 7 (i.e 23 ≡ 2 (mod 7))

Therefore 23 satisfies all parts of the question. When you divide it by 3 you get a remainder of 2, when you divide it by 5 you get a remainder of 3, and when you divide it by 7 you get a remainder of 2. So, 23 is our answer.

The second method is quite a bit more complicated - but is a better method when dealing with large numbers.

Method 2

1) We rewrite the problem in terms of modulus.

$x \equiv 2 \pmod 3$

$x \equiv 3 \pmod 5$

$x \equiv 2 \pmod 7$

We assign a = 2, b = 3, c = 2.

2) We give the values $m_1 = 3$ (because the first line is mod 3), $m_2 = 5$ (because the second line is mod 5), $m_3 = 2$ (because the third line is mod 7).

3) We calculate $M = m_1m_2m_3 = 3x5x7 = 105$

4) We calculate $M_1 = M/m_1 = 105/3 = 35$
$M_2 = M/m_2 = 105/5 = 21$
$M_3 = M/m_3 = 105/7 = 15$

4) We then note that $M_1 \equiv 2 \pmod 3$, $M_2 \equiv 1 \pmod 5$, $M_3 \equiv 1 \pmod 7$

5) We then look for the multiplicative inverse of M_1. This is the number which when multiplied by M_1 will give an answer of 1 (mod 3). This number is 2 because $2x2 = 4 \equiv 1 \pmod 3$. We assign A = 2

We then look for the multiplicative inverse of M_2. This is the number which when multiplied by M_2 will give an answer of 1 (mod 5). This number is 1 because $1x1 = 1 \equiv 1 \pmod 3$. We assign B = 1.

We then look for the multiplicative inverse of M_3. This is the number which when multiplied by M_3 will give an answer of 1 (mod 7). This number is 1 because $1x1 = 1 \equiv 1 \pmod 7$. We assign C = 1.

6) We now put all this together:

$x = aM_1A + bM_2B + cM_3C \pmod M$

$x = 2x35x2 + 3x21x1 + 2x15x1 = 233 \equiv 23 \pmod{105}$

That last method may seem a lot slower - but when working with large numbers is actually a lot quicker. So there we go - that's the method that Sun Zi noted more than 1500 years ago. This topic whilst seemingly quite abstract is a good introduction to number theory - the branch of mathematics which deals with the properties of whole numbers.

Mathematical Proof and Paradox

http://www.youtube.com/watch?v=omyUncKI7oU

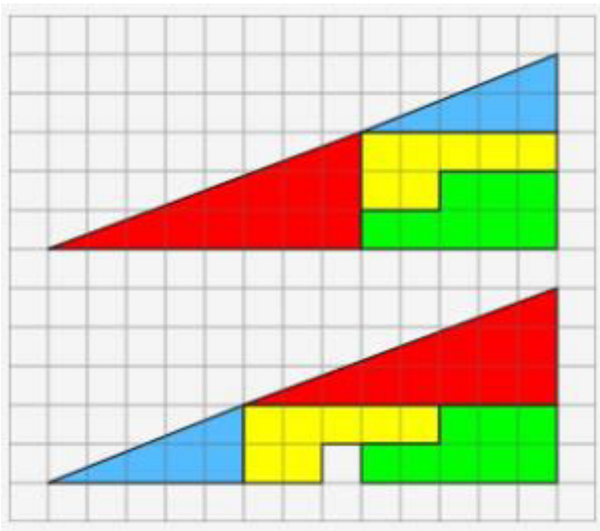
This classic youtube clip "proves" how 25/5 = 14, and does it three different ways. Maths is a powerful method for providing proof - but we need to be careful that each step is based on correct assumptions.

One of the most well known fake proofs is as follows:

let $a = b$
Then $a^2 = ab$
$a^2 - b^2 = ab - b^2$
$(a-b)(a+b) = b(a-b)$
$a+b = b$ (divide by $a-b$)
$b+b = b$ (as $a = b$)
$2b = b$
$2 = 1$

Can you spot the step that causes the proof to be incorrect?

Another well known maths problem that appears to prove the impossible is the following:



This was created by magician Paul Curry - and is called Curry's Paradox. You can work out the areas of all the 4 different coloured shapes on both triangles, and yet by simply rearranging them you created a different area.

A third "proof" shows that -1 = 1:

Let $a = b = -1$
$a^2 = b^2$
$2a^2 = 2b^2$
$a^2 = 2b^2 - a^2$
$a = \sqrt{(2b^2 - a^2)}$
$a = \sqrt{(2(-1)^2 - (-1)^2)}$

$a = \sqrt{(1)}$
$-1 = 1$

And finally a proof that 1= 0. This last proof was used by Italian mathematician Guido Ubaldus as an example of a proof of God because it showed how something could appear from nothing.

$0 = 0 + 0 + 0 + 0$ ......
$0 = (1-1) + (1-1) + (1-1) + (1-1)$ ......
$0 = 1-1+1-1+1$....
$0 = 1 + (-1+1) + (-1+1) +$ ....
$0 = 1$

So, maths is a powerful tool for convincing people of an argument - but you always need to make sure that the maths is accurate!

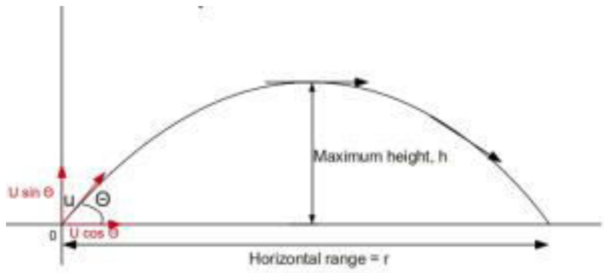
1) We divide by (a-b) in the 5th line. As a = b, then (a-b) = 0. We can't divide by zero!
2) Neither of the "triangles" are in fact triangles - the hypotenuse is not actually straight. This discrepancy allows for the apparent paradox.
3) In the second to last line we square root 1, but this has 2 possible answers, 1 or -1. As a is already defined as a = -1 then there is no contradiction.
4) This is very similar to the Cesaro Summation problem which exercised mathematicians for centuries. The infinite summation of 0 + 0 + 0 + 0 ... is not the same as the infinite summation 1 - 1 + 1 – 1 + 1 ....

War Maths - Projectile Motion

http://www.youtube.com/watch?v=lMILWzE9f0k

Despite maths having a reputation for being a somewhat bookish subject, it is also an integral part of the seamier side of human nature and has been used by armies to give their side an advantage in wars throughout the ages. Military officers all need to have a firm grasp of kinematics and projectile motion - so let's look at some War Maths.

Cannons have been around since the 1200s - and these superseded other siege weapon projectiles such as catapults which fired large rocks and burning tar into walled cities. Mankind has been finding ever more ingenious ways of firing projectiles for the best part of two thousand years.



The motion of a cannon ball can be modeled as long as we know the initial velocity and angle of elevation. If the initial velocity is V_i and the angle of elevation is θ, then we can split this into vector components in the x and y direction:

$V_{xi} = V_i\cos\theta$ (V_{xi} is the horizontal component of the initial velocity V_i)
$V_{yi} = V_i\sin\theta$ (V_{yi} is the vertical component of the initial velocity V_i)

Next we know that gravity will affect the motion of the cannonball in the y direction only - and that gravity can be incorporated using g (around 9.8 m/s^2) which gives gravitational acceleration. Therefore we can create 2 equations giving the changing velocity in both the x direction (V_x) and y direction (V_y):

$V_x = V_i\cos\theta$
$V_y = V_i\sin\theta - gt$

To now find the distance traveled we use our knowledge from kinematics - ie. that when we integrate velocity we get distance. Therefore we integrate both equations with respect to time:

$S_x = x = (V_i\cos\theta)t$
$S_y = y = (V_i\sin\theta)t - 0.5gt^2$

We now have all the information needed to calculate cannon ball projectile questions. For example if a cannon aims at an angle of 60 degrees with an initial velocity of 100 m/s, how far will the cannon ball travel?

Step (1) We find out when the cannon ball reaches maximum height:

$V_y = V_i\sin\theta - gt = 0$
$100\sin60 - 9.8(t) = 0$
$t \approx 8.83$ seconds

Step (2) We now use the fact that a parabola is symmetric around the maximum - so that after $2(8.83) \approx 17.7$ seconds it will hit the ground. Therefore substitute 17.7 seconds into the equation for $S_x = (V_i\cos\theta)t$.

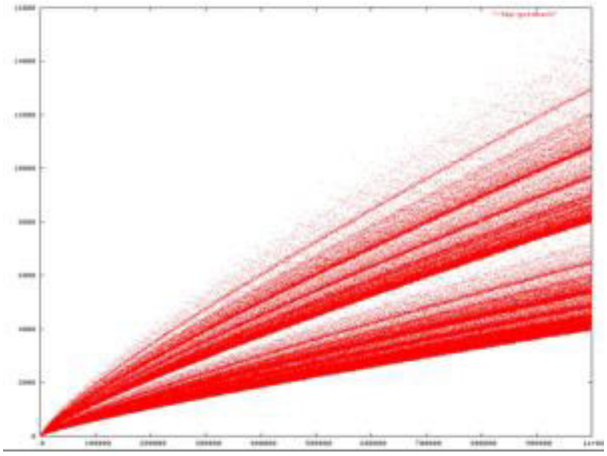
$S_x = (V_i\cos\theta)t$
$S_x = (100\cos60).17.7$
$S_x \approx 885$ metres

So the range of the cannon ball is just under 1km. You can use this JAVA app to model the motion of cannon balls under different initial conditions and also factor in air resistance.

http://www.youtube.com/watch?v=wAm-rHXyeEM

There are lots of other uses of projectile motion - the game Angry Birds is based on the same projectile principles as shooting a cannon, as is stunt racing - such as Evel Knieval's legendary motorbike jumps

The Goldbach Conjecture



The Goldbach Conjecture is one of the most famous problems in mathematics. It has remained unsolved for over 250 years - after being proposed by German mathematician Christian Goldbach in 1742. Anyone who could provide a proof would certainly go down in history as one of the true great mathematicians. The conjecture itself is deceptively simple:

"Every even integer greater than 2 can be written as the sum of 2 prime numbers."

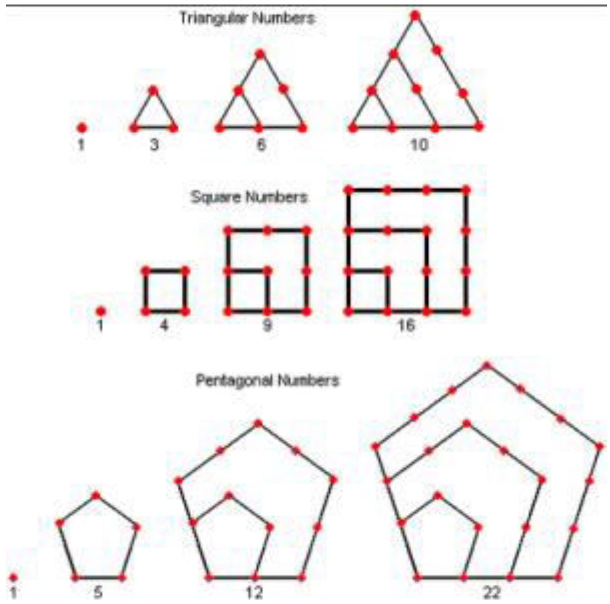
It's easy enough to choose some values and see that it *appears* to be true:

4: 2+2
6: 3+3
8: 3+5
10: 3+7 or 5+5

But unfortunately that's not enough to *prove* it's true - after all, how do we know the next number can also be written as 2 primes? The only way to prove the conjecture using this method would be to check every even number. Unfortunately there's an infinite number of these!

Super-fast computers have now checked all the first $4x10^{17}$ even numbers ($4x10^{17}$ is a number so mind bogglingly big it would take about 45 trillion years to write out, writing 1 digit every second). So far they have found that every single even number greater than 2 can indeed be written as the sum of 2 primes.

So, if this doesn't constitute a proof, then what might? Well, mathematicians have noticed that the greater the even number, the more likely it will have different prime sums. For example 10 can be written as either 3+7 or 5+5. As the even numbers get larger, they can be written with larger combinations of primes. The graph at the top of the page shows this. The x axis plots the even numbers, and the y axis plots the number of different ways of making those even numbers with primes. As the even numbers get larger, the cone widens - showing ever more possible combinations. That would suggest that the conjecture gets ever more likely to be true as the even numbers get larger.



A similar problem from Number Theory (the study of whole numbers) was proposed by legendary mathematician Fermat in the 1600s. He was interested in the links between numbers and geometry - and noticed some interesting patterns between triangular numbers, square numbers and pentagonal numbers:

Every integer (whole number) is either a triangular number or a sum of 2 or 3 triangular numbers. Every integer is a square number or a sum of 2, 3 or 4 square numbers. Every integer is a pentagonal number or a sum of 2, 3, 4 or 5 pentagonal numbers.

There are lots of things to investigate with this. Does this pattern continue with hexagonal numbers? Can you find a formula for triangular numbers or pentagonal numbers? Why does this relationship hold?

The Riemann Hypothesis Explained

This is quite a complex topic probably only accessible for high achieving HL IB students, but nevertheless it's still a fascinating introduction to one of the most important (and valuable) unsolved problems in pure mathematics.

Firstly, the Riemann Hypothesis is concerned with the Riemann zeta function. This function is defined in many ways, but probably the most useful for us is this version:

$$\zeta(s) \;=\; \left(1-2^{1-s}\right)^{-1} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}.$$

In other words the Riemann zeta function consists of a sum to infinity multiplied by an external bracket. s is a complex number of the form $s = \sigma + it$. This formula is valid for Re(s) > 0 . This means that the real part of the complex number must be positive.

Now, the Riemann Hypothesis is concerned with finding the roots of the Riemann zeta function - ie. what values of complex number s cause the function to be zero. However the equation above is only valid for Re(s) > 0. To check for roots where Re(s) is less than or equal to 0 we can use an alternative representation of the Riemann zeta function:

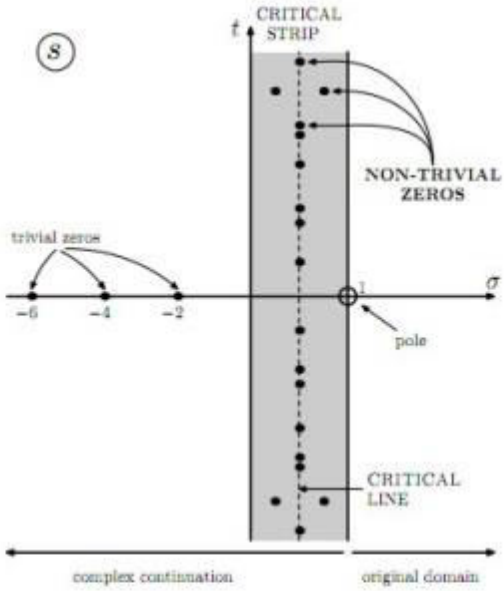
$$\zeta(s) = 2^s \pi^{s-1} \;\sin\left(\frac{\pi s}{2}\right)\; \Gamma(1-s)\; \zeta(1-s),$$

which shows that the zeta function is zero whenever s = -2,-4,-6.... as for these values the sine term becomes zero. (s = 0 has no solution in this representation because it leaves us with a zeta function of 1 in the far right term - which produces a singularity). These values are called the *trivial zeroes* of the zeta function.

The other, non-trivial zeroes of the Riemann zeta function are more difficult to find - and the search for them leads to the Riemann hypothesis:

The non-trivial zeroes of the Riemann zeta function have a real part of s equal to 1/2

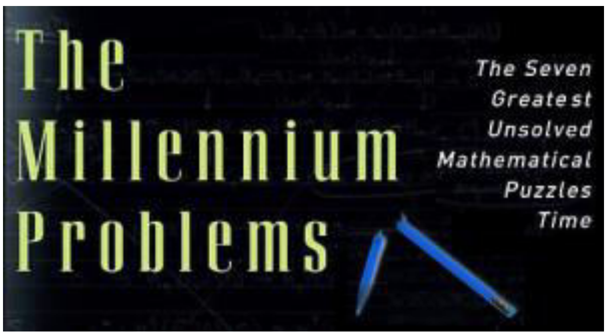
In other words, $\zeta(s)$ has non-trivial zeroes only when s is in the form s = 1/2 + *it*. This is probably easier to understand in graphical form. Below we have s plotted in the complex plane:



We can see that when s = 1 the function is not defined. This is because when s =1 in the original equation for the zeta function we get a singularity as this causes the bracket to the left of the summation to reduce to 1/0. All the non-trivial zeroes for the zeta function are known to lie in the grey boxed, "critical strip" - and the Riemann hypothesis is that they all lie on the dotted line where the real value is 1/2.

This hypothesis, made by German mathematician Bernhard Riemann in 1859 is still unsolved over 150 years later - despite some of the greatest mathematical minds of the 2oth century attempting the problem. Indeed it is considered by many mathematicians to be the most important unresolved question in pure mathematics. Mathematician David Hilbert who himself collected 23 great unsolved mathematical problems together in 1900 stated,

"If I were to awaken after having slept for a thousand years, my first question would be: has the Riemann hypothesis been proven?"



The problem is today listed as one of the Clay Institute's Millennium Prize Problems - anyone who can prove it will win $1 million and will quite probably go down in history as one of the greatest mathematicians of all time.

One solution for s which gives a zero of the zeta function is 0.5 + 14.134725142 i. Another one is 0.5 + 21.022039639 i. These both satisfy the Riemann Hypothesis by having a real part of 1/2. Indeed, to date, *10 trillion* (10,000,000,000,000) non trivial solutions have been found - and they *all* have a real part of 1/2. But this is not a proof that it is true for all roots - and so the problem remains unsolved.

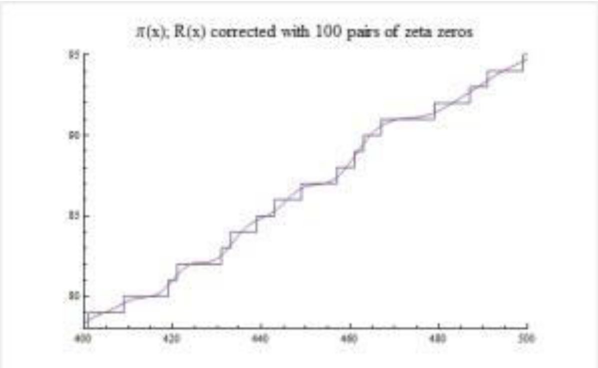
So, why is this such an important problem? Well, because there is a connection between the Riemann zeta function and distribution of prime numbers. The function below on the left is another way of representing the Riemann zeta function and the function on the right is an infinite product including all prime numbers:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}},$$

where:

$$\prod_{p \text{ prime}} \frac{1}{1-p^{-s}} = \frac{1}{1-2^{-s}} \cdot \frac{1}{1-3^{-s}} \cdot \frac{1}{1-5^{-s}} \cdot \frac{1}{1-7^{-s}} \cdot \frac{1}{1-11^{-s}} \cdots \frac{1}{1-p^{-s}} \cdots .$$

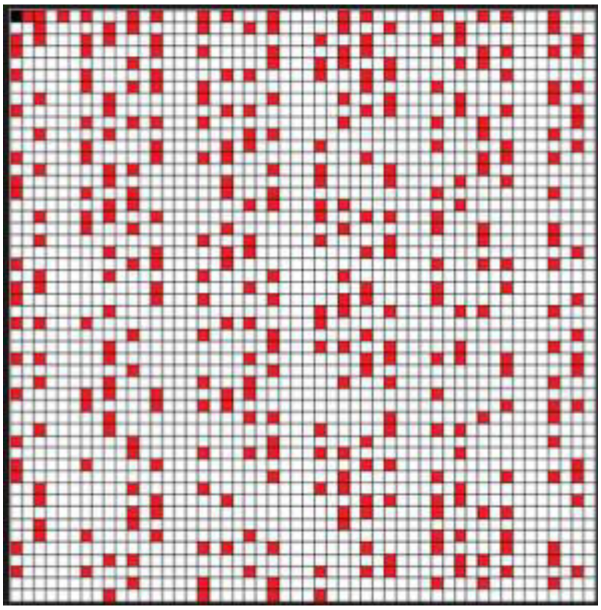
Understanding the Riemann zeta function will help mathematicians unlock some of the mysteries of the prime numbers - which are the building blocks of number theory (the study of integers). For example looking at the graph below (drawn by a Wolfram Mathlab probject) we can see the function pi(x) plotted against a function which uses both the Riemann zeta function and the distribution of its zeroes. pi(x) is blue graph and shows the number of primes less than or equal to x.



With the number of primes on the y axis, we can see that out of the first 420 numbers there are approximately 80 primes. What is remarkable about the red line is that it so accurately tracks the progress of the prime numbers.

If you are interested in reading more on this the Wikipedia page on the Riemann zeta function goes into a lot more detail. A more lighthearted introduction to the topic is given by the paper, "A Friendly Introduction to the Riemann Hypothesis"

How Are Prime Numbers Distributed? Twin Primes Conjecture



Thanks to a great post on the [Teaching Mathematics blog] about getting students to conduct an open ended investigation on consecutive numbers, I tried this with my year 10s - with some really interesting results. My favourites were these conjectures:

1) In a set of any 10 consecutive numbers, there will be no more than 5 primes. (And the only set of 5 primes is 2,3,5,7,11)
2) There is only 1 example of 3 consecutive odd numbers all being primes – 3,5,7

(You can prove both in a relatively straightforward manner by considering that a span of 3 consecutive odd numbers will always contain a multiple of 3)

Twin Prime Conjecture

These are particularly interesting because the study of the distribution of prime numbers is very much a live mathematical topic that mathematicians still work on today. Indeed studying the distribution of primes and trying to prove the twin prime conjecture are important areas of research in number theory.

The twin prime conjecture is one of those nice mathematical problems (like Fermat's Last Theorem) which is very easy to understand and explain:

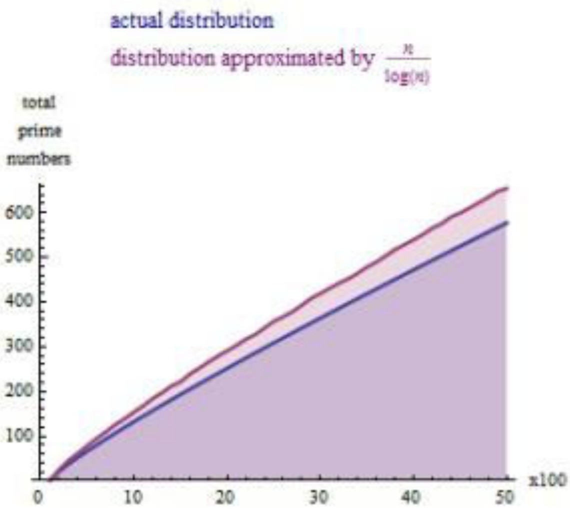
It is conjectured that there are infinitely many twin primes - ie. pairs of prime numbers which are 2 away from each other. For example 3 and 5 are twin primes, as are 11 and 13. Whilst it is easy to state the problem it is very difficult to prove.

However, this year there has been a major breakthrough in the quest to answer this problem. Chinese mathematician Yitang Zhang has proved that there are infinitely many prime pairs with gap N for some N less than 70,000,000.

This may at first glance not seem very impressive - after all to prove the conjecture we need to prove there are infinitely many prime pairs with gap N = 2. 70,000,000 is a long way away! Nevertheless this mathematical method gives a building block for other mathematicians to tighten this bound. Already that bound has been reduced to N <60,744 and is being reduced almost daily.

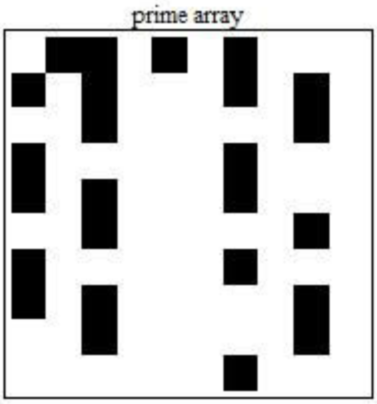
Prime Number Distribution

Associated with research into twin primes is also a desire to understand the distribution of prime numbers. Wolfram have a nice demonstration showing the cumulative distribution of prime numbers (x axis shows total integers x100)

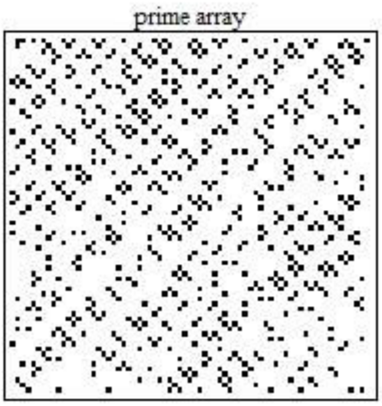


Indeed, if you choose at random an integer from the first N numbers, the probability that it is prime is approximately given by 1/ln(N).

We can see other patterns by looking at prime arrays:



This array is for the first 100 integers - counting from top left to right. Each black square represents a prime number. The array below shows the first 5000 integers. We can see that prime numbers start to "thin out" as the numbers get larger.



The desire to understand the distribution of the prime numbers is intimately tied up with the Riemann Hypothesis - which is one of the million dollar maths problems. Despite being conjectured by Bernhard Riemann over 150 years ago it has still to be proven and so remains one of the most important unanswered questions in pure mathematics.

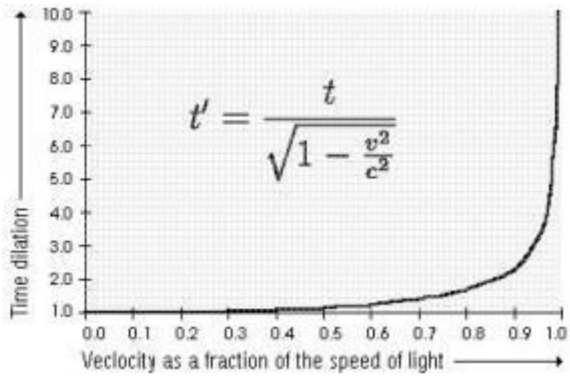
For more reading on twin primes and Yitang Zhang's discovery, there is a great (and detailed) article in Wired on this topic.

Time Travel and the Speed of Light

This is one of my favourite videos from the legendary Carl Sagan. He explains the consequences of near to speed of light travel.

http://www.youtube.com/watch?v=Uy7rrrCQh2w

This topic fits quite well into a number of mathematical topics - from graphing, to real life uses of equations, to standard form and unit conversions. It also challenges our notion of time as we usually experience it and therefore leads onto some interesting questions about the nature of reality. Below we can see the time dilation graph:



This clearly shows that for low speeds there is very little time dilation, but when we start getting to within 90% of the speed of light, that there is a very significant time dilation effect. For more accuracy we can work out the exact dilation using the formula given - where v is the speed traveled, c is the speed of light, t is the time experienced in the observer's own frame of reference (say, by looking at his watch) and t' is the time experienced in a different, stationary time frame (say on Earth) . Putting some numbers in for real life examples:

1) A long working air steward spends a cumulative total of 5 years in the air - flying at an average speed of 900km/h. How much longer will he live (from a stationary viewpoint) compared to if he had been a bus driver?

2) Voyager 1, launched in 1977 and now currently about 1.8×10^10 km away from Earth is traveling at around 17km/s. How far does this craft travel in 1 hour? What would the time dilation be for someone onboard since 1977?

3) I built a spacecraft capable of traveling at 95% the speed of light. I said goodbye to my twin sister and hopped aboard, flew for a while before returning to Earth. If I experienced 10 years on the space craft, how much younger will I be than my twin?

Scroll to the bottom for the answers

Marcus De Sautoy also presents an interesting Horizon documentary on the speed of light, its history and the CERN experiments last year that suggested that some particles may have traveled faster than light:

http://www.youtube.com/watch?v=tUR7hhkhrto

There is a lot of scope for extra content on this topic - for example, looking at the distance of some stars visible in the night sky. For example, red super-giant star Belelgeuse is around 600 light years from Earth.

(How many kilometres is that?) When we look at Betelgeuse we are actually looking 600 years "back in time" - so does it make sense to use time as a frame of reference for existence?

Answers

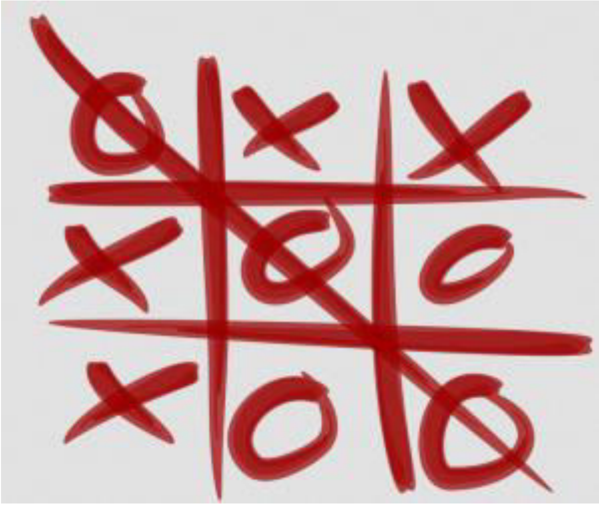
1) Convert 900km/h into km/s = 0.25km/s. Now substitute this value into the equation, along with the speed of light at 300,000km/s....and even using Google's computer calculator we get a difference so negligible that the denominator rounds to 1.

2) With units already in km/s we substitute the values in - and using a powerful calculator find that denominator is 0.99999999839. Therefore someone traveling on the ship for what their watch recorded as 35 years would actually have been recorded as leaving Earth 35.0000000562 years ago. Which is about 1.78seconds! So still not much effect.

3) This time we get a denominator of 0.3122498999 and so the time experienced by my twin will be 32 years. In effect my sister will have aged 22 years more than me on my return. Amazing!

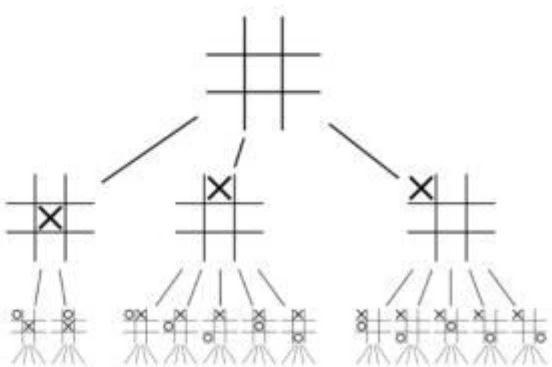
Games and Codes Topic

Game Theory and Tic Tac Toe



The game of Noughts and Crosses or Tic Tac Toe is well known throughout the world and variants are thought to have been played over 2000 years ago in Rome. It's a very simple game - the first person to get 3 in a row wins. In fact it's so simple that it has been "solved" - before any move has been played we already know it should result in a draw (as long as the participants play optimal moves).

The way to solve Noughts and Crosses is to use combinatorial Game Theory - which is a branch of mathematics that allows us to analyses all different outcomes of an event.



This is the start of the game tree for Noughts and Crosses. We can expand this game tree to cover every possible outcome for the game. Once this complete tree is drawn, any participant can work through this tree to see what is their optimal move at any one time from any position.

An upper bound for the number of positions and number of different games is given by:

9^3= 19,683. This is the total number of possible game positions in a 3x3 grid - as every square will either be a O, X or blank.

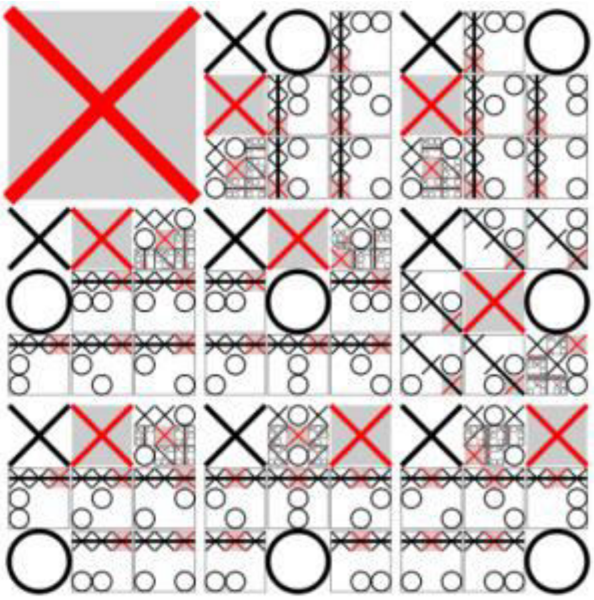
9! = 362,880. This is the total number of ways that positions can be filled on the grid. (First you have 9 choices of squares, then there are 8 choices of squares etc). This counts each X and O as distinct from other X and Os.

9 choose 5 = 126. This is the number of *different* combinations of filling the grid with 5 Xs and 4 Os.

However the analysis of this game tree can be significantly simplified by realising that many different positions are simply reflections or rotations of each other. By looking only for distinct positions (positions that are isometric under refection and rotation) we can, for example, see that there are actually only three distinct starting moves - as shown in the diagram above.

http://www.youtube.com/watch?v=wdDF7_vfLcE

James Grime from Numberphile takes us through how to answer a related question, "*How many different ways are there to completely fill a Noughts and Crosses board with 5 Xs and 4 Os - not including rotations and reflections?*" The solution above is a little complicated (it makes uses of Group Theory) but it is an excellent introduction to some uses of higher level mathematics.



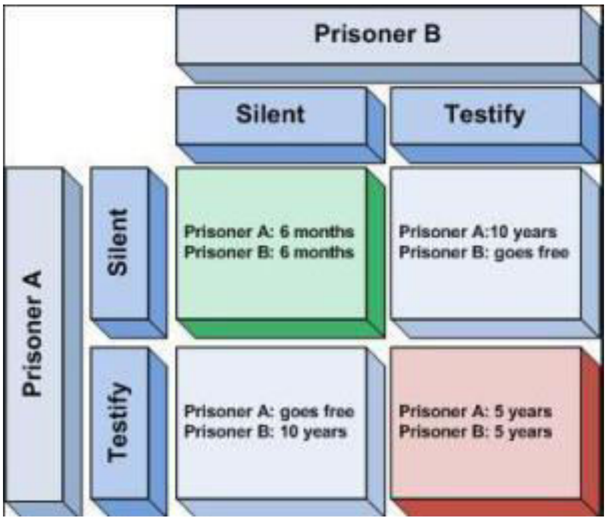
This somewhat horrendous looking graphic actually contains the solution to playing Noughts and Crosses. You can use it to always achieve the optimal outcome for X. It works as follows:
1) The big red X in the top left hand corner represents your best first move. So you make this move first.
2) Next, you see what your opponent does and choose the grid with the big black O in the position they have chosen.
3) This new grid will have a big red X - this is your next optimal move.
4) You then remain in your subsection of the larger grid - and repeat the process.

Does it Pay to be Nice? Game Theory and Evolution

Golden Balls, hosted by Jasper Carrot, is based on a version of the Prisoner's Dilemma. For added interest, try and predict what the 2 contestants are going to do. Any psychological cues to pick up on?

http://www.youtube.com/watch?v=p3Uos2fzIJ0

Game theory is an interesting branch of mathematics with links across a large number of disciplines - from politics to economics to biology and psychology. The most well known example is that of the Prisoner's Dilemma. (Illustrated below). Two prisoners are taken into custody and held in separate rooms. During interrogation they are told that if they testify to everything (ie betray their partner) then they will go free and their partner will get 10 years. However, if they both testify they will both get 5 years, and if they both remain silent then they will both get 6 months in jail.



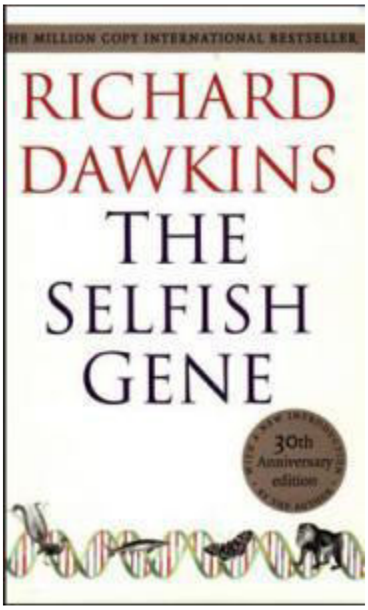
So, what is the optimum strategy for prisoner A? In this version he should testify - because whichever strategy his partner chooses this gives prisoner A the best possible outcome. Looking at it in reverse, if prisoner B testifies, then prisoner A would have been best testifying (gets 5 years rather than 10).   If prisoner B remains silent, then prisoner A would have been best testifying (goes free rather than 6 months).

This brings in an interesting moral dilemma - ie. even if the prisoner and his partner are innocent they are is placed in a situation where it is in his best interest to testify against their partner - thus increasing the likelihood of an innocent man being sent to jail. This situation represents a form of plea bargaining - which is more common in America than Europe.

Part of the dilemma arises because if both men know that the optimum strategy is to testify, then they both end up with lengthy 5 year jail sentences. If only they can trust each other to be altruistic rather than selfish - and both remain silent, then they get away with only 6 months each.   So does mathematics provide an amoral framework? i.e. in this case mathematically optimum strategies are not "nice," but selfish.

	back off	strike
back off	-1 , -1	-∞ , -∞
strike	-∞ , -∞	-∞ , -∞

Game theory became quite popular during the Cold War, as the matrix above represented the state of the nuclear stand-off. The threat of Mutually Assured Destruction (MAD) meant that neither the Americans or the Russians had any incentive to strike, because that would inevitably lead to a retaliatory strike - with catastrophic consequences. The above matrix uses negative infinity to represent the worst possible outcome, whilst both sides not striking leads to a positive pay off. Such a game has a very strong Nash Equilibrium - ie. there is no incentive to deviate from the non strike policy. Could the optimal maths strategy here be said to be responsible for saving the world?



Game theory can be extended to evolutionary biology - and is covered in Richard Dawkin's The Selfish Gene in some detail. Basically whilst it is an optimum strategy to be selfish in a single round of the prisoner's dilemma, any iterated games (ie repeated a number of times) actually tend towards a co-operative strategy. If someone is nasty to you on round one (ie by testifying) then you can punish them the next time. So with the threat of punishment, a mutually co-operative strategy is superior.

You can actually play the iterated Prisoner Dilemma game as an applet on the website Game Theory. Alternatively pairs within a class can play against each other.

An interesting extension is this applet, also on Game Theory, which models the evolution of 2 populations - residents and invaders. You can set different responses - and then see what happens to the respective populations. This is a good reflection of interactions in real life - where species can choose to live co-cooperatively, or to fight for the same resources.

The first stop for anyone interested in more information about Game Theory should be the Maths Illuminated website - which has an entire teacher unit on the subject - complete with different sections, a video and pdf documents. There's also a great article on Plus Maths - Does it Pay to be Nice? all about this topic. There are a lot of different games which can be modeled using game theory - and many are listed here . These include the Stag Hunt, Hawk/ Dove and the Peace War game. Some of these have direct applicability to population dynamics, and to the geo-politics of war versus peace.

Knight's Tour Puzzles

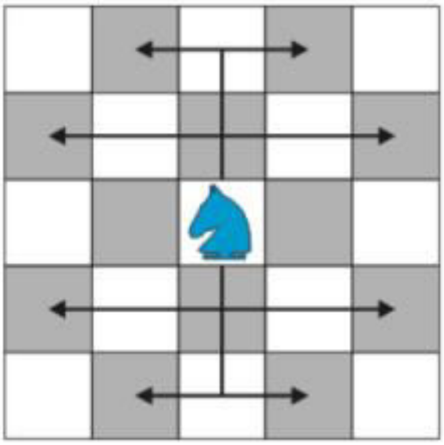
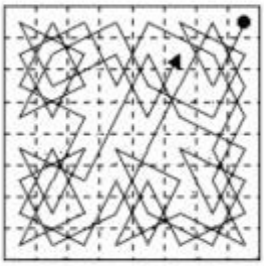


Figure 1: Legal moves for a knight

The Knight's Tour is a mathematical puzzle that has endured over 1000 years. The question is simple enough - a knight (which can move as illustrated above) wants to visit all the squares on a chess board only once. What paths can it take? You can vary the problem by requiring that the knight starts and finishes on the same square (a closed tour) and change the dimensions of the board.

The first recorded solution (as explained in [this]() excellent pdf exploration of the Knight's Tour by Ben Hill and Kevin Tostado) is shown below:

60	11	56	07	54	03	42	01
57	08	59	62	31	64	53	04
12	61	10	55	06	41	02	43
09	58	13	32	63	30	05	52
34	17	36	23	40	27	44	29
37	14	33	20	47	22	51	26
18	35	16	39	24	49	28	45
15	38	19	48	21	46	25	50

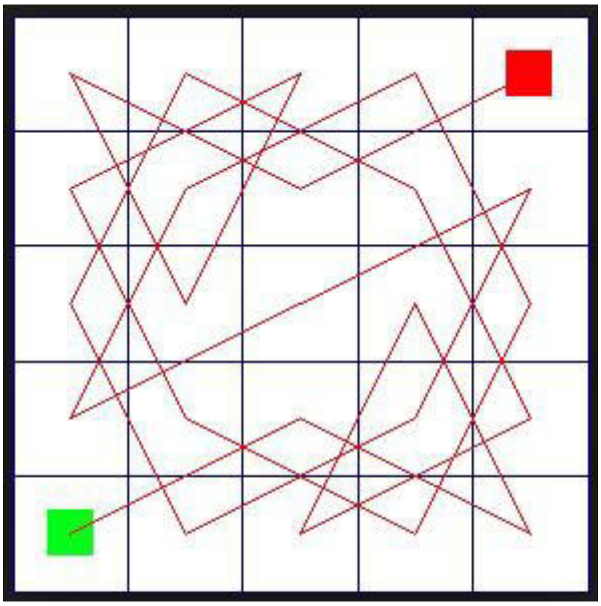


The numbers refer to the sequence of moves that the knight takes. So, in this case the knight will start in the top right hand corner (01), before hopping to number 02. Following the numbers around produces the pattern on the right. This particular knight's tour is closed as it starts and finishes at the same square and incredibly can be dated back to the chess enthusiast al-Adli ar-Rumi circa 840 AD.

Despite this puzzle being well over 1000 years old, and despite modern computational power it is still unknown as to how many distinct knight's tours there are for an 8x8 chess board. The number of distinct paths are as follows:

1x1 grid: 1,
2x2 grid: 0,
3x3 grid: 0,
4x4 grid: 0,
5x5 grid: 1728,
6x6 grid: 6,637,920,
7x7 grid: 165,575,218,320
8x8 grid: unknown

We can see just how rapidly this sequence grows by going from 6x6 to 7x7 - so the answer for the 8x8 grid must be huge. Below is one of the 1728 solutions to the 5x5 knight's tour:



You might be wondering if this has any applications beyond being a diverting puzzle, well Euler - one of the world's true great mathematicians - used knight's tours in his study of graph theory. Graph theory is an entire branch of mathematics which models connections between objects.

Knight's tours have also been used for cryptography:

PUZZLE

sor	to	king	good	say	luck	loy	eth
and	moth	a	soon	dis	our	to	bad
place	ry	church	his	force	is	hat	al
er	queen	him	wight	he	to	may	truth
man	his	and	and	chess	es	knight	op's
a	sneer	the	and	un	lawn	of	tates
cas	that	at	less	pawn	no	bish	lant
eth	faith	tles	hath	the	gal	in	love

TOUR SOLUTION

14	55	22	37	12	51	18	35
23	38	13	54	17	36	11	50
56	15	40	21	52	9	34	19
39	24	53	16	33	20	49	10
2	57	28	41	8	61	32	47
25	42	1	60	29	48	7	62
58	3	44	27	64	5	46	31
43	26	59	4	45	30	63	6

VERSE SOLUTION

The man that have no love of chess
Is, truth to say, a sorry wight,
Disloyal to his king and queen.
A faithless and ungallant knight;
He hateth our good mother church,
And sneereth at the bishop's lawn;
May bad luck force him soon to place
His castles and estates in pawn!

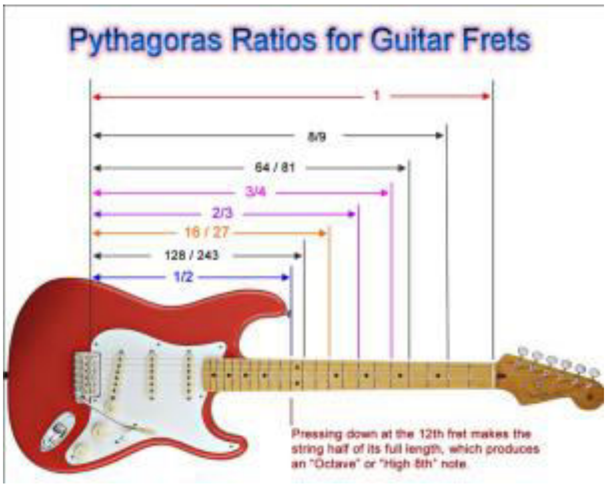
This code is from the 1870s and exploits the huge number of possible knight's tours for an 8x8 chess board. You would require that the recipient of your code knew the tour solution (bottom left) in advance. With this solution key you can read the words in order - first by finding where 1 is in the puzzle (row 6 column 3) - and seeing that this equates to the word "the". Next we see that 2 equates to "man" and so on. Without the solution key you would be faced with an unimaginably large number of possible combinations - making cracking the code virtually impossible.

If you are interested in looking at some more of the maths behind the knight's tour problem then the paper by Ben Hill and Kevin Tostado referenced above go into some more details. In particular we have the following rules:

An m x n chessboard with m less than or equal to n has a knight's tour unless one or more of these three conditions hold:

1) m and n are both odd
2) m = 1, 2 or 4
3) m = 3 and n = 4,6,8

Maths and Music



Western music has its roots in the harmonics discovered by Pythagoras - himself a keen musician - over 2000 years ago. Pythagoras noticed that certain string ratios would produce sounds that were in harmony with each other. The simplest example is illustrated above with an electric guitar. When a string is played, and then that same string pressed half-way along its length (in the guitar's case the 12th fret), then we get the same note - this is a whole octave.

If you were to then half the distance again you would get another octave (which explains why guitar frets get smaller and smaller near the base of the instrument - the frets mark ratios relative to the whole string).

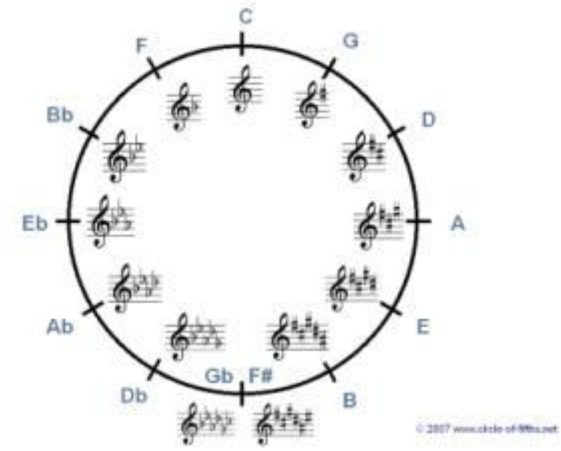
The ratio **1: 1/2** shows the ratio of an octave. A full length string: half length string. We can multiply both sides by 2 to remove the fraction to get, 2: 1. This is the octave ratio.

All the other harmonies that are the basis of Western music can also be understood through similar ratios. The chord sequence E, A, B - which is the standard progression for blues and modern music comprises of the base note (in this case E), along with the perfect fourth (A) and the major fifth (B) of the base note.

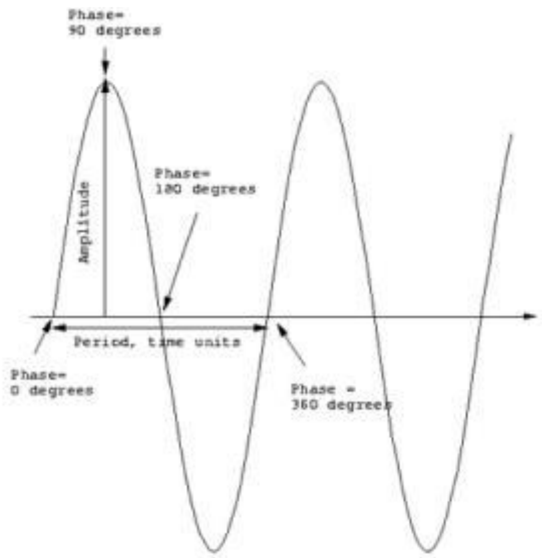
Looking at the guitar fret we can see that the perfect fourth (A), which occurs on the fifth fret, has the ratio **1: 3/4**. That is 1 whole string: 3/4 of the whole string. We can simplify this to get 4:3.

The major fifth (B) which occurs on the seventh fret has the ratio **1: 2/3** which simplifies to 3:2.

The other most likely note used in the key of E would then also be either G (the minor third) which has a ratio of 6:5, or G sharp (the major third) which has a ratio of 5:4).



It's interesting that we find these particular whole number ratios pleasing to listen to - indeed these are the notes that often sound "right" when playing through songs. It's also helpful to look at the circle of fifths - which shows all notes which are in the ratio 3:2 with each other. Moving around the circle again produces music which sounds nice. For an example of this, starting at C, the progression C,G,D,A,E is the one used by Jimmy Hendrix in the classic song, Hey Joe

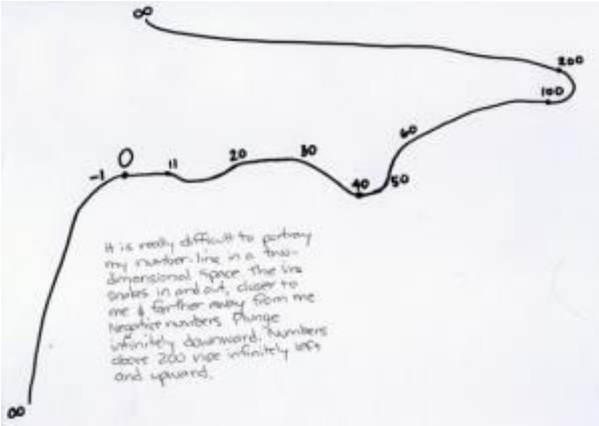


There are lots of other areas to explore when looking at the relationship between maths and music - one of which is looking at how we can model the wave frequencies of notes using modified sine/cosine curves. The IB have included a piece of coursework on this as an example for the new exploration topics.

Another interesting exploration is looking at the strange properties of the Harmonic Sequence - which is the sequence 1, 1/2, 1/3, 1/4... This sequence like many of those found in music is said to be in harmonic progression . There are some interesting paradoxes related to the harmonic sequence - and a variety of methods of proving that the sum of this sequence (the series) actually diverges to infinity - even though you would intuitively expect it to converge. The video below provides a taster on this topic:

http://www.youtube.com/watch?v=x8-3OlB3G0M

Synesthesia - Do Your Numbers Have Colour?



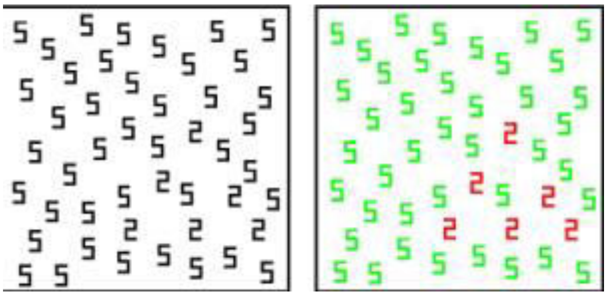
Synesthesia is another topic which provides insights into how people perceive numbers - and how a synesthetic's perception of the mathematical world is distinctly different to everyone else's.

Those with synesthesia have a cross-wiring of brain activity between 2 of their senses - so for example they may hear sounds when they see images, sounds may invoke taste sensations, or numbers may be perceived as colours.

Daniel Tammet, an autistic savant with remarkable memory abilities (he can remember pi to 22 thousand places and learn a new language to fluency in one week). He also has number synesthesia which means that he "sees" numbers as each having their own distinct colour. This also allows him to multiply two numbers in his head almost instantaneously by "seeing" the two colours merge into a third one.

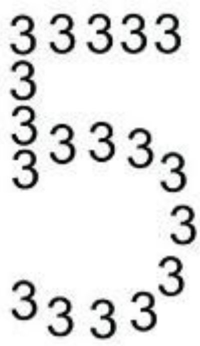
http://www.youtube.com/watch?NR=1&v=AbASOcqc1Ss&feature=endscreen

Dr Ramachandran (of phantom limb fame) has written a fascinating academic article looking at synesthesia - and estimates that as many as 1 in 200 people may have some form of it. A simple test of grapheme colour synesthesia (where people perceive numbers with colours) is the graphic below:



For people without synesthesia, locating the 2s from graphic on the left is a slow process, but for people with synesthesia, they can immediately see the 2s as standing out - like the graphic on the right. This test is easily able to distinguish that this type of synesthesia is real.

Those with grapheme synesthesia also report that the image below *changes* colour - depending on whether they look at the whole image (ie. a five) or concentrate on how it is made of smaller constituent parts (of threes):



What is truly remarkable about synesthesia is what it reveals about our brain's innate capacity for mathematical calculations far beyond what average people can achieve. Francois Galton, the 19th Century polymath who first documented the condition (which he himself had) described how synesthetics often also experienced a tangible number line in their mind - that was not straight but curved and bent and in which some numbers were closer that others (an example is at the top of the page). This allowed him, and others like Temmet, to perform lightening fast mental calculations of unimaginable complexity. In the above video Daniel is able to divide 13 by 97 in a matter of seconds to over 30 decimal places.

Numberphile have also made a short video in which they interview a lady with synesthesia:

http://www.youtube.com/watch?v=dNy23tJMTzQ

Could one day we all unlock this potential? And what does this condition tell us about whether numbers exist in any tangible sense? Do they exist in a more real sense for a grapheme synesthic than someone else?

Benford's Law - Using Maths to Catch Fraudsters

http://www.youtube.com/watch?v=vIsDjbhbADY

Benford's Law is a very powerful and counter-intuitive mathematical rule which determines the distribution of leading digits (ie the first digit in any number). You would probably expect that distribution would be equal - that a number 9 occurs as often as a number 1. But this, whilst intuitive, is false for a large number of datasets. Accountants looking for fraudulant activity and investigators looking for falsified data use Benford's Law to catch criminals.

The probability function for Benford's Law is:

A set of numbers is said to satisfy Benford's law if the leading digit d ($d \in \{1, \ldots, 9\}$) occurs with probability

$$P(d) = \log_{10}(d+1) - \log_{10}(d) = \log_{10}\left(1 + \frac{1}{d}\right).$$

d	$P(d)$	Relative size of $P(d)$
1	30.1%	
2	17.6%	
3	12.5%	
4	9.7%	
5	7.9%	
6	6.7%	
7	5.8%	
8	5.1%	
9	4.6%	

This clearly shows that a 1 is by far the most likely number to occur - and that you have nearly a 60% chance of the leading digit being 3,2 or 1. Any criminal trying to make up data who didn't know this law would be easily caught out.

Scenario for students 1:

You are a corrupt bank manager who is secretly writing cheques to your own account. You can write any cheques for any amount – but you want it to appear natural so as not to arouse suspicion. Write yourself 20 cheque amounts. Try not to get caught!

Look at the following fraudulent cheques that were written by an Arizona manager - can you see why he was caught?

The table lists the checks that a manager in the office of the Arizona State Treasurer wrote to divert funds for his own use. The vendors to whom the checks were issued were fictitious.

Date of Check	Amount
October 9, 1992	$ 1,927.48
	27,902.31
October 14, 1992	86,241.90
	72,117.46
	81,321.75
	97,473.96
October 19, 1992	93,249.11
	89,658.17
	87,776.89
	92,105.83
	79,949.16
	87,602.93
	96,879.27
	91,806.47
	84,991.67
	90,831.83
	93,766.67
	88,338.72
	94,639.49
	83,709.28
	96,412.21
	88,432.86
	71,552.16
TOTAL	$ 1,878,687.58

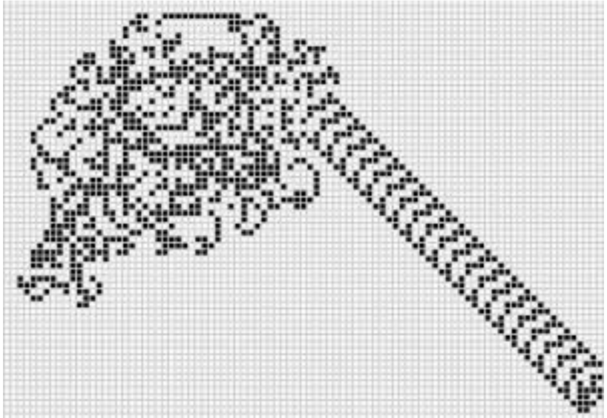
Scenario for students 2:

Use the formula for the probability density function to find the probability of the respective leading digits. Look at the leading digits for the first 50 Fibonacci numbers. Does the law hold?

(1) = 1	(26) = 121393
(2) = 1	(27) = 196418
(3) = 2	(28) = 317811
(4) = 3	(29) = 514229
(5) = 5	(30) = 832040
(6) = 8	(31) = 1346269
(7) = 13	(32) = 2178309
(8) = 21	(33) = 3524578
(9) = 34	(34) = 5702887
(10) = 55	(35) = 9227465
(11) = 89	(36) = 14930352
(12) = 144	(37) = 24157817
(13) = 233	(38) = 39088169
(14) = 377	(39) = 63245986
(15) = 610	(40) = 102334155
(16) = 987	(41) = 165580141
(17) = 1597	(42) = 267914296
(18) = 2584	(43) = 433494437
(19) = 4181	(44) = 701408733
(20) = 6765	(45) = 1134903170
(21) = 10946	(46) = 1836311903
(22) = 17711	(47) = 2971215073
(23) = 28657	(48) = 4807526976
(24) = 46368	(49) = 7778742049
(25) = 75025	(50) = 12586269025

There is also an excellent Numberphile video on Benford's Law. Wikipedia has a lot more on the topic, as have the Journal of Accountancy.

The Game of Life



This is another fascinating branch of mathematics - which uses computing to illustrate complexity (and order) in nature. Langton's Ant shows how very simple initial rules (ie a deterministic system) can have very unexpected consequences. Langton's Ant follows two simple rules:
1) At a white square, turn 90° right, flip the color of the square, move forward one unit
2) At a black square, turn 90° left, flip the color of the square, move forward one unit.

The ant exists on an infinite grid - and is able to travel N,S,E or W. You might expect the pattern generated to either appear completely random, or to replicate a fixed pattern. What actually happens is you have a chaotic pattern for around 10,000 iterations - and then all of a sudden a diagonal "highway" emerges - and then continues forever. In other words there is emergent behavior - order from chaos. What is even more remarkable is that you can populate the initial starting grid with any number of black squares - and you will still end up with the same emergent pattern of an infinitely repeating diagonal highway.

See a JAVA app demonstration (this uses a flat screen where exiting the end of one side allows you to return elsewhere - so this will ultimately lead to disruption of the highway pattern)

Such cellular automatons are a way of using computational power to try and replicate the natural world - The Game of Life is another well known automaton which starts of with very simple rules - designed to replicate (crudely) bacterial population growth. Small changes to the initial starting conditions result in wildly different outcomes - and once again you see patterns emerging from apparent random behavior. Such automatons can themselves be used as "computers" to calculate the solution to problems. One day could we design a computer program that replicates life itself? Could that then be said to be alive?

Cracking RSA Code - The World's Most Important Code?

RSA code is the basis of all important data transfer. Encrypted data that needs to be sent between two parties, such as banking data or secure communications relies on the techniques of RSA code. RSA code was invented in 1978 by three mathematicians (Rivest, Shamir and Adleman). Cryptography relies on numerous mathematical techniques from Number Theory - which until the 1950s was thought to be a largely theoretical pursuit with few practical applications. Today RSA code is absolutely essential to keeping digital communications safe.

To encode a message using the RSA code follow the steps below:

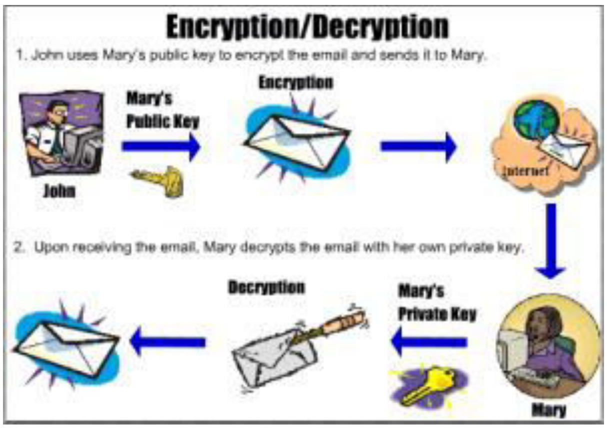
1) Choose 2 prime numbers p and q (let's say p=7 and q=5)

2) Multiply these 2 numbers together (5x7 = 35). This is the public key (m) - which you can let everyone know. So m = 35.

3) Now we need to use an encryption key (e). Let's say that e = 5. e is also made public. (There are restrictions as to what values e can take - e must actually be relatively prime to (p-1)(q-1))

4) Now we are ready to encode something. First we can assign 00 = A, 01 = B, 02 = C, 03 = D, 04 = E etc. all the way to 25 = Z. So the word CODE is converted into: 02, 14, 03, 04.

5) We now use the formula: $C = y^e \pmod m$ where y is the letter we want to encode. So for the letters CODE we get: $C = 02^5 = 32 \pmod{35}$. $C = 14^5 = 537824$ which is equivalent to 14 (mod 35). $C = 03^5 = 33 \pmod{35}$. $C = 04^5 = 1024$ which is equivalent to 09 (mod 35). (Mod 35 simply mean we look at the remainder when we divide by 35). Make use of an online modulus calculator! So our coded word becomes: 32 14 33 09.



This form of public key encryption forms the backbone of the internet and the digital transfer of information. It is so powerful because it is very quick and easy for computers to decode if they know the original prime numbers used, and exceptionally difficult to crack if you try and guess the prime numbers. Because of the value of using very large primes there is a big financial reward on offer for finding them. The world's current largest prime number is over 17 million digits long and was found in February 2013. Anyone who can find a prime 100 million digits long will win $100,000.

To decode the message 11 49 41 we need to do the following:

1) In RSA encryption we are given both m and e. These are public keys. For example we are given that m = 55 and e = 27. We need to find the two prime numbers that multiply to give 55. These are p = 5 and q = 11.

2) Calculate (p-1)(q-1). In this case this is (5-1)(11-1) = 40. Call this number theta.

3) Calculate a value d such that de = 1 (mod theta). We already know that e is 27. Therefore we want 27d = 1 (mod 40). When d = 3 we have 27x3 = 81 which is 1 (mod 40). So d = 3.

4) Now we can decipher using the formula: y = C^d (mod m), where C is the codeword. So for the cipher text 11 49 41: $y = 11^3 = 08 \pmod{55}$. $y = 49^3 = 04 \pmod{55}$. $y = 41^3 = 6 \pmod{55}$.

5) We then convert these numbers back to letters using A = 00, B = 01 etc. This gives the decoded word as: LEG.

Cracking ISBN and Credit Card Codes

ISBN 10: 1-932698-18-3
ISBN 13: 978-1-932698-18-3

9 781932 698183

ISBN codes are used on all books published worldwide. It's a very powerful and useful code, because it has been designed so that if you enter the wrong ISBN code the computer will immediately know – so that you don't end up with the wrong book. There is lots of information stored in this number. The first numbers tell you which country published it, the next the identity of the publisher, then the book reference.

Here is how it works:

Look at the 10 digit ISBN number. The first digit is 1 so do 1x1. The second digit is 9 so do 2x9. The third digit is 3 so do 3x3. We do this all the way until 10x3. We then add all the totals together. If we have a proper ISBN number then we can divide this final number by 11. If we have made a mistake we can't. This is a very important branch of coding called error detection and error correction. We can use it to still interpret codes even if there have been errors made.
If we do this for the barcode above we should get 286. 286/11 = 26 so we have a genuine barcode.

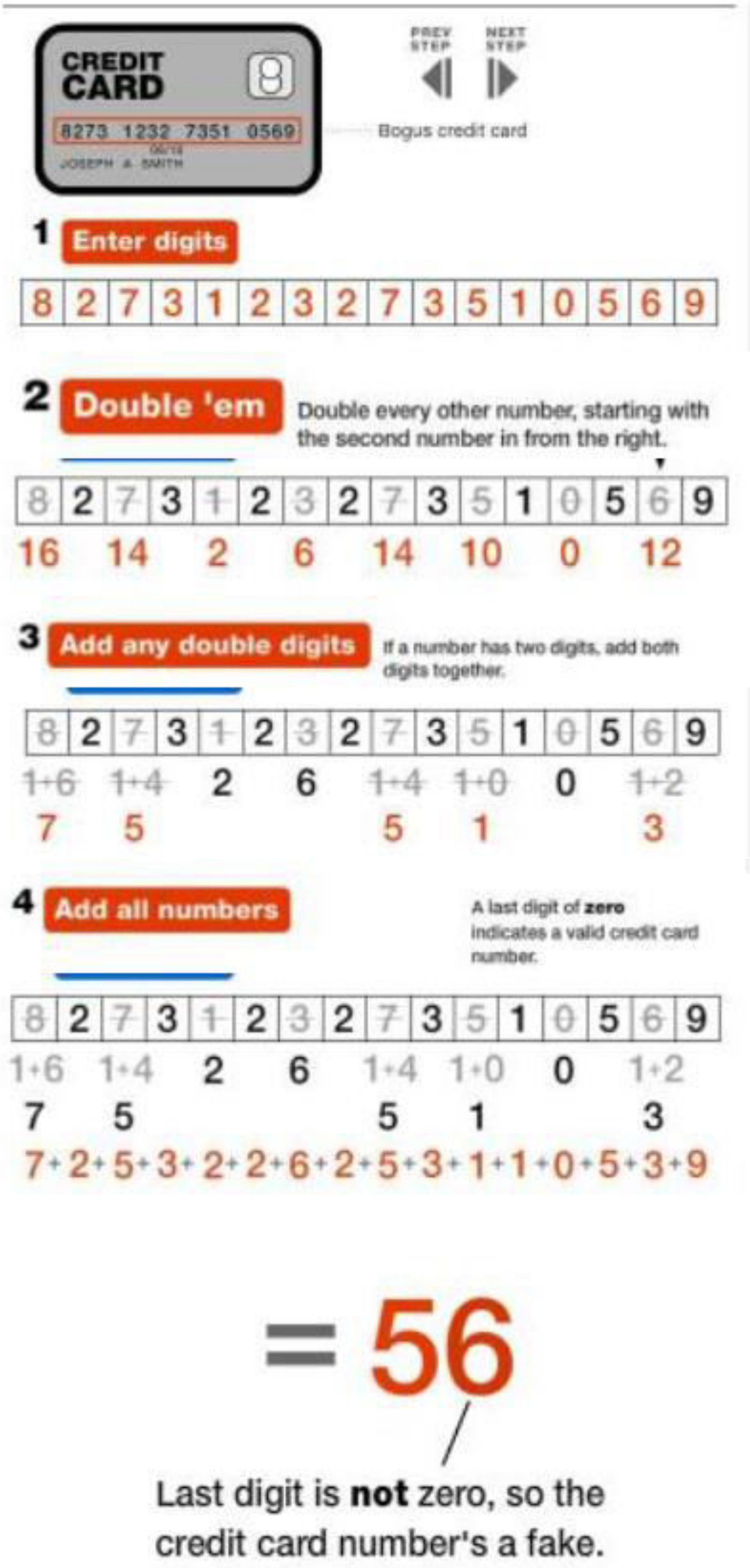
Check whether the following are ISBNs

1) 0-13165332-6
2) 0-1392-4191-4
3) 07-028761-4

Challenge (harder!) : The following ISBN code has a number missing, what is it?
1) 0-13-1?9139-9

Answers in white text at the bottom, highlight to reveal!

Credit cards use a different algorithm - but one based on the same principle - that if someone enters a digit incorrectly the computer can immediately know that this credit card does not exist. This is obviously very important to prevent bank errors. The method is a little more complicated than for the ISBN code and is given below from computing site Hacktrix:


CREDIT CARD
8273 1232 7351 0569
JOSEPH A SMITH
PREV STEP
NEXT STEP
Bogus credit card
1 Enter digits
8 2 7 3 1 2 3 2 7 3 5 1 0 5 6 9
2 Double 'em
Double every other number, starting with the second number in from the right.
8 2 7 3 1 2 3 2 7 3 5 1 0 5 6 9
16 14 2 6 14 10 0 12
3 Add any double digits
If a number has two digits, add both digits together.
8 2 7 3 1 2 3 2 7 3 5 1 0 5 6 9
1+6 1+4 2 6 1+4 1+0 0 1+2
7 5 5 1 3
4 Add all numbers
A last digit of zero indicates a valid credit card number.
8 2 7 3 1 2 3 2 7 3 5 1 0 5 6 9
1+6 1+4 2 6 1+4 1+0 0 1+2
7 5 5 1 3
7+2+5+3+2+2+6+2+5+3+1+1+0+5+3+9
= 56
Last digit is not zero, so the credit card number's a fake.

You can download a worksheet for this method here. Try and use this algorithm to validate which of the following 3 numbers are genuine credit cards:

1) 5184 8204 5526 6425

2) 5184 8204 5526 6427

3) 5184 8204 5526 6424

Answers:

ISBN:
1) Yes
2) No
3) No
1) 3 – using x as the missing number we end up with $5x + 7 = 0 \mod 11$. So $5x = 4 \mod 11$. When $x = 3$ this is solved.
Credit Card: The second one is genuine

NASA, Aliens and Binary Codes from the Star

The Drake Equation was intended by astronomer Frank Drake to spark a dialogue about the odds of intelligent life on other planets. He was one of the founding members of SETI - the Search for Extra Terrestrial Intelligence - which has spent the past 50 years scanning the stars looking for signals that could be messages from other civilisations.

In the following video, Carl Sagan explains about the Drake Equation:

http://www.youtube.com/watch?v=MlikCebQSlY

The Drake equation is:

$$N = R^* \cdot f_p \cdot n_e \cdot f_\ell \cdot f_i \cdot f_c \cdot L$$

where:

N = the number of civilizations in our galaxy with which communication might be possible (i.e. which are on our current past light cone);
R = the average number of star formation per year in our galaxy*
fp = the fraction of those stars that have planets
ne = the average number of planets that can potentially support life per star that has planets
fl = the fraction of planets that could support life that actually develop life at some point
fi = the fraction of planets with life that actually go on to develop intelligent life (civilizations)
fc = the fraction of civilizations that develop a technology that releases detectable signs of their existence into space
L = the length of time for which such civilizations release detectable signals into space

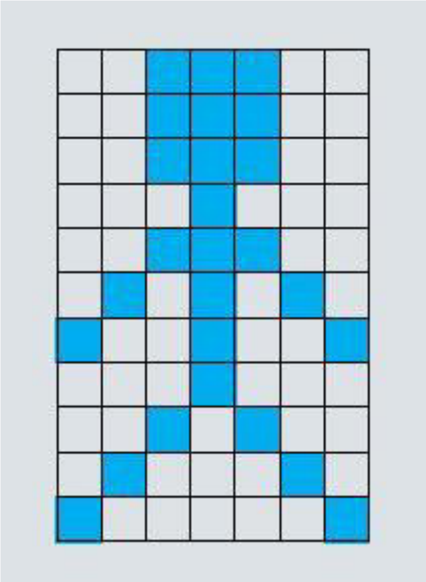
The desire to encode and decode messages is a very important branch of mathematics - with direct application to all digital communications - from mobile phones to TVs and the internet.

All data content can be encoded using binary strings. A very simple code could be to have 1 signify "black" and 0 to signify "white" - and then this could then be used to send a picture. Data strings can be sent which are the product of 2 primes - so that the recipient can know the dimensions of the rectangle in which to fill in the colours.

If this sounds complicated, an example from the excellent Maths Illuminated handout on codes:

0011100001110000111000001000001110001010101001001000100000101000100
0101000001

If this mystery message was received from space, how could we interpret it? Well, we would start by noticing that it is 77 digits long - which is the product of 2 prime numbers, 7 and 11. Prime numbers are universal and so we would expect any advanced civilisation to know about their properties. This gives us either a 7x11 or 11x7 rectangular grid to fill in. By trying both possibilities we see that an 11x7 grid gives the message below.



More examples can be downloaded from the Maths Illuminated section on Primes (go to the facilitator pdf).

A puzzle to try:

"If the following message was received from outer space, what would we conjecture that the aliens sending it looked like?"

0011000 0011000 1111111 1011001 0011001 0111100 0100100 0100100 0100100 1100110

Hint: also 77 digits long.

This is an excellent example of the universality of mathematics in communicating across all languages and indeed species. Prime strings and binary represent an excellent means of communicating data that all advanced civilisations would easily understand.

Answer:

Arrange the code into a rectangular array - ie a 11 rows by 7 columns rectangle. The first 7 numbers represent the 7 boxes in the first row etc. A 0 represents white and 1 represents black. Filling in the boxes and we end up with an alien with 2 arms and 2 legs - though with one arm longer than the other.