LIMITS AND IMPORTANT THEOREMS

THE ALGEBRA OF LIMITS

If the sequence $\{a_n\}$ converges to a limit *a* and the sequence $\{b_n\}$ converges to a limit *b*, then:

$$\lim_{n\to\infty}(pa_n + qb_n) = p\lim_{n\to\infty}a_n + q\lim_{n\to\infty}b_n = pa + qb$$

$$\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty}a_n \cdot \lim_{n\to\infty}b_n = ab$$

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty}a_n}{\lim_{n\to\infty}b_n} = \frac{a}{b}, b \neq 0$$

If the sequence $\{a_n\}$ diverges, then for any constant $c \in \mathbb{R}, (c \neq 0)$

$$\lim_{n\to\infty}\left(\frac{c}{a_n}\right) = 0$$

$$\lim_{n\to\infty}\left(\frac{a_n}{c}\right) = \infty$$

THE SQUEEZE THEOREM

If we have sequences $\{a_n\}$, $\{b_n\}$,and $\{c_n\}$ such that $a_n \leq b_n \leq c_n$, for all $n \in \mathbb{Z}^+$ and

$$\lim_{n\to\infty}a_n = \lim_{n\to\infty}c_n = L < \infty$$

then

$$\lim_{n\to\infty}b_n = L$$

In other words, if we can find two sequences that coverage to the same limit and squeeze another sequence between them, then that sequence must also converge to the same limit.

EXAMPLE USE THE SQUEEZE THEOREM TO FIND $\lim\limits_{n\to\infty}\dfrac{\sin n}{n}$

Start by bounding the sin n $\quad -1 \le \sin n \le 1$

Then, $\quad \dfrac{-1}{n} \le \dfrac{\sin n}{n} \le \dfrac{1}{n}$ for all $n \in \mathbb{Z}^+$

Since, $\quad \lim\limits_{n\to\infty}\dfrac{-1}{n} = \lim\limits_{n\to\infty}\dfrac{1}{n} = 0$

By the Squeeze Theorem $\quad \lim\limits_{n\to\infty}\dfrac{\sin n}{n} = 0$

When using the Squeeze Theorem, it can be difficult to choose the sequences $\{a_n\}$,and $\{c_n\}$. Here is a common way of doing this.

EXAMPLE: Show that $\lim\limits_{n\to\infty}\dfrac{n!}{n^n} = 0$.

Because $\dfrac{n!}{n^n} > 0$ for all $n \in \mathbb{Z}^+$ we are looking for a sequence that is always at least as large as $\dfrac{n!}{n^n}$ but which tends toward 0 as $n \to \infty$ so that we can squeeze $\dfrac{n!}{n^n}$.
Start by writing the terms of the sequence by hand

$$\begin{aligned}\frac{n!}{n^n} &= \frac{n}{n}\cdot\frac{(n-1)}{n}\cdot\frac{(n-2)}{n}\cdot\frac{(n-3)}{n}\cdots\frac{3}{n}\cdot\frac{2}{n}\cdot\frac{1}{n}\\ &< \frac{n}{n}\cdot\frac{n}{n}\cdot\frac{n}{n}\cdot\frac{n}{n}\cdots\frac{n}{n}\cdot\frac{n}{n}\cdot\frac{1}{n} = \frac{1}{n}\end{aligned}$$

$$0 < \frac{n!}{n^n} < \frac{1}{n} \text{ for all } n \in \mathbb{Z}^+$$

Since $\lim\limits_{n\to\infty}\dfrac{1}{n} = 0$, $\lim\limits_{n\to\infty}\dfrac{n!}{n^n} = 0$ by the Squeeze Theorem.

For the limit of a *function, f(x),* to exist, the limit of the function from the right and from the left both have to exist and coincide for the limit to exist at that point. If they do then

$$\lim_{n\to x_0} f(x) = \lim_{n\to x_{0^+}} f(x) = \lim_{n\to x_{0^-}} f(x)$$

EXAMPLE: Show that $\lim_{x\to 0}\cos\left(\frac{1}{x}\right)$ does not exist.

If we can find two sequences, x_n and y_n that tend to zero but for which $f(x_n)$ and $f(y_n)$ tend to different limits, then the limit of $f(x)$ does not exist.

Let $f(x) = \cos\left(\frac{1}{x}\right)$

The sequence $a_n = 2\pi, 4\pi, 6\pi, ..., 2n\pi$ has $\cos a_n = 1$ and

$b_n = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ..., \frac{(2n-1)\pi}{2}$ with $\cos b_n = 0$

Take $x_n = \frac{1}{a_n}$ and $y_n = \frac{1}{b_n}$

Then, $f(x_n) = \cos\left(\frac{1}{\frac{1}{2n\pi}}\right) = \cos(2n\pi) = 1$ for all *n*

Therefore, $\lim_{x_n\to 0} f(x_n) = 1$

But, $f(y_n) = \cos\left(\frac{1}{\frac{2}{(2n-1)\pi}}\right) = \cos\left(\frac{(2n-1)\pi}{2}\right) = 0$ for all *n*.

Therefore, $\lim_{y_n\to 0} f(y_n) = 0$

As $\lim_{x_n\to 0} f(x_n) \neq \lim_{y_n\to 0} f(y_n)$, the limit does not exist.

As for sequences, the Squeeze Theorem also holds true for functions.

EXAMPLE Show that $\lim_{x\to 0} x\cos\left(\frac{1}{x}\right)=0$

$$\left|x\cos\left(\frac{1}{x}\right)\right|=|x|\cdot\left|\cos\left(\frac{1}{x}\right)\right|\le|x|$$

$$-|x|\le x\cos\left(\frac{1}{x}\right)\le|x|$$

So, by the Squeeze Theorem, $\lim_{x\to 0}(-|x|)=0$ and $\lim_{x\to 0}(|x|)=0$ so, $\lim_{x\to 0} x\cos\left(\frac{1}{x}\right)=0$.

EXAMPLE of an Algebraic technique.

Evaluate $\lim_{x\to -2}\left(\frac{(x+2)^2}{\sqrt{x^2+4x+13}-3}\right)$

$$\frac{(x+2)^2}{\sqrt{x^2+4x+13}-3}=\frac{(x+2)^2}{\sqrt{(x+2)^2+9}-3}$$

$$=\frac{(x+2)^2}{\sqrt{(x+2)^2+9}-3}\cdot\frac{\sqrt{(x+2)^2+9}+3}{\sqrt{(x+2)^2+9}+3}$$

$$=\frac{(x+2)^2\sqrt{(x+2)^2+9}+3}{(x+2)^2+9-9}$$

$$=\sqrt{(x+2)^2+9}+3$$

$$\lim_{x\to -2}\sqrt{(x+2)^2+9}+3=6$$