Differential Equations

Differential equations are the family of equations used to describe dynamic phenomena. They are complex equations where the unknown is a function and the equation describes a relation between: this function, its derivative, the independent variable and some constants.

Classifications of Differential Equations

Ordinary Differential Equations are those that relate a function of one variable with: its derivatives; an independent variable; and constants. The **order** of a differential equation is the order of the highest derivative involved in the equation.

$$\frac{dy}{dx} + y = 4 \text{ is a first order differential equation}$$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = y \text{ is a second order differential equation}$$

$$\frac{d^3y}{dx^3} + 5x\frac{dy}{dx} = e^x y \text{ is a third order differential equation}$$

Equations two and three can be classified according to the type of coefficients – equation 2 has constant coefficients but in equation three, the coefficients are dependent upon *x* and are called variable coefficients.

Classify each of the following differential equations.

$y' - 4y = 5$

First order with constant coefficients

$y''' + 5y'' - xy' = 4y + 3$

Third order with variable Coefficients

Another important classification has to do with what is done with the function $y = y(x)$ and its derivatives: Ordinary Differential Equations can also be classified as linear or non-linear. For example, $xy' + \frac{2}{x}y = x^2$ and $\frac{2}{x}y' + 3xy = e^x$ are **first order linear** differential equations because they are of the form

$$a(x)\frac{dy}{dx} + b(x)y = c(x)$$

where both the function and its derivative are multiplied only by functions of *x*.

A non-linear equation involves expressions such as $y^2, (y')^2, \sin(xy), e^y$

i. $(y')^2 + 3y = 4$ is a first order non-linear differential equation

ii. $y' + 4x\sin(y) = 2$ is a first order non-linear differential equation

iii. $y'' + y' = e^{xy}$ is a second order non-linear differential equation

What is a solution of a differential equation? If a differential equation is of order *n*, meaning the equation contains terms involving $y^n, y^{n-1}, y^{n-2}, ..., y'$, it can be differentiated at least *n* times in an interval, *I*. When its derivatives and the function itself are substituted into the equation, both sides of the equation are equal for all values of *x* in the interval *I*.

EXAMPLE: Verify that the function $f(x) = 2\sin(x) + 3\cos(x)$ is a solution of the differential equation $y'' + y = 0$ for all values of *x*.

$$f'(x) = 2\cos(x) - 3\sin(x)$$
$$f''(x) = -2\sin(x) - 3\cos(x)$$
$$-2\sin(x) - 3\cos(x) + 2\sin(x) + 3\cos(x) = 0$$

EXAMPLE: Show that the equation $x^2 + y^2 = 1$ is an ***implicit*** solution of $\dfrac{dy}{dx} = \dfrac{xy}{x^2 - 1}$

$$2x + 2yy' = 0$$
$$y' = -\frac{x}{y}$$
$$y' = -\frac{x}{y} \cdot \frac{y}{y} = -\frac{xy}{y^2} = -\frac{xy}{1 - x^2}$$

EXAMPLE: For example, find all functions *y* that satisfy the equation

$$\frac{dy}{dx} = \sec^2 x + 2x + 5$$

$$\frac{dy}{dx} = \sec^2 x + 2x + 5$$
$$dy = \left(\sec^2 x + 2x + 5\right)dx$$
$$\int dy = \int \left(\sec^2 x + 2x + 5\right)dx$$
$$y = \tan x + x^2 + 5x + c$$

This solution is the general solution of the differential equation. We cannot find a unique solution to a differential equation unless we are given more information. If the general solution to a first order differential equation is continuous, the only additional information needed is the value of the function at a single point, called an *initial condition.* This may also be called a boundary condition. A differential equation with an initial condition is called an *initial value problem.*

EXAMPLE: Find the particular solution to the equation $\frac{dy}{dx} = e^x - 6x^2$ whose graph passes through the point (1, 0).

$$\frac{dy}{dx} = e^x - 6x^2$$
$$dy = \left(e^x - 6x^2\right)dx$$
$$\int dy = \int \left(e^x - 6x^2\right)dx$$
$$y = e^x - 2x^3 + c$$
$$f(1) = 0$$
$$e^1 - 2(1)^3 + c = 0$$
$$c = 2 - e$$
$$y = e^x - 2x^3 + 2 - e$$

EXAMPLE: Show that $y = \ln(3x) + k$ is a general solution of the differential equation $y' = \frac{1}{x}$.

$$y' = \frac{3}{3x} = \frac{1}{x}$$
$$\frac{dy}{dx} = \frac{3}{3x}$$
$$dy = \frac{3}{3x}dx$$
$$\int dy = \int \frac{3}{3x}dx$$
$$y = \ln(3x) + k$$

Find the value of *k* and determine which function of the family $y = \ln(x) + k$ verifies the boundary condition $y(1) = 2$

$$y = \ln(3x) + k$$
$$y(1) = 2$$
$$2 = \ln(3 \cdot 1) + k \Rightarrow k = 2 - \ln(3)$$
$$\therefore y = \ln(3x) + 2 - \ln(3) \Rightarrow y = \ln(x) + 2$$

Differential Equations with separated variables

A simple class of differential equations are called first order differential equations with separated variables. They are of the form $y' = f(x)$ and usually arise from problems about rates of change, where the rate of change y' only depends on the independent variable *x*.

EXAMPLE: Find the general solution of the differential equation $y' = x^2 + 1$. Hence find the solution that satisfies the initial condition $y(0) = 1$.

$$y' = x^2 + 1 \Rightarrow y = \frac{x^3}{3} + x + C$$
$$y(0) = 1 \Rightarrow C = 1$$
$$\therefore y = \frac{x^3}{3} + x + 1 \text{ satisifies the initial condition}$$

EXAMPLE: Find the particular solution of the differential equation that satisifies the condition given: $\frac{dy}{dx} = \cos(x)e^{\sin(x)}$ and $y(0) = 0$

$$\text{Let } u = \sin(x)$$
$$\Rightarrow u' = \cos(x) \Rightarrow \frac{dy}{dx} = u'e^u$$
$$y = \int u'e^u \, dx = e^u + c$$
$$y = e^{\sin(x)} + c$$
$$0 = e^0 + c$$
$$c = -1$$
$$y = e^{\sin(x)} - 1$$

EXAMPLE: Find the general solution of the differential equation $\frac{dy}{dx} = x\sin(2x)$.

$$\frac{dy}{dx} = x\sin(2x)$$

Let $u = x, du = dx$

Let $dv = \sin(2x)dx,\ v = -\frac{\cos(2x)}{2}$

$$\int x\sin(2x)dx = -\frac{1}{2}x\cos(2x) - \int -\frac{\cos(2x)}{2}dx$$

$$= -\frac{x\cos(2x)}{2} + \frac{\sin(2x)}{4} + C$$