Indeterminate Forms and L'Hospital's Rule

THEOREM (L'Hospital's Rule): Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.) Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Indeterminate Forms of Type $\dfrac{0}{0}$ and $\dfrac{\infty}{\infty}$

EXAMPLES:

1. Find $\displaystyle\lim_{x\to\infty} \frac{5x-2}{7x+3}$.

Solution 1: We have

$$\lim_{x\to\infty} \frac{5x-2}{7x+3} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{\frac{5x-2}{x}}{\frac{7x+3}{x}} = \lim_{x\to\infty} \frac{\frac{5x}{x} - \frac{2}{x}}{\frac{7x}{x} + \frac{3}{x}} = \lim_{x\to\infty} \frac{5 - \frac{2}{x}}{7 + \frac{3}{x}} = \frac{5-0}{7+0} = \frac{5}{7}$$

Solution 2: We have

$$\lim_{x\to\infty} \frac{5x-2}{7x+3} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{(5x-2)'}{(7x+3)'} = \lim_{x\to\infty} \frac{(5x)'-2'}{(7x)'+3'} = \lim_{x\to\infty} \frac{5x'-2'}{7x'+3'}$$

$$= \lim_{x\to\infty} \frac{5\cdot 1 - 0}{7\cdot 1 + 0} = \lim_{x\to\infty} \frac{5}{7} = \frac{5}{7}$$

In short,

$$\lim_{x\to\infty} \frac{5x-2}{7x+3} = \lim_{x\to\infty} \frac{(5x-2)'}{(7x+3)'} = \lim_{x\to\infty} \frac{5}{7} = \frac{5}{7}$$

2. Find $\displaystyle\lim_{x\to\infty} \frac{x^5+x^4+x^3+x^2+x+1}{2x^5+x^4+x^3+x^2+x+1}$.

2. Find $\lim\limits_{x\to\infty} \dfrac{x^5+x^4+x^3+x^2+x+1}{2x^5+x^4+x^3+x^2+x+1}$.

Solution 1: We have

$$\lim_{x\to\infty} \frac{x^5+x^4+x^3+x^2+x+1}{2x^5+x^4+x^3+x^2+x+1} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{\frac{x^5+x^4+x^3+x^2+x+1}{x^5}}{\frac{2x^5+x^4+x^3+x^2+x+1}{x^5}}$$

$$= \lim_{x\to\infty} \frac{\frac{x^5}{x^5}+\frac{x^4}{x^5}+\frac{x^3}{x^5}+\frac{x^2}{x^5}+\frac{x}{x^5}+\frac{1}{x^5}}{\frac{2x^5}{x^5}+\frac{x^4}{x^5}+\frac{x^3}{x^5}+\frac{x^2}{x^5}+\frac{x}{x^5}+\frac{1}{x^5}} = \lim_{x\to\infty} \frac{1+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^4}+\frac{1}{x^5}}{2+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^4}+\frac{1}{x^5}}$$

$$= \frac{1+0+0+0+0+0}{2+0+0+0+0+0} = \frac{1}{2}$$

Solution 2: We have

$$\lim_{x\to\infty} \frac{x^5+x^4+x^3+x^2+x+1}{2x^5+x^4+x^3+x^2+x+1} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{(x^5+x^4+x^3+x^2+x+1)'}{(2x^5+x^4+x^3+x^2+x+1)'}$$

$$= \lim_{x\to\infty} \frac{5x^4+4x^3+3x^2+2x+1}{10x^4+4x^3+3x^2+2x+1} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{(5x^4+4x^3+3x^2+2x+1)'}{(10x^4+4x^3+3x^2+2x+1)'}$$

$$= \lim_{x\to\infty} \frac{20x^3+12x^2+6x+2}{40x^3+12x^2+6x+2} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{(20x^3+12x^2+6x+2)'}{(40x^3+12x^2+6x+2)'}$$

$$= \lim_{x\to\infty} \frac{60x^2+24x+6}{120x^2+24x+6} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{(60x^2+24x+6)'}{(120x^2+24x+6)'}$$

$$= \lim_{x\to\infty} \frac{120x+24}{240x+24} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{(120x+24)'}{(240x+24)'} = \lim_{x\to\infty} \frac{120}{240} = \frac{120}{240} = \frac{1}{2}$$

3. Find $\lim\limits_{x\to-2} \dfrac{x+2}{\ln(x+3)}$.

Solution: We have

$$\lim_{x\to-2} \frac{x+2}{\ln(x+3)} = \left[\frac{0}{0}\right] = \lim_{x\to-2} \frac{(x+2)'}{(\ln(x+3))'} = \left\{ \lim_{x\to-2} \frac{(x+2)'}{\frac{1}{x+3}\cdot(x+3)'} = \lim_{x\to-2} \frac{x'+2'}{\frac{1}{x+3}\cdot(x'+3')} \right.$$

$$\left. = \lim_{x\to-2} \frac{1+0}{\frac{1}{x+3}\cdot(1+0)} \right\} = \lim_{x\to-2} \frac{1}{\frac{1}{x+3}} = \lim_{x\to-2} \frac{1\cdot(x+3)}{\frac{1}{x+3}\cdot(x+3)} = \lim_{x\to-2} \frac{x+3}{1} = \lim_{x\to-2}(x+3)$$

$$= -2+3 = 1$$

In short,

$$\lim_{x\to-2} \frac{x+2}{\ln(x+3)} = \lim_{x\to-2} \frac{(x+2)'}{(\ln(x+3))'} = \lim_{x\to-2} \frac{1}{(x+3)^{-1}} = \lim_{x\to-2}(x+3) = 1$$

4. Find $\lim\limits_{x\to\infty} \dfrac{3^x}{x^2+x-1}$.

4. Find $\displaystyle\lim_{x\to\infty}\frac{3^x}{x^2+x-1}$.

Solution: We have

$$\lim_{x\to\infty}\frac{3^x}{x^2+x-1}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{(3^x)'}{(x^2+x-1)'}=\lim_{x\to\infty}\frac{3^x\ln 3}{2x+1}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{(3^x\ln 3)'}{(2x+1)'}$$
$$=\lim_{x\to\infty}\frac{\ln 3\,(3^x)'}{2}=\lim_{x\to\infty}\frac{\ln 3\cdot 3^x\cdot\ln 3}{2}=\infty\text{ (D.N.E.)}$$

5. Find $\displaystyle\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}$.

Solution: We have

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{(\ln x)'}{(x^{1/2})'}=\lim_{x\to\infty}\frac{x^{-1}}{\frac{1}{2}x^{-1/2}}=\lim_{x\to\infty}\frac{x^{-1}\cdot x}{\frac{1}{2}x^{-1/2}\cdot x}=\lim_{x\to\infty}\frac{1}{\frac{1}{2}x^{1/2}}=0$$

6. Find $\displaystyle\lim_{x\to 0}\frac{5x-\tan 5x}{x^3}$.

Solution 1: We have

$$\lim_{x\to 0}\frac{5x-\tan 5x}{x^3}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{(5x-\tan 5x)'}{(x^3)'}=\lim_{x\to 0}\frac{5-\sec^2 5x\cdot(5x)'}{3x^2}=\lim_{x\to 0}\frac{5-\sec^2 5x\cdot 5}{3x^2}$$
$$=\lim_{x\to 0}\frac{5(1-\sec^2 5x)}{3x^2}=\frac{5}{3}\lim_{x\to 0}\frac{1-\sec^2 5x}{x^2}$$

Since $\displaystyle\lim_{x\to 0}\frac{1-\sec^2 5x}{x^2}$ is an indeterminate form of type $\dfrac{0}{0}$, we can use L'Hospital's Rule again. But it is easier to do trigonometry instead. Note that $1-\sec^2 5x=-\tan^2 5x$. Therefore

$$\frac{5}{3}\lim_{x\to 0}\frac{1-\sec^2 5x}{x^2}=\frac{5}{3}\lim_{x\to 0}\frac{-\tan^2 5x}{x^2}=-\frac{5}{3}\lim_{x\to 0}\frac{\tan^2 5x}{x^2}=-\frac{5}{3}\lim_{x\to 0}\frac{\frac{\sin^2 5x}{\cos^2 5x}}{x^2}=-\frac{5}{3}\lim_{x\to 0}\frac{\frac{\sin^2 5x}{1}}{x^2}$$
$$=-\frac{5}{3}\lim_{x\to 0}\frac{\sin^2 5x}{x^2}=\left[\frac{0}{0}\right]=-\frac{5}{3}\lim_{x\to 0}\left(\frac{\sin 5x}{x}\right)^2=-\frac{5}{3}\lim_{x\to 0}\left(5\cdot\frac{\sin 5x}{5x}\right)^2=-\frac{5}{3}\left(5\lim_{x\to 0}\frac{\sin 5x}{5x}\right)^2$$
$$=-\frac{5}{3}\left(5\lim_{5x\to 0}\frac{\sin 5x}{5x}\right)^2=\left[\lim_{u\to 0}\frac{\sin u}{u}=1\right]=-\frac{5}{3}\,(5\cdot 1)^2=-\frac{5}{3}\cdot 25=-\frac{125}{3}$$

In short,

$$\lim_{x\to 0}\frac{5x-\tan 5x}{x^3}=\lim_{x\to 0}\frac{(5x-\tan 5x)'}{(x^3)'}=\frac{5}{3}\lim_{x\to 0}\frac{1-\sec^2 5x}{x^2}=-\frac{5}{3}\lim_{x\to 0}\frac{\tan^2 5x}{x^2}=-\frac{5}{3}\lim_{x\to 0}\frac{\sin^2 5x}{x^2}$$
$$=-\frac{5}{3}\lim_{x\to 0}\left(5\cdot\frac{\sin 5x}{5x}\right)^2=-\frac{5}{3}(5\cdot 1)^2=-\frac{125}{3}$$

Solution 2??? (**WRONG!**): We have

$$\lim_{x\to 0}\frac{5x-\tan 5x}{x^3}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{5x-\dfrac{\sin 5x}{\cos 5x}}{x^3}\overset{???}{=}\lim_{x\to 0}\frac{5x-\dfrac{\sin 5x}{1}}{x^3}=\lim_{x\to 0}\frac{5x-\sin 5x}{x^3}=\left[\frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{(5x-\sin 5x)'}{(x^3)'}=\lim_{x\to 0}\frac{5-5\cos 5x}{3x^2}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{(5-5\cos 5x)'}{(3x^2)'}=\lim_{x\to 0}\frac{25\sin 5x}{6x}=\frac{125}{6}$$

7. Find $\lim\limits_{x\to\infty}\dfrac{x+\sin x}{x+\cos x}$.

Solution 1: We have

$$\lim_{x\to\infty}\frac{x+\sin x}{x+\cos x}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{\frac{x+\sin x}{x}}{\frac{x+\cos x}{x}}=\lim_{x\to\infty}\frac{\frac{x}{x}+\frac{\sin x}{x}}{\frac{x}{x}+\frac{\cos x}{x}}=\lim_{x\to\infty}\frac{1+\frac{\sin x}{x}}{1+\frac{\cos x}{x}}$$

It is easy to show that $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}=0$ and $\lim\limits_{x\to\infty}\dfrac{\cos x}{x}=0$ by the Squeeze Theorem. Therefore

$$\lim_{x\to\infty}\frac{x+\sin x}{x+\cos x}=\lim_{x\to\infty}\frac{1+\frac{\sin x}{x}}{1+\frac{\cos x}{x}}=\frac{1+0}{1+0}=1$$

Solution 2(???): We have

$$\lim_{x\to\infty}\frac{x+\sin x}{x+\cos x}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{(x+\sin x)'}{(x+\cos x)'}=\lim_{x\to\infty}\frac{x'+(\sin x)'}{x'+(\cos x)'}=\lim_{x\to\infty}\frac{1+\cos x}{1-\sin x}$$

One can show, however, that $\lim\limits_{x\to\infty}\dfrac{1+\cos x}{1-\sin x}$ does not exist. In fact, we first note that $1+\cos x$ and $1-\sin x$ may attain any value between 0 and 2. From this one can deduce that $\dfrac{1+\cos x}{1-\sin x}$ attains any nonnegative value infinitely often as $x\to\infty$. This means that $\lim\limits_{x\to\infty}\dfrac{1+\cos x}{1-\sin x}$ does not exist, so L'Hospital's Rule can't be applied here.

8. Find $\lim\limits_{x\to\pi^-}\dfrac{\sin x}{1-\cos x}$.

Solution(???): We have

$$\lim_{x\to\pi^-}\frac{\sin x}{1-\cos x}=\lim_{x\to\pi^-}\frac{(\sin x)'}{(1-\cos x)'}=\lim_{x\to\pi^-}\frac{\cos x}{\sin x}=-\infty$$

This is **WRONG**. In fact, although the numerator $\sin x\to 0$ as $x\to\pi^-$, notice that the denominator $(1-\cos x)$ does not approach 0, so L'Hospital's Rule can't be applied here. The required limit is easy to find, because the function is continuous at π and the denominator is nonzero here:

$$\lim_{x\to\pi^-}\frac{\sin x}{1-\cos x}=\frac{\sin\pi}{1-\cos\pi}=\frac{0}{1-(-1)}=0$$

9. Find $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$.

Solution(???): We have

$$\lim_{x\to 0}\frac{\sin x}{x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{(\sin x)'}{x'}=\lim_{x\to 0}\frac{\cos x}{1}=\frac{\cos 0}{1}=\frac{1}{1}=1$$

The answer is correct, but the solution is **WRONG**. Indeed, the above proof is based on the formula $(\sin x)'=\cos x$. But this result was deduced from the fact that $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$ (see Appendix A). So, the solution is wrong because it is based on Circular Reasoning which is a logical fallacy. However, one can apply L'Hospital's Rule to modifications of this limit (see Appendix B).

Indeterminate Forms of Type $\infty - \infty$ and $0 \cdot \infty$

EXAMPLES:

10. Find $\lim_{x\to\infty}(x - \ln x)$.

Solution 1: We have

$$\lim_{x\to\infty}(x - \ln x) = [\infty - \infty] = \lim_{x\to\infty}\left(x \cdot 1 - x \cdot \frac{\ln x}{x}\right) = \lim_{x\to\infty} x\left(1 - \frac{\ln x}{x}\right)$$

Note that

$$\lim_{x\to\infty}\frac{\ln x}{x} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty}\frac{(\ln x)'}{x'} = \lim_{x\to\infty}\frac{x^{-1}}{1} = \lim_{x\to\infty}\frac{1}{x} = 0$$

therefore

$$\lim_{x\to\infty}(x - \ln x) = \lim_{x\to\infty} x\left(1 - \frac{\ln x}{x}\right) = \lim_{x\to\infty} x\,(1-0) = \infty \text{ (D.N.E.)}$$

Solution 2: We have

$$\lim_{x\to\infty}(x - \ln x) = [\infty - \infty] = \lim_{x\to\infty}(\ln(e^x) - \ln x) = \lim_{x\to\infty}\ln\left(\frac{e^x}{x}\right)$$

$$= \ln\left(\lim_{x\to\infty}\frac{e^x}{x}\right) = \left[\frac{\infty}{\infty}\right] = \ln\left(\lim_{x\to\infty}\frac{(e^x)'}{x'}\right) = \ln\left(\lim_{x\to\infty}\frac{e^x}{1}\right) = \infty \text{ (D.N.E.)}$$

11. Find $\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$.

11. Find $\lim\limits_{x\to 1}\left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right)$.

Solution: We have

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=[\infty-\infty]=\lim_{x\to 1}\left(\frac{1\cdot(x-1)}{\ln x\cdot(x-1)}-\frac{\ln x\cdot 1}{\ln x\cdot(x-1)}\right)$$

$$=\lim_{x\to 1}\frac{x-1-\ln x}{\ln x(x-1)}=\left[\frac{0}{0}\right]=\lim_{x\to 1}\frac{(x-1-\ln x)'}{(\ln x(x-1))'}$$

$$=\lim_{x\to 1}\frac{x'-1'-(\ln x)'}{(\ln x)'\cdot(x-1)+\ln x\cdot(x-1)'}=\lim_{x\to 1}\frac{1-\dfrac{1}{x}}{\dfrac{x-1}{x}+\ln x}$$

$$=\lim_{x\to 1}\frac{\left(1-\dfrac{1}{x}\right)x}{\left(\dfrac{x-1}{x}+\ln x\right)x}=\lim_{x\to 1}\frac{1\cdot x-\dfrac{1}{x}\cdot x}{\dfrac{x-1}{x}\cdot x+\ln x\cdot x}$$

$$=\lim_{x\to 1}\frac{x-1}{x-1+x\ln x}=\left[\frac{0}{0}\right]=\lim_{x\to 1}\frac{(x-1)'}{(x-1+x\ln x)'}=\lim_{x\to 1}\frac{x'-1'}{x'-1'+x'\ln x+x(\ln x)'}$$

$$=\lim_{x\to 1}\frac{1-0}{1-0+1\cdot\ln x+x\cdot\dfrac{1}{x}}=\lim_{x\to 1}\frac{1}{2+\ln x}=\frac{1}{2+0}=\frac{1}{2}$$

In short,

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to 1}\frac{x-1-\ln x}{\ln x\cdot(x-1)}=\lim_{x\to 1}\frac{(x-1-\ln x)'}{(\ln x\cdot(x-1))'}=\lim_{x\to 1}\frac{1-\dfrac{1}{x}}{\dfrac{x-1}{x}+\ln x}$$

$$=\lim_{x\to 1}\frac{x-1}{x-1+x\ln x}=\lim_{x\to 1}\frac{(x-1)'}{(x-1+x\ln x)'}=\lim_{x\to 1}\frac{1}{2+\ln x}=\frac{1}{2}$$

12. Find $\lim\limits_{x\to 0^+}(\sin x\ln x)$.

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Solution 1: We have

$$\lim_{x\to 0^+}(\sin x\ln x)=[0\cdot\infty]=\left\{\lim_{x\to 0^+}\frac{\sin x\ln x}{1}=\lim_{x\to 0^+}\frac{(\sin x)^{-1}\cdot\sin x\ln x}{(\sin x)^{-1}\cdot 1}\right\}=\lim_{x\to 0^+}\frac{\ln x}{(\sin x)^{-1}}=\left[\frac{\infty}{\infty}\right]$$

$$=\lim_{x\to 0^+}\frac{(\ln x)'}{((\sin x)^{-1})'}=\lim_{x\to 0^+}\frac{x^{-1}}{-(\sin x)^{-2}\cdot(\sin x)'}=\lim_{x\to 0^+}\frac{x^{-1}}{-(\sin x)^{-2}\cos x}$$

$$=\lim_{x\to 0^+}\frac{x^{-1}\cdot x\sin^2 x}{-(\sin x)^{-2}\cos x\cdot x\sin^2 x}=-\lim_{x\to 0^+}\frac{\sin^2 x}{x\cos x}$$

We can now proceed in two different ways. Either

$$-\lim_{x\to 0^+}\frac{\sin^2 x}{x\cos x}=\left[\frac{0}{0}\right]=-\lim_{x\to 0^+}\left(\frac{\sin x}{x}\cdot\frac{\sin x}{\cos x}\right)$$

$$=-\lim_{x\to 0^+}\frac{\sin x}{x}\cdot\lim_{x\to 0^+}\frac{\sin x}{\cos x}=-1\cdot\frac{\sin 0}{\cos 0}=-1\cdot\frac{0}{1}=0$$

or

$$-\lim_{x\to 0^+}\frac{\sin^2 x}{x\cos x}=\left[\frac{0}{0}\right]=-\lim_{x\to 0^+}\frac{(\sin^2 x)'}{(x\cos x)'}=-\lim_{x\to 0^+}\frac{2\sin x\cos x}{x'\cos x+x(\cos x)'}$$

$$=-\lim_{x\to 0^+}\frac{2\sin x\cos x}{\cos x-x\sin x}=-\frac{2\sin 0\cos 0}{\cos 0-0\cdot\sin 0}=-\frac{2\cdot 0\cdot 1}{1-0\cdot 0}=0$$

In short,

$$\lim_{x\to 0^+}(\sin x\ln x)=\lim_{x\to 0^+}\frac{\ln x}{(\sin x)^{-1}}=\lim_{x\to 0^+}\frac{(\ln x)'}{((\sin x)^{-1})'}=\lim_{x\to 0^+}\frac{x^{-1}}{-(\sin x)^{-2}\cos x}$$

$$=-\lim_{x\to 0^+}\frac{\sin^2 x}{x\cos x}=-\lim_{x\to 0^+}\left(\frac{\sin x}{x}\cdot\frac{\sin x}{\cos x}\right)=-1\cdot 0=0$$

Solution 2: We have

$$\lim_{x\to 0^+}(\sin x\ln x)=[0\cdot\infty]=\left\{\lim_{x\to 0^+}\frac{\sin x\ln x}{1}=\lim_{x\to 0^+}\frac{\sin x\ln x\cdot\ln^{-1}x}{1\cdot\ln^{-1}x}\right\}=\lim_{x\to 0^+}\frac{\sin x}{\ln^{-1}x}=\left[\frac{0}{0}\right]$$

$$=\lim_{x\to 0^+}\frac{(\sin x)'}{\left(\ln^{-1}x\right)'}=\lim_{x\to 0^+}\frac{\cos x}{-\ln^{-2}x\cdot(\ln x)'}=\lim_{x\to 0^+}\frac{\cos x}{-\ln^{-2}x\cdot x^{-1}}=\lim_{x\to 0^+}\frac{\ln^2 x\cdot\cos x}{-\ln^2 x\cdot\ln^{-2}x\cdot x^{-1}}$$

$$=\lim_{x\to 0^+}\frac{\ln^2 x\cdot\cos x}{-x^{-1}}=\lim_{x\to 0^+}\frac{\ln^2 x}{-x^{-1}}\cdot\lim_{x\to 0^+}\cos x=\lim_{x\to 0^+}\frac{\ln^2 x}{-x^{-1}}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to 0^+}\frac{\left(\ln^2 x\right)'}{\left(-x^{-1}\right)'}$$

$$=\lim_{x\to 0^+}\frac{2\ln x\cdot(\ln x)'}{x^{-2}}=\lim_{x\to 0^+}\frac{2\ln x\cdot x^{-1}}{x^{-2}}=\lim_{x\to 0^+}\frac{2\ln x\cdot x^{-1}\cdot x}{x^{-2}\cdot x}=\lim_{x\to 0^+}\frac{2\ln x}{x^{-1}}=\left[\frac{\infty}{\infty}\right]$$

$$=\lim_{x\to 0^+}\frac{(2\ln x)'}{\left(x^{-1}\right)'}=\lim_{x\to 0^+}\frac{2x^{-1}}{-x^{-2}}=\lim_{x\to 0^+}\frac{2x^{-1}\cdot x^2}{-x^{-2}\cdot x^2}=\lim_{x\to 0^+}\frac{2x}{-1}=\frac{2\cdot 0}{-1}=0$$

Indeterminate Forms of Type ∞^0, 0^0 and 1^∞

13. Find $\lim\limits_{x\to\infty} x^{1/x}$.

Solution (version 1): Note that $\lim\limits_{x\to\infty} x^{1/x}$ is ∞^0 type of an indeterminate form. Put

$$y = x^{1/x}$$

then

$$\ln y = \ln x^{1/x} = \frac{1}{x}\ln x = \frac{\ln x}{x}$$

We have

$$\lim_{x\to\infty}\frac{\ln x}{x} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty}\frac{(\ln x)'}{x'} = \lim_{x\to\infty}\frac{x^{-1}}{1} = \lim_{x\to\infty}\frac{1}{x} = 0$$

Therefore

$$\lim_{x\to\infty} x^{1/x} = e^0 = 1$$

Solution (version 2): We have

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln x^{1/x}} = \lim_{x\to\infty} e^{\frac{\ln x}{x}} = e^{\lim\limits_{x\to\infty}\frac{\ln x}{x}} = e^{\lim\limits_{x\to\infty}\frac{(\ln x)'}{x'}} = e^{\lim\limits_{x\to\infty}\frac{x^{-1}}{1}} = e^{\lim\limits_{x\to\infty}\frac{1}{x}} = e^0 = 1$$

14. Find $\lim\limits_{x\to\pi/2}(\tan x)^{2x-\pi}$.

14. Find $\lim\limits_{x\to\pi/2}(\tan x)^{2x-\pi}$.

Solution: Note that $\lim\limits_{x\to\pi/2}(\tan x)^{2x-\pi}$ is ∞^0 type of an indeterminate form. Put

$$y=(\tan x)^{2x-\pi}$$

then

$$\ln y=\ln\left((\tan x)^{2x-\pi}\right)=(2x-\pi)\ln(\tan x)$$
$$=\left\{\frac{(2x-\pi)\ln(\tan x)}{1}=\frac{(2x-\pi)^{-1}\cdot(2x-\pi)\ln(\tan x)}{(2x-\pi)^{-1}\cdot 1}\right\}=\frac{\ln(\tan x)}{(2x-\pi)^{-1}}$$

We have

$$\lim_{x\to\pi/2}\frac{\ln(\tan x)}{(2x-\pi)^{-1}}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\pi/2}\frac{[\ln(\tan x)]'}{[(2x-\pi)^{-1}]'}=\lim_{x\to\pi/2}\frac{\frac{1}{\tan x}\cdot(\tan x)'}{(-1)(2x-\pi)^{-2}\cdot(2x-\pi)'}$$

$$=\lim_{x\to\pi/2}\frac{\frac{1}{\tan x}\cdot\sec^2 x}{(-1)(2x-\pi)^{-2}\cdot 2}=-\frac{1}{2}\lim_{x\to\pi/2}\frac{\frac{1}{\frac{\sin x}{\cos x}}\cdot\frac{1}{\cos^2 x}}{(2x-\pi)^{-2}}=-\frac{1}{2}\lim_{x\to\pi/2}\frac{\frac{1}{\frac{\sin x}{\cos x}\cdot\cos^2 x}}{(2x-\pi)^{-2}}$$

$$=-\frac{1}{2}\lim_{x\to\pi/2}\frac{\frac{1}{\sin x\cos x}}{(2x-\pi)^{-2}}=-\frac{1}{2}\lim_{x\to\pi/2}\frac{\frac{1}{\sin x\cos x}\cdot\sin x\cos x(2x-\pi)^2}{(2x-\pi)^{-2}\cdot\sin x\cos x(2x-\pi)^2}=-\frac{1}{2}\lim_{x\to\pi/2}\frac{(2x-\pi)^2}{\sin x\cos x}$$

$$=-\frac{1}{2}\lim_{x\to\pi/2}\frac{(2x-\pi)^2}{\sin\left(\frac{\pi}{2}\right)\cos x}=-\frac{1}{2}\lim_{x\to\pi/2}\frac{(2x-\pi)^2}{\cos x}=\left[\frac{0}{0}\right]=-\frac{1}{2}\lim_{x\to\pi/2}\frac{[(2x-\pi)^2]'}{(\cos x)'}$$

$$=-\frac{1}{2}\lim_{x\to\pi/2}\frac{2(2x-\pi)\cdot(2x-\pi)'}{-\sin x}=-\frac{1}{2}\lim_{x\to\pi/2}\frac{2(2x-\pi)\cdot 2}{-\sin x}=-\frac{1}{2}\cdot\frac{2\cdot\left(2\cdot\frac{\pi}{2}-\pi\right)\cdot 2}{-\sin\frac{\pi}{2}}$$

$$=-\frac{1}{2}\cdot\frac{2\cdot 0\cdot 2}{-1}=0$$

Therefore

$$\lim_{x\to\pi/2}(\tan x)^{2x-\pi}=e^0=1$$

In short,

$$y=(\tan x)^{2x-\pi}\quad\Longrightarrow\quad\ln y=(2x-\pi)\ln(\tan x)=\frac{\ln(\tan x)}{(2x-\pi)^{-1}}$$

We have

$$\lim_{x\to\pi/2}\frac{\ln(\tan x)}{(2x-\pi)^{-1}}=\lim_{x\to\pi/2}\frac{[\ln(\tan x)]'}{[(2x-\pi)^{-1}]'}=\lim_{x\to\pi/2}\frac{\frac{1}{\tan x}\cdot\sec^2 x}{(-1)(2x-\pi)^{-2}\cdot 2}=-\frac{1}{2}\lim_{x\to\pi/2}\frac{(2x-\pi)^2}{\sin x\cos x}$$

$$=-\frac{1}{2}\lim_{x\to\pi/2}\frac{(2x-\pi)^2}{\cos x}=\left[\frac{0}{0}\right]=-\frac{1}{2}\lim_{x\to\pi/2}\frac{[(2x-\pi)^2]'}{(\cos x)'}$$

$$=-\frac{1}{2}\lim_{x\to\pi/2}\frac{2(2x-\pi)\cdot 2}{-\sin x}=-\frac{1}{2}\cdot\frac{2\cdot\left(2\cdot\frac{\pi}{2}-\pi\right)\cdot 2}{-\sin\frac{\pi}{2}}=0$$

Therefore $\lim\limits_{x\to\pi/2}(\tan x)^{2x-\pi}=e^0=1$.

15. Find $\lim\limits_{x\to 0^+} x^x$.

Solution: Note that $\lim\limits_{x\to 0^+} x^x$ is 0^0 type of an indeterminate form. Put

$$y = x^x$$

then

$$\ln y = \ln x^x = x\ln x = \left\{\frac{x\ln x}{1} = \frac{x^{-1}\cdot x\ln x}{x^{-1}\cdot 1}\right\} = \frac{\ln x}{x^{-1}}$$

We have

$$\lim_{x\to 0^+}\frac{\ln x}{x^{-1}} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to 0^+}\frac{(\ln x)'}{(x^{-1})'} = \lim_{x\to 0^+}\frac{x^{-1}}{-x^{-2}}$$

$$= \lim_{x\to 0^+}\frac{x^{-1}\cdot x^2}{-x^{-2}\cdot x^2} = \lim_{x\to 0^+}\frac{x}{-1} = \frac{0}{-1} = 0$$

Therefore

$$\lim_{x\to 0^+} x^x = e^0 = 1$$

In short,

$$y = x^x \quad\Longrightarrow\quad \ln y = x\ln x = \frac{\ln x}{x^{-1}}$$

We have

$$\lim_{x\to 0^+}\frac{\ln x}{x^{-1}} = \lim_{x\to 0^+}\frac{(\ln x)'}{(x^{-1})'} = \lim_{x\to 0^+}\frac{x^{-1}}{-x^{-2}} = -\lim_{x\to 0^+} x = 0$$

Therefore

$$\lim_{x\to 0^+} x^x = e^0 = 1$$

16. Find $\lim\limits_{x\to 0^+}(\tan 5x)^x$.

16. Find $\lim\limits_{x\to 0^+}(\tan 5x)^x$.

Solution: Note that $\lim\limits_{x\to 0^+}(\tan 5x)^x$ is 0^0 type of an indeterminate form. Put

$$y = (\tan 5x)^x$$

then

$$\ln y = \ln(\tan 5x)^x = x\ln(\tan 5x) = \left\{\frac{x\ln(\tan 5x)}{1} = \frac{x^{-1}\cdot x\ln(\tan 5x)}{x^{-1}\cdot 1}\right\} = \frac{\ln(\tan 5x)}{x^{-1}}$$

We have

$$\lim_{x\to 0^+}\frac{\ln(\tan 5x)}{x^{-1}} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to 0^+}\frac{(\ln(\tan 5x))'}{(x^{-1})'} = \lim_{x\to 0^+}\frac{\frac{1}{\tan 5x}\cdot(\tan 5x)'}{-x^{-2}}$$

$$= \lim_{x\to 0^+}\frac{\frac{1}{\tan 5x}\cdot\sec^2 5x\cdot(5x)'}{-x^{-2}} = \lim_{x\to 0^+}\frac{\frac{1}{\tan 5x}\cdot\sec^2 5x\cdot 5}{-x^{-2}}$$

$$= \lim_{x\to 0^+}\frac{\frac{1}{\frac{\sin 5x}{\cos 5x}}\cdot\sec^2 5x\cdot 5}{-x^{-2}} = \lim_{x\to 0^+}\frac{\frac{1}{\frac{\sin 5x}{\cos 0}}\cdot\sec^2 0\cdot 5}{-x^{-2}}$$

$$= \lim_{x\to 0^+}\frac{\frac{1}{\sin 5x}\cdot 5}{-x^{-2}} = \lim_{x\to 0^+}\frac{\frac{1}{\sin 5x}\cdot 5\cdot x^2\sin 5x}{-x^{-2}\cdot x^2\sin 5x}$$

$$= \lim_{x\to 0^+}\frac{5x^2}{-\sin 5x} = \left[\frac{0}{0}\right] = \lim_{x\to 0^+}\frac{(5x^2)'}{(-\sin 5x)'}$$

$$= \lim_{x\to 0^+}\frac{10x}{-\cos 5x\cdot(5x)'} = \lim_{x\to 0^+}\frac{10x}{-\cos 5x\cdot 5} = \frac{10\cdot 0}{-\cos 0\cdot 5} = 0$$

Therefore

$$\lim_{x\to 0^+}(\tan 5x)^x = e^0 = 1$$

In short,

$$y = (\tan 5x)^x \quad\Longrightarrow\quad \ln y = x\ln(\tan 5x) = \frac{\ln(\tan 5x)}{x^{-1}}$$

We have

$$\lim_{x\to 0^+}\frac{\ln(\tan 5x)}{x^{-1}} = \lim_{x\to 0^+}\frac{(\ln(\tan 5x))'}{(x^{-1})'} = \lim_{x\to 0^+}\frac{\frac{1}{\tan 5x}\cdot\sec^2 5x\cdot 5}{-x^{-2}} = \lim_{x\to 0^+}\frac{\frac{1}{\frac{\sin 5x}{\cos 5x}}\cdot\sec^2 5x\cdot 5}{-x^{-2}}$$

$$= \lim_{x\to 0^+}\frac{\frac{1}{\sin 5x}\cdot 5}{-x^{-2}} = \lim_{x\to 0^+}\frac{5x^2}{-\sin 5x} = \lim_{x\to 0^+}\frac{(5x^2)'}{(-\sin 5x)'} = \lim_{x\to 0^+}\frac{10x}{-\cos 5x\cdot 5} = 0$$

Therefore

$$\lim_{x\to 0^+}(\tan 5x)^x = e^0 = 1$$

17. Find $\lim\limits_{x\to 0^+}(\sin 2x)^{\tan 3x}$.

Solution: Note that $\lim\limits_{x\to 0^+}(\sin 2x)^{\tan 3x}$ is 0^0 type of an indeterminate form. Put

$$y = (\sin 2x)^{\tan 3x}$$

then

$$\ln y = \ln\left((\sin 2x)^{\tan 3x}\right) = \tan 3x \ln(\sin 2x) = \frac{\sin 3x \ln(\sin 2x)}{\cos 3x}$$

We have

$$\lim_{x\to 0^+}\frac{\sin 3x\ln(\sin 2x)}{\cos 3x} = \frac{\lim\limits_{x\to 0^+}(\sin 3x\ln(\sin 2x))}{\lim\limits_{x\to 0^+}\cos 3x} = \lim_{x\to 0^+}(\sin 3x\ln(\sin 2x)) = [0\cdot\infty]$$

$$= \left\{\lim_{x\to 0^+}\frac{\sin 3x\ln(\sin 2x)}{1} = \lim_{x\to 0^+}\frac{(\sin 3x)^{-1}\cdot\sin 3x\ln(\sin 2x)}{(\sin 3x)^{-1}\cdot 1}\right\} = \lim_{x\to 0^+}\frac{\ln(\sin 2x)}{(\sin 3x)^{-1}} = \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 0^+}\frac{[\ln(\sin 2x)]'}{\left((\sin 3x)^{-1}\right)'} = \lim_{x\to 0^+}\frac{\sin^{-1}2x\cdot(\sin 2x)'}{-(\sin 3x)^{-2}\cdot(\sin 3x)'} = \lim_{x\to 0^+}\frac{\sin^{-1}2x\cdot\cos 2x\cdot(2x)'}{-(\sin 3x)^{-2}\cdot\cos 3x\cdot(3x)'}$$

$$= \lim_{x\to 0^+}\frac{\sin^{-1}2x\cdot\cos 2x\cdot 2}{-(\sin 3x)^{-2}\cdot\cos 3x\cdot 3} = -\frac{2}{3}\lim_{x\to 0^+}\frac{\sin^{-1}2x}{(\sin 3x)^{-2}}\cdot\lim_{x\to 0^+}\frac{\cos 2x}{\cos 3x} = -\frac{2}{3}\lim_{x\to 0^+}\frac{\sin^{-1}2x}{(\sin 3x)^{-2}}$$

$$= -\frac{2}{3}\lim_{x\to 0^+}\frac{\sin^{-1}2x\cdot\sin 2x\cdot\sin^2 3x}{(\sin 3x)^{-2}\cdot\sin 2x\cdot\sin^2 3x} = -\frac{2}{3}\lim_{x\to 0^+}\frac{\sin^2 3x}{\sin 2x} = \left[\frac{0}{0}\right] = -\frac{2}{3}\lim_{x\to 0^+}\frac{(\sin^2 3x)'}{(\sin 2x)'}$$

$$= -\frac{2}{3}\lim_{x\to 0^+}\frac{2\sin 3x\cdot(\sin 3x)'}{\cos 2x\cdot(2x)'} = -\frac{2}{3}\lim_{x\to 0^+}\frac{2\sin 3x\cdot\cos 3x\cdot(3x)'}{\cos 2x\cdot 2}$$

$$= -\frac{2}{3}\lim_{x\to 0^+}\frac{2\sin 3x\cdot\cos 3x\cdot 3}{\cos 2x\cdot 2} = -\frac{2}{3}\cdot\frac{2\sin 0\cdot\cos 0\cdot 3}{\cos 0\cdot 2} = -\frac{2}{3}\cdot\frac{2\cdot 0\cdot 1\cdot 3}{1\cdot 2} = 0$$

Therefore

$$\lim_{x\to 0^+}(\sin 2x)^{\tan 3x} = e^0 = 1$$

In short,

$$y = (\sin 2x)^{\tan 3x} \quad\Longrightarrow\quad \ln y = \tan 3x\ln(\sin 2x) = \frac{\sin 3x\ln(\sin 2x)}{\cos 3x}$$

We have

$$\lim_{x\to 0^+}\frac{\sin 3x\ln(\sin 2x)}{\cos 3x} = \lim_{x\to 0^+}(\sin 3x\ln(\sin 2x)) = \lim_{x\to 0^+}\frac{\ln(\sin 2x)}{(\sin 3x)^{-1}} = \lim_{x\to 0^+}\frac{[\ln(\sin 2x)]'}{\left((\sin 3x)^{-1}\right)'}$$

$$= \lim_{x\to 0^+}\frac{\sin^{-1}2x\cdot\cos 2x\cdot 2}{-(\sin 3x)^{-2}\cdot\cos 3x\cdot 3} = -\frac{2}{3}\lim_{x\to 0^+}\frac{\sin^{-1}2x}{(\sin 3x)^{-2}} = -\frac{2}{3}\lim_{x\to 0^+}\frac{\sin^2 3x}{\sin 2x} = -\frac{2}{3}\lim_{x\to 0^+}\frac{(\sin^2 3x)'}{(\sin 2x)'}$$

$$= -\frac{2}{3}\lim_{x\to 0^+}\frac{2\sin 3x\cdot\cos 3x\cdot 3}{\cos 2x\cdot 2} = -\frac{2}{3}\cdot\frac{2\sin 0\cdot\cos 0\cdot 3}{\cos 0\cdot 2} = 0$$

Therefore $\lim\limits_{x\to 0^+}(\sin 2x)^{\tan 3x} = e^0 = 1$.

18. Find $\lim\limits_{x\to\infty}\left(\dfrac{x+1}{x+2}\right)^x$.

Solution 1: Note that $\lim\limits_{x\to\infty}\left(\dfrac{x+1}{x+2}\right)^x$ is 1^∞ type of an indeterminate form. Put $y=\left(\dfrac{x+1}{x+2}\right)^x$, then

$$\ln y=\ln\left(\left(\frac{x+1}{x+2}\right)^x\right)=x\ln\left(\frac{x+1}{x+2}\right)=\left\{\frac{x\ln\left(\frac{x+1}{x+2}\right)}{1}=\frac{x^{-1}\cdot x\ln\left(\frac{x+1}{x+2}\right)}{x^{-1}\cdot 1}\right\}=\frac{\ln\left(\frac{x+1}{x+2}\right)}{x^{-1}}$$

We have

$$\lim_{x\to\infty}\frac{\ln\left(\frac{x+1}{x+2}\right)}{x^{-1}}=\left[\frac{0}{0}\right]=\lim_{x\to\infty}\frac{\left(\ln\left(\frac{x+1}{x+2}\right)\right)'}{(x^{-1})'}=\lim_{x\to\infty}\frac{\frac{1}{\frac{x+1}{x+2}}\cdot\left(\frac{x+1}{x+2}\right)'}{-x^{-2}}$$

$$=\lim_{x\to\infty}\frac{\frac{x+2}{x+1}\cdot\frac{(x+1)'(x+2)-(x+1)(x+2)'}{(x+2)^2}}{-x^{-2}}=\lim_{x\to\infty}\frac{\frac{x+2}{x+1}\cdot\frac{1\cdot(x+2)-1\cdot(x+1)}{(x+2)^2}}{-x^{-2}}$$

$$=\lim_{x\to\infty}\frac{\frac{x+2}{x+1}\cdot\frac{x+2-x-1}{(x+2)^2}}{-x^{-2}}=\lim_{x\to\infty}\frac{\frac{x+2}{x+1}\cdot\frac{1}{(x+2)^2}}{-x^{-2}}=\lim_{x\to\infty}\frac{\frac{1}{(x+1)(x+2)}}{-x^{-2}}$$

$$=\lim_{x\to\infty}\frac{\frac{1}{(x+1)(x+2)}\cdot x^2(x+1)(x+2)}{-x^{-2}\cdot x^2(x+1)(x+2)}=\lim_{x\to\infty}\frac{x^2}{-(x+1)(x+2)}=\left[\frac{\infty}{\infty}\right]=-\lim_{x\to\infty}\frac{x^2}{x^2+3x+1}$$

$$=-\lim_{x\to\infty}\frac{\frac{x^2}{x^2}}{\frac{x^2+3x+1}{x^2}}=-\lim_{x\to\infty}\frac{\frac{x^2}{x^2}}{\frac{x^2}{x^2}+\frac{3x}{x^2}+\frac{1}{x^2}}=-\lim_{x\to\infty}\frac{1}{1+\frac{3}{x}+\frac{1}{x^2}}=-\frac{1}{1+0+0}=-1$$

Therefore

$$\lim_{x\to\infty}\left(\frac{x+1}{x+2}\right)^x=e^{-1}$$

In short,

$$y=\left(\frac{x+1}{x+2}\right)^x\quad\Longrightarrow\quad\ln y=x\ln\left(\frac{x+1}{x+2}\right)=\frac{\ln\left(\frac{x+1}{x+2}\right)}{x^{-1}}$$

We have

$$\lim_{x\to\infty}\frac{\ln\left(\frac{x+1}{x+2}\right)}{x^{-1}}=\lim_{x\to\infty}\frac{\left(\ln\left(\frac{x+1}{x+2}\right)\right)'}{(x^{-1})'}=\lim_{x\to\infty}\frac{\frac{x+2}{x+1}\cdot\frac{1}{(x+2)^2}}{-x^{-2}}=-\lim_{x\to\infty}\frac{x^2}{(x+1)(x+2)}$$

$$=-\lim_{x\to\infty}\frac{x^2}{x^2+3x+1}=-1$$

Therefore $\lim\limits_{x\to\infty}\left(\dfrac{x+1}{x+2}\right)^x=e^{-1}$.

Solution 2: We have

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{x+1}{x+2}\right)^x &= \lim_{x\to\infty}\left(\frac{x+2-1}{x+2}\right)^x\\ &= \lim_{x\to\infty}\left(\frac{x+2}{x+2}-\frac{1}{x+2}\right)^x\\ &= \lim_{x\to\infty}\left(1-\frac{1}{x+2}\right)^x\\ &= \lim_{x\to\infty}\left(1+\frac{1}{-x-2}\right)^{(-x-2)\frac{x}{-x-2}}\\ &= \lim_{x\to\infty}\left[\left(1+\frac{1}{-x-2}\right)^{-x-2}\right]^{\frac{x}{-x-2}}\end{aligned}$$

Since $\lim\limits_{u\to\pm\infty}\left(1+\dfrac{1}{u}\right)^u = e$, it follows that $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{-x-2}\right)^{-x-2} = e$. Therefore

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{x+1}{x+2}\right)^x &= \lim_{x\to\infty}\left[\left(1+\frac{1}{-x-2}\right)^{-x-2}\right]^{\frac{x}{-x-2}}\\ &= \lim_{x\to\infty} e^{\frac{x}{-x-2}}\\ &= e^{\lim\limits_{x\to\infty}\frac{x}{-x-2}}\\ &= e^{\lim\limits_{x\to\infty}\frac{x'}{(-x-2)'}}\\ &= e^{-1}\end{aligned}$$

<u>COMPARE</u>: We have

$$\lim_{x\to\infty}\left(\frac{x+1}{3x+2}\right)^x = 0$$

since

$$\lim_{x\to\infty}\frac{x+1}{3x+2} = \frac{1}{3} \quad \text{and} \quad \lim_{x\to\infty}\left(\frac{1}{3}\right)^x = 0$$

Similarly,

$$\lim_{x\to\infty}\left(\frac{3x+1}{x+2}\right)^x = \infty \text{ (D.N.E.)}$$

since

$$\lim_{x\to\infty}\frac{3x+1}{x+2} = 3 \quad \text{and} \quad \lim_{x\to\infty} 3^x = \infty \text{ (D.N.E.)}$$

19. Find $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x$.

19. Find $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x$.

Solution: We have

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

by definition of e. Note, that we can't use the approach described in Example 13, since it is based of the formula $(\ln x)' = \dfrac{1}{x}$. But this result was deduced from the fact that $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x = e$ (see Appendix A). This is Circular Reasoning which is a logical fallacy. However, one can apply L'Hospital's Rule to modifications of this limit.

20. Find $\lim\limits_{x\to 0^+}(1+\sin 7x)^{\cot 5x}$.

Solution: Note that $\lim\limits_{x\to 0^+}(1+\sin 7x)^{\cot 5x}$ is 1^∞ type of an indeterminate form. Put

$$y = (1+\sin 7x)^{\cot 5x}$$

then

$$\ln y = \ln\left((1+\sin 7x)^{\cot 5x}\right) = \cot 5x\ln(1+\sin 7x) = \frac{\cos 5x\ln(1+\sin 7x)}{\sin 5x}$$

We have

$$\lim_{x\to 0^+}\frac{\cos 5x\ln(1+\sin 7x)}{\sin 5x} = \lim_{x\to 0^+}\cos 5x\cdot\lim_{x\to 0^+}\frac{\ln(1+\sin 7x)}{\sin 5x}$$

$$= \lim_{x\to 0^+}\frac{\ln(1+\sin 7x)}{\sin 5x} = \left[\frac{0}{0}\right] = \lim_{x\to 0^+}\frac{[\ln(1+\sin 7x)]'}{(\sin 5x)'} = \lim_{x\to 0^+}\frac{\frac{1}{1+\sin 7x}\cdot(1+\sin 7x)'}{\cos 5x\cdot(5x)'}$$

$$= \lim_{x\to 0^+}\frac{\frac{1}{1+\sin 7x}\cdot\cos 7x\cdot(7x)'}{\cos 5x\cdot 5} = \lim_{x\to 0^+}\frac{\frac{1}{1+\sin 7x}\cdot\cos 7x\cdot 7}{\cos 5x\cdot 5} = \frac{\frac{1}{1+0}\cdot 1\cdot 7}{1\cdot 5} = \frac{7}{5}$$

Therefore

$$\lim_{x\to 0^+}(1+\sin 7x)^{\cot 5x} = e^{7/5}$$

In short,

$$y = (1+\sin 7x)^{\cot 5x} \quad\Longrightarrow\quad \ln y = \cot 5x\ln(1+\sin 7x) = \frac{\cos 5x\ln(1+\sin 7x)}{\sin 5x}$$

We have

$$\lim_{x\to 0^+}\frac{\cos 5x\ln(1+\sin 7x)}{\sin 5x} = \lim_{x\to 0^+}\frac{\ln(1+\sin 7x)}{\sin 5x} = \lim_{x\to 0^+}\frac{[\ln(1+\sin 7x)]'}{(\sin 5x)'}$$

$$= \lim_{x\to 0^+}\frac{\frac{1}{1+\sin 7x}\cdot\cos 7x\cdot 7}{\cos 5x\cdot 5} = \frac{\frac{1}{1+0}\cdot 1\cdot 7}{1\cdot 5} = \frac{7}{5}$$

Therefore $\lim\limits_{x\to 0^+}(1+\sin 7x)^{\cot 5x} = e^{7/5}$.

Appendix A

THEOREM: The function $f(x) = \sin x$ is differentiable and

$$f'(x) = \cos x$$

Proof: We have

$$(\sin x)' = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

[We use $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$]

$$= \lim_{h\to 0} \frac{\sin x\cos h + \cos x \sin h - \sin x}{h} = \lim_{h\to 0}\left(\frac{\sin x\cos h - \sin x}{h} + \frac{\cos x\sin h}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\sin x(\cos h - 1)}{h} + \cos x \cdot \frac{\sin h}{h}\right) = \sin x \lim_{h\to 0}\frac{\cos h - 1}{h} + \cos x \lim_{h\to 0}\frac{\sin h}{h}$$

$$= \sin x \cdot 0 + \cos x \cdot 1 = \cos x$$

THEOREM: The function $f(x) = \log_a x$ is differentiable and

$$f'(x) = \frac{1}{x\ln a}$$

Proof: We have

$$\frac{d}{dx}(\log_a x) = \lim_{h\to 0}\frac{\log_a(x+h) - \log_a x}{h} = \lim_{h\to 0}\left[\frac{1}{h}\log_a\left(\frac{x+h}{x}\right)\right]$$

$$= \lim_{h\to 0}\left[\frac{1}{h}\log_a\left(1+\frac{h}{x}\right)\right] = \lim_{h\to 0}\left[\frac{1}{x}\cdot\frac{x}{h}\log_a\left(1+\frac{h}{x}\right)\right]$$

$$= \frac{1}{x}\lim_{h\to 0}\left[\log_a\left(1+\frac{h}{x}\right)^{x/h}\right] = \frac{1}{x}\lim_{h\to 0}\left[\log_a\left(1+\frac{h}{x}\right)^{1/(h/x)}\right]$$

$$= \frac{1}{x}\log_a\left[\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{1/(h/x)}\right] = \left[\lim_{u\to 0}(1+u)^{1/u} = e\right] = \frac{1}{x}\log_a e = \frac{1}{x}\frac{\ln e}{\ln a} = \frac{1}{x\ln a}$$

Appendix B

1. Find $\lim_{x\to 0}\frac{\sin x}{2x}$.

Solution 1: We have

$$\lim_{x\to 0}\frac{\sin x}{2x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{(\sin x)'}{(2x)'}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{\cos 0}{2}=\frac{1}{2}$$

Solution 2: We have

$$\lim_{x\to 0}\frac{\sin x}{2x}=\left[\frac{0}{0}\right]=\frac{1}{2}\lim_{x\to 0}\frac{\sin x}{x}=\frac{1}{2}\cdot 1=\frac{1}{2}$$

2. Find $\lim_{x\to 0}\frac{\sin 2x}{x}$.

Solution 1: We have

$$\lim_{x\to 0}\frac{\sin 2x}{x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{(\sin 2x)'}{x'}=\lim_{x\to 0}\frac{2\cos 2x}{1}=\frac{2\cos 0}{1}=2$$

Solution 2: We have

$$\lim_{x\to 0}\frac{\sin 2x}{x}=\left[\frac{0}{0}\right]=2\lim_{x\to 0}\frac{\sin 2x}{2x}=2\cdot 1=2$$

3. Find $\lim_{x\to 0}\frac{\sin 2x}{5x}$.

Solution 1: We have

$$\lim_{x\to 0}\frac{\sin 2x}{5x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{(\sin 2x)'}{(5x)'}=\lim_{x\to 0}\frac{2\cos 2x}{5}=\frac{2\cos 0}{5}=\frac{2}{5}$$

Solution 2: We have

$$\lim_{x\to 0}\frac{\sin 2x}{5x}=\left[\frac{0}{0}\right]=\frac{2}{5}\lim_{x\to 0}\frac{\sin 2x}{2x}=\frac{2}{5}\cdot 1=\frac{2}{5}$$

4. Find $\lim_{x\to 0}\frac{\sin 2x}{\sin 5x}$.

Solution 1: We have

$$\lim_{x\to 0}\frac{\sin 2x}{\sin 5x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{(\sin 2x)'}{(\sin 5x)'}=\lim_{x\to 0}\frac{2\cos 2x}{5\cos 5x}=\frac{2\cos 0}{5\cos 0}=\frac{2}{5}$$

Solution 2: We have

$$\lim_{x\to 0}\frac{\sin 2x}{\sin 5x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{\frac{\sin 2x}{2x}\cdot 2x}{\frac{\sin 5x}{5x}\cdot 5x}=\lim_{x\to 0}\frac{1\cdot 2x}{1\cdot 5x}=\lim_{x\to 0}\frac{2}{5}=\frac{2}{5}$$