IMPROPER INTEGRALS 2

Integrals of the form $\int_a^\infty f(x)dx$ are known as improper integrals.

The improper integral $\int_a^\infty f(x)dx$ is ***convergent*** if the limit $\lim_{b\to\infty}\int_a^b f(x)dx = \lim_{b\to\infty}\{I(b)\} - I(a)$ exists and is finite. Otherwise the integral ***diverges.***

EXAMPLE: Evaluate $\int_0^\infty e^{-3x}dx$

$$\int_0^\infty e^{-3x}dx = \lim_{b\to\infty}\int_0^b e^{-3x}dx$$
$$= \lim_{b\to\infty}\left[-\frac{1}{3}e^{-3x}\right]_0^b$$
$$= \lim_{b\to\infty}\left[-\frac{1}{3}e^{-3x} + \frac{1}{3}\right]$$
$$= \lim_{b\to\infty}\left(-\frac{1}{3}e^{-3x}\right) + \frac{1}{3}$$
$$= \frac{1}{3}$$

Therefore, the integral is convergent.

EXAMPLE: Determine for which values of $p \in \Re$, $\int_1^\infty x^p dx$ is convergent.

$$\int_1^\infty x^p dx = \lim_{b \to \infty} \int_1^b x^p dx$$
$$= \begin{cases} \lim_{b\to\infty} \left[\frac{x^{p+1}}{p+1} \right]_1^b, p \neq -1 \\ \lim_{b\to\infty} \left[\ln x \right]_1^b, p = -1 \end{cases}$$
$$= \begin{cases} \lim_{b\to\infty} \left[\frac{b^{p+1}}{p+1} - \frac{1^{p+1}}{p+1} \right], p \neq -1 \\ \lim_{b\to\infty} \left[\ln b - \ln 1 \right], p = -1 \end{cases}$$
$$= \begin{cases} \lim_{b\to\infty} \left[\frac{b^{p+1}-1}{p+1} \right], p \neq -1 \\ \lim_{b\to\infty} \left[\ln b \right], p = -1 \end{cases}$$
$$= \begin{cases} \infty, p > -1 \\ -\frac{1}{p+1}, p < -1 \\ \infty, p = -1 \end{cases}$$

i.e. $\int_1^\infty x^p dx$ converges only if $p < -1$ or, equivalently $\int_1^\infty \frac{1}{x^p} dx$ converges for $p > 1$.

Sometimes, the integration is difficult, if not impossible in terms of the standard functions, so we need another method for determining if the integral is convergent without having to evaluate it explicitly.

Comparison Test for improper integrals

If $0 \le f(x) \le g(x)$ for all $x \ge a$ then:

$\int_a^\infty f(x)dx$ is convergent if $\int_a^\infty g(x)dx$ is convergent.

$\int_a^\infty g(x)dx$ is divergent if $\int_a^\infty f(x)dx$ is divergent.

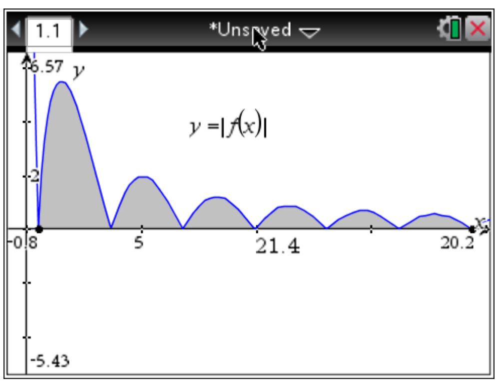
EXAMPLE: Show that $\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx$ is convergent.

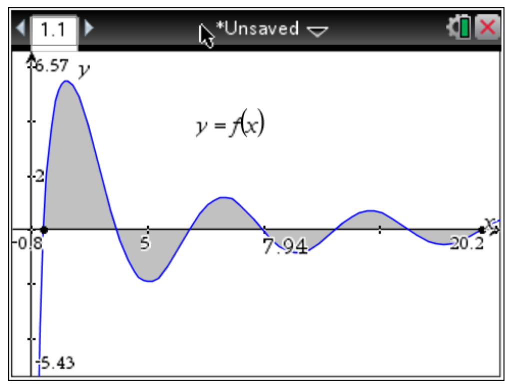
Because this integration is complicated we consider using the Comparison Test.

$\Rightarrow \frac{\sqrt{x}}{1+x^2} \le \frac{\sqrt{x}}{x^2} = \frac{1}{x^{3/2}}$. Then, since $\int_0^\infty \frac{1}{x^{3/2}}dx$ converges, $\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx$ converges by the Comparison Test for improper integrals.

The next result is also useful, often in conjunction with the Comparison Test.

If $\int_0^\infty |f(x)|dx$ converges, then so does $\int_0^\infty f(x)dx$





EXAMPLE: Show that $\int_1^\infty \frac{\cos x}{1+x^3}dx$ converges. If we can show that $\int_1^\infty \left|\frac{\cos x}{1+x^3}\right|dx$ converges, we can also conclude that $\int_1^\infty \frac{\cos x}{1+x^3}dx$ converges.

$$|\cos x| \le 1 \text{ and } 1+x^3 > x^3$$

$$\therefore \left|\frac{\cos x}{1+x^3}\right| \le \frac{1}{x^3}$$

Since $\int_1^\infty \frac{1}{x^3}dx$ converges, so does $\int_1^\infty \left|\frac{\cos x}{1+x^3}\right|dx$ by the Comparison Text. Hence, $\int_1^\infty \frac{\cos x}{1+x^3}dx$ converges.

EXAMPLE: Evaluate the convergent improper integral $\int_1^{\infty} xe^{-x}dx$. This problem will require integration by parts.

$$\int_1^{\infty} xe^{-x}dx = \lim_{b\to\infty}\int_1^{b} xe^{-x}dx$$

Let $u = x$ and $du = dx$

Let $dv = e^{-x}$ so $v = -e^{-x}$

$$= \lim_{b\to\infty}\left[-xe^{-x}\right]_1^b - \int_1^b e^{-x}dx$$

$$= \lim_{b\to\infty}\left(\left[-xe^{-x}\right]_1^b - \left[e^{-x}\right]_1^b\right)$$

$$= \lim_{b\to\infty}\left\{\left(-be^{-b} + e^{-1}\right) - \left(e^{-b} - e^{-1}\right)\right\}$$

$$= 2e^{-1} - \lim_{b\to\infty}\left(\frac{1+b}{e^b}\right)$$

This is a limit of the indeterminate form $\dfrac{\infty}{\infty}$

Applying l'Hopital's Rule

$$\lim_{b\to\infty}\left(\frac{1+b}{e^b}\right) = \lim_{b\to\infty}\left(\frac{1}{e^b}\right) = 0$$

$$\therefore \int_1^{\infty} xe^{-x}dx = 2e^{-1}$$

We have established $\lim_{x\to\infty}\left(\frac{x}{e^x}\right) = \lim_{x\to\infty}\left(\frac{1}{e^x}\right) = 0$ through the application of L'Hopital's Rule. Using the same method we can establish and use the following limits.

$$\lim_{x\to\infty}\left(\frac{x^p}{e^x}\right) = 0$$

$$\lim_{x\to\infty}\left(\frac{x^p}{\ln x}\right) = \infty$$