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Mathematics
for the international student
Mathematics SL
second edition

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John Owen
Michael Haese
Robert Haese
Sandra Haese
Mark Humphries

for use with
IB Diploma
Programme

MATHEMATICS FOR THE INTERNATIONAL STUDENT

Mathematics SL second edition

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The textbook and its accompanying CD have been developed independently of the International Baccalaureate Organization (IBO). The textbook and CD are in no way connected with, or endorsed by, the IBO.

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FOREWORD

Mathematics for the International Student: Mathematics SL has been written to embrace the syllabus for the two-year Mathematics SL Course, which is one of the courses of study in the IB Diploma Programme. It is not our intention to define the course. Teachers are encouraged to use other resources. We have developed this book independently of the International Baccalaureate Organization (IBO) in consultation with many experienced teachers of IB Mathematics. The text is not endorsed by the IBO.

The second edition builds on the strength of the first edition. Chapters are arranged to follow the same order as the chapters in our *Mathematics HL (Core) second edition*, making it easier for teachers who have combined classes of SL and HL students.

Syllabus references are given at the beginning of each chapter. The new edition reflects the Mathematics SL syllabus more closely, with several sections from the first edition being consolidated in this second edition for greater teaching efficiency. Topics such as Pythagoras' theorem, coordinate geometry, and right angled triangle trigonometry, which appeared in Chapters 7 and 10 in the first edition, are now in the 'Background Knowledge' at the beginning of the book and accessible as printable pages on the CD.

Changes have been made in response to the introduction of a calculator-free examination paper. A large number of questions have been added and categorised as 'calculator' or 'non calculator'. In particular, the final chapter contains over 150 examination-style questions.

Comprehensive graphics calculator instructions are given for Casio fx-9860G, TI-84 Plus and TI-*n*spire in an introductory chapter (see p. 17) and, occasionally, where additional help may be needed, more detailed instructions are available as printable pages on the CD. The extensive use of graphics calculators and computer packages throughout the book enables students to realise the importance, application, and appropriate use of technology. No single aspect of technology has been favoured. It is as important that students work with a pen and paper as it is that they use their calculator or graphics calculator, or use a spreadsheet or graphing package on computer.

This package is language rich and technology rich. The combination of textbook and interactive Student CD will foster the mathematical development of students in a stimulating way. Frequent use of the interactive features on the CD is certain to nurture a much deeper understanding and appreciation of mathematical concepts. The CD also offers **Self Tutor** for every worked example. **Self Tutor** is accessed via the CD – click anywhere on any worked example to hear a teacher's voice explain each step in that worked example. This is ideal for catch-up and revision, or for motivated students who want to do some independent study outside school hours.

For students who may not have a good understanding of the necessary background knowledge for this course, we have provided printable pages of information, examples, exercises, and answers on the Student CD – see 'Background knowledge' (p. 12). To access these pages, click on the 'Background knowledge' icon when running the CD.

The interactive features of the CD allow immediate access to our own specially designed geometry software, graphing software and more. Teachers are provided with a quick and easy way to demonstrate concepts, and students can discover for themselves and re-visit when necessary.

continued next page

It is not our intention that each chapter be worked through in full. Time constraints may not allow for this. Teachers must select exercises carefully, according to the abilities and prior knowledge of their students, to make the most efficient use of time and give as thorough coverage of work as possible. Investigations throughout the book will add to the discovery aspect of the course and enhance student understanding and learning. Many investigations are suitable for portfolio assignments.

In this changing world of mathematics education, we believe that the contextual approach shown in this book, with the associated use of technology, will enhance the students' understanding, knowledge and appreciation of mathematics, and its universal application.

We welcome your feedback.

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EMM JTO PMH

RCH SHH MAH

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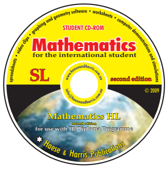
The publishers wish to make it clear that acknowledging these individuals does not imply any endorsement of this book by any of them, and all responsibility for the content rests with the authors and publishers.

USING THE INTERACTIVE STUDENT CD



The interactive CD is ideal for independent study.

Students can revisit concepts taught in class and undertake their own revision and practice. The CD also has the text of the book, allowing students to leave the textbook at school and keep the CD at home.

By clicking on the relevant icon, a range of interactive features can be accessed:

- Self Tutor
- Graphics calculator instructions
- Background knowledge (as printable pages)
- Interactive links to spreadsheets, graphing and geometry software, computer demonstrations and simulations

INTERACTIVE LINK

Graphics calculator instructions: where additional help may be needed, detailed instructions are available on the CD, as printable pages. Click on the relevant icon for TI-$n$spire, TI-84 Plus or Casio fx-9860G.

TI-$n$spire
TI-84
Casio

SELF TUTOR is an exciting feature of this book.

The Self Tutor icon on each worked example denotes an active link on the CD.

> Simply 'click' on the Self Tutor (or anywhere in the example box) to access the worked example, with a teacher's voice explaining each step necessary to reach the answer.
>
> Play any line as often as you like. See how the basic processes come alive using movement and colour on the screen.
>
> Ideal for students who have missed lessons or need extra help.

Example 1 **Self Tutor**

Find **a** the vector **b** the parametric **c** the Cartesian equation of the line passing through the point A(1, 5) with direction $\begin{pmatrix}3\\2\end{pmatrix}$.

a $\mathbf{a} = \overrightarrow{\text{OA}} = \begin{pmatrix}1\\5\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}3\\2\end{pmatrix}$

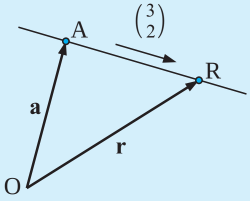
But $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ $\quad\therefore\quad \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\5\end{pmatrix} + t\begin{pmatrix}3\\2\end{pmatrix}$, $t \in \mathbb{R}$

b From **a**, $x = 1 + 3t$ and $y = 5 + 2t$, $t \in \mathbb{R}$

c Now $t = \dfrac{x-1}{3} = \dfrac{y-5}{2}$

$\therefore\quad 2x - 2 = 3y - 15$

$\therefore\quad 2x - 3y = -13 \qquad$ {general form}



See **Chapter 13**, **Lines and planes in space**, p. 356

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SYMBOLS AND NOTATION USED IN THIS BOOK

$\mathbb{N}$	the set of positive integers and zero, $\{0, 1, 2, 3, ...\}$
$\mathbb{Z}$	the set of integers, $\{0, \pm 1, \pm 2, \pm 3, ...\}$
$\mathbb{Z}^+$	the set of positive integers, $\{1, 2, 3, ...\}$
$\mathbb{Q}$	the set of rational numbers
$\mathbb{Q}^+$	the set of positive rational numbers, $\{x \mid x > 0,\ x \in \mathbb{Q}\}$
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of positive real numbers, $\{x \mid x > 0,\ x \in \mathbb{R}\}$
$\{x_1, x_2, ...\}$	the set with elements $x_1, x_2, ...$
$n(A)$	the number of elements in set A
$\{x \mid ....$	the set of all x such that
$\in$	is an element of
$\notin$	is not an element of
$\varnothing$	the empty (null) set
U	the universal set
$\cup$	union
$\cap$	intersection
$\subset$	is a proper subset of
$\subseteq$	is a subset of
A'	the complement of the set A
$a^{\frac{1}{n}},\ \sqrt[n]{a}$	a to the power of $\frac{1}{n}$, nth root of a (if $a \geqslant 0$ then $\sqrt[n]{a} \geqslant 0$)
$a^{\frac{1}{2}},\ \sqrt{a}$	a to the power $\frac{1}{2}$, square root of a (if $a \geqslant 0$ then $\sqrt{a} \geqslant 0$)
$\lvert x \rvert$	the modulus or absolute value of x $\lvert x \rvert = \begin{cases} x \text{ for } x \geqslant 0 & x \in \mathbb{R} \\ -x \text{ for } x < 0 & x \in \mathbb{R} \end{cases}$
$\equiv$	identity or is equivalent to
$\approx$	is approximately equal to
$>$	is greater than
$\geq$ or $\geqslant$	is greater than or equal to
$<$	is less than
$\leq$ or $\leqslant$	is less than or equal to
$\ngtr$	is not greater than
$\nless$	is not less than

$a \mid b$	a divides b
u_n	the nth term of a sequence or series
d	the common difference of an arithmetic sequence
r	the common ratio of a geometric sequence
S_n	the sum of the first n terms of a sequence, $u_1 + u_2 + ... + u_n$
S_∞ or S	the sum to infinity of a sequence, $u_1 + u_2 + ...$
$\sum_{i=1}^{n} u_i$	$u_1 + u_2 + ... + u_n$
$\binom{n}{r}$	the r^{th} binomial coefficient, $r = 0, 1, 2, ...$ in the expansion of $(a+b)^n$
$f : A \to B$	f is a function under which each element of set A has an image in set B
$f : x \mapsto y$	f is a function which maps x onto y
$f(x)$	the image of x under the function f
f^{-1}	the inverse function of the function f
$f \circ g$	the composite function of f and g
$\lim\limits_{x \to a} f(x)$	the limit of $f(x)$ as x tends to a
$\frac{dy}{dx}$	the derivative of y with respect to x
$f'(x)$	the derivative of $f(x)$ with respect to x
$\frac{d^2y}{dx^2}$	the second derivative of y with respect to x
$f''(x)$	the second derivative of $f(x)$ with respect to x
$\int y\,dx$	the indefinite integral of y with respect to x
$\int_a^b y\,dx$	the definite integral of y with respect to x between the limits $x = a$ and $x = b$
e^x	exponential function of x
$\log_a x$	logarithm to the base a of x
$\ln x$	the natural logarithm of x, $\log_e x$
sin, cos, tan	the circular functions

$\mathrm{A}(x, y)$	the point A in the plane with Cartesian coordinates x and y
[AB]	the line segment with end points A and B
AB	the length of [AB]
(AB)	the line containing points A and B
$\widehat{A}$	the angle at A
$\widehat{\mathrm{CAB}}$ or $\mathrm{C}\widehat{\mathrm{A}}\mathrm{B}$	the angle between [CA] and [AB]
$\Delta\mathrm{ABC}$	the triangle whose vertices are A, B and C
$\parallel$	is parallel to
$\perp$	is perpendicular to
$\mathbf{v}$	the vector $\mathbf{v}$
$\overrightarrow{\mathrm{AB}}$	the vector represented in magnitude and direction by the directed line segment from A to B
$\mathbf{a}$	the position vector $\overrightarrow{\mathrm{OA}}$
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors in the directions of the Cartesian coordinate axes
$\lvert\mathbf{a}\rvert$	the magnitude of vector $\mathbf{a}$
$\left\lvert\overrightarrow{\mathrm{AB}}\right\rvert$	the magnitude of $\overrightarrow{\mathrm{AB}}$
$\mathbf{v} \bullet \mathbf{w}$	the scalar product of $\mathbf{v}$ and $\mathbf{w}$
$\mathbf{A}^{-1}$	the inverse of the non-singular matrix $\mathbf{A}$
$\det\mathbf{A}$ or $\lvert\mathbf{A}\rvert$	the determinant of the square matrix $\mathbf{A}$
$\mathbf{I}$	the identity matrix
$\mathrm{P}(A)$	probability of event A
$\mathrm{P}(A')$	probability of the event ‘not A’
$\mathrm{P}(A \mid B)$	probability of the event A given B

$x_1, x_2, ...$	observations of a variable
$f_1, f_2, ...$	frequencies with which the observations $x_1, x_2, x_3, ...$ occur
$\mathrm{P}(X = x)$	the probability distribution function of the discrete random variable X
$\mathrm{E}(X)$	the expected value of the random variable X
μ	population mean
σ	population standard deviation
σ^2	population variance
$\overline{x}$	sample mean
s_n^2	sample variance
s_n	standard deviation of the sample
s_{n-1}^2	unbiased estimate of the population variance
$\mathrm{B}(n, p)$	binomial distribution with parameters n and p
$\mathrm{N}(\mu, \sigma^2)$	normal distribution with mean μ and variance σ^2
$X \sim \mathrm{B}(n, p)$	the random variable X has a binomial distribution with parameters n and p
$X \sim \mathrm{N}(\mu, \sigma^2)$	the random variable X has a normal distribution with mean μ and variance σ^2
Φ	the cumulative distribution function of the standardised normal variable with distribution $\mathrm{N}(0, 1)$

BACKGROUND KNOWLEDGE

Before starting this course you can make sure that you have a good understanding of the necessary background knowledge. Click on the icon alongside to obtain a printable set of exercises and answers on this background knowledge.

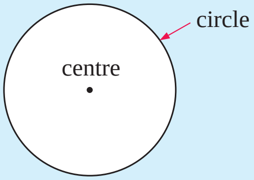
BACKGROUND KNOWLEDGE

Click on the icon to access printable facts about number sets.

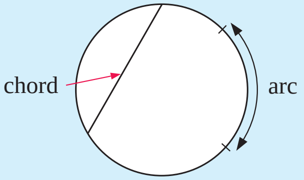
NUMBER SETS

SUMMARY OF CIRCLE PROPERTIES

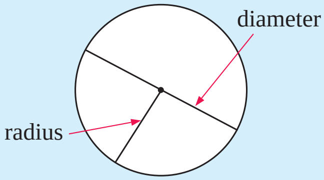
- A **circle** is a set of points which are equidistant from a fixed point, which is called its **centre**.
- The circumference is the distance around the entire circle boundary.



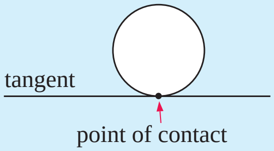
- An **arc** of a circle is any continuous part of the circle.
- A **chord** of a circle is a line segment joining any two points of a circle.



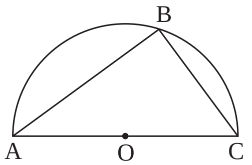
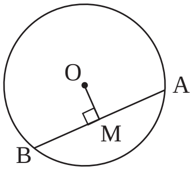
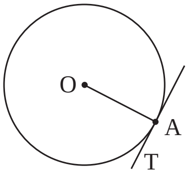
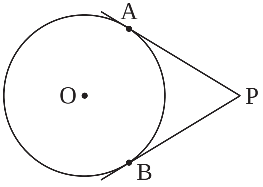
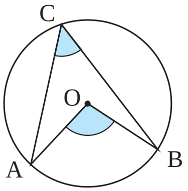
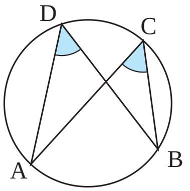
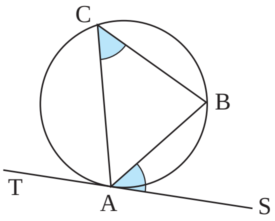
- A **semi-circle** is a half of a circle.
- A **diameter** of a circle is any chord passing through its centre.
- A **radius** of a circle is any line segment joining its centre to any point on the circle.



- A **tangent** to a circle is any line which touches the circle in exactly one point.



Click on the appropriate icon to revisit these well known theorems.

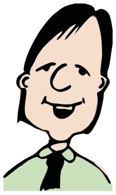
Name of theorem	Statement	Diagram
Angle in a semi-circle	The angle in a semi-circle is a right angle.	 $A\hat{B}C = 90^o$ GEOMETRY PACKAGE
Chords of a circle	The perpendicular from the centre of a circle to a chord bisects the chord.	 $AM = BM$ GEOMETRY PACKAGE
Radius-tangent	The tangent to a circle is perpendicular to the radius at the point of contact.	 $O\hat{A}T = 90^o$ GEOMETRY PACKAGE
Tangents from an external point	Tangents from an external point are equal in length.	 $AP = BP$ GEOMETRY PACKAGE
Angle at the centre	The angle at the centre of a circle is twice the angle on the circle subtended by the same arc.	 $A\hat{O}B = 2\times A\hat{C}B$ GEOMETRY PACKAGE
Angles subtended by the same arc	Angles subtended by an arc on the circle are equal in size.	 $A\hat{D}B = A\hat{C}B$ GEOMETRY PACKAGE
Angle between a tangent and a chord	The angle between a tangent and a chord at the point of contact is equal to the angle subtended by the chord in the alternate segment.	 $B\hat{A}S = B\hat{C}A$ GEOMETRY PACKAGE

SUMMARY OF MEASUREMENT FACTS

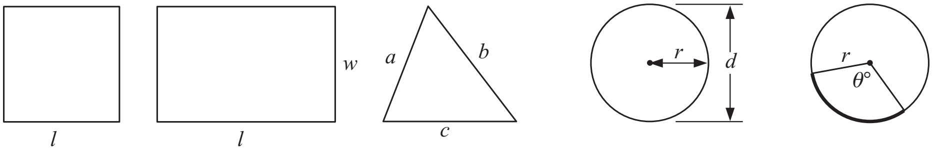
PERIMETER FORMULAE

The distance around a closed figure is its **perimeter**.

For some shapes we can derive a formula for perimeter. The formulae for the most common shapes are given below:

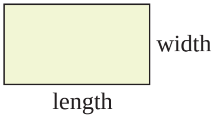
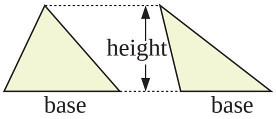
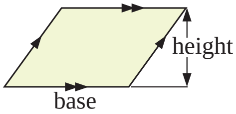
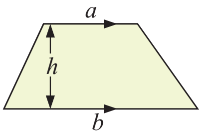
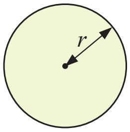
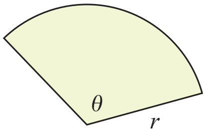


The length of an arc is a fraction of the circumference of a circle.



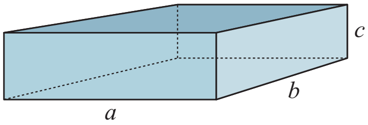
square	**rectangle**	**triangle**	**circle**	**arc**
$P = 4l$	$P = 2(l + w)$	$P = a + b + c$	$C = 2\pi r$ or $C = \pi d$	$l = \left(\frac{\theta}{360}\right) 2\pi r$

AREA FORMULAE

Shape	*Figure*	*Formula*
Rectangle		**Area** $=$ **length** $\times$ **width**
Triangle		**Area** $= \frac{1}{2}$**base** $\times$ **height**
Parallelogram		**Area** $=$ **base** $\times$ **height**
Trapezium or **Trapezoid**		**Area** $= \left(\frac{a+b}{2}\right) \times h$
Circle		**Area** $= \pi r^2$
Sector		**Area** $= \left(\frac{\theta}{360}\right) \times \pi r^2$

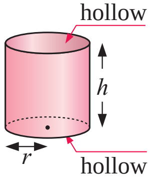
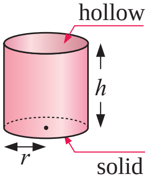
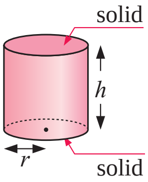
SURFACE AREA FORMULAE

RECTANGULAR PRISM

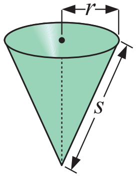
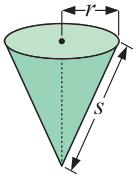


$$A = 2(ab + bc + ac)$$

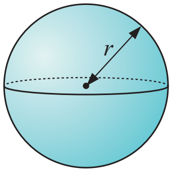
CYLINDER

Object	*Outer surface area*
Hollow cylinder  	$A = 2\pi rh$ (no ends)
Open can  	$A = 2\pi rh + \pi r^2$ (one end)
Solid cylinder  	$A = 2\pi rh + 2\pi r^2$ (two ends)

CONE

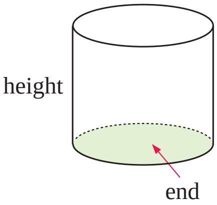
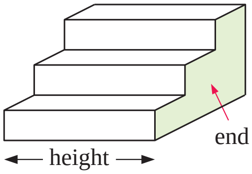
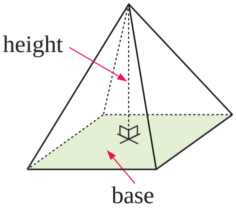
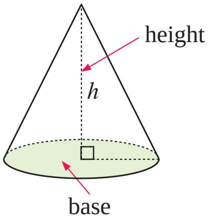
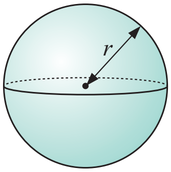
Object	*Outer surface area*
Open cone  	$A = \pi rs$ (no base)
Solid cone  	$A = \pi rs + \pi r^2$ (solid)

SPHERE



$$A = 4\pi r^2$$

VOLUME FORMULAE

Object	*Figure*	*Volume*
Solids of uniform cross-section	 	**Volume of uniform solid** **= area of end × length**
Pyramids and cones	 	**Volume of a pyramid or cone** $= \frac{1}{3}$**(area of base × height)**
Spheres		**Volume of a sphere** $= \frac{4}{3}\pi r^3$

Graphics calculator instructions

Contents:

In this course it is assumed that you have a **graphics calculator**. If you learn how to operate your calculator successfully, you should experience little difficulty with future arithmetic calculations.

There are many different brands (and types) of calculators. Different calculators do not have exactly the same keys. It is therefore important that you have an instruction booklet for your calculator, and use it whenever you need to.

However, to help get you started, we have included here some basic instructions for the **Casio fx-9860G**, the **Texas Instruments TI-84 plus** and the **Texas Instruments TI-*n*spire** calculators. Note that instructions given may need to be modified slightly for other models.

The instructions have been divided into three sections, one for each of the calculator models.

A CASIO FX-9860G

BASIC FUNCTIONS

GROUPING SYMBOLS (BRACKETS)

The Casio has bracket keys that look like $\boxed{(}$ and $\boxed{)}$.

Brackets are regularly used in mathematics to indicate an expression which needs to be evaluated before other operations are carried out.

For example, to evaluate $2 \times (4+1)$ we type 2 $\boxed{\times}$ $\boxed{(}$ 4 $\boxed{+}$ 1 $\boxed{)}$ $\boxed{\text{EXE}}$.

We also use brackets to make sure the calculator understands the expression we are typing in.

For example, to evaluate $\frac{2}{4+1}$ we type 2 $\boxed{\div}$ $\boxed{(}$ 4 $\boxed{+}$ 1 $\boxed{)}$ $\boxed{\text{EXE}}$.

If we typed 2 $\boxed{\div}$ 4 $\boxed{+}$ 1 $\boxed{\text{EXE}}$ the calculator would think we meant $\frac{2}{4}+1$.

In general, it is a good idea to place brackets around any complicated expressions which need to be evaluated separately.

POWER KEYS

The Casio has a power key that looks like $\boxed{\wedge}$. We type the base first, press the power key, then enter the index or exponent.

For example, to evaluate 25^3 we type 25 $\boxed{\wedge}$ 3 $\boxed{\text{EXE}}$.

Numbers can be squared on the Casio using the special key $\boxed{x^2}$.

For example, to evaluate 25^2 we type 25 $\boxed{x^2}$ $\boxed{\text{EXE}}$.

ROOTS

To enter roots on the Casio we need to use the secondary function key $\boxed{\text{SHIFT}}$.

We enter square roots by pressing $\boxed{\text{SHIFT}}$ $\boxed{x^2}$.

For example, to evaluate $\sqrt{36}$ we press $\boxed{\text{SHIFT}}$ $\boxed{x^2}$ 36 $\boxed{\text{EXE}}$.

If there is a more complicated expression under the square root sign you should enter it in brackets.

For example, to evaluate $\sqrt{18 \div 2}$ we press $\boxed{\text{SHIFT}}$ $\boxed{x^2}$ $\boxed{(}$ 18 $\boxed{\div}$ 2 $\boxed{)}$ $\boxed{\text{EXE}}$.

Cube roots are entered by pressing $\boxed{\text{SHIFT}}$ $\boxed{(}$.

For example, to evaluate $\sqrt[3]{8}$ we press $\boxed{\text{SHIFT}}$ $\boxed{(}$ 8 $\boxed{\text{EXE}}$.

Higher roots are entered by pressing $\boxed{\text{SHIFT}}$ $\boxed{\wedge}$.

For example, to evaluate $\sqrt[4]{81}$ we press 4 $\boxed{\text{SHIFT}}$ $\boxed{\wedge}$ 81 $\boxed{\text{EXE}}$.

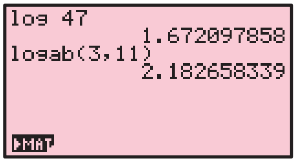
LOGARITHMS

We can perform operations involving logarithms in base 10 using the $\boxed{\text{log}}$ button.

To evaluate $\log(47)$ press $\boxed{\text{log}}$ 47 $\boxed{\text{EXE}}$.

To evaluate $\log_3 11$, press $\boxed{\text{SHIFT}}$ **4 (CATALOG)**, and select **logab(** . You can use the alpha keys to navigate the catalog, so in this example press $\boxed{\rightarrow}$ to jump to "L".

Press 3 $\boxed{,}$ 11 $\boxed{)}$ $\boxed{\text{EXE}}$.



```
log 47
              1.672097858
logab(3,11)
              2.182658339
```

INVERSE TRIGONOMETRIC FUNCTIONS

The inverse trigonometric functions $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ are the secondary functions of $\boxed{\text{sin}}$, $\boxed{\text{cos}}$ and $\boxed{\text{tan}}$ respectively. They are accessed by using the secondary function key $\boxed{\text{SHIFT}}$.

For example, if $\cos x = \frac{3}{5}$, then $x = \cos^{-1}\left(\frac{3}{5}\right)$.

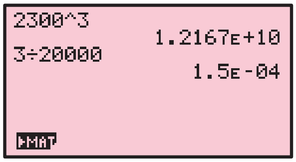
To calculate this, press $\boxed{\text{SHIFT}}$ $\boxed{\text{cos}}$ $\boxed{(}$ 3 $\boxed{\div}$ 5 $\boxed{)}$ $\boxed{\text{EXE}}$.

SCIENTIFIC NOTATION

If a number is too large or too small to be displayed neatly on the screen, it will be expressed in scientific notation, which is the form $a \times 10^k$ where $1 \leqslant a < 10$ and k is an integer.

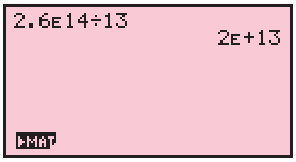
To evaluate 2300^3, press 2300 $\boxed{\wedge}$ 3 $\boxed{\text{EXE}}$. The answer displayed is 1.2167E+10, which means 1.2167×10^{10}.

To evaluate $\frac{3}{20\,000}$, press 3 $\boxed{\div}$ 20 000 $\boxed{\text{EXE}}$. The answer displayed is 1.5E−04, which means 1.5×10^{-4}.



```
2300^3
              1.2167E+10
3÷20000
                 1.5E-04
```

You can enter values in scientific notation using the EXP key. For example, to evaluate $\frac{2.6\times10^{14}}{13}$, press 2.6 EXP 14 ÷ 13 EXE . The answer is 2×10^{13}.



```
2.6E14÷13
                 2E+13
```

SECONDARY FUNCTION AND ALPHA KEYS

The **shift function** of each key is displayed in yellow above the key. It is accessed by pressing the SHIFT key followed by the key corresponding to the desired shift function.

For example, to calculate $\sqrt{36}$, press SHIFT x^2 ($\sqrt{}$) 36 EXE .

The **alpha function** of each key is displayed in red above the key. It is accessed by pressing the ALPHA key followed by the key corresponding to the desired letter. The main purpose of the alpha keys is to store values which can be recalled later.

MEMORY

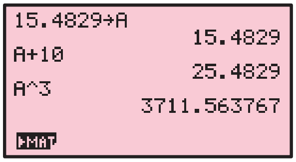
Utilising the memory features of your calculator allows you to recall calculations you have performed previously. This not only saves time, but also enables you to maintain accuracy in your calculations.

SPECIFIC STORAGE TO MEMORY

Values can be stored into the variable letters A, B, ..., Z. Storing a value in memory is useful if you need that value multiple times.

Suppose we wish to store the number 15.4829 for use in a number of calculations. To store this number in variable A, type in the number then press → ALPHA X,θ,T (**A**) EXE .

We can now add 10 to this value by pressing ALPHA X,θ,T + 10 EXE , or cube this value by pressing ALPHA X,θ,T ∧ 3 EXE .



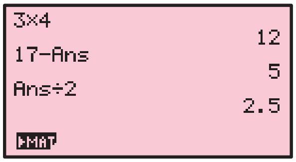
```
15.4829→A
               15.4829
A+10
               25.4829
A^3
           3711.563767
```

ANS VARIABLE

The variable **Ans** holds the most recent evaluated expression, and can be used in calculations by pressing SHIFT (−) . For example, suppose you evaluate 3 × 4, and then wish to subtract this from 17. This can be done by pressing 17 − SHIFT (−) EXE .

```
3×4
                    12
17-Ans
                     5
```

If you start an expression with an operator such as + , − , etc, the previous answer **Ans** is automatically inserted ahead of the operator. For example, the previous answer can be halved simply by pressing ÷ 2 EXE .



```
3×4
                    12
17-Ans
                     5
Ans÷2
                   2.5
```

If you wish to view the answer in fractional form, press F↔D .

RECALLING PREVIOUS EXPRESSIONS

Pressing the left cursor key allows you to edit the most recently evaluated expression, and is useful if you wish to repeat a calculation with a minor change, or if you have made an error in typing.

Suppose you have evaluated $100 + \sqrt{132}$. If you now want to evaluate $100 + \sqrt{142}$, instead of retyping the command, it can be recalled by pressing the left cursor key. Move the cursor between the 3 and the 2, then press DEL 4 to remove the 3 and change it to a 4. Press EXE to re-evaluate the expression.

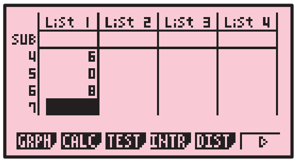
LISTS

Lists enable us to store sets of data, which we can then analyse and compare.

CREATING A LIST

Selecting **STAT** from the Main Menu takes you to the **list editor** screen.

To enter the data {2, 5, 1, 6, 0, 8} into **List 1**, start by moving the cursor to the first entry of **List 1**. Press 2 EXE 5 EXE ...... and so on until all the data is entered.



```
     List 1 List 2 List 3 List 4
SUB
4         6
5         0
6         8
7
```

DELETING LIST DATA

To delete a list of data from the list editor screen, move the cursor to anywhere on the list you wish to delete, then press F6 (▷) F4 (**DEL-A**) F1 (**Yes**).

REFERENCING LISTS

Lists can be referenced using the List function, which is accessed by pressing SHIFT **1**.

For example, if you want to add 2 to each element of **List 1** and display the results in **List 2**, move the cursor to the heading of **List 2** and press SHIFT **1** (**List**) 1 + 2 EXE .

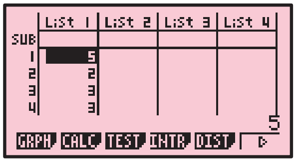
Casio models without the List function can do this by pressing OPTN F1 (**LIST**) F1 (**List**) 1 + 2 EXE .

STATISTICS

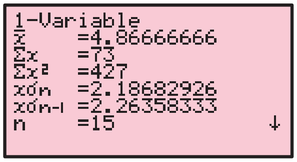
Your graphics calculator is a useful tool for analysing data and creating statistical graphs.

We will first produce descriptive statistics and graphs for the data set 5 2 3 3 6 4 5 3 7 5 7 1 8 9 5.

Enter the data into **List 1**. To obtain the descriptive statistics, press F6 (▷) until the **GRPH** icon is in the bottom left corner of the screen, then press F2 (**CALC**) F1 (**1 VAR**).

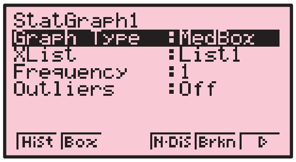


```
     List 1 List 2 List 3 List 4
SUB
1         5
2         2
3         3
4         3
                              5
```

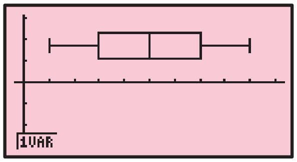


```
1-Variable
x̄     =4.86666666
Σx    =73
Σx²   =427
xσn   =2.18682926
xσn-1 =2.26358333
n     =15
                  ↓
```

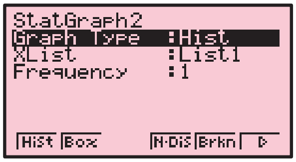
To obtain a boxplot of the data, press EXIT EXIT F1 (**GRPH**) F6 (**SET**), and set up **StatGraph 1** as shown. Press EXIT F1 (**GPH1**) to draw the boxplot.



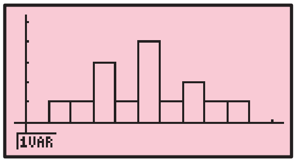
```
StatGraph1
Graph Type  :MedBox
XList       :List1
Frequency   :1
Outliers    :Off
```



To obtain a vertical bar chart of the data, press EXIT F6 (**SET**) F2 (**GPH 2**), and set up **StatGraph 2** as shown. Press EXIT F2 (**GPH 2**) to draw the bar chart (set Start to 0, and Width to 1).

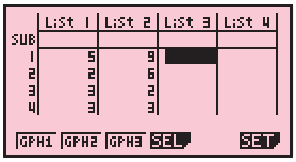


```
StatGraph2
Graph Type  :Hist
XList       :List1
Frequency   :1
```

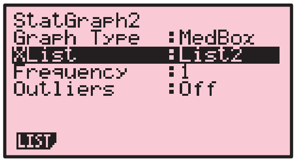


We will now enter a second set of data, and compare it to the first.

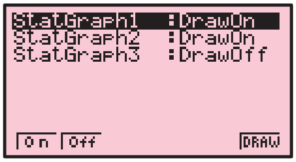
Enter the data set 9 6 2 3 5 5 7 5 6 7 6 3 4 4 5 8 4 into **List 2**, then press F6 (**SET**) F2 (**GPH2**) and set up **StatGraph 2** to draw a boxplot of this data set as shown. Press EXIT F4 (**SEL**), and turn on both **StatGraph 1** and **StatGraph 2**. Press F6 (**DRAW**) to draw the side-by-side boxplots.



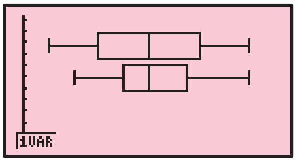
```
    List 1  List 2  List 3  List 4
SUB
  1      5       9
  2      2       6
  3      3       2
  4      3       3
```



```
StatGraph2
Graph Type  :MedBox
XList       :List2
Frequency   :1
Outliers    :Off
```



```
StatGraph1  :DrawOn
StatGraph2  :DrawOn
StatGraph3  :DrawOff
```

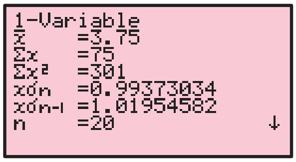


STATISTICS FROM GROUPED DATA

To obtain descriptive statistics for the data in the table alongside, enter the data values into **List 1**, and the frequency values into **List 2**.

Data	*Frequency*
2	3
3	4
4	8
5	5

```
1Var XList  :List1
1Var Freq   :List2
2Var XList  :List1
2Var YList  :List2
2Var Freq   :1
```

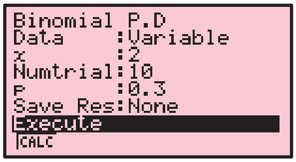


```
1-Variable
x̄      =3.75
Σx     =75
Σx²    =301
xσn    =0.99373034
xσn-1  =1.01954582
n      =20
```

Press F2 (**CALC**) F6 (**SET**), and change the **1 Var Freq** variable to **List 2**. Press EXIT F1 (**1Var**) to view the statistics.

BINOMIAL PROBABILITIES

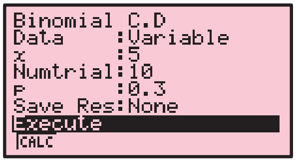
To find $\mathrm{P}(X=2)$ for $X \sim \mathrm{B}(10, 0.3)$, select **STAT** from the Main Menu and press F5 (**DIST**) F5 (**BINM**) F1 (**Bpd**). Set up the screen as shown. Go to **Execute** and press EXE to display the result, which is 0.233.



```
Binomial P.D
Data    :Variable
x       :2
Numtrial:10
p       :0.3
Save Res:None
Execute
```

To find $P(X \leqslant 5)$ for $X \sim B(10, 0.3)$, select **STAT** from the Main Menu and press $\boxed{\text{F5}}$ **(DIST)** $\boxed{\text{F5}}$ **(BINM)** $\boxed{\text{F2}}$ **(Bcd)**. Set up the screen as shown.

Go to **Execute** and press $\boxed{\text{EXE}}$ to display the result, which is 0.953.



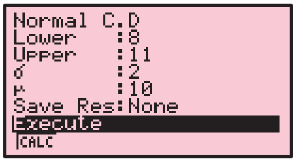
```
Binomial C.D
Data    :Variable
x       :5
Numtrial:10
p       :0.3
Save Res:None
Execute
```

NORMAL PROBABILITIES

Suppose X is normally distributed with mean 10 and standard deviation 2.

To find $P(8 \leqslant X \leqslant 11)$, select **STAT** from the Main Menu and press $\boxed{\text{F5}}$ **(DIST)** $\boxed{\text{F1}}$ **(NORM)** $\boxed{\text{F2}}$ **(Ncd)**.

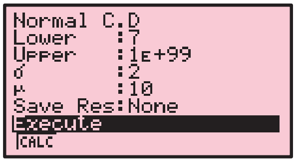
Set up the screen as shown. Go to **Execute** and press $\boxed{\text{EXE}}$ to display the result, which is 0.533.



```
Normal C.D
Lower   :8
Upper   :11
σ       :2
μ       :10
Save Res:None
Execute
```

To find $P(X \geqslant 7)$, select **STAT** from the Main Menu and press $\boxed{\text{F5}}$ **(DIST)** $\boxed{\text{F1}}$ **(NORM)** $\boxed{\text{F2}}$ **(Ncd)**.

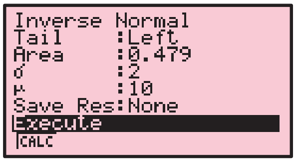
Set up the screen as shown. Go to **Execute** and press $\boxed{\text{EXE}}$ to display the result, which is 0.933.



```
Normal C.D
Lower   :7
Upper   :1E+99
σ       :2
μ       :10
Save Res:None
Execute
```

To find a such that $P(X \leqslant a) = 0.479$, select **STAT** from the Main Menu and press $\boxed{\text{F5}}$ **(DIST)** $\boxed{\text{F1}}$ **(NORM)** $\boxed{\text{F3}}$ **(InvN)**.

Set up the screen as shown. Go to **Execute** and press $\boxed{\text{EXE}}$ to display the result, which is $a \approx 9.89$.

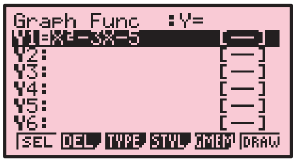


```
Inverse Normal
Tail    :Left
Area    :0.479
σ       :2
μ       :10
Save Res:None
Execute
```

WORKING WITH FUNCTIONS

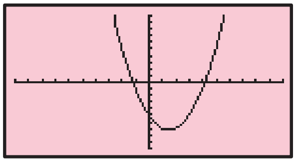
GRAPHING FUNCTIONS

Selecting **GRAPH** from the Main Menu takes you to the Graph Function screen, where you can store functions to graph. Delete any unwanted functions by scrolling down to the function and pressing $\boxed{\text{DEL}}$ $\boxed{\text{F1}}$ **(Yes)**.

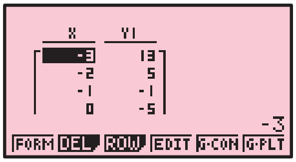


```
Graph Func  :Y=
Y1=X²-3X-5        [—]
Y2:               [—]
Y3:               [—]
Y4:               [—]
Y5:               [—]
Y6:               [—]
```

To graph the function $y = x^2 - 3x - 5$, move the cursor to **Y1** and press $\boxed{\text{X,}\theta\text{,T}}$ $\boxed{x^2}$ $\boxed{-}$ 3 $\boxed{\text{X,}\theta\text{,T}}$ $\boxed{-}$ 5 $\boxed{\text{EXE}}$. This stores the function into **Y1**. Press $\boxed{\text{F6}}$ **(DRAW)** to draw a graph of the function.



To view a table of values for the function, press $\boxed{\text{MENU}}$ and select **TABLE**. The function is stored in **Y1**, but not selected. Press $\boxed{\text{F1}}$ **(SEL)** to select the function, and $\boxed{\text{F6}}$ **(TABL)** to view the table. You can adjust the table settings by pressing $\boxed{\text{EXIT}}$ and then $\boxed{\text{F5}}$ **(SET)** from the Table Function screen.



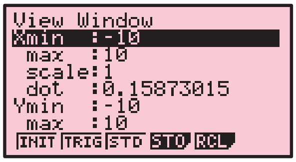
X	Y1
-3	13
-2	5
-1	-1
0	-5

-3

ADJUSTING THE VIEWING WINDOW

When graphing functions it is important that you are able to view all the important features of the graph. As a general rule it is best to start with a large viewing window to make sure all the features of the graph are visible. You can then make the window smaller if necessary.

The viewing window can be adjusted by pressing SHIFT F3 (**V-Window**). You can manually set the minimum and maximum values of the x and y axes, or press F3 (**STD**) to obtain the standard viewing window $-10 \leqslant x \leqslant 10$, $-10 \leqslant y \leqslant 10$.

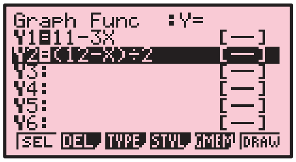


```
View Window
Xmin  :-10
 max  :10
 scale:1
 dot  :0.15873015
Ymin  :-10
 max  :10
```

FINDING POINTS OF INTERSECTION

It is often useful to find the points of intersection of two graphs, for instance, when you are trying to solve simultaneous equations.

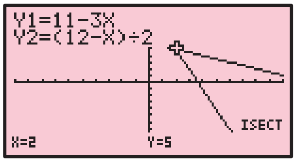
We can solve $y = 11 - 3x$ and $y = \frac{12 - x}{2}$ simultaneously by finding the point of intersection of these two lines. Select **GRAPH** from the Main Menu, then store $11 - 3x$ into **Y1** and $\frac{12 - x}{2}$ into **Y2**. Press F6 (**DRAW**) to draw a graph of the functions.



```
Graph Func  :Y=
Y1=11-3X          [—]
Y2=(12-X)÷2       [—]
Y3:               [—]
Y4:               [—]
Y5:               [—]
Y6:               [—]
```

To find their point of intersection, press F5 (**G-Solv**) F5 (**ISCT**). The solution $x = 2$, $y = 5$ is given.

If there is more than one point of intersection, the remaining points of intersection can be found by pressing ▶ .

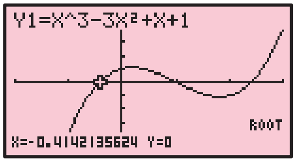


```
Y1=11-3X
Y2=(12-X)÷2
X=2          Y=5
```

FINDING x-INTERCEPTS

In the special case when you wish to solve an equation of the form $f(x) = 0$, this can be done by graphing $y = f(x)$ and then finding when this graph cuts the x-axis.

To solve $x^3 - 3x^2 + x + 1 = 0$, select **GRAPH** from the Main Menu and store $x^3 - 3x^2 + x + 1$ into **Y1**. Press F6 (**DRAW**) to draw the graph.



```
Y1=X^3-3X²+X+1
X=-0.4142135624 Y=0
```

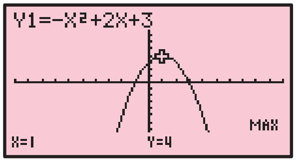
To find where this function cuts the x-axis, press F5 (**G-Solv**) F1 (**ROOT**). The first solution $x \approx -0.414$ is given.

Press ▶ to find the remaining solutions $x = 1$ and $x \approx 2.41$.

TURNING POINTS

To find the turning point or vertex of $y = -x^2 + 2x + 3$, select **GRAPH** from the Main Menu and store $-x^2 + 2x + 3$ into **Y1**. Press F6 (**DRAW**) to draw the graph.

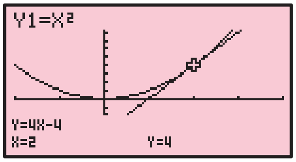
From the graph, it is clear that the vertex is a maximum, so to find the vertex press F5 (**G-Solv**) F2 (**MAX**). The vertex is (1, 4).



```
Y1=-X²+2X+3
X=1          Y=4
```

FINDING THE TANGENT TO A FUNCTION

To find the equation of the tangent to $y = x^2$ when $x = 2$, we first press SHIFT MENU (**SET UP**), and change the **Derivative** setting to **On**. Draw the graph of $y = x^2$, then press SHIFT F4 (**Sketch**) F2 (**Tang**).



```
Y1=X²
Y=4X-4
X=2          Y=4
```

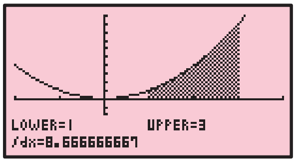
Press 2 EXE EXE to draw the tangent at $x = 2$.

The tangent has gradient 4, and equation $y = 4x - 4$.

DEFINITE INTEGRALS

To calculate $\int_1^3 x^2\,dx$, we first draw the graph of $y = x^2$. Press F5 (**G-Solv**) F6 F3 ($\int$ ***dx***) to select the integral tool. Press 1 EXE 3 EXE to specify the lower and upper bounds of the integral.

So, $\int_1^3 x^2\,dx = 8\frac{2}{3}$.



```
LOWER=1          UPPER=3
∫dx=8.666666667
```

MATRICES

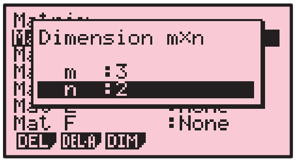
Matrices are easily stored in a graphics calculator. This is particularly valuable if we need to perform a number of operations with the same matrices.

STORING MATRICES

To store the matrix $\begin{pmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 0 \end{pmatrix}$, select **RUN·MAT** from the Main Menu, and press F1 (**▶MAT**). This is where you define matrices and enter their elements.

```
Matrix
Mat A        :None
Mat B        :None
Mat C        :None
Mat D        :None
Mat E        :None
Mat F        :None
```

To define the matrix as matrix **A**, make sure **Mat A** is highlighted, and press F3 (**DIM**) 3 EXE 2 EXE EXE . This indicates that the matrix has 3 rows and 2 columns.

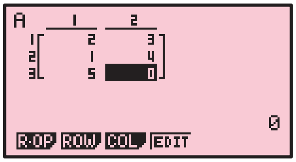


```
Dimension m×n

  m   :3
  n   :2
Mat F        :None
```

Enter the elements of the matrix, pressing EXE after each entry.

Press EXIT twice to return to the home screen when you are done.



```
A      1      2
 1[     2      3]
 2[     1      4]
 3[     5      0]
                      0
```

MATRIX OPERATIONS

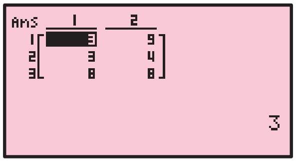
To find the sum of $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 6 \\ 2 & 0 \\ 3 & 8 \end{pmatrix}$, we first define matrices **A** and **B**.

To find **A** + **B**, press OPTN F2 (**MAT**) F1 (**Mat**) ALPHA X,θ,T (**A**) to enter matrix **A**, then + , then F1 (**Mat**) ALPHA log (**B**) to enter matrix **B**.

```
Mat A+Mat B
```

Press $\boxed{\text{EXE}}$ to display the result.

Operations of subtraction, scalar multiplication, and matrix multiplication can be performed in a similar manner.



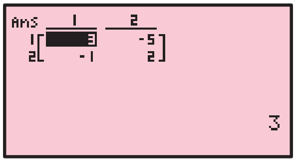
```
Ans    1      2
  1[       3      9]
  2[       3      4]
  3[       8      8]
                          3
```

INVERTING MATRICES

To find the inverse of $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$, we first define matrix $\mathbf{A}$.

Press $\boxed{\text{OPTN}}$ $\boxed{\text{F2}}$ **(MAT)** $\boxed{\text{F1}}$ **(Mat)** $\boxed{\text{ALPHA}}$ $\boxed{\text{X},\theta,\text{T}}$ **(A)** to enter matrix **A**, then $\boxed{\text{SHIFT}}$ $\boxed{)}$ $(\boldsymbol{x}^{-1})$ $\boxed{\text{EXE}}$.

So, $\mathbf{A}^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$.



```
Ans    1      2
  1[       3     -5]
  2[      -1      2]
                          3
```

B TEXAS INSTRUMENTS TI-84 PLUS

BASIC FUNCTIONS

GROUPING SYMBOLS (BRACKETS)

The TI-84 Plus has bracket keys that look like $\boxed{(}$ and $\boxed{)}$.

Brackets are regularly used in mathematics to indicate an expression which needs to be evaluated before other operations are carried out.

For example, to evaluate $2 \times (4 + 1)$ we type 2 $\boxed{\times}$ $\boxed{(}$ 4 $\boxed{+}$ 1 $\boxed{)}$ $\boxed{\text{ENTER}}$.

We also use brackets to make sure the calculator understands the expression we are typing in.

For example, to evaluate $\frac{2}{4+1}$ we type 2 $\boxed{\div}$ $\boxed{(}$ 4 $\boxed{+}$ 1 $\boxed{)}$ $\boxed{\text{ENTER}}$.

If we typed 2 $\boxed{\div}$ 4 $\boxed{+}$ 1 $\boxed{\text{ENTER}}$ the calculator would think we meant $\frac{2}{4} + 1$.

In general, it is a good idea to place brackets around any complicated expressions which need to be evaluated separately.

POWER KEYS

The TI-84 Plus has a power key that looks like $\boxed{\wedge}$. We type the base first, press the power key, then enter the index or exponent.

For example, to evaluate 25^3 we type 25 $\boxed{\wedge}$ 3 $\boxed{\text{ENTER}}$.

Numbers can be squared on the TI-84 Plus using the special key $\boxed{x^2}$.

For example, to evaluate 25^2 we type 25 $\boxed{x^2}$ $\boxed{\text{ENTER}}$.

ROOTS

To enter roots on the TI-84 Plus we need to use the secondary function key 2nd .

We enter square roots by pressing 2nd x^2 .

For example, to evaluate $\sqrt{36}$ we press 2nd x^2 36) ENTER .

The end bracket is used to tell the calculator we have finished entering terms under the square root sign.

Cube roots are entered by pressing MATH **4:** $\sqrt[3]{\ }$ (.

To evaluate $\sqrt[3]{8}$ we press MATH **4** : 8) ENTER .

Higher roots are entered by pressing MATH **5**: $\sqrt[x]{\ }$.

To evaluate $\sqrt[4]{81}$ we press 4 MATH **5** : 81 ENTER .

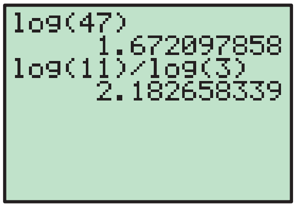
LOGARITHMS

We can perform operations involving logarithms in base 10 using the log button.

To evaluate $\log(47)$, press log 47) ENTER .

Since $\log_a b = \dfrac{\log b}{\log a}$, we can use the base 10 logarithm to calculate logarithms in other bases.

To evaluate $\log_3 11$, we note that $\log_3 11 = \dfrac{\log 11}{\log 3}$, so we press log 11) ÷ log 3) ENTER .



```
log(47)
        1.672097858
log(11)/log(3)
        2.182658339
```

INVERSE TRIGONOMETRIC FUNCTIONS

The inverse trigonometric functions $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ are the secondary functions of SIN , COS and TAN respectively. They are accessed by using the secondary function key 2nd .

For example, if $\cos x = \frac{3}{5}$, then $x = \cos^{-1}\left(\frac{3}{5}\right)$.

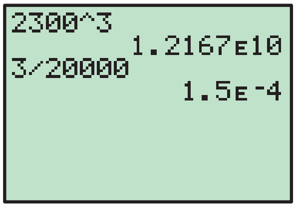
To calculate this, press 2nd COS 3 ÷ 5) ENTER .

SCIENTIFIC NOTATION

If a number is too large or too small to be displayed neatly on the screen, it will be expressed in scientific notation, which is the form $a \times 10^k$ where $1 \leqslant a < 10$ and k is an integer.

To evaluate 2300^3, press 2300 ∧ 3 ENTER . The answer displayed is 1.2167E10, which means 1.2167×10^{10}.

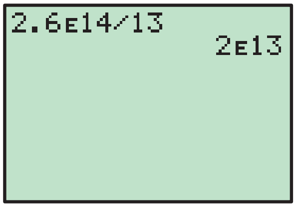
To evaluate $\frac{3}{20\,000}$, press 3 ÷ 20 000 ENTER . The answer displayed is 1.5E−4, which means 1.5×10^{-4}.



```
2300^3
          1.2167E10
3/20000
             1.5E-4
```

You can enter values in scientific notation using the EE function, which is accessed by pressing 2nd , .

For example, to evaluate $\frac{2.6\times10^{14}}{13}$, press 2.6 2nd , 14 ÷ 13 ENTER . The answer is 2×10^{13}.



```
2.6E14/13
             2E13
```

SECONDARY FUNCTION AND ALPHA KEYS

The **secondary function** of each key is displayed in blue above the key. It is accessed by pressing the 2nd key, followed by the key corresponding to the desired secondary function. For example, to calculate $\sqrt{36}$, press 2nd x^2 ($\sqrt{\ }$) 36) ENTER .

The **alpha function** of each key is displayed in green above the key. It is accessed by pressing the ALPHA key followed by the key corresponding to the desired letter. The main purpose of the alpha keys is to store values into memory which can be recalled later.

MEMORY

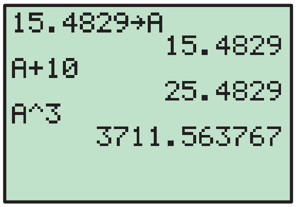
Utilising the memory features of your calculator allows you to recall calculations you have performed previously. This not only saves time, but also enables you to maintain accuracy in your calculations.

SPECIFIC STORAGE TO MEMORY

Values can be stored into the variable letters A, B, ..., Z. Storing a value in memory is useful if you need that value multiple times.

Suppose we wish to store the number 15.4829 for use in a number of calculations. To store this number in variable A, type in the number, then press STO▶ ALPHA MATH (**A**) ENTER .

We can now add 10 to this value by pressing ALPHA MATH + 10 ENTER , or cube this value by pressing ALPHA MATH ∧ 3 ENTER .

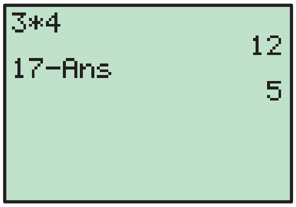


```
15.4829→A
          15.4829
A+10
          25.4829
A^3
      3711.563767
```

ANS VARIABLE

The variable **Ans** holds the most recent evaluated expression, and can be used in calculations by pressing 2nd (−) .

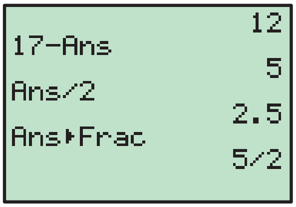
For example, suppose you evaluate 3 × 4, and then wish to subtract this from 17. This can be done by pressing 17 − 2nd (−) ENTER .



```
3*4
               12
17-Ans
                5
```

If you start an expression with an operator such as +, −, etc, the previous answer **Ans** is automatically inserted ahead of the operator. For example, the previous answer can be halved simply by pressing ÷ 2 ENTER.

If you wish to view the answer in fractional form, press MATH **1** ENTER.



```
                12
17-Ans
                 5
Ans/2
               2.5
Ans▸Frac
               5/2
```

RECALLING PREVIOUS EXPRESSIONS

The **ENTRY** function recalls previously evaluated expressions, and is used by pressing 2nd ENTER.

This function is useful if you wish to repeat a calculation with a minor change, or if you have made an error in typing.

Suppose you have evaluated $100 + \sqrt{132}$. If you now want to evaluate $100 + \sqrt{142}$, instead of retyping the command, it can be recalled by pressing 2nd ENTER.

The change can then be made by moving the cursor over the 3 and changing it to a 4, then pressing ENTER.

If you have made an error in your original calculation, and intended to calculate $1500+\sqrt{132}$, again you can recall the previous command by pressing 2nd ENTER.

Move the cursor to the first 0.

You can insert the 5, rather than overwriting the 0, by pressing 2nd DEL **(INS)** 5 ENTER.

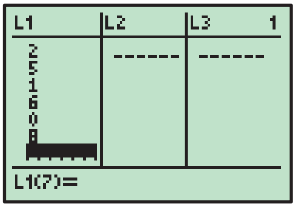
LISTS

Lists are used for a number of purposes on the calculator. They enable us to store sets of data, which we can then analyse and compare.

CREATING A LIST

Press STAT **1** to access the **list editor** screen.

To enter the data {2, 5, 1, 6, 0, 8} into **List 1**, start by moving the cursor to the first entry of **L1**. Press 2 ENTER 5 ENTER ...... and so on until all the data is entered.



```
L1     |L2     |L3     1
2      |------ |------
5      |       |
1      |       |
6      |       |
0      |       |
8      |       |
-------|       |
L1(7)=
```

DELETING LIST DATA

To delete a list of data from the list editor screen, move the cursor to the heading of the list you want to delete then press CLEAR ENTER.

REFERENCING LISTS

Lists can be referenced by using the secondary functions of the keypad numbers 1-6.

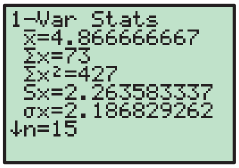
For example, suppose you want to add 2 to each element of **List1** and display the results in **List2**. To do this, move the cursor to the heading of **L₂** and press `2nd` **1** `+` 2 `ENTER`.

STATISTICS

Your graphics calculator is a useful tool for analysing data and creating statistical graphs. We will first produce descriptive statistics and graphs for the data set:
5 2 3 3 6 4 5 3 7 5 7 1 8 9 5.

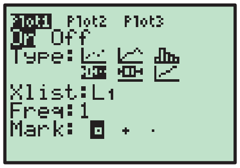
Enter the data set into **List 1**. To obtain descriptive statistics of the data set, press `STAT` `▶` **1:1-Var Stats** `2nd` **1** (**L₁**) `ENTER`.

```
1-Var Stats L1
```

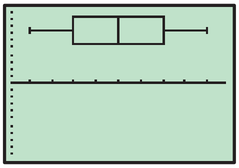


```
1-Var Stats
 x̄=4.866666667
 Σx=73
 Σx²=427
 Sx=2.263583337
 σx=2.186829262
↓n=15
```

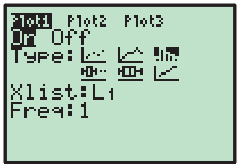
To obtain a boxplot of the data, press `2nd` `Y=` (**STAT PLOT**) **1** and set up **Statplot1** as shown. Press `ZOOM` **9:ZoomStat** to graph the boxplot with an appropriate window.



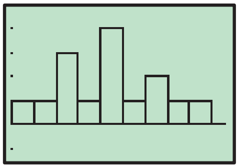
```
Plot1 Plot2 Plot3
On Off
Type:
Xlist:L1
Freq:1
Mark: ▫ + ·
```



To obtain a vertical bar chart of the data, press `2nd` `Y=` **1**, and change the type of graph to a vertical bar chart as shown. Press `ZOOM` **9:ZoomStat** to draw the bar chart. Press `WINDOW` and set the **Xscl** to 1, then `GRAPH` to redraw the bar chart.

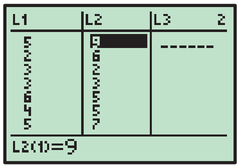


```
Plot1 Plot2 Plot3
On Off
Type:
Xlist:L1
Freq:1
```

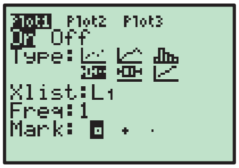


We will now enter a second set of data, and compare it to the first.

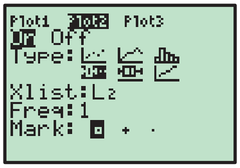
Enter the data set 9 6 2 3 5 5 7 5 6 7 6 3 4 4 5 8 4 into **List 2**, press `2nd` `Y=` **1**, and change the type of graph back to a boxplot as shown. Move the cursor to the top of the screen and select **Plot2**. Set up **Statplot2** in the same manner, except set the **XList** to **L₂**. Press `ZOOM` **9:ZoomStat** to draw the side-by-side boxplots.



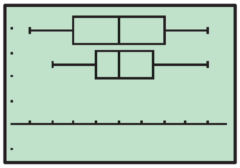
```
L1      L2      L3     2
 5      9       ------
 2      6
 3      2
 3      3
 6      5
 4      5
 5      7
L2(1)=9
```



```
Plot1 Plot2 Plot3
On Off
Type:
Xlist:L1
Freq:1
Mark: ▫ + ·
```



```
Plot1 Plot2 Plot3
On Off
Type:
Xlist:L2
Freq:1
Mark: ▫ + ·
```

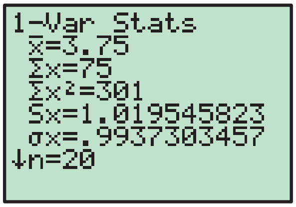


STATISTICS FROM GROUPED DATA

To obtain descriptive statistics for the data in the table alongside, enter the data values into **List 1**, and the frequency values into **List 2**.

Press `STAT` `▶` **1 : 1-Var Stats** `2nd` **1** (**L₁**) `,` `2nd` **2** (**L₂**) `ENTER`.

Data	*Frequency*
2	3
3	4
4	8
5	5

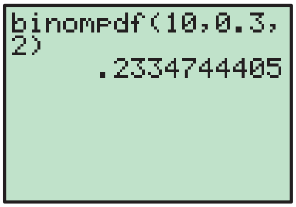


```
1-Var Stats
 x̄=3.75
 Σx=75
 Σx²=301
 Sx=1.019545823
 σx=.9937303457
↓n=20
```

BINOMIAL PROBABILITIES

To find $P(X = 2)$ for $X \sim B(10, 0.3)$, press $\boxed{\text{2nd}}$ $\boxed{\text{VARS}}$ (**DISTR**) and select **A:binompdf(**. Press 10 $\boxed{,}$ 0.3 $\boxed{,}$ 2 $\boxed{)}$ $\boxed{\text{ENTER}}$.

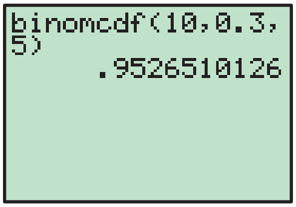
So, $P(X = 2) \approx 0.233$.



```
binompdf(10,0.3,
2)
          .2334744405
```

To find $P(X \leqslant 5)$ for $X \sim B(10, 0.3)$, press $\boxed{\text{2nd}}$ $\boxed{\text{VARS}}$ (**DISTR**) and select **B:binomcdf(**. Press 10 $\boxed{,}$ 0.3 $\boxed{,}$ 5 $\boxed{)}$ $\boxed{\text{ENTER}}$.

So, $P(X \leqslant 5) \approx 0.953$.



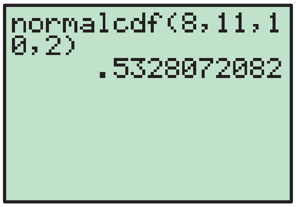
```
binomcdf(10,0.3,
5)
          .9526510126
```

NORMAL PROBABILITIES

Suppose X is normally distributed with mean 10 and standard deviation 2.

To find $P(8 \leqslant X \leqslant 11)$, press $\boxed{\text{2nd}}$ $\boxed{\text{VARS}}$ (**DISTR**), and select **2:normalcdf(**. Press 8 $\boxed{,}$ 11 $\boxed{,}$ 10 $\boxed{,}$ 2 $\boxed{)}$ $\boxed{\text{ENTER}}$.

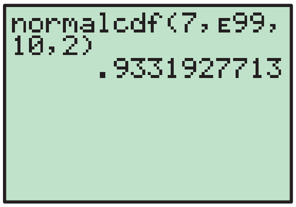
So, $P(8 \leqslant X \leqslant 11) \approx 0.533$.



```
normalcdf(8,11,1
0,2)
          .5328072082
```

To find $P(X \geqslant 7)$, press $\boxed{\text{2nd}}$ $\boxed{\text{VARS}}$ (**DISTR**), and select **2:normalcdf(**. Press 7 $\boxed{,}$ $\boxed{\text{2nd}}$ $\boxed{,}$ (**EE**) 99 $\boxed{,}$ 10 $\boxed{,}$ 2 $\boxed{)}$ $\boxed{\text{ENTER}}$.

So, $P(X \geqslant 7) \approx 0.933$.

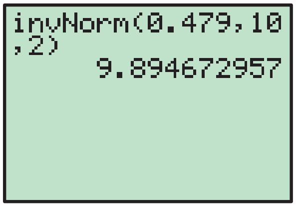


```
normalcdf(7,E99,
10,2)
          .9331927713
```

To find a such that $P(X \leqslant a) = 0.479$, press $\boxed{\text{2nd}}$ $\boxed{\text{VARS}}$ (**DISTR**), and select **3:invNorm(**.

Press 0.479 $\boxed{,}$ 10 $\boxed{,}$ 2 $\boxed{)}$ $\boxed{\text{ENTER}}$.

So, $a \approx 9.89$.

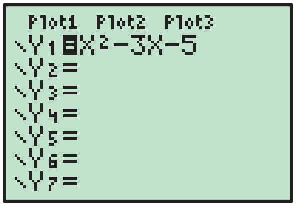


```
invNorm(0.479,10
,2)
          9.894672957
```

WORKING WITH FUNCTIONS

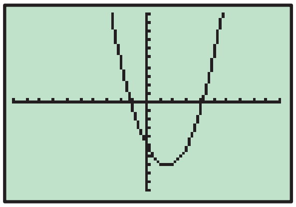
GRAPHING FUNCTIONS

Pressing $\boxed{\text{Y=}}$ selects the **Y=** editor, where you can store functions to graph. Delete any unwanted functions by scrolling down to the function and pressing $\boxed{\text{CLEAR}}$.

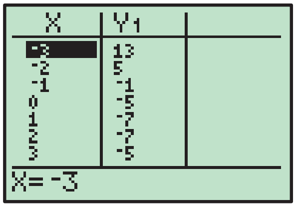


```
 Plot1  Plot2  Plot3
\Y1=X²-3X-5
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

To graph the function $y = x^2 - 3x - 5$, move the cursor to **Y1**, and press X,T,θ,n x^2 − 3 X,T,θ,n − 5 ENTER. This stores the function into **Y1**. Press GRAPH to draw a graph of the function.



To view a table of values for the function, press 2nd GRAPH (**TABLE**). The starting point and interval of the table values can be adjusted by pressing 2nd WINDOW (**TBLSET**).



X	Y1
-3	13
-2	5
-1	-1
0	-5
1	-7
2	-7
3	-5

X=-3

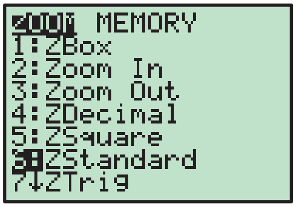
ADJUSTING THE VIEWING WINDOW

When graphing functions it is important that you are able to view all the important features of the graph. As a general rule it is best to start with a large viewing window to make sure all the features of the graph are visible. You can then make the window smaller if necessary.

Some useful commands for adjusting the viewing window include:

ZOOM **0:ZoomFit** : This command scales the y-axis to fit the minimum and maximum values of the displayed graph within the current x-axis range.

ZOOM **6:ZStandard** : This command returns the viewing window to the default setting of $-10 \leqslant x \leqslant 10$, $-10 \leqslant y \leqslant 10$.



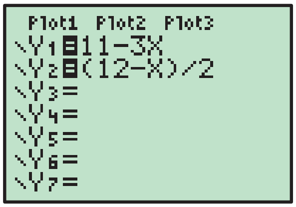
If neither of these commands are helpful, the viewing window can be adjusted manually by pressing WINDOW and setting the minimum and maximum values for the x and y axes.

FINDING POINTS OF INTERSECTION

It is often useful to find the points of intersection of two graphs, for instance, when you are trying to solve simultaneous equations.

We can solve $y = 11 - 3x$ and $y = \dfrac{12 - x}{2}$ simultaneously by finding the point of intersection of these two lines.

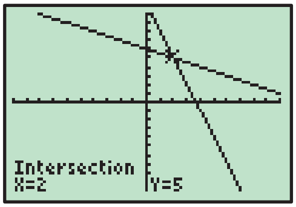
Press Y=, then store $11 - 3x$ into **Y1** and $\dfrac{12 - x}{2}$ into **Y2**. Press GRAPH to draw a graph of the functions.



Plot1 Plot2 Plot3
\Y1=11-3X
\Y2=(12-X)/2
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=

To find their point of intersection, press 2nd TRACE (**CALC**) **5:intersect**. Press ENTER twice to specify the functions **Y1** and **Y2** as the functions you want to find the intersection of, then use the arrow keys to move the cursor close to the point of intersection and press ENTER once more.

The solution $x = 2$, $y = 5$ is given.



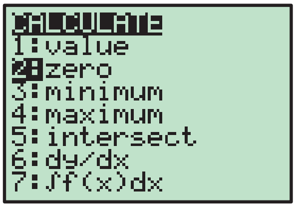
Intersection
X=2 Y=5

FINDING x-INTERCEPTS

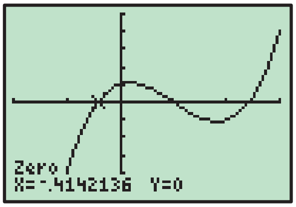
In the special case when you wish to solve an equation of the form $f(x) = 0$, this can be done by graphing $y = f(x)$ and then finding when this graph cuts the x-axis.

For example, to solve $x^3 - 3x^2 + x + 1 = 0$, press Y= and store $x^3 - 3x^2 + x + 1$ into **Y1**. Then press GRAPH .

To find where this function first cuts the x-axis, press 2nd TRACE (CALC) **2:zero**. Move the cursor to the left of the first zero and press ENTER , then move the cursor to the right of the first zero and press ENTER . Finally, move the cursor close to the first zero and press ENTER once more. The solution $x \approx -0.414$ is given.



```
CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
```



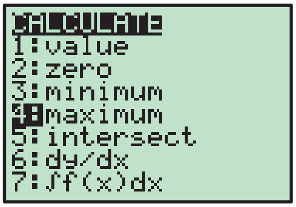
```
Zero
X=-.4142136  Y=0
```

Repeat this process to find the remaining solutions $x = 1$ and $x \approx 2.414$.

TURNING POINTS

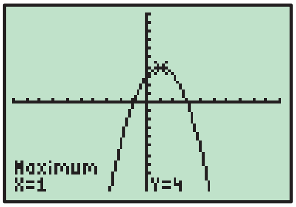
To find the turning point or vertex of $y = -x^2 + 2x + 3$, press Y= and store $-x^2 + 2x + 3$ into **Y1**. Press GRAPH to draw the graph.

From the graph, it is clear that the vertex is a maximum, so press 2nd TRACE (CALC) **4:maximum**.



```
CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
```

Move the cursor to the left of the vertex and press ENTER , then move the cursor to the right of the vertex and press ENTER . Finally, move the cursor close to the vertex and press ENTER once more. The vertex is $(1, 4)$.

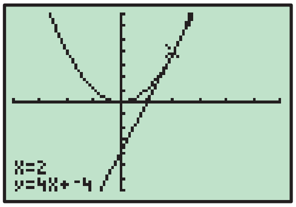


```
Maximum
X=1          Y=4
```

FINDING THE TANGENT TO A FUNCTION

To find the equation of the tangent to $y = x^2$ when $x = 2$, we first draw the graph of $y = x^2$. Press 2nd PRGM (**DRAW**) **5:Tangent**, then press 2 ENTER to draw the tangent at $x = 2$.

The tangent has gradient 4, and equation $y = 4x - 4$.

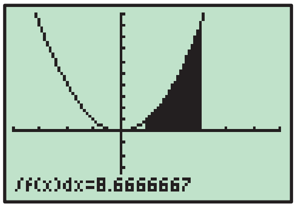


```
X=2
y=4X+-4
```

DEFINITE INTEGRALS

To calculate $\int_1^3 x^2\,dx$, we first draw the graph of $y = x^2$. Press 2nd TRACE **(CALC)** and select **7:**$\boldsymbol{\int f(x)\,dx}$. Press 1 ENTER 3 ENTER to specify the lower and upper limits of the integral.

So, $\int_1^3 x^2\,dx = 8\frac{2}{3}$.



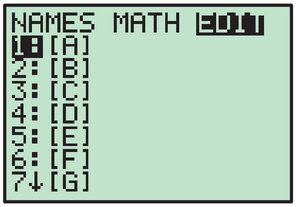
```
∫f(x)dx=8.6666667
```

MATRICES

Matrices are easily stored in a graphics calculator. This is particularly valuable if we need to perform a number of operations with the same matrices.

STORING MATRICES

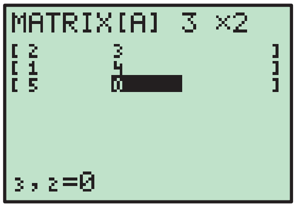
To store the matrix $\begin{pmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 0 \end{pmatrix}$, press 2nd x^{-1} **(MATRIX)** to display the matrices screen, and use ▶ to select the **EDIT** menu. This is where you define matrices and enter the elements.



```
NAMES MATH EDIT
1:[A]
2:[B]
3:[C]
4:[D]
5:[E]
6:[F]
7↓[G]
```

Press **1** to select **1:[A]**. Press 3 ENTER 2 ENTER to define matrix **A** as a 3×2 matrix.

Enter the elements of the matrix, pressing ENTER after each entry.



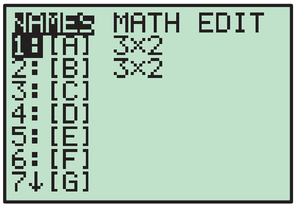
```
MATRIX[A] 3 ×2
[ 2      3        ]
[ 1      4        ]
[ 5      0        ]

3,2=0
```

Press 2nd MODE **(QUIT)** when you are done.

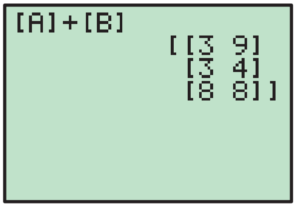
MATRIX OPERATIONS

To find the sum of $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 6 \\ 2 & 0 \\ 3 & 8 \end{pmatrix}$, we first define matrices **A** and **B**.



```
NAMES MATH EDIT
1:[A] 3×2
2:[B] 3×2
3:[C]
4:[D]
5:[E]
6:[F]
7↓[G]
```

To find $\mathbf{A} + \mathbf{B}$, press 2nd x^{-1} **1** to enter matrix **A**, then + , then 2nd x^{-1} **2** to enter matrix **B**. Press ENTER to display the results.



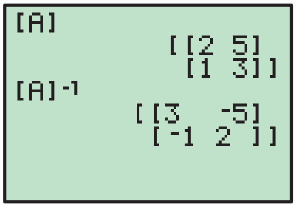
```
[A]+[B]
               [[3  9]
                [3  4]
                [8  8]]
```

Operations of subtraction, scalar multiplication and matrix multiplication can be performed in a similar manner.

INVERTING MATRICES

To find the inverse of $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$, we first define matrix **A**.

Press 2nd x^{-1} **1** to enter matrix **A**, then press x^{-1} ENTER .



```
[A]
               [[2  5]
                [1  3]]
[A]⁻¹
             [[3   -5]
              [-1  2 ]]
```

C TEXAS INSTRUMENTS TI-nSPIRE

GETTING STARTED

Pressing ⌂ takes you to the **home screen**, where you can choose which application you wish to use.

The TI-$n$spire organises any work done into **pages**. Every time you start a new application, a new page is created. You can navigate back and forth between the pages you have worked on by pressing ctrl ◄ or ctrl ►

From the home screen, press **1** to open the Calculator application. This is where most of the basic calculations are performed.

SECONDARY FUNCTION KEY

The secondary function of each key is displayed in grey above the primary function. It is accessed by pressing the ctrl key followed by the key corresponding to the desired secondary function.

BASIC FUNCTIONS

GROUPING SYMBOLS (BRACKETS)

The TI-$n$spire has bracket keys that look like (and) .

Brackets are regularly used in mathematics to indicate an expression which needs to be evaluated before other operations are carried out.

For example, to evaluate $2 \times (4+1)$ we type 2 × (4 + 1) enter .

We also use brackets to make sure the calculator understands the expression we are typing in.

For example, to evaluate $\frac{2}{4+1}$ we type 2 ÷ (4 + 1) enter .

If we typed 2 ÷ 4 + 1 enter the calculator would think we meant $\frac{2}{4}+1$.

In general, it is a good idea to place brackets around any complicated expressions which need to be evaluated separately.

POWER KEYS

The TI-$n$spire has a power key that looks like ^ . We type the base first, press the power key, then enter the index or exponent.

For example, to evaluate 25^3 we type 25 ^ 3 enter .

Numbers can be squared on the TI-$n$spire using the special key x^2 .

For example, to evaluate 25^2 we type 25 x^2 enter .

ROOTS

To enter roots on the TI-*n*spire we need to use the secondary function $\boxed{\text{ctrl}}$.

We enter square roots by pressing $\boxed{\text{ctrl}}$ $\boxed{x^2}$.

For example, to evaluate $\sqrt{36}$ we press $\boxed{\text{ctrl}}$ $\boxed{x^2}$ 36 $\boxed{\text{enter}}$.

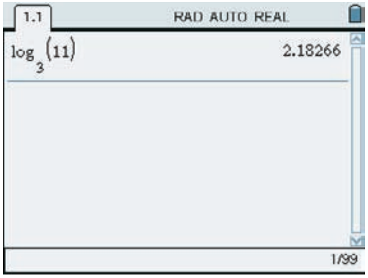
Higher roots are entered by pressing $\boxed{\text{ctrl}}$ $\boxed{\wedge}$, which creates the $\sqrt[\square]{\square}$ template. Enter the root number in the first entry, and the number you wish to find the root of in the second entry. Use the arrow keys to navigate around the template.

LOGARITHMS

We can perform operations involving logarithms using the **log** function, which is accessed by pressing $\boxed{\text{ctrl}}$ $\boxed{10^x}$.

Put the base number in the first entry, and the number you wish to find the logarithm of in the second entry.

For example, to find $\log_3 11$ press $\boxed{\text{ctrl}}$ $\boxed{10^x}$ 3 $\boxed{\blacktriangleright}$ 11 $\boxed{\text{enter}}$.

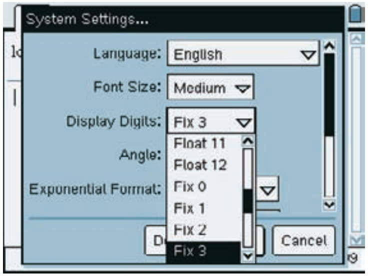


$\log_3(11)$ 2.18266

ROUNDING OFF

You can instruct the TI-*n*spire to round off values to a fixed number of decimal places.

For example, to round to 3 decimal places, from the home screen select **8 : System Info > 2 : System Settings....** Press the $\boxed{\text{tab}}$ button until the **Display Digits** drop-box is highlighted. Use the $\boxed{\blacktriangledown}$ key to select *Fix 3*, then press $\boxed{\text{enter}}$ three times to return to the page you were working on.



Language: English
Font Size: Medium
Display Digits: Fix 3
Angle:
Exponential Format:

To unfix the number of decimal places, select *Float* from the **Display Digits** drop-box.

DECIMAL EXPANSION OF FRACTIONS

If you press 5 $\boxed{\div}$ 8 $\boxed{\text{enter}}$, the calculator will simply return the fraction $\frac{5}{8}$. To find the decimal expansion of $\frac{5}{8}$, press 5 $\boxed{\div}$ 8 $\boxed{\text{ctrl}}$ $\boxed{\text{enter}}$.

INVERSE TRIGONOMETRIC FUNCTIONS

The inverse trigonometric functions $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ are the secondary functions of $\boxed{\text{sin}}$, $\boxed{\text{cos}}$ and $\boxed{\text{tan}}$ respectively. They are accessed by using the secondary function key $\boxed{\text{ctrl}}$.

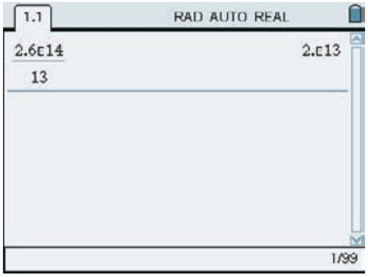
For example, if $\cos x = \frac{3}{5}$, then $x = \cos^{-1}\left(\frac{3}{5}\right)$.

To calculate this, press $\boxed{\text{ctrl}}$ $\boxed{\text{cos}}$ 3 $\boxed{\div}$ 5 $\boxed{)}$ $\boxed{\text{enter}}$.

SCIENTIFIC NOTATION

You can enter very large or very small values by expressing them in scientific notation, which is in the form $a \times 10^k$ where $1 \leqslant a < 10$ and k is an integer. You can do this using the EE button.

For example, to evaluate $\dfrac{2.6 \times 10^{14}}{13}$, press 2.6 EE 14 ÷ 13 enter . The answer is 2×10^{13}.



```
2.6E14
  13                                   2.E13
```

MEMORY

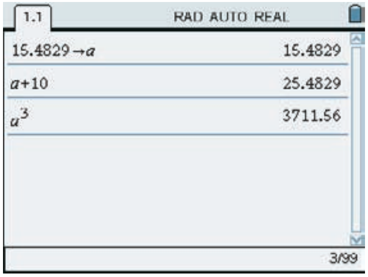
Utilising the memory features of your calculator allows you to recall calculations you have performed previously. This not only saves time, but also enables you to maintain accuracy in your calculations.

SPECIFIC STORAGE TO MEMORY

Values can be stored into the variable letters A, B, ..., Z. Storing a value in memory is useful if you need that value multiple times.

Suppose we wish to store the number 15.4829 for use in a number of calculations. To store this number in variable A, type in the number then press ctrl var **(sto▶)** A enter .

We can now add 10 to this value by pressing A + 10 enter or cube this value by pressing A ^ 3 enter .



```
15.4829→a                              15.4829
a+10                                   25.4829
a^3                                    3711.56
```

ANS VARIABLE

The variable **Ans** holds the most recent evaluated expression, and can be used in calculations by pressing ctrl (−) .

For example, suppose you evaluate 3 × 4, and then wish to subtract this from 17. This can be done by pressing 17 − ctrl (−) enter .

If you start an expression with an operator such as + , − , etc, the previous answer **Ans** is automatically inserted ahead of the operator. For example, the previous answer can be doubled simply by pressing × 2 enter .

RECALLING PREVIOUS EXPRESSIONS

Pressing the ▲ key allows you to recall and edit previously evaluated expressions.

To recall a previous expression, press the ▲ key until the desired expression is highlighted, and press enter . The expression then appears in the current entry line, where it can be edited.

LISTS

Lists are used for a number of purposes on the calculator. They enable us to store sets of data, which we can then analyse and compare.

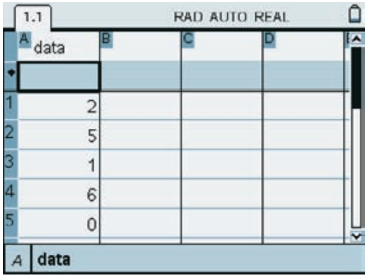
In order to perform many of the operations involving data lists on the TI-*n*spire, you need to **name** your lists as you define them.

CREATING A LIST

Selecting **3 : Lists & Spreadsheet** from the home screen takes you to the **list editor** screen.

To enter the data {2, 5, 1, 6, 0, 8} into **List A**, start by moving the cursor to the first entry of **List A**. Press 2 enter 5 enter ...... and so on until all the data is entered.

Move the cursor to the heading of **List A** and use the green alphabet keys to enter a name for the list, for example *data*.



	A data	B	C	D
◆				
1	2			
2	5			
3	1			
4	6			
5	0			

A data

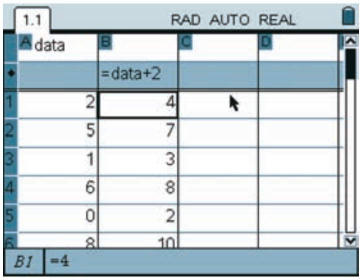
DELETING LIST DATA

To delete a list of data from the list editor screen, move the cursor to the heading of that list, and press ▲ to highlight the whole column. Press ← to delete the list.

REFERENCING LISTS

Once you have named a list, you can use that name to reference the list in other operations.

Suppose you want to add 2 to each element of the *data* list we created in the above example, and display the results in **List B**. Move the cursor to the shaded row of **List B**, and press = *data* + 2 enter .



	A data	B	C	D
◆		=data+2		
1	2	4		
2	5	7		
3	1	3		
4	6	8		
5	0	2		
6	8	10		

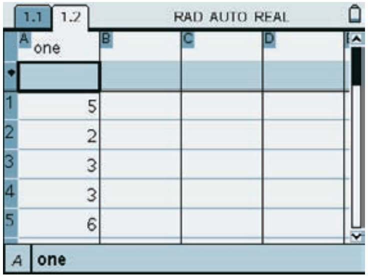
B1 =4

STATISTICS

Your graphics calculator is a useful tool for analysing data and creating statistical graphs.

We will first produce descriptive statistics and graphs for the data set:
5 2 3 3 6 4 5 3 7 5 7 1 8 9 5.

Enter the data into **List A** and name this list *one*.

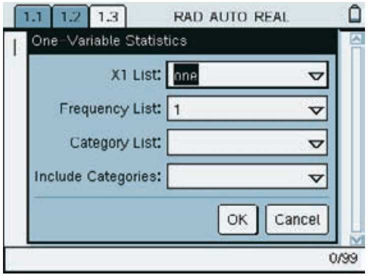


	A one	B	C	D
◆				
1	5			
2	2			
3	3			
4	3			
5	6			

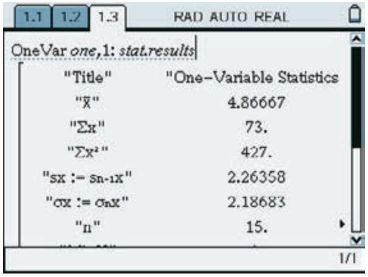
A one

To obtain the descriptive statistics, press ⌂ **1 : Calculator** to open the Calculator application, press menu , then select **5 : Statistics** > **1 : Statistical Calculations** > **1 : One-Variable Statistics**.

Press enter to choose 1 list. Select *one* from the **X1 List** drop-box, and press enter .



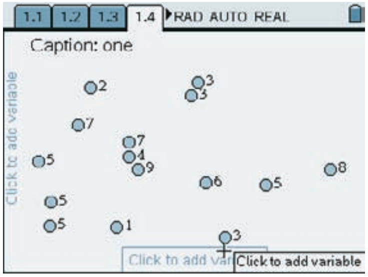
X1 List: one
Frequency List: 1
Category List:
Include Categories:



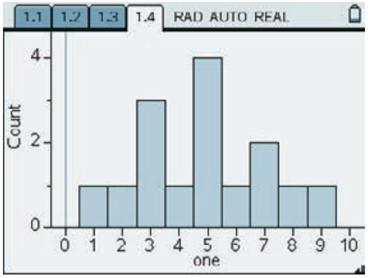
OneVar *one*,1: *stat.results*

"Title"	"One-Variable Statistics
"x̄"	4.86667
"Σx"	73.
"Σx²"	427.
"sx := sn-1x"	2.26358
"σx := σnx"	2.18683
"n"	15.

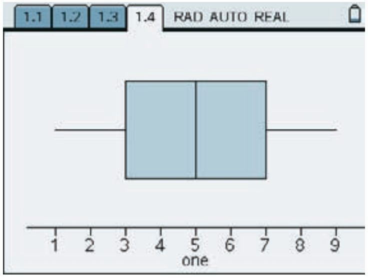
We will now draw some statistical graphs for the data. Press ⌂ **5 : Data & Statistics** to open the Data and Statistics application. You will see the data points randomly scattered on the screen. Move the cursor to the bottom of the screen until the "Click to add variable" box appears. Press enter , then select *one*. The points will order themselves into a dotplot along the horizontal axis.



To obtain a vertical bar chart of the data, press menu , then select **1 : Plot Type** >... **3 : Histogram**.

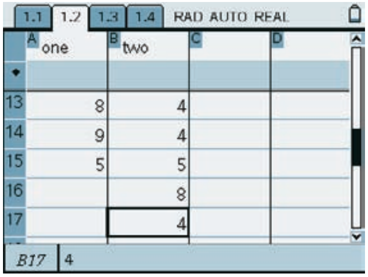


To obtain a boxplot of the data, press menu , then select **1 : Plot Type** > **2 : Box Plot**.



We will now add a second set of data, and compare it to the first.

Use the ctrl ◄ command to return to the page containing the list of data.
Enter the set 9 6 2 3 5 5 7 5 6 7 6 3 4 4 5 8 4 into **List B**, and name the list *two*.

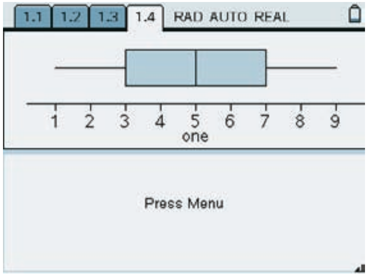


	A one	B two	C	D
◆				
13	8	4		
14	9	4		
15	5	5		
16		8		
17		4		

B17 4

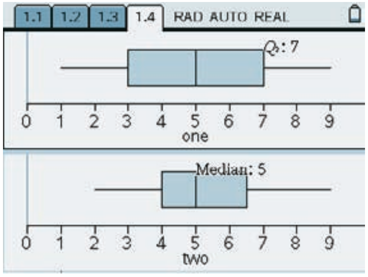
Use the ctrl ► command to navigate back to the page containing the boxplot of *one*.

To view boxplots of both data sets, we need to split the page into two sections. Press ctrl ⌂ , then select **5 : Page Layout** > **2 : Select Layout** > **3 : Layout 3**. Press ctrl tab to navigate between the sections, then press menu **5** to open a new Data and Statistics application. Draw another boxplot in this section, this time using the data in *two*.



In order to compare the boxplots, the horizontal scales should be the same. Press menu , then select **5 : Window/Zoom > 1 : Window Settings**.

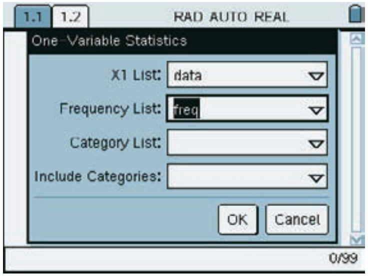
Set the X range from 0 to 10. Press ctrl tab , and do the same with the other boxplot.



STATISTICS FROM GROUPED DATA

To obtain descriptive statistics for the data in the table alongside, enter the data values into **List A**, and label the list *data*. Enter the frequency values into **List B**, and label the list *freq*.

Data	*Frequency*
2	3
3	4
4	8
5	5



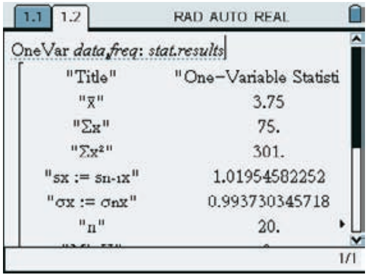
X1 List: data

Frequency List: freq

Category List:

Include Categories:

Press ⌂ **1 : Calculator** to open the Calculator application. Press menu , then select **5 : Statistics > 1 : Stat Calculations .... > 1 : One-Variable Statistics**. Press enter to choose 1 list. Select *data* from the **X1 List** drop-box and *freq* from the **Frequency List** drop-box, then press enter .

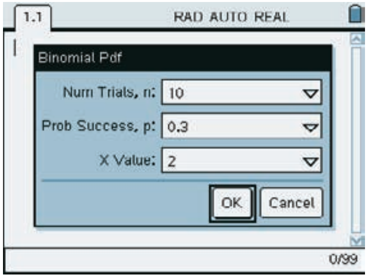


```
OneVar data,freq: stat.results
   "Title"          "One-Variable Statisti
   "x̄"                     3.75
   "Σx"                    75.
   "Σx²"                   301.
 "sx := sn-1x"        1.01954582252
 "σx := σnx"          0.993730345718
   "n"                     20.
```

BINOMIAL PROBABILITIES

To find $P(X = 2)$ for $X \sim B(10,\ 0.3)$, press ⌂ **1 : Calculator** to open the Calculator application. Press menu then select **4 : Probability > 5 : Distributions ... > D : Binomial Pdf**.

Press 10 tab 0.3 tab 2 tab enter . The result is 0.233 .



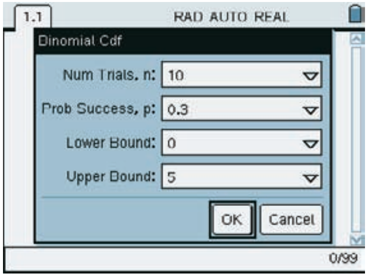
Num Trials, n: 10

Prob Success, p: 0.3

X Value: 2

To find $P(X \leqslant 5)$ for $X \sim B(10,\ 0.3)$, press menu for the Calculator application and select **4 : Probability > 5 : Distributions ... > E : Binomial Cdf**.

Press 10 tab 0.3 tab 0 tab 5 tab enter . The result is 0.953 .



Num Trials, n: 10

Prob Success, p: 0.3

Lower Bound: 0

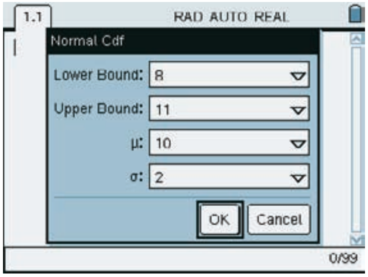
Upper Bound: 5

NORMAL PROBABILITIES

Suppose X is normally distributed with mean 10 and standard deviation 2.

To find $\mathrm{P}(8 \leqslant X \leqslant 11)$, press menu from the Calculator application and select
4 : Probability > 5 : Distributions .... > 2 : Normal Cdf.

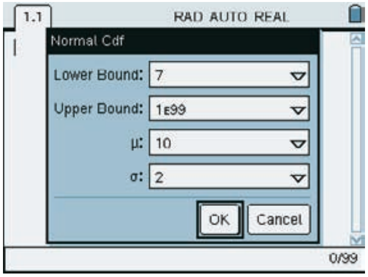
Press 8 tab 11 tab 10 tab 2 tab enter . The result is 0.533 .



Normal Cdf
Lower Bound: 8
Upper Bound: 11
μ: 10
σ: 2

To find $\mathrm{P}(X \geqslant 7)$, press menu from the Calculator application and select
4 : Probability > 5 : Distributions .... > 2 : Normal Cdf.

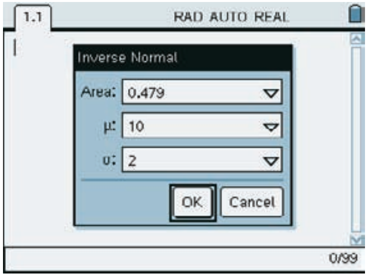
Set up the screen as shown, and select **OK**. The result is 0.933 .



Normal Cdf
Lower Bound: 7
Upper Bound: 1E99
μ: 10
σ: 2

To find a such that $\mathrm{P}(X \leqslant a) = 0.479$, press menu from the Calculator application and select
4 : Probability > 5 : Distributions .... > 3 : Inverse Normal.

Press 0.479 tab 10 tab 2 tab enter . The solution is $a \approx 9.89$.



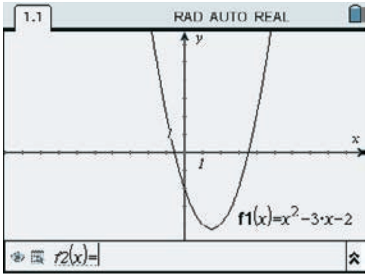
Inverse Normal
Area: 0.479
μ: 10
σ: 2

WORKING WITH FUNCTIONS

GRAPHING FUNCTIONS

Selecting **2 : Graphs & Geometry** from the home screen opens the graphing application, where you can graph functions.

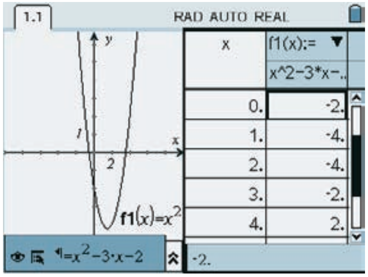
To graph the function $y = x^2 - 3x - 2$, press X x^2 − 3 X − 2 enter .



f2(x)=

To view a table of values for the function, press menu , then select **2 : View > 9 : Add Function Table**.

The page splits into 2, with the graph on one side and the table of values on the other side.



x	f1(x):= x^2-3*x-..
0.	-2.
1.	-4.
2.	-4.
3.	-2.
4.	2.

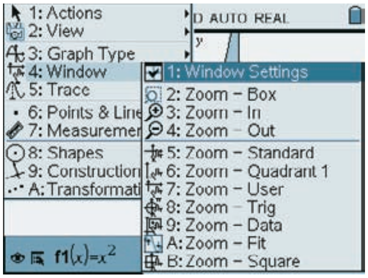
f1(x)=x²−3·x−2

-2.

ADJUSTING THE VIEWING WINDOW

When graphing functions it is important that you are able to view all the important features of the graph. As a general rule it is best to start with a large viewing window to make sure all the features of the graph are visible. You can then make the window smaller if necessary.

To adjust the viewing window, press menu then select **4 : Window** . The most useful options are:



1 : Window Settings : With this option you can set the minimum and maximum values for the x and y axes manually.

5 : Zoom-Standard : This option returns the viewing window to the default setting of $-10 \leqslant x \leqslant 10$, $-\frac{20}{3} \leqslant y \leqslant \frac{20}{3}$.

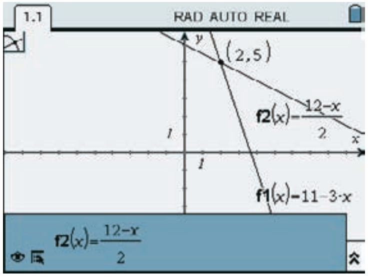
A : Zoom-Fit : This option scales the y-axis to fit the minimum and maximum values of the displayed graph within the current x-axis range.

Pressing ctrl G removes the function entry line, allowing you to use the full screen to view the graph. Press ctrl G again to bring the function entry line back.

FINDING POINTS OF INTERSECTION

When locating points on a graph, it is advised that you set the accuracy level to *Float 4* for ease of reading.

It is often useful to find the points of intersection of two graphs, for instance, when you are trying to solve simultaneous equations.



For example, we can solve $y = 11-3x$ and $y = \dfrac{12-x}{2}$ simultaneously by finding the point of intersection of these two lines.

Store $y = 11 - 3x$ into $f_1(x)$ and $y = \dfrac{12-x}{2}$ into $f_2(x)$. Press menu , then select

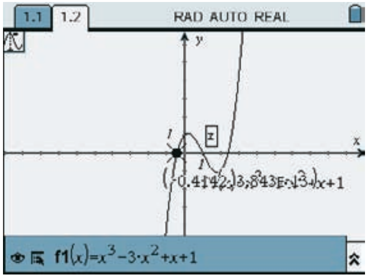
6 : Points & Lines > **3 : Intersection Point(s)**.

Move the cursor over one of the lines and press enter , then move the cursor over the other line and press enter . The intersection point $(2,\ 5)$ is displayed. So, the solution to the equations is $x = 2$, $y = 5$.

FINDING x-INTERCEPTS

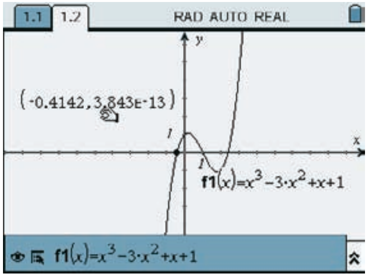
In the special case when you wish to solve an equation of the form $f(x) = 0$, this can be done by graphing $y = f(x)$ and then finding when the graph cuts the x-axis.

For example, to solve $x^3 - 3x^2 + x + 1 = 0$, store $x^3 - 3x^2 + x + 1$ into $f_1(x)$.



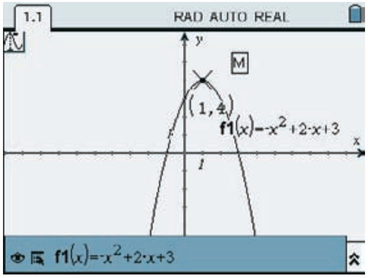
To find where this graph cuts the x-axis, press menu, then select **5 : Trace > 1 : Graph Trace**. Use the ◄ key to move the cursor along the curve until you reach the zero. A z will appear on the screen.

If you need to move the coordinates to make them easier to read, press enter to paste the coordinates to the screen, then esc to exit the trace command. Move the cursor over the coordinates, and hold the ✋ key until the hand closes. Use the cursor keys to drag the coordinates, then press esc.



The solution $x \approx -0.414$ is given. Repeat this process to find the remaining solutions $x = 1$ and $x \approx 2.414$.

TURNING POINTS

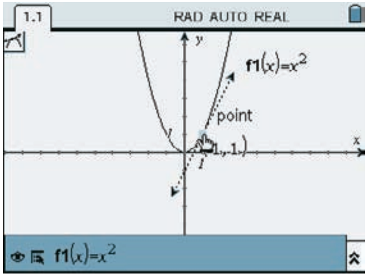


To find the turning point or vertex of $y = -x^2 + 2x + 3$, store $-x^2 + 2x + 3$ into $f_1(x)$.

Press menu, then select **5 : Trace > 1 : Graph Trace**.

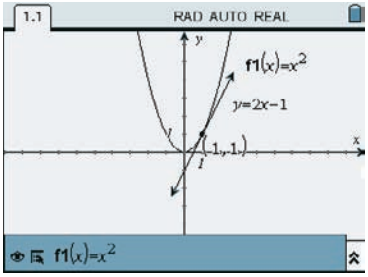
Use the ► key to move the cursor along the curve until the vertex is reached. An M will appear on the screen to indicate a local maximum. The vertex is at (1, 4).

FINDING THE TANGENT TO A FUNCTION



To find the equation of the tangent to $y = x^2$ when $x = 1$, we first draw the graph of $y = x^2$. Press menu then select **5 : Trace > 1 : Graph Trace**. Press 1 enter, and the point (1, 1) is located. Press enter esc to store the point.

Press menu, then select **6 : Points & Lines > 7 : Tangent**. Move the cursor towards the point (1, 1) until the word *point* appears, then press enter to draw the tangent.

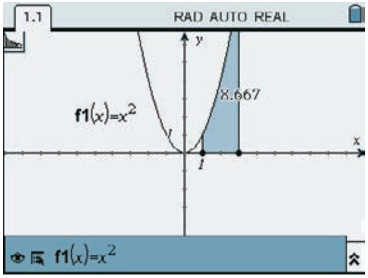


To find the equation of the tangent, press esc to exit the tangent command, and move the cursor over the tangent until the word *line* appears. Press ctrl menu and select **6 : Coordinates and Equations**. The tangent has gradient 2, and equation $y = 2x - 1$.

DEFINITE INTEGRALS

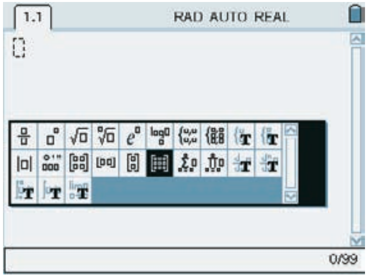
To calculate $\int_1^3 x^2\,dx$, we first draw the graph of $y = x^2$. Press menu , then select **7 : Measurement** > **5 : Integral**. Move the cursor over the graph of $y = x^2$ so that it is highlighted, and press enter . Press 1 enter 3 enter to specify the limits of the integral.

So, $\int_1^3 x^2\,dx = 8\frac{2}{3}$.



MATRICES

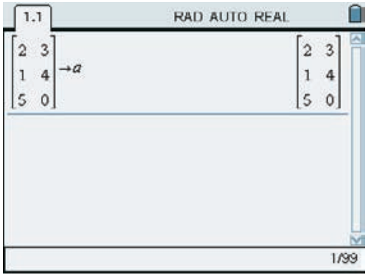
Matrices are easily stored in a graphics calculator. This is particularly valuable if we need to perform a number of operations with the same matrices.



STORING MATRICES

To store the matrix $\begin{pmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 0 \end{pmatrix}$, press ctrl × from the Calculator screen, then select the ▦ template. Press 3 tab 2 enter to define a 3×2 matrix.

Enter the elements of the matrix, pressing tab after each entry. To define this matrix as matrix **A**, press ctrl var (**sto▶**) A enter .



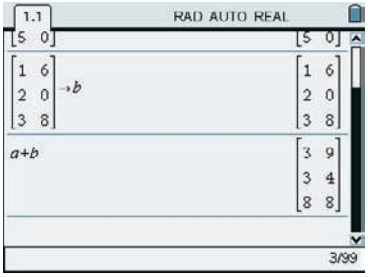
$\begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 0 \end{bmatrix} \to a \qquad \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 0 \end{bmatrix}$

MATRIX OPERATIONS

To find the sum of $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 6 \\ 2 & 0 \\ 3 & 8 \end{pmatrix}$ we first define matrices **A** and **B**.

To find $\mathbf{A} + \mathbf{B}$, press A + B enter .

Operations of subtraction, scalar multiplication and matrix multiplication can be performed in a similar manner.



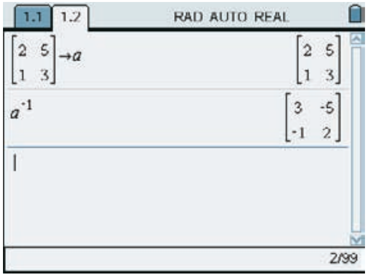
$\begin{bmatrix} 1 & 6 \\ 2 & 0 \\ 3 & 8 \end{bmatrix} \to b \qquad \begin{bmatrix} 1 & 6 \\ 2 & 0 \\ 3 & 8 \end{bmatrix}$

$a+b \qquad \begin{bmatrix} 3 & 9 \\ 3 & 4 \\ 8 & 8 \end{bmatrix}$

INVERTING MATRICES

To find the inverse of $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$, we first define matrix **A**.

Press A ^ (−) 1 enter .



$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \to a \qquad \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

$a^{-1} \qquad \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

Chapter 1

Functions

Syllabus reference: 2.1, 2.2, 2.3, 2.4

Contents:

A RELATIONS AND FUNCTIONS

The charges for parking a car in a short-term car park at an airport are given in the table alongside.

Car park charges	
Period (h)	*Charge*
0 - 1 hours	\$5.00
1 - 2 hours	\$9.00
2 - 3 hours	\$11.00
3 - 6 hours	\$13.00
6 - 9 hours	\$18.00
9 - 12 hours	\$22.00
12 - 24 hours	\$28.00

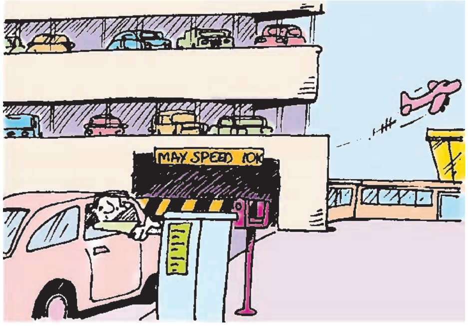
There is an obvious relationship between the time spent in the car park and the cost. The cost is *dependent* on the length of time the car is parked.

Looking at this table we might ask: How much would be charged for *exactly* one hour? Would it be \$5 or \$9?

To make the situation clear, and to avoid confusion, we could adjust the table and draw a graph. We need to indicate that 2-3 hours really means a time over 2 hours up to and including 3 hours, i.e., $2 < t \leqslant 3$.

We now have:

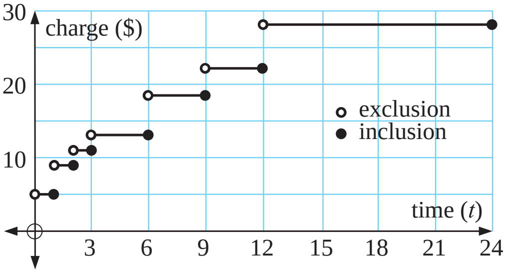
Car park charges	
Period	*Charge*
$0 < t \leqslant 1$ hours	\$5.00
$1 < t \leqslant 2$ hours	\$9.00
$2 < t \leqslant 3$ hours	\$11.00
$3 < t \leqslant 6$ hours	\$13.00
$6 < t \leqslant 9$ hours	\$18.00
$9 < t \leqslant 12$ hours	\$22.00
$12 < t \leqslant 24$ hours	\$28.00



In mathematical terms, we have a relationship between the two variables *time* and *cost*, so the schedule of charges is an example of a **relation**.

A relation may consist of a finite number of ordered pairs, such as $\{(1, 5), (-2, 3), (4, 3), (1, 6)\}$ or an infinite number of ordered pairs.

The parking charges example is clearly the latter as any real value of time (t hours) in the interval $0 < t \leqslant 24$ is represented.



The set of possible values of the variable on the horizontal axis is called the **domain** of the relation.

For example:
- $\{t \mid 0 < t \leqslant 24\}$ is the domain for the car park relation
- $\{-2, 1, 4\}$ is the domain of $\{(1, 5), (-2, 3), (4, 3), (1, 6)\}$.

The set which describes the possible y-values is called the **range** of the relation.

For example:
- the range of the car park relation is $\{5, 9, 11, 13, 18, 22, 28\}$
- the range of $\{(1, 5), (-2, 3), (4, 3), (1, 6)\}$ is $\{3, 5, 6\}$.

We will now look at relations and functions more formally.

RELATIONS

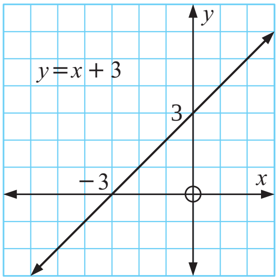
A **relation** is any set of points on the Cartesian plane.

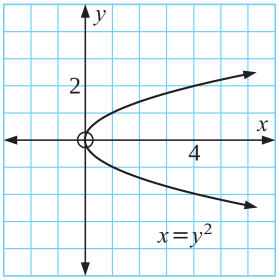
A relation is often expressed in the form of an **equation** connecting the **variables** x and y.

For example $y = x + 3$ and $x = y^2$ are the equations of two relations.

These equations generate sets of ordered pairs.

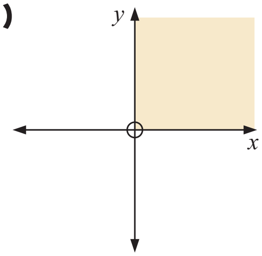
Their graphs are:





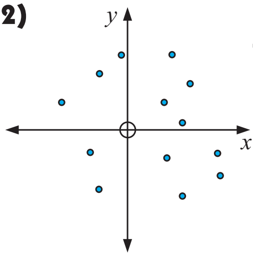
However, a relation may not be able to be defined by an equation. Below are two examples which show this:

(1)



All points in the first quadrant are a relation. $x > 0, y > 0$

(2)



These 13 points form a relation.

FUNCTIONS

A **function**, sometimes called a **mapping**, is a relation in which no two different ordered pairs have the same x-coordinate or first member.

We can see from the above definition that a function is a special type of relation.

TESTING FOR FUNCTIONS

Algebraic Test:

If a relation is given as an equation, and the substitution of any value for x results in one and only one value of y, we have a function.

For example:
- $y = 3x - 1$ is a function, as for any value of x there is only one value of y
- $x = y^2$ is not a function since if $x = 4$ then $y = \pm 2$.

Geometric Test or Vertical Line Test:

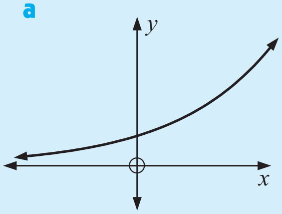
If we draw all possible vertical lines on the graph of a relation, the relation:

- is a function if each line cuts the graph no more than once
- is not a function if at least one line cuts the graph more than once.

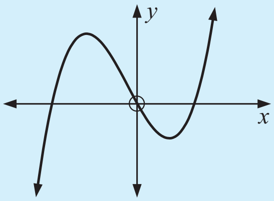
Example 1 **Self Tutor**

Which of the following relations are functions?

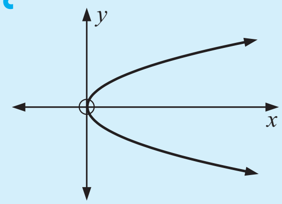
a



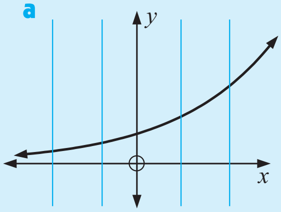
b



c

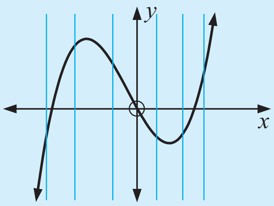


a



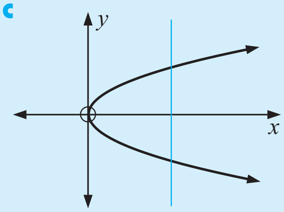
a function

b



a function

c



not a function

DEMO

GRAPHICAL NOTE

- If a graph contains a small **open circle** such as —o—, this point is **not included**.
- If a graph contains a small **filled-in circle** such as ——●, this point **is included**.
- If a graph contains an **arrow head** at an end such as ——→ , then the graph continues indefinitely in that general direction, or the shape may repeat as it has done previously.

EXERCISE 1A

1 Which of the following sets of ordered pairs are functions? Give reasons.

a $\{(1, 3), (2, 4), (3, 5), (4, 6)\}$

b $\{(1, 3), (3, 2), (1, 7), (-1, 4)\}$

c $\{(2, -1), (2, 0), (2, 3), (2, 11)\}$

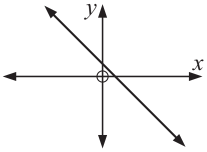
d $\{(7, 6), (5, 6), (3, 6), (-4, 6)\}$

e $\{(0, 0), (1, 0), (3, 0), (5, 0)\}$

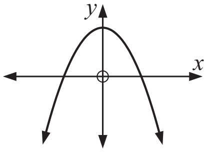
f $\{(0, 0), (0, -2), (0, 2), (0, 4)\}$

2 Use the vertical line test to determine which of the following relations are functions:

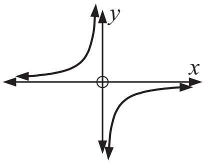
a



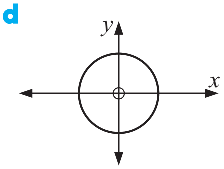
b



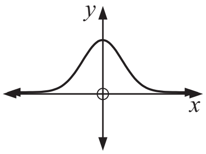
c



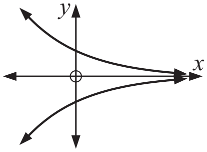
d



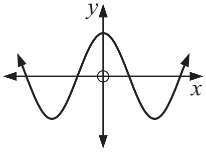
e



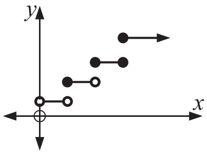
f



g



h



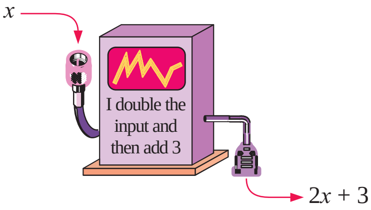
3 Will the graph of a straight line always be a function? Give evidence.

4 Give algebraic evidence to show that the relation $x^2 + y^2 = 9$ is not a function.

B FUNCTION NOTATION

Function machines are sometimes used to illustrate how functions behave.

For example:



So, if 4 is fed into the machine, $2(4) + 3 = 11$ comes out.

The above 'machine' has been programmed to perform a particular function.

If f is used to represent that particular function we can write:

f is the function that will convert x into $2x + 3$.

So, f would convert 2 into $2(2) + 3 = 7$ and
-4 into $2(-4) + 3 = -5$.

This function can be written as:

$$\underset{\text{function } f}{\boldsymbol{f}} \;\underset{\text{such that}}{\boldsymbol{:}}\; \underbrace{\boldsymbol{x \mapsto 2x+3}}_{x \text{ is converted into } 2x+3}$$

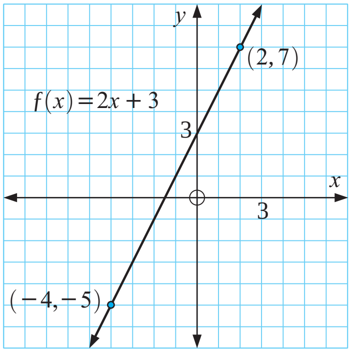
Two other equivalent forms we use are: $f(x) = 2x + 3$ or $y = 2x + 3$

> $f(x)$ is the value of y for a given value of x, so $y = f(x)$.

Notice that for $f(x) = 2x + 3$, $f(2) = 2(2) + 3 = 7$ and $f(-4) = 2(-4) + 3 = -5$.

Consequently, $f(2) = 7$ indicates that the point $(2, 7)$ lies on the graph of the function.

Likewise, $f(-4) = -5$ indicates that the point $(-4, -5)$ also lies on the graph.



Note:

- $f(x)$ is read as "f of x".
- f is the function which converts x into $f(x)$, so we write $f : x \mapsto f(x)$.
- $y = f(x)$ is sometimes called the **image** of x.

Example 2 **Self Tutor**

If $f : x \mapsto 2x^2 - 3x$, find the value of: **a** $f(5)$ **b** $f(-4)$

$f(x) = 2x^2 - 3x$

a $f(5) = 2(5)^2 - 3(5)$ {replacing x with (5)}
$= 2 \times 25 - 15$
$= 35$

b $f(-4) = 2(-4)^2 - 3(-4)$ {replacing x with (-4)}
$= 2(16) + 12$
$= 44$

Example 3 **Self Tutor**

If $f(x) = 5 - x - x^2$, find in simplest form: **a** $f(-x)$ **b** $f(x+2)$

a $f(-x) = 5 - (-x) - (-x)^2$ {replacing x with $(-x)$}
$= 5 + x - x^2$

b $f(x+2) = 5 - (x+2) - (x+2)^2$ {replacing x with $(x+2)$}
$= 5 - x - 2 - [x^2 + 4x + 4]$
$= 3 - x - x^2 - 4x - 4$
$= -x^2 - 5x - 1$

EXERCISE 1B

1 If $f : x \mapsto 3x + 2$, find the value of:
a $f(0)$ **b** $f(2)$ **c** $f(-1)$ **d** $f(-5)$ **e** $f(-\frac{1}{3})$

2 If $f : x \mapsto 3x - x^2 + 2$, find the value of:
a $f(0)$ **b** $f(3)$ **c** $f(-3)$ **d** $f(-7)$ **e** $f(\frac{3}{2})$

3 If $g : x \mapsto x - \dfrac{4}{x}$, find the value of:
a $g(1)$ **b** $g(4)$ **c** $g(-1)$ **d** $g(-4)$ **e** $g(-\frac{1}{2})$

4 If $f(x) = 7 - 3x$, find in simplest form:
a $f(a)$ **b** $f(-a)$ **c** $f(a+3)$ **d** $f(b-1)$ **e** $f(x+2)$ **f** $f(x+h)$

5 If $F(x) = 2x^2 + 3x - 1$, find in simplest form:
a $F(x+4)$ **b** $F(2-x)$ **c** $F(-x)$ **d** $F(x^2)$ **e** $F(x^2-1)$ **f** $F(x+h)$

6 Suppose $G(x) = \dfrac{2x+3}{x-4}$.
a Evaluate: **i** $G(2)$ **ii** $G(0)$ **iii** $G(-\frac{1}{2})$

b Find a value of x such that $G(x)$ does not exist.

c Find $G(x+2)$ in simplest form.

d Find x if $G(x) = -3$.

7 f represents a function. What is the difference in meaning between f and $f(x)$?

8 The value of a photocopier t years after purchase is given by $V(t) = 9650 - 860t$ euros.

a Find $V(4)$ and state what $V(4)$ means.

b Find t when $V(t) = 5780$ and explain what this represents.

c Find the original purchase price of the photocopier.

9 On the same set of axes draw the graphs of three different functions $f(x)$ such that $f(2) = 1$ and $f(5) = 3$.

10 Find a linear function $f(x) = ax + b$ for which $f(2) = 1$ and $f(-3) = 11$.

11 Find constants a and b if $f(x) = ax + \frac{b}{x}$, $f(1) = 1$, and $f(2) = 5$.

12 Given $T(x) = ax^2 + bx + c$, find a, b and c if $T(0) = -4$, $T(1) = -2$ and $T(2) = 6$.

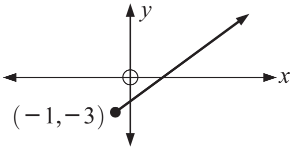
C DOMAIN AND RANGE

The **domain** of a relation is the set of permissible values that x may have.

The **range** of a relation is the set of permissible values that y may have.

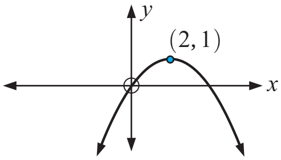
For example:

(1)



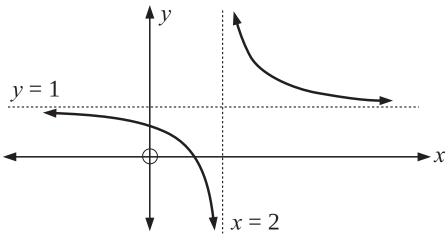
All values of $x \geqslant -1$ are permissible.
So, the domain is $\{x \mid x \geqslant -1\}$.
All values of $y \geqslant -3$ are permissible.
So, the range is $\{y \mid y \geqslant -3\}$.

(2)



x can take any value.
So, the domain is $\{x \mid x \in \mathbb{R}\}$.
y cannot be > 1.
So, the range is $\{y \mid y \leqslant 1\}$.

(3)



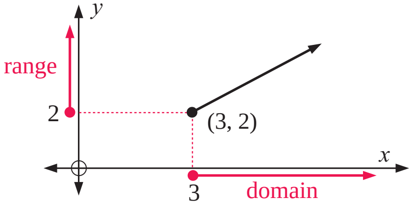
x can take all values except $x = 2$.
So, the domain is $\{x \mid x \neq 2\}$.
Likewise, the range is $\{y \mid y \neq 1\}$.

Click on the icon to obtain software for finding the domain and range of different functions.

DOMAIN AND RANGE

The domain and range of a relation are often described using **interval notation**.

For example:



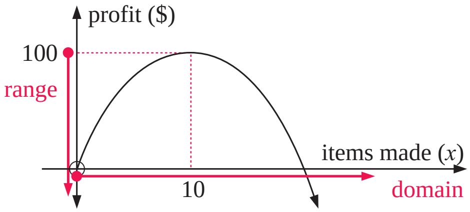
The domain consists of all real x such that $x \geqslant 3$ and we write this as

$$\{\underbrace{x}_{\text{the set of all } x}\ \underbrace{|}_{\text{such that}}\ \underbrace{x \geqslant 3}_{x \geqslant 3}\}$$

Likewise the range would be $\{y \mid y \geqslant 2\}$.

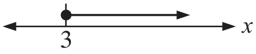
For the profit function alongside:

- the domain is $\{x \mid x \geqslant 0\}$
- the range is $\{y \mid y \leqslant 100\}$.

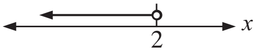


Intervals have corresponding graphs. For example:

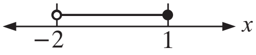
$\{x \mid x \geqslant 3\}$ is read 'the set of all x such that x is greater than or equal to 3' and has number line graph



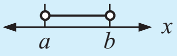
$\{x \mid x < 2\}$ has number line graph



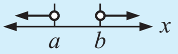
$\{x \mid -2 < x \leqslant 1\}$ has number line graph



$\{x \mid x \leqslant 0 \text{ or } x > 4\}$ has number line graph



for numbers *between* a and b we write $a < x < b$.

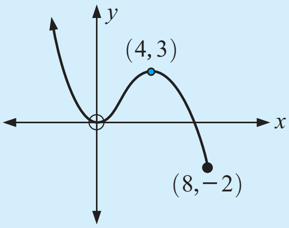


for numbers '*outside*' a and b we write $x < a$ or $x > b$.

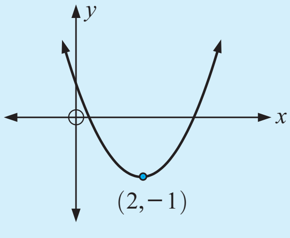
Example 4 **Self Tutor**

For each of the following graphs state the domain and range:

a



b



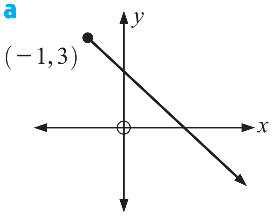
a Domain is $\{x \mid x \leqslant 8\}$
Range is $\{y \mid y \geqslant -2\}$

b Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \geqslant -1\}$

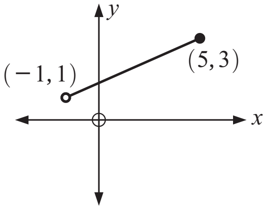
EXERCISE 1C

1 For each of the following graphs, find the domain and range:

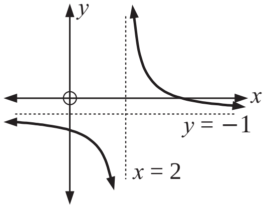
a



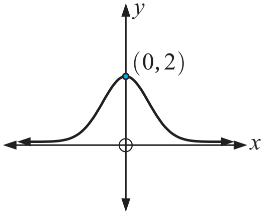
b



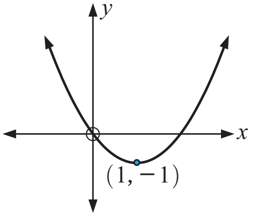
c



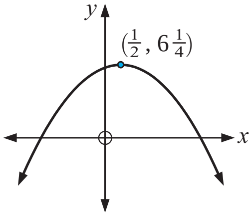
d



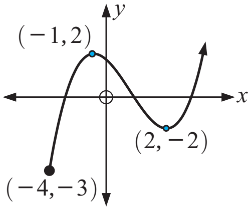
e



f



g



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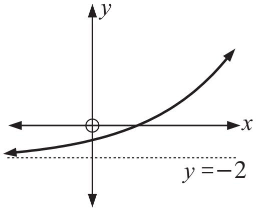
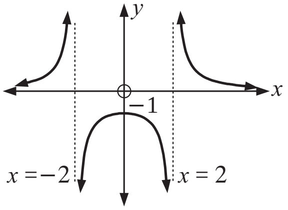


i



2 For each of the following functions, state the values of x for which $f(x)$ is undefined:

a $f(x)=\sqrt{x+6}$

b $f: x \mapsto \dfrac{1}{x^2}$

c $f(x)=\dfrac{-7}{\sqrt{3-2x}}$

3 Find the domain and range of each of the following functions:

a $f: x \mapsto 2x-1$

b $f: x \mapsto 3$

c $f: x \mapsto \sqrt{x^2+4}$

d $y=\sqrt{x^2-4}$

e $y=\dfrac{5}{x-2}$

f $f(x)=\sqrt{2-x}$

g $f(x)=\dfrac{3}{\sqrt{2x-5}}$

h $f: x \mapsto 2+\dfrac{3}{5-x}$

4 Use technology to help sketch graphs of the following functions. Find the domain and range of each.

DOMAIN AND RANGE

a $f(x)=\sqrt{x}$

b $f: x \mapsto \dfrac{1}{x^2}$

c $f: x \mapsto \sqrt{4-x}$

d $y=x^2-7x+10$

e $f: x \mapsto 5x-3x^2$

f $f: x \mapsto x+\dfrac{1}{x}$

g $y=\dfrac{x+4}{x-2}$

h $y=x^3-3x^2-9x+10$

i $f: x \mapsto \dfrac{3x-9}{x^2-x-2}$

j $y=x^2+x^{-2}$

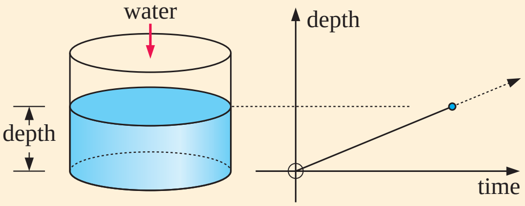
k $y=x^3+\dfrac{1}{x^3}$

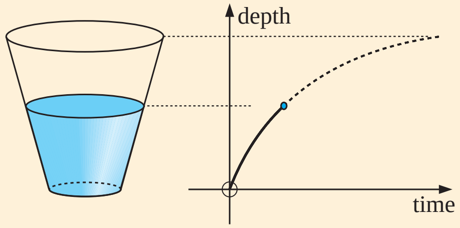
l $f: x \mapsto x^4+4x^3-16x+3$

INVESTIGATION 1 FLUID FILLING FUNCTIONS

When water is added at a **constant rate** to a cylindrical container, the depth of water in the container is a linear function of time. This is because the volume of water added is directly proportional to the time taken to add it. If water was not added at a constant rate the direct proportionality would not exist.

The depth-time graph for a cylindrical container is shown alongside. In this investigation we explore the changes in the graph for different shaped containers such as the conical vase.



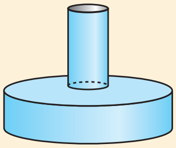


DEMO

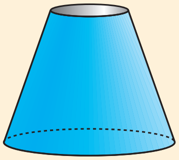
What to do:

1 For each of the following containers, draw a depth-time graph as water is added:

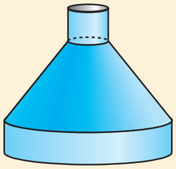
a



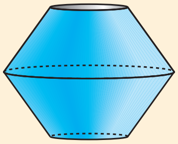
b



c



d



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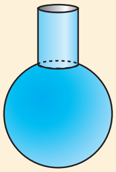
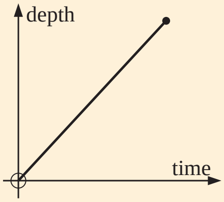


2 Use the water filling demonstration to check your answers to question **1**.

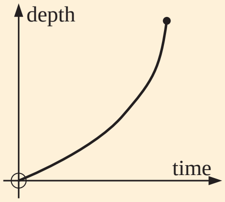
3 Write a brief report on the connection between the shape of a vessel and the corresponding shape of its depth-time graph. You may wish to discuss this in parts. For example, first examine cylindrical containers, then conical, then other shapes. Gradients of curves must be included in your report.

4 Draw possible containers as in question **1** which have the following depth-time graphs:

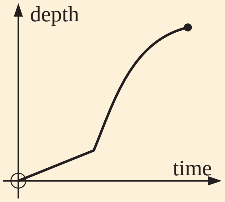
a



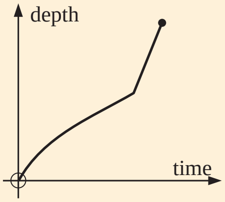
b



c



d



D COMPOSITE FUNCTIONS

Given $f : x \mapsto f(x)$ and $g : x \mapsto g(x)$, the **composite function** of f and g will convert x into $f(g(x))$.

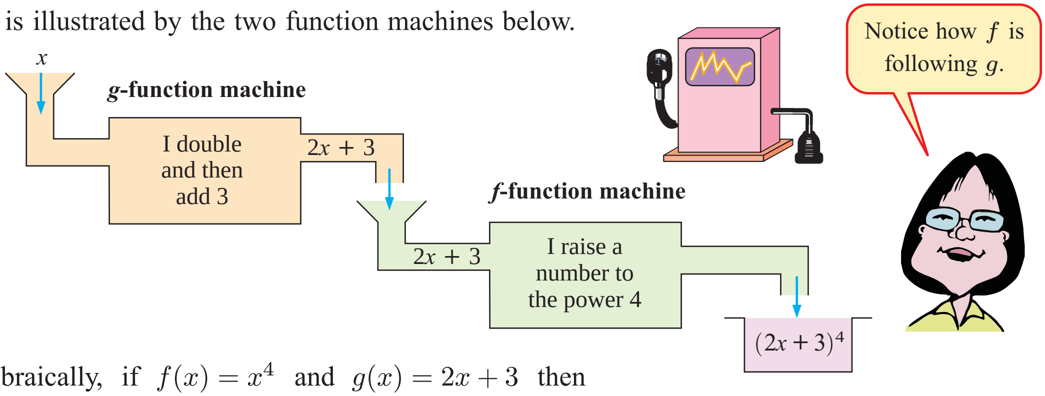
$f \circ g$ is used to represent the composite function of f and g. It means "f following g".

$$\boldsymbol{(f \circ g)(x) = f(g(x))} \qquad \text{or} \qquad \boldsymbol{f \circ g : x \mapsto f(g(x))}.$$

Consider $f : x \mapsto x^4$ and $g : x \mapsto 2x + 3$.

$f \circ g$ means that g converts x to $2x + 3$ and then f converts $(2x + 3)$ to $(2x + 3)^4$.

This is illustrated by the two function machines below.



Algebraically, if $f(x) = x^4$ and $g(x) = 2x + 3$ then

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(2x+3) && \{g \text{ operates on } x \text{ first}\} \\ &= (2x+3)^4 && \{f \text{ operates on } g(x) \text{ next}\}\end{aligned}$$

and

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x^4) \\ &= 2(x^4) + 3 \\ &= 2x^4 + 3\end{aligned}$$

So, in general, $f(g(x)) \neq g(f(x))$.

Example 5 Self Tutor

Given $f : x \mapsto 2x + 1$ and $g : x \mapsto 3 - 4x$, find in simplest form:

a $(f \circ g)(x)$ **b** $(g \circ f)(x)$

$f(x) = 2x + 1$ and $g(x) = 3 - 4x$

a
$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(3 - 4x) \\ &= 2(3 - 4x) + 1 \\ &= 6 - 8x + 1 \\ &= 7 - 8x\end{aligned}$$

b
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(2x + 1) \\ &= 3 - 4(2x + 1) \\ &= 3 - 8x - 4 \\ &= -8x - 1\end{aligned}$$

In the example above you should have observed how we can substitute an expression into a function.

If $f(x) = 2x + 1$ then $f(\Delta) = 2(\Delta) + 1$

and so $f(3 - 4x) = 2(3 - 4x) + 1$.

EXERCISE 1D

1 Given $f : x \mapsto 2x + 3$ and $g : x \mapsto 1 - x$, find in simplest form:

a $(f \circ g)(x)$ **b** $(g \circ f)(x)$ **c** $(f \circ g)(-3)$

2 Given $f : x \mapsto x^2$ and $g : x \mapsto 2 - x$, find $(f \circ g)(x)$ and $(g \circ f)(x)$.
Find also the domain and range of $f \circ g$ and $g \circ f$.

3 Given $f : x \mapsto x^2 + 1$ and $g : x \mapsto 3 - x$, find in simplest form:

a $(f \circ g)(x)$ **b** $(g \circ f)(x)$ **c** x if $(g \circ f)(x) = f(x)$

4 **a** If $ax + b = cx + d$ for all values of x, show that $a = c$ and $b = d$.
Hint: If it is true for all x, it is true for $x = 0$ and $x = 1$.

b Given $f(x) = 2x + 3$ and $g(x) = ax + b$ and that $(f \circ g)(x) = x$ for all values of x, deduce that $a = \frac{1}{2}$ and $b = -\frac{3}{2}$.

c Is the result in **b** true if $(g \circ f)(x) = x$ for all x?

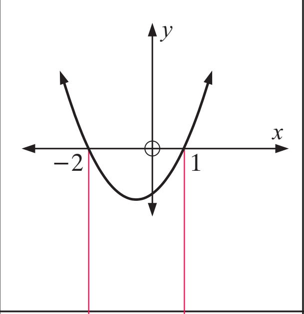
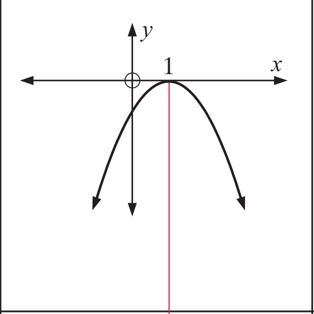
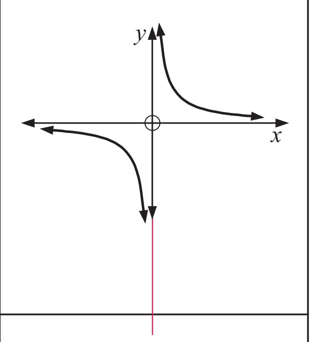
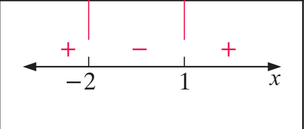
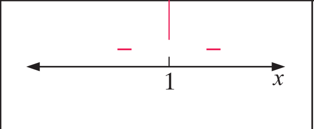
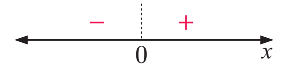
E SIGN DIAGRAMS

Sometimes we do not wish to draw a time-consuming graph of a function but wish to know when the function is positive, negative, zero or undefined. A *sign diagram* enables us to do this and is relatively easy to construct.

A **sign diagram** consists of:

- a **horizontal line** which is really the x-axis
- **positive** (+) and **negative** (−) signs indicating that the graph is **above** and **below** the x-axis respectively
- **critical values**, the numbers written below the line, which are the graph's x-intercepts and points where it is undefined.

Consider the three functions given below.

Function	$y = (x + 2)(x - 1)$	$y = -2(x - 1)^2$	$y = \frac{4}{x}$
Graph			
Sign diagram			

Notice that:

- A sign change occurs about a critical value for single factors such as $(x+2)$ and $(x-1)$. This indicates **cutting** of the x-axis.
- No sign change occurs about the critical value for squared factors such as $(x-1)^2$. This indicates **touching** of the x-axis.
- 

shows a function is **undefined** at $x = 0$.

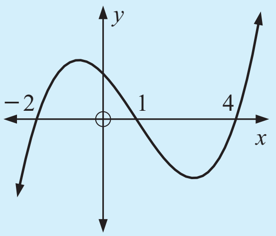
In general:

- when a factor has an **odd power** there is a change of sign about that critical value
- when a factor has an **even power** there is no sign change about that critical value.

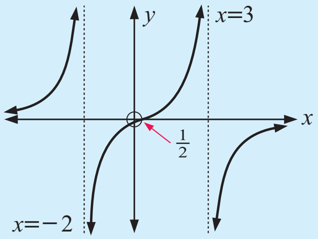
Example 6 **Self Tutor**

Draw sign diagrams for:

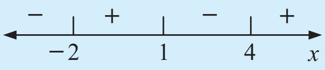
a



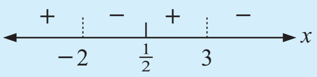
b



a



b



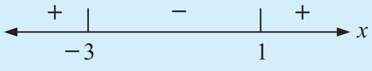
Example 7 **Self Tutor**

Draw a sign diagram for: **a** $(x+3)(x-1)$ **b** $2(2x+5)(3-x)$

a $(x+3)(x-1)$ has critical values of -3 and 1.

We try any number > 1.
When $x = 2$ we have
$(5)(1) > 0$, so we
put a $+$ sign here.

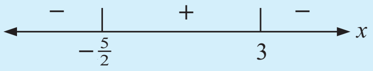
As the factors are 'single' the signs alternate.



b $2(2x+5)(3-x)$ has critical values of $-\frac{5}{2}$ and 3.

We try any number > 3.
When $x = 5$ we have
$2(15)(-2) < 0$, so we
put a $-$ sign here.

As the factors are 'single' the signs alternate.



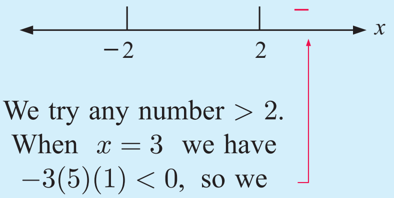
Example 8 **Self Tutor**

Draw a sign diagram for: **a** $12 - 3x^2$ **b** $-4(x-3)^2$

a $12 - 3x^2 = -3(x^2 - 4)$

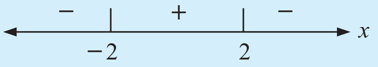
$= -3(x+2)(x-2)$

which has critical values of -2 and 2.

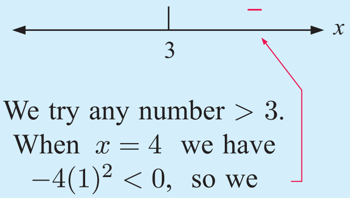


We try any number > 2.
When $x = 3$ we have
$-3(5)(1) < 0$, so we
put a $-$ sign here.

As the factors are 'single' the signs alternate.

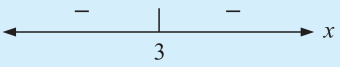


b $-4(x-3)^2$
has a critical value of 3.



We try any number > 3.
When $x = 4$ we have
$-4(1)^2 < 0$, so we
put a $-$ sign here.

As the factor is 'squared' the signs do not change.

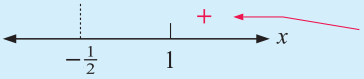


Example 9 **Self Tutor**

Draw a sign diagram for $\dfrac{x-1}{2x+1}$.

$\dfrac{x-1}{2x+1}$ is zero when $x = 1$ and undefined when $x = -\frac{1}{2}$.

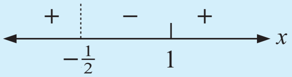
So,



When $x = 10$, $\dfrac{x-1}{2x+1} = \frac{9}{21} > 0$

Since $(x-1)$ and $(2x+1)$ are single factors, the signs alternate.

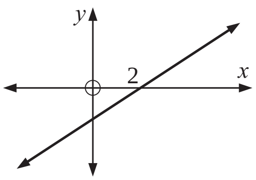
$\therefore$ the sign diagram is



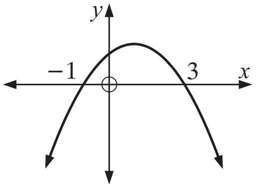
EXERCISE 1E

1 From the graphs below, draw corresponding sign diagrams:

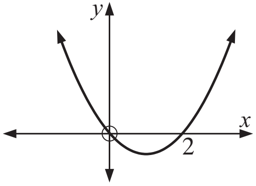
a



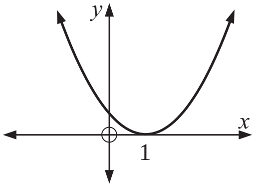
b



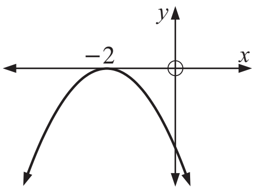
c



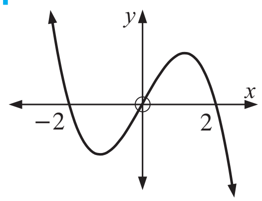
d



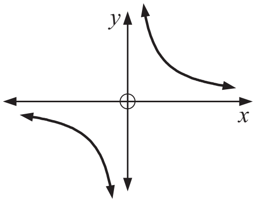
e



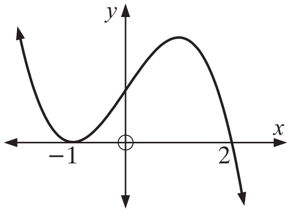
f



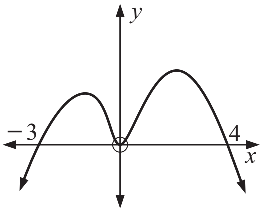
g



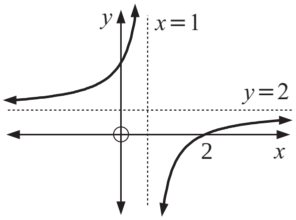
h



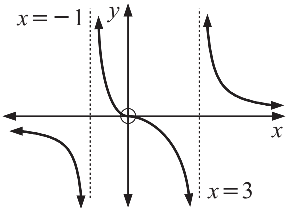
i



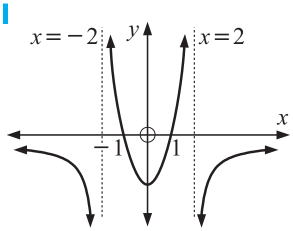
j



k



l



2 Draw sign diagrams for:

a $(x+4)(x-2)$
b $x(x-3)$
c $x(x+2)$
d $-(x+1)(x-3)$
e $(2x-1)(3-x)$
f $(5-x)(1-2x)$
g x^2-9
h $4-x^2$
i $5x-x^2$
j x^2-3x+2
k $2-8x^2$
l $6x^2+x-2$
m $6-16x-6x^2$
n $-2x^2+9x+5$
o $-15x^2-x+2$

3 Draw sign diagrams for:

a $(x+2)^2$
b $(x-3)^2$
c $-(x+2)^2$
d $-(x-4)^2$
e x^2-2x+1
f $-x^2+4x-4$
g $4x^2-4x+1$
h $-x^2-6x-9$
i $-4x^2+12x-9$

4 Draw sign diagrams for:

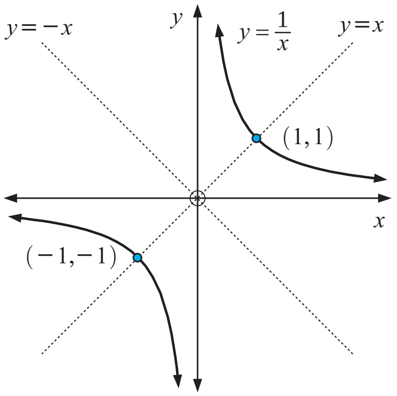
a $\dfrac{x+2}{x-1}$
b $\dfrac{x}{x+3}$
c $\dfrac{2x+3}{4-x}$
d $\dfrac{4x-1}{2-x}$
e $\dfrac{3x}{x-2}$
f $\dfrac{-8x}{3-x}$
g $\dfrac{(x-1)^2}{x}$
h $\dfrac{4x}{(x+1)^2}$
i $\dfrac{(x+2)(x-1)}{3-x}$
j $\dfrac{x(x-1)}{2-x}$
k $\dfrac{x^2-4}{-x}$
l $\dfrac{3-x}{2x^2-x-6}$

F THE RECIPROCAL FUNCTION

The **reciprocal function** is defined as $x \mapsto \frac{1}{x}$ or $f(x) = \frac{1}{x}$.

The reciprocal function is the simplest example of a **rational** function.

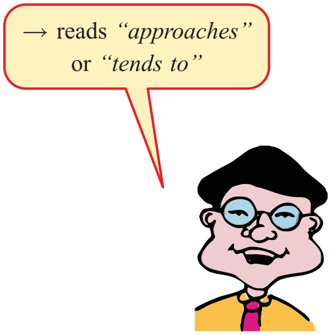
Its graph is:



Notice that:

- $f(x) = \frac{1}{x}$ is undefined when $x = 0$
- The graph of $f(x) = \frac{1}{x}$ exists in the first and third quadrants only.
- $f(x) = \frac{1}{x}$ is symmetric about $y = x$ and $y = -x$.
- as $x \to \infty$, $f(x) \to 0$ (from above)
  as $x \to -\infty$, $f(x) \to 0$ (from below)
  as $x \to 0$ (from the right), $f(x) \to \infty$
  as $x \to 0$ (from the left), $f(x) \to -\infty$
- $f(x) = \frac{1}{x}$ is **asymptotic** to the x-axis and to the y-axis.

  The graph gets closer to the axes as it gets further from the origin.



GRAPHING PACKAGE

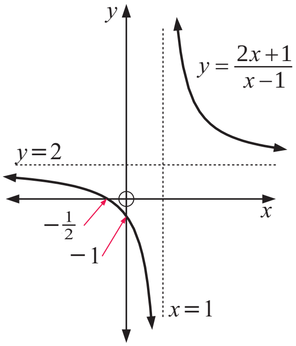
EXERCISE 1F

1 Sketch the graphs of $f(x) = \frac{1}{x}$, $g(x) = \frac{2}{x}$, $h(x) = \frac{4}{x}$ on the same set of axes. Comment on any similarities and differences.

2 Sketch the graphs of $f(x) = -\frac{1}{x}$, $g(x) = -\frac{2}{x}$, $h(x) = -\frac{4}{x}$ on the same set of axes. Comment on any similarities and differences.

G ASYMPTOTES OF OTHER RATIONAL FUNCTIONS

The function $f(x) = \dfrac{2x+1}{x-1}$ has the graph given below.



Notice that at $x = 1$, $f(x)$ is undefined.

Since the graph approaches the vertical line $x = 1$, we say that $x = 1$ is a **vertical asymptote**.

Notice that: $f(1.001) = 3002$ and $f(0.999) = -2998$

so we can write: as $x \to 1$ (from the left), $f(x) \to -\infty$
as $x \to 1$ (from the right), $f(x) \to \infty$

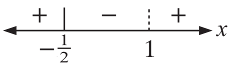
or alternatively, as $x \to 1^-$, $f(x) \to -\infty$
as $x \to 1^+$, $f(x) \to \infty$.

We also notice that $f(1000) = \dfrac{2001}{999} \approx 2.003$ and $f(-1000) = \dfrac{-1999}{-1001} \approx 1.997$

This indicates that $y = 2$ is a **horizontal asymptote** and we write:

as $x \to \infty$, $y \to 2$ (from above) *or* as $x \to \infty$, $y \to 2^+$
as $x \to -\infty$, $y \to 2$ (from below) as $x \to -\infty$, $y \to 2^-$.

Notice that as $|x| \to \infty$, $f(x) \to 2$.

The sign diagram of $y = \dfrac{2x+1}{x-1}$ is  and can be used to discuss the function near its vertical asymptote without having to graph the function.

INVESTIGATION 2 FINDING ASYMPTOTES

Use the **graphing package** supplied or a **graphics calculator** to examine the following functions for asymptotes:

GRAPHING PACKAGE

a $y = -1 + \dfrac{3}{x-2}$ **b** $y = \dfrac{3x+1}{x+2}$ **c** $y = \dfrac{2x-9}{3-x}$

DISCUSSION

Can a function cross a vertical asymptote?

Further examples of asymptotic behaviour are seen in exponential, logarithmic and some trigonometric functions.

EXERCISE 1G

1 For the following functions:

- **i** determine the equations of the asymptotes
- **ii** discuss the behaviour of the function as it approaches its asymptotes
- **iii** find the axes intercepts
- **iv** sketch the graph of the function.

a $f : x \mapsto \dfrac{3}{x-2}$ **b** $y = 2 - \dfrac{3}{x+1}$ **c** $f : x \mapsto \dfrac{x+3}{x-2}$ **d** $f(x) = \dfrac{3x-1}{x+2}$

H INVERSE FUNCTIONS

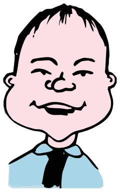
The operations of + and −, × and ÷, squaring and finding the square root, are **inverse operations** as one undoes what the other does.

For example, $x+3-3=x$, $x \times 3 \div 3 = x$ and $\sqrt{8^2} = 8$.

A function $y = f(x)$ *may or may not* have an inverse function.

f^{-1} is **not** the reciprocal of f.

$f^{-1}(x) \neq \dfrac{1}{f(x)}$



If $y = f(x)$ has an **inverse function**, this new function:

- is denoted $f^{-1}(x)$
- must indeed be a function, and so must satisfy the vertical line test
- is the reflection of $y = f(x)$ in the line $y = x$
- satisfies $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$.

The function $y = x$, defined as $f : x \mapsto x$, is the **identity function**.

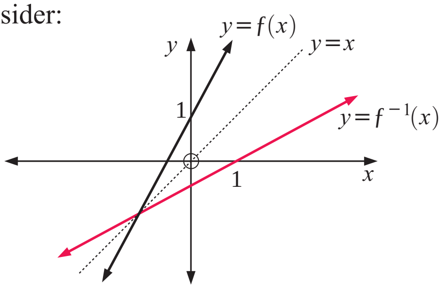
If (x, y) lies on f, then (y, x) lies on f^{-1}. Reflecting the function in the line $y = x$ has the algebraic effect of interchanging x and y.

For example, $f : y = 5x + 2$ becomes $f^{-1} : x = 5y + 2$.

The domain of f^{-1} is equal to the range of f.

The range of f^{-1} is equal to the domain of f.

Consider:

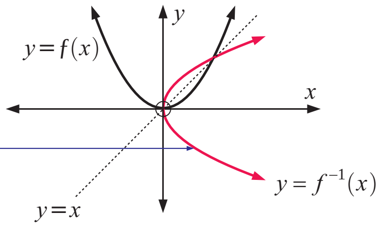


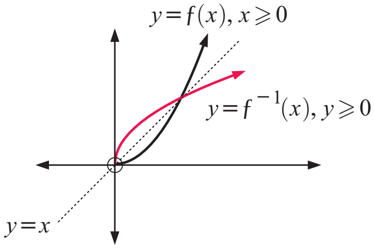
$y = f^{-1}(x)$ is the inverse of $y = f(x)$ as:

- it is also a function
- it is the reflection of $y = f(x)$ in the line $y = x$.

This is the reflection of $y = f(x)$ in $y = x$, but it is *not* the inverse function of $y = f(x)$ as it fails the vertical line test.

In this case the function $y = f(x)$ does not have an inverse.





Now consider the same function $y = f(x)$ but with the restricted domain $x \geqslant 0$.

The function does now have an inverse function, as shown alongside.

$y = f(x)$ subject to $x \leqslant 0$ would also have an inverse function.

Example 10

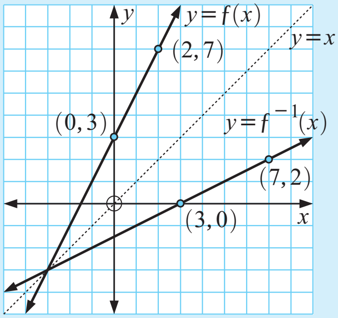
Self Tutor

Consider $f : x \mapsto 2x + 3$.

- **a** On the same axes, graph f and its inverse function f^{-1}.
- **b** Find $f^{-1}(x)$ using:
  - **i** coordinate geometry and the gradient of $f^{-1}(x)$ from **a**
  - **ii** variable interchange.
- **c** Check that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$

a $f(x) = 2x + 3$ passes through $(0, 3)$ and $(2, 7)$.

$\therefore$ $f^{-1}(x)$ passes through $(3, 0)$ and $(7, 2)$.



b **i** This line has gradient $\dfrac{2-0}{7-3} = \dfrac{1}{2}$.

So, its equation is $\dfrac{y-0}{x-3} = \dfrac{1}{2}$

$$\therefore \quad y = \frac{x-3}{2}$$

$$\text{So,} \quad f^{-1}(x) = \frac{x-3}{2}$$

ii f is $y = 2x + 3$, so f^{-1} is $x = 2y + 3$

$$\therefore \quad x - 3 = 2y$$

$$\therefore \quad \frac{x-3}{2} = y$$

$$\text{So,} \quad f^{-1}(x) = \frac{x-3}{2}$$

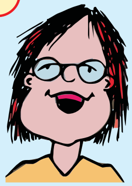
c $(f \circ f^{-1})(x)$

$$= f(f^{-1}(x))$$
$$= f\left(\frac{x-3}{2}\right)$$
$$= 2\left(\frac{x-3}{2}\right) + 3$$
$$= x$$

and $(f^{-1} \circ f)(x)$

$$= f^{-1}(f(x))$$
$$= f^{-1}(2x+3)$$
$$= \frac{(2x+3)-3}{2}$$
$$= \frac{2x}{2}$$
$$= x$$

If f includes point (a, b) then f^{-1} includes point (b, a).



The function $f(x) = \frac{1}{x}$, $x \neq 0$, is called the **reciprocal function**.

It is said to be a **self-inverse function** as $f = f^{-1}$.

This is because the graph of $y = \frac{1}{x}$ is symmetrical about the line $y = x$.

> Any function with a graph which is symmetrical about the line $y = x$ must be a **self-inverse function**.

EXERCISE 1H

1 For each of the following functions f:

- **i** on the same axes graph $y = x$, f and f^{-1}
- **ii** find $f^{-1}(x)$ using coordinate geometry and **i**
- **iii** find $f^{-1}(x)$ using variable interchange.

a $f : x \mapsto 3x + 1$

b $f : x \mapsto \frac{x+2}{4}$

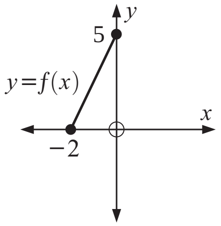
2 For each of the following functions f:

- **i** find $f^{-1}(x)$
- **ii** sketch $y = f(x)$, $y = f^{-1}(x)$ and $y = x$ on the same set of axes
- **iii** show that $f^{-1} \circ f = f \circ f^{-1} = x$, the identity function.

a $f : x \mapsto 2x + 5$ **b** $f : x \mapsto \frac{3-2x}{4}$ **c** $f : x \mapsto x + 3$

3 Copy the graphs of the following functions and in each case include the graphs of $y = x$ and $y = f^{-1}(x)$:

a



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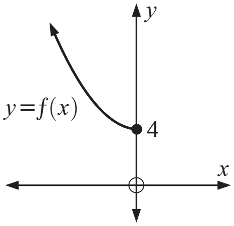
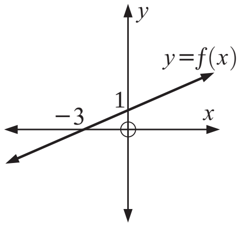
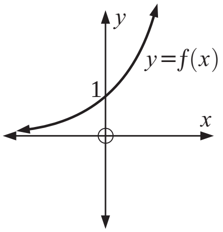


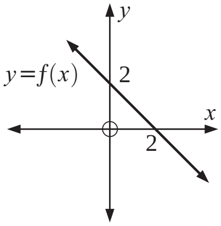
c



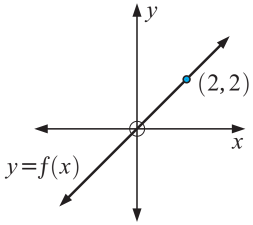
d



e



f



4 For the graph of $y = f(x)$ given in **3 a**, state:

- **a** the domain of $f(x)$
- **b** the range of $f(x)$
- **c** the domain of $f^{-1}(x)$
- **d** the range of $f^{-1}(x)$.

5 If the domain of $H(x)$ is $\{x \mid -2 \leqslant x < 3\}$, state the range of $H^{-1}(x)$.

6 Sketch the graph of $f : x \mapsto x^3$ and its inverse function $f^{-1}(x)$.

7 **a** Show that $f : x \mapsto \dfrac{1}{x}$ has an inverse function for all $x \neq 0$.

b Find f^{-1} algebraically and show that f is a self-inverse function.

8 Show that $f : x \mapsto \dfrac{3x-8}{x-3}$, $x \neq 3$ is a self-inverse function by:

a reference to its graph **b** using algebra.

9 Consider the function $f(x) = \frac{1}{2}x - 1$.

a Find $f^{-1}(x)$. **b** Find: **i** $(f \circ f^{-1})(x)$ **ii** $(f^{-1} \circ f)(x)$.

10 Consider the functions $f : x \mapsto 2x + 5$ and $g : x \mapsto \dfrac{8-x}{2}$.

a Find $g^{-1}(-1)$. **b** Solve for x if $(f \circ g^{-1})(x) = 9$.

11 Consider the functions $f : x \mapsto 5^x$ and $g : x \mapsto \sqrt{x}$.

a Find: **i** $f(2)$ **ii** $g^{-1}(4)$. **b** Solve the equation $(g^{-1} \circ f)(x) = 25$.

12 Given $f : x \mapsto 2x$ and $g : x \mapsto 4x - 3$, show that $(f^{-1} \circ g^{-1})(x) = (g \circ f)^{-1}(x)$.

13 Which of these functions is a self-inverse function, so $f^{-1}(x) = f(x)$?

a $f(x) = 2x$ **b** $f(x) = x$ **c** $f(x) = -x$

d $f(x) = \dfrac{2}{x}$ **e** $f(x) = -\dfrac{6}{x}$

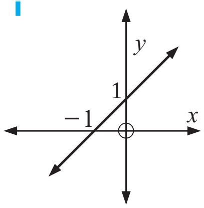
14 The **horizontal line test** says:

For a function to have an inverse function, no horizontal line can cut its graph more than once.

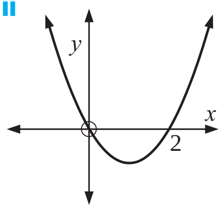
a Explain why this is a valid test for the existence of an inverse function.

b Which of the following functions have an inverse function?

i



ii



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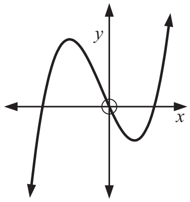


REVIEW SET 1A NON-CALCULATOR

1 If $f(x) = 2x - x^2$ find: **a** $f(2)$ **b** $f(-3)$ **c** $f(-\frac{1}{2})$

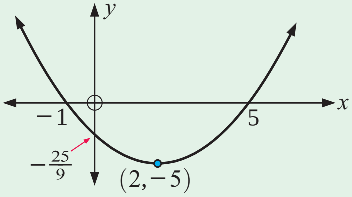
2 If $f(x) = ax + b$ where a and b are constants, find a and b for $f(1) = 7$ and $f(3) = -5$.

3 If $g(x) = x^2 - 3x$, find in simplest form: **a** $g(x+1)$ **b** $g(x^2 - 2)$

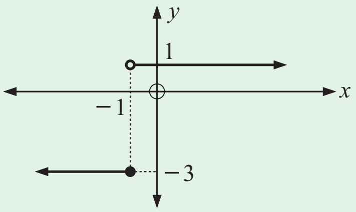
4 For each of the following graphs determine:

i the range and domain
ii the x and y-intercepts
iii whether it is a function.

a



b



5 Draw a sign diagram for:

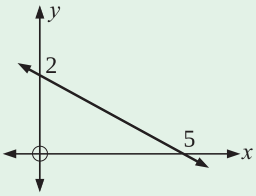
a $(3x+2)(4-x)$

b $\dfrac{x-3}{x^2+4x+4}$

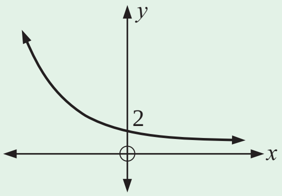
6 If $f(x) = ax + b$, $f(2) = 1$ and $f^{-1}(3) = 4$, find a and b.

7 Copy the following graphs and draw the inverse function on the same set of axes:

a



b



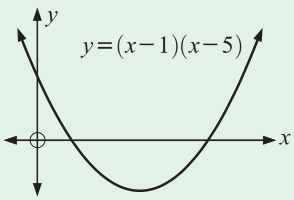
8 Find $f^{-1}(x)$ given that $f(x)$ is: **a** $4x+2$ **b** $\dfrac{3-5x}{4}$

9 Given $f : x \mapsto 3x+6$ and $h : x \mapsto \dfrac{x}{3}$, show that $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$.

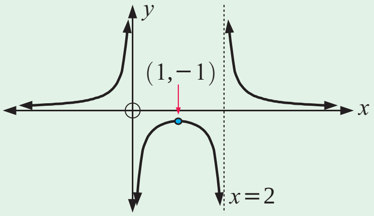
REVIEW SET 1B CALCULATOR

1 For each of the following graphs, find the domain and range:

a



b



2 If $f(x) = 2x - 3$ and $g(x) = x^2 + 2$, find: **a** $(f \circ g)(x)$ **b** $(g \circ f)(x)$

3 Draw a sign diagram for:

a $\dfrac{x^2-6x-16}{x-3}$

b $\dfrac{x+9}{x+5} + x$

4 Consider $f(x) = \dfrac{1}{x^2}$.

- **a** For what value of x is $f(x)$ meaningless?
- **b** Sketch the graph of this function using technology.
- **c** State the domain and range of the function.

5 Consider the function $f(x) = \dfrac{ax+3}{x-b}$.

- **a** Find a and b given that $y = f(x)$ has asymptotes with equations $x = -1$ and $y = 2$.
- **b** Write down the domain and range of $f^{-1}(x)$.

6 Consider the function $f : x \mapsto \dfrac{4x+1}{2-x}$.

- **a** Determine the equations of the asymptotes.
- **b** Discuss the behaviour of the function as it approaches its asymptotes.
- **c** Determine the axes intercepts.
- **d** Sketch the graph.

7 Consider the functions $f(x) = 3x + 1$ and $g(x) = \dfrac{2}{x}$.

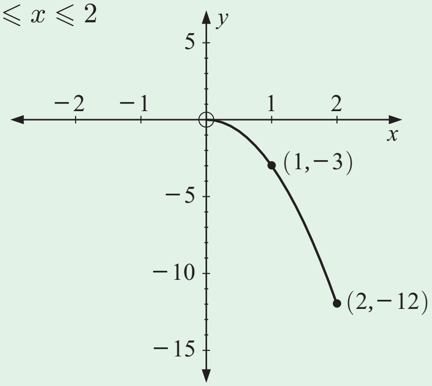
- **a** Find $(g \circ f)(x)$.
- **b** Given $(g \circ f)(x) = -4$, solve for x.
- **c** Let $h(x) = (g \circ f)(x)$, $x \neq -\frac{1}{3}$.
  - **i** Write down the equations of the asymptotes for the graph of $h(x)$.
  - **ii** Sketch the graph of $h(x)$ for $-3 \leqslant x \leqslant 2$.
  - **iii** State the domain and range of $h(x)$.

8 Consider $f : x \mapsto 2x - 7$.

- **a** On the same set of axes graph $y = x$, f and f^{-1}.
- **b** Find $f^{-1}(x)$ using variable interchange.
- **c** Show that $f \circ f^{-1} = f^{-1} \circ f = x$, the identity function.

9 The graph of the function $f(x) = -3x^2$, $0 \leqslant x \leqslant 2$ is shown alongside.

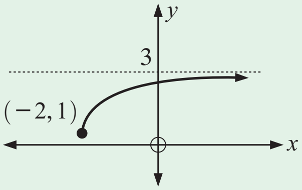
- **a** Sketch the graph of $y = f^{-1}(x)$.
- **b** State the range of f^{-1}.
- **c** Solve:
  - **i** $f(x) = -10$ **ii** $f^{-1}(x) = 1$



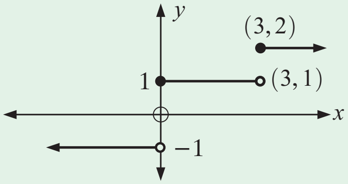
REVIEW SET 1C

1 For each of the following graphs, find the domain and range:

a



b



2 If $h(x) = 7 - 3x$:

a find in simplest form $h(2x-1)$ **b** find x if $h(2x-1) = -2$.

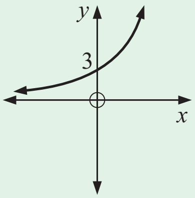
3 If $f(x) = 1 - 2x$ and $g(x) = \sqrt{x}$, find in simplest form:

a $(f \circ g)(x)$ **b** $(g \circ f)(x)$

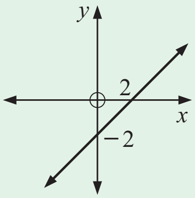
4 Find a, b and c if $f(0) = 5$, $f(-2) = 21$ and $f(3) = -4$ and $f(x) = ax^2+bx+c$.

5 Copy the following graphs and draw the graph of each inverse function on the same set of axes:

a



b



6 For each of the following functions $f(x)$ find $f^{-1}(x)$:

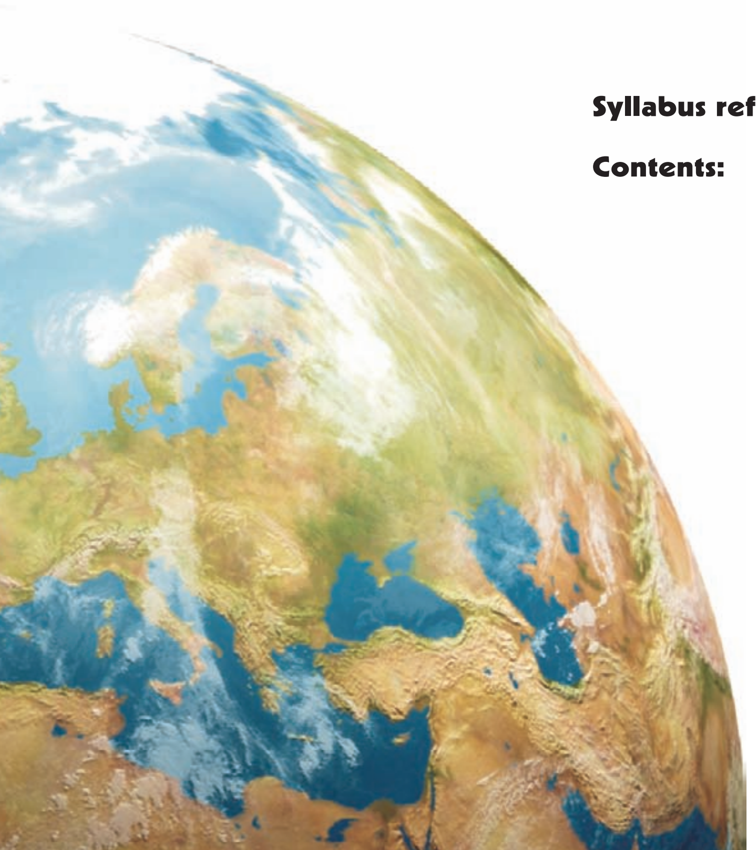
a $f(x) = 7 - 4x$ **b** $f(x) = \dfrac{3+2x}{5}$

7 Given $f : x \mapsto 5x - 2$ and $h : x \mapsto \dfrac{3x}{4}$, show that $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$.

8 Given $f(x) = 2x + 11$ and $g(x) = x^2$, find $(g \circ f^{-1})(3)$.

Chapter 2

Sequences and series



Syllabus reference: 1.1

Contents:

OPENING PROBLEM

A circular stadium consists of sections as illustrated, with aisles in between. The diagram shows the 13 tiers of concrete steps for the final section, Section K. Seats are to be placed along every concrete step, with each seat being 0.45 m wide. The arc AB at the front of the first row, is 14.4 m long, while the arc CD at the back of the back row, is 20.25 m long.

Things to think about:

1 How wide is each concrete step?

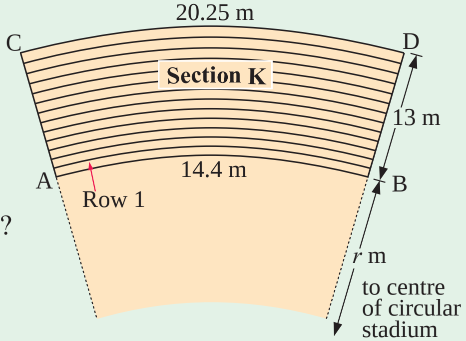
2 What is the length of the arc of the back of Row 1, Row 2, Row 3, and so on?

3 How many seats are there in Row 1, Row 2, Row 3, ..., Row 13?

4 How many sections are there in the stadium?

5 What is the total seating capacity of the stadium?

6 What is the radius of the 'playing surface'?



To solve problems like the **Opening Problem** and many others, we need to study **sequences** and their sums which are called **series**.

A NUMBER PATTERNS

In mathematics it is important that we can:

- **recognise** a pattern in a set of numbers,
- **describe** the pattern in words, and
- **continue** the pattern.

A list of numbers where there is a pattern is called a **number sequence**.
The numbers in the sequence are said to be its **members** or its **terms**.

For example, 3, 7, 11, 15, ... form a number sequence.

The first term is 3, the second term is 7, the third term is 11, and so on.

We can describe this pattern in words:

"*The sequence starts at 3 and each term is 4 more than the previous one.*"

Thus, the fifth term is 19, the sixth term is 23, and so on.

Example 1 **Self Tutor**

Describe the sequence: 14, 17, 20, 23, ... and write down the next two terms.

The sequence starts at 14 and each term is 3 more than the previous term.
The next two terms are 26 and 29.

EXERCISE 2A

1 Write down the first four terms of the sequence if you start with:

a 4 and add 9 each time **b** 45 and subtract 6 each time

c 2 and multiply by 3 each time **d** 96 and divide by 2 each time.

2 For each of the following write a description of the sequence and find the next 2 terms:

a 8, 16, 24, 32, ... **b** 2, 5, 8, 11, ... **c** 36, 31, 26, 21, ...

d 96, 89, 82, 75, ... **e** 1, 4, 16, 64, ... **f** 2, 6, 18, 54, ...

g 480, 240, 120, 60, ... **h** 243, 81, 27, 9, ... **i** 50 000, 10 000, 2000, 400, ...

3 Describe the following number patterns and write down the next 3 terms:

a 1, 4, 9, 16, ... **b** 1, 8, 27, 64, ... **c** 2, 6, 12, 20, ...

4 Find the next two terms of:

a 95, 91, 87, 83, ... **b** 5, 20, 80, 320, ... **c** 1, 16, 81, 256, ...

d 2, 3, 5, 7, 11, ... **e** 2, 4, 7, 11, ... **f** 3, 4, 6, 8, 12, ...

B SEQUENCES OF NUMBERS

A **number sequence** is a set of numbers defined by a rule that is valid for positive integers.

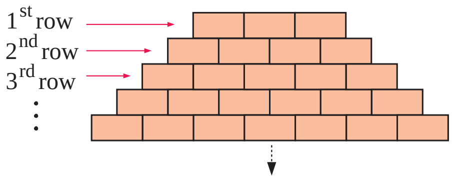
A number sequence is a function whose domain is the set of positive integers.

Sequences may be defined in one of the following ways:

- listing the first few terms and assuming that the pattern represented continues indefinitely
- giving a description in words
- using a formula which represents the **general term** (or ***n*th term**).

Consider the illustrated tower of bricks. The first row has three bricks, the second row has four bricks, and the third row has five bricks.

If u_n represents the number of bricks in row n (from the top) then $u_1 = 3$, $u_2 = 4$, $u_3 = 5$, $u_4 = 6$, ...



This sequence can be specified by:

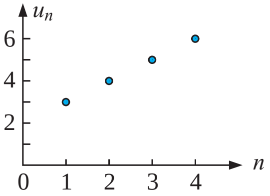
- **listing terms** 3, 4, 5, 6, ...
- **using words** "The top row has three bricks and each successive row under it has one more brick."
- **using an explicit formula** $u_n = n + 2$ is the **general term** or ***n*th term** formula for $n = 1, 2, 3, 4, 5, ...$

Check: $u_1 = 1 + 2 = 3$ ✓ $u_2 = 2 + 2 = 4$ ✓
$u_3 = 3 + 2 = 5$ ✓

- **a pictorial or graphical representation**

Early members of a sequence can be graphed with each represented by a dot.

The dots *must not* be joined because n must be an integer.



THE GENERAL TERM

u_n, T_n, t_n, A_n, and so on can all be used to represent the **general term** or ***n*th term** of a sequence. The general term is defined for $n = 1, 2, 3, 4, 5, 6, ...$

$\{u_n\}$ represents the sequence that can be generated by using u_n as the ***n*th term**.

$\{u_n\}$ is a function, where $n \mapsto u_n$, $n \in \mathbb{Z}^+$.

For example, $\{2n+1\}$ generates the sequence $3, 5, 7, 9, 11, ...$

EXERCISE 2B

1 List the first *five* terms of the sequence:

a $\{2n\}$ **b** $\{2n+2\}$ **c** $\{2n-1\}$ **d** $\{2n-3\}$

e $\{2n+3\}$ **f** $\{2n+11\}$ **g** $\{3n+1\}$ **h** $\{4n-3\}$

2 List the first *five* terms of the sequence:

a $\{2^n\}$ **b** $\{3 \times 2^n\}$ **c** $\{6 \times (\frac{1}{2})^n\}$ **d** $\{(-2)^n\}$

3 List the first *five* terms of the sequence $\{15 - (-2)^n\}$.

C ARITHMETIC SEQUENCES

An **arithmetic sequence** is a sequence in which each term differs from the previous one by the same fixed number.

For example:

- the tower of bricks in the previous section forms an arithmetic sequence where the difference between terms is 1
- $2, 5, 8, 11, 14, ...$ is arithmetic as $5 - 2 = 8 - 5 = 11 - 8 = 14 - 11, ...$
- $31, 27, 23, 19, ...$ is arithmetic as $27 - 31 = 23 - 27 = 19 - 23, ...$

ALGEBRAIC DEFINITION

$\{u_n\}$ is **arithmetic** $\Leftrightarrow$ $u_{n+1} - u_n = d$ for all positive integers n where d is a constant called the **common difference**.

The symbol $\Leftrightarrow$ is read as 'if and only if'. It implies that if $\{u_n\}$ is arithmetic then $u_{n+1} - u_n$ is a constant *and* if $u_{n+1} - u_n$ is a constant then $\{u_n\}$ is arithmetic.

THE NAME 'ARITHMETIC'

If a, b and c are any consecutive terms of an arithmetic sequence then

$$\begin{aligned} b-a &= c-b \qquad \{\text{equating common differences}\}\\ \therefore\ 2b &= a+c\\ \therefore\ b &= \frac{a+c}{2}\end{aligned}$$

So, the middle term is the **arithmetic mean** of the terms on either side of it.

THE GENERAL TERM FORMULA

Suppose the first term of an arithmetic sequence is u_1 and the common difference is d.

Then $u_2 = u_1 + d$, $u_3 = u_1 + 2d$, $u_4 = u_1 + 3d$, and so on.

$$\text{Hence}\quad u_n = u_1 + \underbrace{(n-1)}_{}d$$

The coefficient of d is one less than the subscript.

For an **arithmetic sequence** with **first term u_1** and **common difference d** the **general term** or **nth term** is $\boldsymbol{u_n = u_1 + (n-1)d}$.

Example 2

Self Tutor

Consider the sequence 2, 9, 16, 23, 30, ...

a Show that the sequence is arithmetic.

b Find the formula for the general term u_n.

c Find the 100th term of the sequence.

d Is **i** 828 **ii** 2341 a member of the sequence?

a
$$\begin{aligned} 9-2 &= 7\\ 16-9 &= 7\\ 23-16 &= 7\\ 30-23 &= 7\end{aligned}$$

So, assuming that the pattern continues, consecutive terms differ by 7.

$\therefore$ the sequence is arithmetic with $u_1 = 2$, $d = 7$.

b $u_n = u_1 + (n-1)d$ $\quad\therefore\ u_n = 2 + 7(n-1)$

$\therefore\ u_n = 7n - 5$

c If $n = 100$, $u_{100} = 7(100) - 5 = 695$.

d **i**
$$\begin{aligned} \text{Let}\quad u_n &= 828\\ \therefore\ 7n-5 &= 828\\ \therefore\ 7n &= 833\\ \therefore\ n &= 119\end{aligned}$$

$\therefore$ 828 is a term of the sequence. In fact it is the 119th term.

ii
$$\begin{aligned} \text{Let}\quad u_n &= 2341\\ \therefore\ 7n-5 &= 2341\\ \therefore\ 7n &= 2346\\ \therefore\ n &= 335\tfrac{1}{7}\end{aligned}$$

which is not possible as n must be an integer.

$\therefore$ 2341 cannot be a term.

Example 3 **Self Tutor**

Find k given that $3k+1$, k and -3 are consecutive terms of an arithmetic sequence.

Since the terms are consecutive, $k-(3k+1)=-3-k$ {equating differences}

$$\begin{aligned} \therefore\quad k-3k-1&=-3-k \\ \therefore\quad -2k-1&=-3-k \\ \therefore\quad -1+3&=-k+2k \\ \therefore\quad k&=2 \end{aligned}$$

Example 4 **Self Tutor**

Find the general term u_n for an arithmetic sequence with $u_3=8$ and $u_8=-17$.

$u_3=8$ $\quad\therefore\quad u_1+2d=8$ .... (1) $\quad$ {using $u_n=u_1+(n-1)d$}

$u_8=-17$ $\quad\therefore\quad u_1+7d=-17$ .... (2)

We now solve (1) and (2) simultaneously:

$$\begin{aligned} -u_1-2d&=-8 \quad &&\{\text{multiplying both sides of (1) by } -1\} \\ u_1+7d&=-17 \\ \hline \therefore\quad 5d&=-25 &&\{\text{adding the equations}\} \\ \therefore\quad d&=-5 \end{aligned}$$

So, in (1):

$$\begin{aligned} u_1+2(-5)&=8 \\ \therefore\quad u_1-10&=8 \\ \therefore\quad u_1&=18 \\ \text{Now}\quad u_n&=u_1+(n-1)d \\ \therefore\quad u_n&=18-5(n-1) \\ \therefore\quad u_n&=18-5n+5 \\ \therefore\quad u_n&=23-5n \end{aligned}$$

Check:

$$\begin{aligned} u_3&=23-5(3) \\ &=23-15 \\ &=8 \quad \checkmark \\ u_8&=23-5(8) \\ &=23-40 \\ &=-17 \quad \checkmark \end{aligned}$$

Example 5 **Self Tutor**

Insert four numbers between 3 and 12 so that all six numbers are in arithmetic sequence.

If the numbers are $3,\ 3+d,\ 3+2d,\ 3+3d,\ 3+4d,\ 12$

$$\begin{aligned} \text{then}\quad 3+5d&=12 \\ \therefore\quad 5d&=9 \\ \therefore\quad d&=\tfrac{9}{5}=1.8 \end{aligned}$$

So, we have 3, 4.8, 6.6, 8.4, 10.2, 12.

EXERCISE 2C

1 Find the 10th term of each of the following arithmetic sequences:

a 19, 25, 31, 37, ... **b** 101, 97, 93, 89, ... **c** 8, $9\frac{1}{2}$, 11, $12\frac{1}{2}$, ...

2 Find the 15th term of each of the following arithmetic sequences:

a 31, 36, 41, 46, ... **b** 5, -3, -11, -19, ...

c a, $a+d$, $a+2d$, $a+3d$, ...

3 Consider the sequence 6, 17, 28, 39, 50, ...

a Show that the sequence is arithmetic. **b** Find the formula for its general term.

c Find its 50th term. **d** Is 325 a member?

e Is 761 a member?

4 Consider the sequence 87, 83, 79, 75, ...

a Show that the sequence is arithmetic. **b** Find the formula for its general term.

c Find the 40th term. **d** Which term of the sequence is -297?

5 A sequence is defined by $u_n = 3n - 2$.

a Prove that the sequence is arithmetic. **Hint:** Find $u_{n+1} - u_n$.

b Find u_1 and d. **c** Find the 57th term.

d What is the least term of the sequence which is greater than 450?

6 A sequence is defined by $u_n = \dfrac{71-7n}{2}$.

a Prove that the sequence is arithmetic. **b** Find u_1 and d. **c** Find u_{75}.

d For what values of n are the terms of the sequence less than -200?

7 Find k given the consecutive arithmetic terms:

a 32, k, 3 **b** k, 7, 10 **c** $k+1$, $2k+1$, 13

d $k-1$, $2k+3$, $7-k$ **e** k, k^2, k^2+6 **f** 5, k, k^2-8

8 Find the general term u_n for an arithmetic sequence given that:

a $u_7 = 41$ and $u_{13} = 77$ **b** $u_5 = -2$ and $u_{12} = -12\frac{1}{2}$

c the seventh term is 1 and the fifteenth term is -39

d the eleventh and eighth terms are -16 and $-11\frac{1}{2}$ respectively.

9 **a** Insert three numbers between 5 and 10 so that all five numbers are in arithmetic sequence.

b Insert six numbers between -1 and 32 so that all eight numbers are in arithmetic sequence.

10 Consider the finite arithmetic sequence 36, $35\frac{1}{3}$, $34\frac{2}{3}$, ..., -30.

a Find u_1 and d. **b** How many terms does the sequence have?

11 An arithmetic sequence starts 23, 36, 49, 62, ... What is the first term of the sequence to exceed 100 000?

D GEOMETRIC SEQUENCES

A sequence is **geometric** if each term can be obtained from the previous one by multiplying by the same non-zero constant.

For example: 2, 10, 50, 250, ... is a geometric sequence as

$2 \times 5 = 10$ and $10 \times 5 = 50$ and $50 \times 5 = 250$.

Notice that $\frac{10}{2} = \frac{50}{10} = \frac{250}{50} = 5$, so each term divided by the previous one gives the same constant.

Algebraic definition:

$\{u_n\}$ is **geometric** $\Leftrightarrow$ $\dfrac{u_{n+1}}{u_n} = r$ for all positive integers n
where r is a **constant** called the **common ratio**.

For example:
- 2, 10, 50, 250, ... is geometric with $r = 5$.
- 2, -10, 50, -250, ... is geometric with $r = -5$.

THE NAME 'GEOMETRIC'

If a, b and c are any consecutive terms of a geometric sequence then $\dfrac{b}{a} = \dfrac{c}{b}$.

$\therefore$ $b^2 = ac$ and so $b = \pm\sqrt{ac}$ where $\sqrt{ac}$ is the **geometric mean** of a and c.

THE GENERAL TERM

Suppose the first term of a geometric sequence is u_1 and the common ratio is r.

Then $u_2 = u_1\, r$, $u_3 = u_1\, r^2$, $u_4 = u_1\, r^3$, and so on.

Hence $u_n = u_1\, r^{\underbrace{n-1}}$

The power of r is one less than the subscript.

For a **geometric sequence** with **first term** u_1 and **common ratio** r, the **general term** or n**th term** is $u_n = u_1 r^{n-1}$.

Example 6 **Self Tutor**

For the sequence 8, 4, 2, 1, $\frac{1}{2}$, ...

a Show that the sequence is geometric.

b Find the general term u_n.

c Hence, find the 12th term as a fraction.

a $\dfrac{4}{8} = \frac{1}{2}$ $\quad \dfrac{2}{4} = \frac{1}{2}$ $\quad \dfrac{1}{2} = \frac{1}{2}$ $\quad \dfrac{\frac{1}{2}}{1} = \frac{1}{2}$

So, assuming the pattern continues, consecutive terms have a common ratio of $\frac{1}{2}$.

$\therefore$ the sequence is geometric with $u_1 = 8$ and $r = \frac{1}{2}$.

b $u_n = u_1 r^{n-1}$ $\quad \therefore \quad u_n = 8\left(\frac{1}{2}\right)^{n-1}$ *or*

$$\begin{aligned} u_n &= 2^3 \times (2^{-1})^{n-1} \\ &= 2^3 \times 2^{-n+1} \\ &= 2^{3+(-n+1)} \\ &= 2^{4-n} \end{aligned}$$

c

$$\begin{aligned} u_{12} &= 8 \times \left(\tfrac{1}{2}\right)^{11} \\ &= \tfrac{1}{256} \end{aligned}$$

Example 7 **Self Tutor**

$k - 1$, $2k$ and $21 - k$ are consecutive terms of a geometric sequence. Find k.

Since the terms are geometric, $\quad \dfrac{2k}{k-1} = \dfrac{21-k}{2k} \quad$ {equating rs}

$$\begin{aligned} \therefore \quad 4k^2 &= (21-k)(k-1) \\ \therefore \quad 4k^2 &= 21k - 21 - k^2 + k \\ \therefore \quad 5k^2 - 22k + 21 &= 0 \\ \therefore \quad (5k-7)(k-3) &= 0 \quad \text{and so} \quad k = \tfrac{7}{5} \text{ or } 3 \end{aligned}$$

Check: If $k = \frac{7}{5}$ the terms are: $\frac{2}{5}, \frac{14}{5}, \frac{98}{5}$. ✓ $\{r = 7\}$

If $k = 3$ the terms are: $2, 6, 18$. ✓ $\{r = 3\}$

Example 8 **Self Tutor**

A geometric sequence has $u_2 = -6$ and $u_5 = 162$. Find its general term.

$$u_2 = u_1 r = -6 \quad \text{.... (1)}$$

and $$u_5 = u_1 r^4 = 162 \quad \text{.... (2)}$$

So, $$\frac{u_1 r^4}{u_1 r} = \frac{162}{-6} \quad \{(2) \div (1)\}$$

$$\begin{aligned} \therefore \quad r^3 &= -27 \\ \therefore \quad r &= \sqrt[3]{-27} \\ \therefore \quad r &= -3 \end{aligned}$$

and so in (1) $$u_1(-3) = -6$$

$$\therefore \quad u_1 = 2$$

Thus $u_n = 2 \times (-3)^{n-1}$.

Example 9 **Self Tutor**

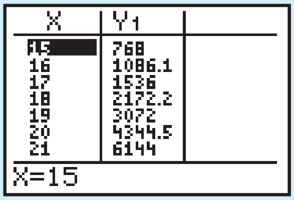
Find the first term of the sequence $6, 6\sqrt{2}, 12, 12\sqrt{2}, \ldots.$ which exceeds 1400.

The sequence is geometric with $u_1 = 6$ and $r = \sqrt{2}$

$$\therefore \quad u_n = 6 \times (\sqrt{2})^{n-1}.$$

Next we need to find n such that $u_n > 1400$.

Using a graphics calculator with $Y_1 = 6 \times (\sqrt{2})\hat{\ }(n-1)$, we view a *table of values*:



```
  X     Y1
 15    768
 16    1086.1
 17    1536
 18    2172.2
 19    3072
 20    4344.5
 21    6144
X=15
```

So, the first term to exceed 1400 is u_{17} where $u_{17} = 1536$.

EXERCISE 2D.1

1 For the geometric sequence with first two terms given, find b and c:

a $2, 6, b, c, \ldots..$ **b** $10, 5, b, c, \ldots..$ **c** $12, -6, b, c, \ldots..$

2 Find the 6th term in each of the following geometric sequences:

a $3, 6, 12, 24, \ldots..$ **b** $2, 10, 50, \ldots..$ **c** $512, 256, 128, \ldots..$

3 Find the 9th term in each of the following geometric sequences:

a $1, 3, 9, 27, \ldots..$ **b** $12, 18, 27, \ldots..$

c $\frac{1}{16}, -\frac{1}{8}, \frac{1}{4}, -\frac{1}{2}, \ldots..$ **d** $a, ar, ar^2, \ldots..$

4 **a** Show that the sequence $5, 10, 20, 40, \ldots..$ is geometric.

b Find u_n and hence find the 15th term.

5 **a** Show that the sequence $12, -6, 3, -\frac{3}{2}, \ldots..$ is geometric.

b Find u_n and hence find the 13th term (as a fraction).

6 Show that the sequence $8, -6, 4.5, -3.375, \ldots.$ is geometric. Hence find the 10th term as a decimal.

7 Show that the sequence $8, 4\sqrt{2}, 4, 2\sqrt{2}, \ldots.$ is geometric. Hence find, in simplest form, the general term u_n.

8 Find k given that the following are consecutive terms of a geometric sequence:

a $7, k, 28$ **b** $k, 3k, 20-k$ **c** $k, k+8, 9k$

9 Find the general term u_n of the geometric sequence which has:

a $u_4 = 24$ and $u_7 = 192$ **b** $u_3 = 8$ and $u_6 = -1$

c $u_7 = 24$ and $u_{15} = 384$ **d** $u_3 = 5$ and $u_7 = \frac{5}{4}$.

10 **a** Find the first term of the sequence $2, 6, 18, 54, \ldots..$ which exceeds 10 000.

b Find the first term of the sequence $4, 4\sqrt{3}, 12, 12\sqrt{3}, \ldots..$ which exceeds 4800.

c Find the first term of the sequence $12, 6, 3, 1.5, \ldots..$ which is less than 0.0001.

COMPOUND INTEREST

Suppose you invest \$1000 in the bank. You leave the money in the bank for 3 years, and are paid an interest rate of 10% per annum (p.a). The interest is added to your investment each year.

The interest paid on your investment causes the value to *increase* each year.

The percentage increase each year is 10%, so at the end of the year you will have $100\% + 10\% = 110\%$ of the value at its start. This corresponds to a *multiplier* of 1.1 .

After one year your investment is worth $\$1000 \times 1.1 = \$1100$.

After two years it is worth

$$\begin{aligned} & \$1100 \times 1.1 \\ = {} & \$1000 \times 1.1 \times 1.1 \\ = {} & \$1000 \times (1.1)^2 = \$1210 \end{aligned}$$

After three years it is worth

$$\begin{aligned} & \$1210 \times 1.1 \\ = {} & \$1000 \times (1.1)^2 \times 1.1 \\ = {} & \$1000 \times (1.1)^3 = \$1331 \end{aligned}$$

This suggests that if the money is left in your account for n years it would amount to $\$1000 \times (1.1)^n$.

Observe that:

$$\begin{aligned} u_1 &= \$1000 && = \text{initial investment} \\ u_2 &= u_1 \times 1.1 && = \text{amount after 1 year} \\ u_3 &= u_1 \times (1.1)^2 && = \text{amount after 2 years} \\ u_4 &= u_1 \times (1.1)^3 && = \text{amount after 3 years} \\ &\vdots \\ u_{n+1} &= u_1 \times (1.1)^n && = \text{amount after } n \text{ years} \end{aligned}$$

In general, we can use the compound interest formula $\boldsymbol{u_{n+1} = u_1 \times r^n}$

where
$u_1 =$ initial investment
$r =$ growth multiplier
$n =$ number of years
$u_{n+1} =$ amount after n years.

Example 10 **Self Tutor**

\$5000 is invested for 4 years at 7% p.a. compound interest, compounded annually. What will it amount to at the end of this period? Give your answer to the nearest cent.

$$\begin{aligned} u_5 &= u_1 \times r^4 && \text{is the amount after 4 years} \\ &= 5000 \times (1.07)^4 && \{\text{for a 7\% increase 100\% becomes 107\%}\} \\ &\approx 6553.98 && \{5000 \ \boxed{\times}\ 1.07 \ \boxed{\wedge}\ 4 \ \boxed{\text{ENTER}}\ \} \end{aligned}$$

So, it amounts to \$6553.98 .

Example 11 **Self Tutor**

How much should I invest now if I want the maturing value to be €10 000 in 4 years' time, if I am able to invest at 8.5% p.a. compounded annually? Give your answer to the nearest cent.

$u_1 = ?,\quad u_5 = 10\,000,\quad r = 1.085$

$$u_5 = u_1 \times r^4 \qquad \{\text{using}\quad u_{n+1} = u_1 \times r^n\}$$

$$\therefore\quad 10\,000 = u_1 \times (1.085)^4$$

$$\therefore\quad u_1 = \frac{10\,000}{(1.085)^4}$$

$$\therefore\quad u_1 = 7215.74 \qquad \{10\,000 \boxed{\div}\ 1.085 \boxed{\wedge}\ 4 \boxed{\text{ENTER}}\}$$

So, I should invest €7215.74 now.

EXERCISE 2D.2

1 **a** What will an investment of \$3000 at 10% p.a. compound interest amount to after 3 years?

b What part of this is interest?

2 How much compound interest is earned by investing €20 000 at 12% p.a. if the investment is over a 4 year period?

3 **a** What will an investment of ¥30 000 at 10% p.a. compound interest amount to after 4 years?

b What part of this is interest?

4 How much compound interest is earned by investing \$80 000 at 9% p.a., if the investment is over a 3 year period?

5 What will an investment of ¥100 000 amount to after 5 years if it earns 8% p.a. compounded twice annually?

6 What will an investment of £45 000 amount to after 21 months if it earns 7.5% p.a. compounded quarterly?

7 How much money must be invested now if you require \$20 000 for a holiday in 4 years' time and the money can be invested at a fixed rate of 7.5% p.a. compounded annually?

8 What initial investment is required to produce a maturing amount of £15 000 in 60 months' time given that a fixed rate of 5.5% p.a. compounded annually is guaranteed?

9 How much should I invest now if I need a maturing amount of €25 000 in 3 years' time and the money can be invested at a fixed rate of 8% p.a. compounded quarterly?

10 What initial investment is required to produce a maturing amount of ¥40 000 in 8 years' time if your money can be invested at 9% p.a., compounded monthly?

OTHER GEOMETRIC SEQUENCE PROBLEMS

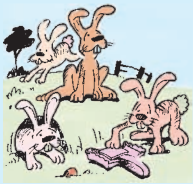
Example 12 **Self Tutor**

The initial population of rabbits on a farm was 50.
The population increased by 7% each week.

a How many rabbits were present after:

  i 15 weeks **ii** 30 weeks?

b How long would it take for the population to reach 500?



There is a fixed percentage increase each week, so the population forms a geometric sequence.

$u_1 = 50$ and $r = 1.07$

$u_2 = 50 \times 1.07 =$ the population after 1 week

a **i** $u_{n+1} = u_1 \times r^n$

$\therefore\ u_{16} = 50 \times (1.07)^{15}$

≈ 137.95

There were 138 rabbits.

ii $u_{31} = 50 \times (1.07)^{30}$

≈ 380.61

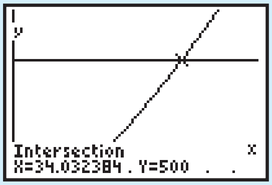
There were 381 rabbits.

b $u_{n+1} = u_1 \times (1.07)^n$ after n weeks

So, we need to find when $50 \times (1.07)^n = 500$.

Trial and error on your calculator gives $n \approx 34$ weeks.

or by finding the **point of intersection** of $Y_1 = 50 \times 1.07$^X and $Y_2 = 500$ on a graphics calculator, the solution is ≈ 34.0 weeks.



```
Intersection
X=34.032384 . Y=500  .  .
```

EXERCISE 2D.3

1 A nest of ants initially contains 500 individuals.
The population is increasing by 12% each week.

  a How many ants will there be after

    i 10 weeks **ii** 20 weeks?

  b Use technology to find how many weeks it will take for the ant population to reach 2000.

2 The animal *Eraticus* is endangered. Since 1992 there has only been one colony remaining, and in 1992 the population of the colony was 555. Since then the population has been steadily decreasing at 4.5% per year. Find:

  a the population in the year 2007

  b the year in which we would expect the population to have declined to 50.

E SERIES

A **series** is the addition of the terms of a sequence.

For the sequence $\{u_n\}$ the corresponding series is $u_1 + u_2 + u_3 + ... + u_n$.

The **sum** of a series is the result when we perform the addition.

Given a series which includes the first n terms of a sequence, its sum is $S_n = u_1 + u_2 + u_3 + ... + u_n$.

Example 13 Self Tutor

Consider the sequence 1, 4, 9, 16, 25, ...

a Write down an expression for S_n.

b Find S_n for $n = 1, 2, 3, 4$ and 5.

a $S_n = 1^2 + 2^2 + 3^2 + 4^2 + ... + n^2$
{all terms are perfect squares}

b $S_1 = 1$
$S_2 = 1 + 4 = 5$
$S_3 = 1 + 4 + 9 = 14$
$S_4 = 1 + 4 + 9 + 16 = 30$
$S_5 = 1 + 4 + 9 + 16 + 25 = 55$

SIGMA NOTATION

$u_1 + u_2 + u_3 + u_4 + ... + u_n$ can be written more compactly using **sigma notation**.

$\sum$, which is called **sigma**, is the equivalent of capital S in the Greek alphabet.

We write $u_1 + u_2 + u_3 + u_4 + ... + u_n$ as $\sum\limits_{k=1}^{n} u_k$.

$\sum\limits_{k=1}^{n} u_k$ reads "the sum of all numbers of the form u_k where $k = 1, 2, 3, ...,$ up to n".

Example 14 Self Tutor

Expand and evaluate: **a** $\sum\limits_{k=1}^{7} (k+1)$ **b** $\sum\limits_{k=1}^{5} \frac{1}{2^k}$

a $\sum\limits_{k=1}^{7} (k+1)$
$= 2 + 3 + 4 + 5 + 6 + 7 + 8$
$= 35$

b $\sum\limits_{k=1}^{5} \frac{1}{2^k}$
$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$
$= \frac{31}{32}$

PROPERTIES OF SIGMA NOTATION

$$\sum_{k=1}^{n}(a_k+b_k)=\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}b_k$$

If c is a constant, $\displaystyle\sum_{k=1}^{n}ca_k=c\sum_{k=1}^{n}a_k$ and $\displaystyle\sum_{k=1}^{n}c=cn$.

EXERCISE 2E.1

1 For the following sequences:

i write down an expression for S_n **ii** find S_5.

a $3, 11, 19, 27, ...$ **b** $42, 37, 32, 27, ...$ **c** $12, 6, 3, 1\frac{1}{2}, ...$

d $2, 3, 4\frac{1}{2}, 6\frac{3}{4}, ...$ **e** $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...$ **f** $1, 8, 27, 64, ...$

2 Expand and evaluate:

a $\displaystyle\sum_{k=1}^{4}(3k-5)$ **b** $\displaystyle\sum_{k=1}^{5}(11-2k)$ **c** $\displaystyle\sum_{k=1}^{7}k(k+1)$ **d** $\displaystyle\sum_{k=1}^{5}10\times2^{k-1}$

3 For $u_n=3n-1$, write $u_1+u_2+u_3+....+u_{20}$ using sigma notation and evaluate the sum.

4 Show that:

a $\displaystyle\sum_{k=1}^{n}c=cn$ **b** $\displaystyle\sum_{k=1}^{n}ca_k=c\sum_{k=1}^{n}a_k$ **c** $\displaystyle\sum_{k=1}^{n}(a_k+b_k)=\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}b_k$

ARITHMETIC SERIES

An **arithmetic series** is the addition of successive terms of an arithmetic sequence.

For example: $21, 23, 25, 27, ..., 49$ is an arithmetic sequence.

So, $21+23+25+27+.....+49$ is an arithmetic series.

SUM OF AN ARITHMETIC SERIES

If the first term is u_1 and the common difference is d, the terms are u_1, u_1+d, u_1+2d, u_1+3d, and so on.

Suppose that u_n is the final term of an arithmetic series.

So, $S_n=u_1+(u_1+d)+(u_1+2d)+....+(u_n-2d)+(u_n-d)+u_n$

But $S_n=u_n+(u_n-d)+(u_n-2d)+....+(u_1+2d)+(u_1+d)+u_1$ {reversing them}

Adding these two expressions vertically we get

$$2S_n=\underbrace{(u_1+u_n)+(u_1+u_n)+(u_1+u_n)+....+(u_1+u_n)+(u_1+u_n)+(u_1+u_n)}_{n\text{ of these}}$$

$\therefore\quad 2S_n=n(u_1+u_n)$

$\therefore\quad S_n=\dfrac{n}{2}(u_1+u_n)$ where $u_n=u_1+(n-1)d$

So,
$$S_n = \frac{n}{2}(u_1 + u_n) \quad or \quad S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

Example 15 🔊 **Self Tutor**

Find the sum of $4 + 7 + 10 + 13 + ...$ to 50 terms.

The series is arithmetic with $u_1 = 4$, $d = 3$ and $n = 50$.

Now $S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$\therefore \quad S_{50} = \frac{50}{2}(2 \times 4 + 49 \times 3)$

$= 3875$

Example 16 🔊 **Self Tutor**

Find the sum of $-6 + 1 + 8 + 15 + ... + 141$.

The series is arithmetic with $u_1 = -6$, $d = 7$ and $u_n = 141$.

First we need to find n.

$$\begin{aligned} \text{Now} \quad u_n &= 141 \\ \therefore \quad u_1 + (n-1)d &= 141 \\ \therefore \quad -6 + 7(n-1) &= 141 \\ \therefore \quad 7(n-1) &= 147 \\ \therefore \quad n - 1 &= 21 \\ \therefore \quad n &= 22 \end{aligned}$$

Using $S_n = \frac{n}{2}(u_1 + u_n)$,

$$\begin{aligned} S_{22} &= \tfrac{22}{2}(-6 + 141) \\ &= 11 \times 135 \\ &= 1485 \end{aligned}$$

EXERCISE 2E.2

1 Find the sum of:

- **a** $3 + 7 + 11 + 15 + ...$ to 20 terms
- **b** $\frac{1}{2} + 3 + 5\frac{1}{2} + 8 + ...$ to 50 terms
- **c** $100 + 93 + 86 + 79 + ...$ to 40 terms
- **d** $50 + 48\frac{1}{2} + 47 + 45\frac{1}{2} + ...$ to 80 terms.

2 Find the sum of:

- **a** $5 + 8 + 11 + 14 + ... + 101$
- **b** $50 + 49\frac{1}{2} + 49 + 48\frac{1}{2} + ... + (-20)$
- **c** $8 + 10\frac{1}{2} + 13 + 15\frac{1}{2} + ... + 83$

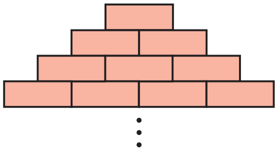
3 Evaluate these arithmetic series:

- **a** $\sum_{k=1}^{10} (2k + 5)$
- **b** $\sum_{k=1}^{15} (k - 50)$
- **c** $\sum_{k=1}^{20} \left(\frac{k+3}{2}\right)$

Hint: List the first 3 terms and the last term.

4 An arithmetic series has seven terms. The first term is 5 and the last term is 53. Find the sum of the series.

5 An arithmetic series has eleven terms. The first term is 6 and the last term is -27. Find the sum of the series.

6 

A bricklayer builds a triangular wall with layers of bricks as shown. If the bricklayer uses 171 bricks, how many layers did he build?

7 Each section of a soccer stadium has 44 rows with 22 seats in the first row, 23 in the second row, 24 in the third row, and so on. How many seats are there in:

a row 44 **b** each section **c** the stadium which has 25 sections?

8 Find the sum of:

a the first 50 multiples of 11

b the multiples of 7 between 0 and 1000

c the integers between 1 and 100 which are not divisible by 3.

9 Prove that the sum of the first n positive integers is $\frac{1}{2}n(n+1)$.

10 Consider the series of odd numbers $1 + 3 + 5 + 7 + ...$

a What is the nth odd number u_n?

b Prove that the sum of the first n odd integers is n^2.

c Check your answer to **b** by finding S_1, S_2, S_3 and S_4.

11 Find the first two terms of an arithmetic sequence if the sixth term is 21 and the sum of the first seventeen terms is 0.

12 Three consecutive terms of an arithmetic sequence have a sum of 12 and a product of -80. Find the terms. **Hint:** Let the terms be $x - d$, x and $x + d$.

13 Five consecutive terms of an arithmetic sequence have a sum of 40. The product of the first, middle and last terms is 224. Find the terms of the sequence.

GEOMETRIC SERIES

A **geometric series** is the addition of successive terms of a geometric sequence.

For example: 1, 2, 4, 8, 16, ... , 1024 is a geometric sequence.
$1 + 2 + 4 + 8 + 16 + ... + 1024$ is a geometric series.

SUM OF A GEOMETRIC SERIES

If the first term is u_1 and the common ratio is r, then the terms are: $u_1, u_1r, u_1r^2, u_1r^3, ...$

So,
$$S_n = u_1 + \underset{\substack{\uparrow \\ u_2}}{u_1r} + \underset{\substack{\uparrow \\ u_3}}{u_1r^2} + \underset{\substack{\uparrow \\ u_4}}{u_1r^3} + ... + \underset{\substack{\uparrow \\ u_{n-1}}}{u_1r^{n-2}} + \underset{\substack{\uparrow \\ u_n}}{u_1r^{n-1}}$$

and for $r \neq 1$,
$$\boldsymbol{S_n = \frac{u_1(r^n - 1)}{r - 1}} \quad \textit{or} \quad \boldsymbol{S_n = \frac{u_1(1 - r^n)}{1 - r}}.$$

Proof: If $S_n = u_1 + u_1r + u_1r^2 + u_1r^3 + ... + u_1r^{n-2} + u_1r^{n-1}$ ....(1)

$$\text{then } rS_n = (u_1r + u_1r^2 + u_1r^3 + u_1r^4 + ... + u_1r^{n-1}) + u_1r^n$$

$$\therefore \quad rS_n = (S_n - u_1) + u_1r^n \qquad \{\text{from } (1)\}$$

$$\therefore \quad rS_n - S_n = u_1r^n - u_1$$

$$\therefore \quad S_n(r-1) = u_1(r^n - 1) \text{ and so } S_n = \frac{u_1(r^n-1)}{r-1} \text{ or } \frac{u_1(1-r^n)}{1-r} \text{ for } r \neq 1.$$

In the case $r = 1$ we have a sequence in which all terms are the same, and $S_n = u_1n$.

Example 17 **Self Tutor**

Find the sum of $2 + 6 + 18 + 54 + ...$ to 12 terms.

The series is geometric with $u_1 = 2$, $r = 3$ and $n = 12$.

$$S_n = \frac{u_1(r^n-1)}{r-1}$$

$$\therefore \quad S_{12} = \frac{2(3^{12}-1)}{3-1}$$

$$= 531\,440$$

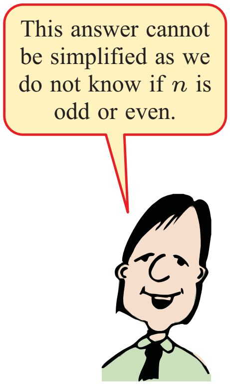
Example 18 **Self Tutor**

Find a formula for S_n for the first n terms of $9 - 3 + 1 - \frac{1}{3} + ...$

The series is geometric with $u_1 = 9$ and $r = -\frac{1}{3}$

$$\text{So,} \quad S_n = \frac{u_1(1-r^n)}{1-r} = \frac{9(1-(-\frac{1}{3})^n)}{\frac{4}{3}}$$

$$\therefore \quad S_n = \tfrac{27}{4}(1-(-\tfrac{1}{3})^n)$$



EXERCISE 2E.3

1 Find the sum of the following series:

- **a** $12 + 6 + 3 + 1.5 + ...$ to 10 terms
- **b** $\sqrt{7} + 7 + 7\sqrt{7} + 49 + ...$ to 12 terms
- **c** $6 - 3 + 1\frac{1}{2} - \frac{3}{4} + ...$ to 15 terms
- **d** $1 - \frac{1}{\sqrt{2}} + \frac{1}{2} - \frac{1}{2\sqrt{2}} + ...$ to 20 terms

2 Find a formula for S_n for the first n terms of:

- **a** $\sqrt{3} + 3 + 3\sqrt{3} + 9 + ...$
- **b** $12 + 6 + 3 + 1\frac{1}{2} + ...$
- **c** $0.9 + 0.09 + 0.009 + 0.0009 + ...$
- **d** $20 - 10 + 5 - 2\frac{1}{2} + ...$

3 Let S_n be the sum of an infinite geometric sequence such that $S_1 = 3$ and $S_2 = 4$.

- **a** State the first term u_1.
- **b** Calculate the common ratio r.
- **c** Calculate u_5.

4 Evaluate these geometric series:

a $\sum_{k=1}^{10} 3 \times 2^{k-1}$ **b** $\sum_{k=1}^{12} (\frac{1}{2})^{k-2}$ **c** $\sum_{k=1}^{25} 6 \times (-2)^k$

5 Each year a salesperson is paid a bonus of $2000 which is banked into the same account. It earns a fixed rate of interest of 6% p.a. with interest being paid annually. The total amount in the account at the end of each year is calculated as follows:

$$A_0 = 2000$$
$$A_1 = A_0 \times 1.06 + 2000$$
$$A_2 = A_1 \times 1.06 + 2000 \quad \text{and so on.}$$

a Show that $A_2 = 2000 + 2000 \times 1.06 + 2000 \times (1.06)^2$.

b Show that $A_3 = 2000[1 + 1.06 + (1.06)^2 + (1.06)^3]$.

c Find the total bank balance after 10 years, assuming there are no fees or charges.

6 Consider $S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ... + \frac{1}{2^n}$.

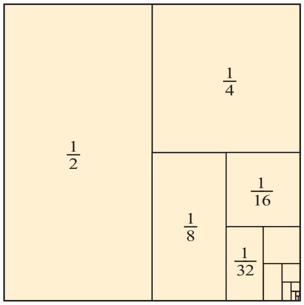
a Find S_1, S_2, S_3, S_4 and S_5 in fractional form.

b From **a** guess the formula for S_n.

c Find S_n using $S_n = \dfrac{u_1(1 - r^n)}{1 - r}$.

d Comment on S_n as n gets very large.

e What is the relationship between the given diagram and **d**?



SUM OF AN INFINITE GEOMETRIC SERIES

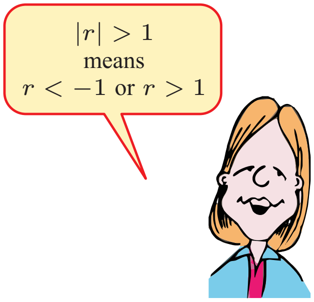
Sometimes it is necessary to consider $S_n = \dfrac{u_1(1 - r^n)}{1 - r}$ when n gets very large.

What happens to S_n in this situation?

If $|r| > 1$, the series is said to be **divergent** and the sum becomes infinitely large.

If $-1 < r < 1$, or in other words $|r| < 1$, then r^n approaches 0 for very large n.

This means that S_n will get closer and closer to $\dfrac{u_1}{1 - r}$.



We say that the series **converges** and we write its sum as

$$\boldsymbol{S = \frac{u_1}{1 - r} \quad \text{for} \quad |r| < 1.}$$

We call this the **limiting sum** of the series.

This result can be used to find the value of recurring decimals.

Example 19 **Self Tutor**

Write $0.\overline{7}$ as a rational number.

$0.\overline{7} = \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \frac{7}{10\,000} + ...$

which is a geometric series with infinitely many terms.

In this case, $u_1 = \frac{7}{10}$ and $r = \frac{1}{10}$

$$\therefore \quad S = \frac{u_1}{1-r} = \frac{\frac{7}{10}}{1-\frac{1}{10}} = \frac{7}{9}$$

So, $0.\overline{7} = \frac{7}{9}$.

EXERCISE 2E.4

1 Consider $0.\overline{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + ...$ which is an infinite geometric series.

- **a** Find: **i** u_1 **ii** r
- **b** Using **a**, show that $0.\overline{3} = \frac{1}{3}$.

2 Write as a rational number: **a** $0.\overline{4}$ **b** $0.\overline{16}$ **c** $0.\overline{312}$

3 Use $S = \dfrac{u_1}{1-r}$ to check your answer to **Exercise 2E.3** question **6d**.

4 Find the sum of each of the following infinite geometric series:

- **a** $18 + 12 + 8 + ...$
- **b** $18.9 - 6.3 + 2.1 - ......$

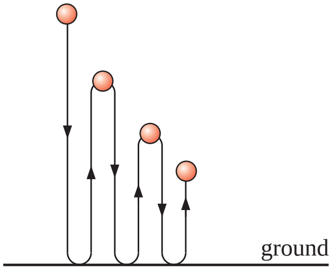
5 Find each of the following:

- **a** $\sum\limits_{k=1}^{\infty} \dfrac{3}{4^k}$
- **b** $\sum\limits_{k=0}^{\infty} 6\left(-\frac{2}{5}\right)^k$

6 The sum of the first three terms of a convergent geometric series is 19. The sum of the series is 27. Find the first term and the common ratio.

7 The second term of a convergent geometric series is $\frac{8}{5}$.
The sum of the series is 10. Show that there are two possible series and find the first term and the common ratio in each case.

8

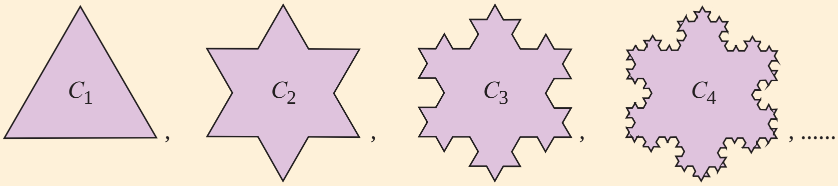


A ball takes 1 second to hit the ground when dropped. It then takes 90% of this time to rebound to its new height and this continues until the ball comes to rest.

- **a** Show that the total time of motion is given by $1 + 2(0.9) + 2(0.9)^2 + 2(0.9)^3 + ...$
- **b** Find S_n for the series in **a**.
- **c** How long does it take for the ball to come to rest?

Note: This diagram is inaccurate as the motion is really up and down on the same spot. It has been separated out to help us visualise what is happening.

INVESTIGATION VON KOCH'S SNOWFLAKE CURVE



In this investigation we consider a **limit curve** named after the Swedish mathematician Niels Fabian Helge von Koch (1870 - 1924).

To draw **Von Koch's Snowflake curve** we:

- start with an equilateral triangle, C_1
- then divide each side into 3 equal parts 
- then on each middle part draw an equilateral triangle 
- then delete the side of the smaller triangle which lies on C_1. 

DEMO

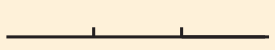
The resulting curve is C_2, and C_3, C_4, C_5, ... are found by 'pushing out' equilateral triangles on each edge of the previous curve as we did with C_1 to get C_2.

We get a sequence of special curves C_1, C_2, C_3, C_4, ... and Von Koch's curve is the limiting case when n is infinitely large.

Your task is to investigate the perimeter and area of Von Koch's curve.

What to do:

1 Suppose C_1 has a perimeter of 3 units. Find the perimeter of C_2, C_3, C_4 and C_5.

Hint:  becomes  so 3 parts become 4 parts.

Remembering that Von Koch's curve is C_n, where n is infinitely large, find the perimeter of Von Koch's curve.

2 Suppose the area of C_1 is 1 unit^2. Explain why the areas of C_2, C_3, C_4 and C_5 are

$A_2 = 1 + \frac{1}{3}$ units^2 $\qquad$ $A_3 = 1 + \frac{1}{3}[1 + \frac{4}{9}]$ units^2

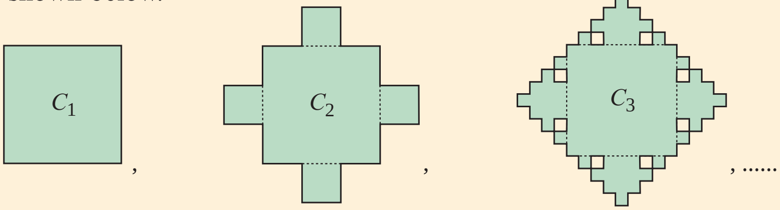
$A_4 = 1 + \frac{1}{3}[1 + \frac{4}{9} + (\frac{4}{9})^2]$ units^2 $\qquad$ $A_5 = 1 + \frac{1}{3}[1 + \frac{4}{9} + (\frac{4}{9})^2 + (\frac{4}{9})^3]$ units^2.

Use your calculator to find A_n where $n = 1, 2, 3, 4, 5, 6, 7$, etc., giving answers which are as accurate as your calculator permits.

What do you think will be the area within Von Koch's snowflake curve?

3 Is there anything remarkable about your answers to **1** and **2**?

4 Similarly, investigate the sequence of curves obtained by 'pushing out' squares on successive curves from the middle third of each side. These are the curves C_1, C_2, C_3, ... shown below.



REVIEW SET 2A NON-CALCULATOR

1 Identify the following sequences as arithmetic, geometric, or neither:

a $7, -1, -9, -17, ...$ **b** $9, 9, 9, 9, ...$ **c** $4, -2, 1, -\frac{1}{2}, ...$

d $1, 1, 2, 3, 5, 8, ...$ **e** the set of all multiples of 4 in ascending order.

2 Find k if $3k$, $k-2$ and $k+7$ are consecutive terms of an arithmetic sequence.

3 Show that $28, 23, 18, 13, ...$ is arithmetic. Hence find u_n and the sum S_n of the first n terms in simplest form.

4 Find k given that 4, k and k^2-1 are consecutive terms of a geometric sequence.

5 Determine the general term of a geometric sequence given that its sixth term is $\frac{16}{3}$ and its tenth term is $\frac{256}{3}$.

6 Insert six numbers between 23 and 9 so that all eight numbers are in arithmetic sequence.

7 Find the formula for u_n, the general term of:

a $86, 83, 80, 77, ...$ **b** $\frac{3}{4}, 1, \frac{7}{6}, \frac{9}{7}, ...$ **c** $100, 90, 81, 72.9, ...$

Hint: One of these sequences is neither arithmetic nor geometric.

8 Write down the expansion of: **a** $\sum_{k=1}^{7} k^2$ **b** $\sum_{k=1}^{8} \dfrac{k+3}{k+2}$

9 Find the sum of each of the following infinite geometric series:

a $18 - 12 + 8 - ......$ **b** $8 + 4\sqrt{2} + 4 + ...$

10 A ball bounces from a height of 2 metres and returns to 80% of its previous height on each bounce. Find the total distance travelled by the ball until it stops bouncing.

11 The sum of the first n terms of a sequence is $\dfrac{3n^2+5n}{2}$.

a Find the nth term. **b** Prove that the sequence is arithmetic.

REVIEW SET 2B CALCULATOR

1 A sequence is defined by $u_n = 6(\frac{1}{2})^{n-1}$.

a Prove that the sequence is geometric.

b Find u_1 and r.

c Find the 16th term to 3 significant figures.

2 **a** Determine the number of terms in the sequence $24, 23\frac{1}{4}, 22\frac{1}{2}, ..., -36$.

b Find the value of u_{35} for the sequence in **a**.

c Find the sum of the terms of the sequence in **a**.

3 Find the sum of:

a $3 + 9 + 15 + 21 + ...$ to 23 terms **b** $24 + 12 + 6 + 3 + ...$ to 12 terms.

4 Find the first term of the sequence 5, 10, 20, 40, ... which exceeds 10 000.

5 What will an investment of €6000 at 7% p.a. compound interest amount to after 5 years if the interest is compounded:

a annually **b** quarterly **c** monthly?

6 The nth term of a sequence is given by the formula $u_n = 5n - 8$.

a Find the value of u_{10}.

b Write down an expression for $u_{n+1} - u_n$ and simplify it.

c Use **b** to explain why the sequence is arithmetic.

d Evaluate $u_{15} + u_{16} + u_{17} + ... + u_{30}$.

7 A geometric sequence has $u_6 = 24$ and $u_{11} = 768$. Determine the general term of the sequence and hence find:

a u_{17} **b** the sum of the first 15 terms.

8 Find the first term of the sequence $24, 8, \frac{8}{3}, \frac{8}{9}, ...$ which is less than 0.001.

9 **a** Determine the number of terms in the sequence $128, 64, 32, 16, ..., \frac{1}{512}$.

b Find the sum of these terms.

10 Find the sum of each of the following infinite geometric series:

a $1.21 - 1.1 + 1 - ......$ **b** $\frac{14}{3} + \frac{4}{3} + \frac{8}{21} + ...$

11 How much should be invested at a fixed rate of 9% p.a. compounded interest if you need it to amount to \$20 000 after 4 years with interest paid monthly?

12 In 2004 there were 3000 koalas on Koala Island. Since then, the population of koalas on the island has increased by 5% each year.

a How many koalas were on the island in 2007?

b In what year will the population first exceed 5000?

REVIEW SET 2C

1 A sequence is defined by $u_n = 68 - 5n$.

a Prove that the sequence is arithmetic. **b** Find u_1 and d.

c Find the 37th term. **d** State the first term of the sequence less than -200.

2 **a** Show that the sequence 3, 12, 48, 192, ... is geometric.

b Find u_n and hence find u_9.

3 Find the general term of the arithmetic sequence with $u_7 = 31$ and $u_{15} = -17$. Hence, find the value of u_{34}.

4 Write in the form $\sum_{k=1}^{n} (.....)$:

a $4 + 11 + 18 + 25 + ...$ for n terms **b** $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + ...$ for n terms.

5 Evaluate: **a** $\sum\limits_{k=1}^{8}\left(\dfrac{31-3k}{2}\right)$ **b** $\sum\limits_{k=1}^{15} 50(0.8)^{k-1}$

6 Find $\sum\limits_{k=7}^{\infty} 5\left(\frac{2}{5}\right)^{k-1}$.

7 How many terms of the series $11+16+21+26+....$ are needed to exceed a sum of 450?

8 £12 500 is invested in an account which pays 8.25% p.a. compounded. Find the value of the investment after 5 years if the interest is compounded:

a half-yearly **b** monthly.

9 The sum of the first two terms of a geometric series is 90. The third term is 24.

a Show that there are two possible series and find the first term and the common ratio in each case.

b Show that both series converge and find their respective sums.

10 Seve is training for a long distance walk. He walks for 10 km in the first week, then each week thereafter he walks an additional 500 m. If he continues this pattern for a year, how far does he walk:

a in the last week **b** in total?

11 **a** Under what conditions will the series $\sum\limits_{k=1}^{\infty} 50(2x-1)^{k-1}$ converge? Explain your answer.

b Find $\sum\limits_{k=1}^{\infty} 50(2x-1)^{k-1}$ if $x=0.3$.

Chapter 3

Exponentials

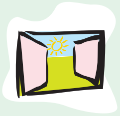
Syllabus reference: 1.2, 2.2, 2.7, 2.8

Contents:

We often deal with numbers that are repeatedly multiplied together. Mathematicians use **indices** or **exponents** to represent such expressions. For example, $5 \times 5 \times 5 = 5^3$.

Indices have many applications in areas such as finance, engineering, physics, electronics, biology and computer science. Problems encountered in these areas may involve situations where quantities increase or decrease over time. Such problems are often examples of **exponential growth** or **decay**.

OPENING PROBLEM LEGEND OF THE AMBALAPPUZHA PAAL PAYASAM

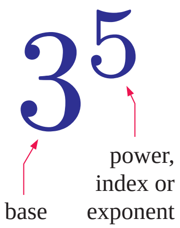


According to Hindu legend, Lord Krishna once appeared as a sage before the king who ruled a region of India, and challenged him to a game of chess. The prize if Lord Krishna won was based on the chessboard: that the king would provide him with a single grain of rice for the first square, two grains of rice for the second square, four grains of rice for the third square, and so on doubling the rice on each successive square on the board. Lord Krishna of course did win, and the king was most unhappy when he realised he owed more rice than there was in the world.

Things to think about:

- **a** Is there a function which describes the number of grains of rice on each square?
- **b** How many grains of rice would there be on the 40th square?
- **c** Using your knowledge of series, find the total number of grains of rice that the king owed.

A INDEX NOTATION



Rather than writing $3 \times 3 \times 3 \times 3 \times 3$, we can write such a product as 3^5.

3^5 reads 'three to the power of five' or 'three with index five'.

Thus $4^3 = 4 \times 4 \times 4$ and $5^6 = 5 \times 5 \times 5 \times 5 \times 5 \times 5$.

If n is a positive integer, then a^n is the product of n factors of a.

$$a^n = \underbrace{a \times a \times a \times a \times \ ...... \ \times a}_{n \text{ factors}}$$

EXERCISE 3A

1 List the first six powers of:

a 2 **b** 3 **c** 4

2 Copy and complete the values of these common powers:

a $5^1 = ...$, $5^2 = ...$, $5^3 = ...$, $5^4 = ...$

b $6^1 = ...$, $6^2 = ...$, $6^3 = ...$, $6^4 = ...$

c $7^1 = ...$, $7^2 = ...$, $7^3 = ...$, $7^4 = ...$

HISTORICAL NOTE

Nicomachus discovered an interesting number pattern involving cubes and sums of odd numbers. Nicomachus was born in Roman Syria (now Jerash, Jordan) around 100 AD. He wrote in Greek and was a Pythagorean.

$$\begin{aligned} 1 &= 1^3 \\ 3+5 = 8 &= 2^3 \\ 7+9+11 = 27 &= 3^3 \\ &\vdots \end{aligned}$$

B EVALUATING POWERS

So far we have only considered **positive** bases raised to a power.

We will now briefly look at **negative** bases. Consider the statements below:

$$\begin{aligned} (-1)^1 &= -1 & (-2)^1 &= -2 \\ (-1)^2 &= -1 \times -1 = 1 & (-2)^2 &= -2 \times -2 = 4 \\ (-1)^3 &= -1 \times -1 \times -1 = -1 & (-2)^3 &= -2 \times -2 \times -2 = -8 \\ (-1)^4 &= -1 \times -1 \times -1 \times -1 = 1 & (-2)^4 &= -2 \times -2 \times -2 \times -2 = 16 \end{aligned}$$

From the patterns above we can see that:

A **negative** base raised to an **odd** power is **negative**.
A **negative** base raised to an **even** power is **positive**.

CALCULATOR USE

Although different calculators vary in the appearance of keys, they all perform operations of raising to powers in a similar manner.

Power keys	$\boxed{x^2}$	squares the number in the display.
	$\boxed{\wedge}$ 3	raises the number in the display to the power 3.
	$\boxed{\wedge}$ 5	raises the number in the display to the power 5.
	$\boxed{\wedge}$ $\boxed{(-)}$ 4	raises the number in the display to the power -4.

Example 1 Self Tutor

Find, using your calculator: **a** 6^5 **b** $(-5)^4$ **c** -7^4

			Answer
a	Press:	6 $\boxed{\wedge}$ 5 $\boxed{\text{ENTER}}$	7776
b	Press:	$\boxed{(}$ $\boxed{(-)}$ 5 $\boxed{)}$ $\boxed{\wedge}$ 4 $\boxed{\text{ENTER}}$	625
c	Press:	$\boxed{(-)}$ 7 $\boxed{\wedge}$ 4 $\boxed{\text{ENTER}}$	-2401

You will need to check if your calculator uses the same key sequence as in the examples. If not, work out the sequence which gives you the correct answers.

Example 2 **Self Tutor**

Find using your calculator, and comment on: **a** 5^{-2} **b** $\frac{1}{5^2}$

Answer

a Press:	$5\ \boxed{\wedge}\ \boxed{(-)}\ 2\ \boxed{\text{ENTER}}$	0.04	The answers indicate
b Press:	$1\ \boxed{\div}\ 5\ \boxed{x^2}\ \boxed{\text{ENTER}}$	0.04	that $5^{-2} = \frac{1}{5^2}$.

EXERCISE 3B

1 Simplify, then use a calculator to check your answer:

a $(-1)^5$ **b** $(-1)^6$ **c** $(-1)^{14}$ **d** $(-1)^{19}$

e $(-1)^8$ **f** -1^8 **g** $-(-1)^8$ **h** $(-2)^5$

i -2^5 **j** $-(-2)^6$ **k** $(-5)^4$ **l** $-(-5)^4$

2 Use your calculator to find the value of the following, recording the entire display:

a 4^7 **b** 7^4 **c** -5^5 **d** $(-5)^5$ **e** 8^6

f $(-8)^6$ **g** -8^6 **h** 2.13^9 **i** -2.13^9 **j** $(-2.13)^9$

3 Use your calculator to find the values of the following:

a 9^{-1} **b** $\frac{1}{9^1}$ **c** 6^{-2} **d** $\frac{1}{6^2}$

e 3^{-4} **f** $\frac{1}{3^4}$ **g** 17^0 **h** $(0.366)^0$

What do you notice?

4 Consider $3^1, 3^2, 3^3, 3^4, 3^5$ .... Look for a pattern and find the last digit of 3^{101}.

5 What is the last digit of 7^{217}?

6 Answer the **Opening Problem** on page **94**.

C INDEX LAWS

The following are **laws of indices** for $m, n \in \mathbb{Z}$:

$a^m \times a^n = a^{m+n}$	To **multiply** numbers with the **same base**, keep the base and **add** the indices.
$\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$	To **divide** numbers with the same base, keep the base and **subtract** the indices.
$(a^m)^n = a^{m \times n}$	When **raising** a **power** to a **power**, keep the base and **multiply** the indices.
$(ab)^n = a^n b^n$	The power of a product is the product of the powers.

$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, $b \neq 0$ The power of a quotient is the quotient of the powers.

$a^0 = 1$, $a \neq 0$ Any non-zero number raised to the power of zero is 1.

$a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$ and in particular $a^{-1} = \frac{1}{a}$, $a \neq 0$.

Example 3

Self Tutor

Simplify using the index laws:

a $3^5 \times 3^4$ **b** $\frac{5^3}{5^5}$ **c** $(m^4)^3$

a $3^5 \times 3^4$
$= 3^{5+4}$
$= 3^9$

b $\frac{5^3}{5^5}$
$= 5^{3-5}$
$= 5^{-2}$
$= \frac{1}{25}$

c $(m^4)^3$
$= m^{4\times 3}$
$= m^{12}$

Example 4

Self Tutor

Write as powers of 2:

a 16 **b** $\frac{1}{16}$ **c** 1 **d** 4×2^n **e** $\frac{2^m}{8}$

a 16
$= 2 \times 2 \times 2 \times 2$
$= 2^4$

b $\frac{1}{16}$
$= \frac{1}{2^4}$
$= 2^{-4}$

c 1
$= 2^0$

d 4×2^n
$= 2^2 \times 2^n$
$= 2^{2+n}$

e $\frac{2^m}{8}$
$= \frac{2^m}{2^3}$
$= 2^{m-3}$

Example 5

Self Tutor

Write in simplest form, without brackets:

a $(-3a^2)^4$ **b** $\left(-\frac{2a^2}{b}\right)^3$

a $(-3a^2)^4$
$= (-3)^4 \times (a^2)^4$
$= 81 \times a^{2\times 4}$
$= 81a^8$

b $\left(-\frac{2a^2}{b}\right)^3$
$= \frac{(-2)^3 \times (a^2)^3}{b^3}$
$= \frac{-8a^6}{b^3}$

Example 6

Self Tutor

Write without negative indices: $\dfrac{a^{-3}b^2}{c^{-1}}$

$a^{-3} = \dfrac{1}{a^3}$ and $\dfrac{1}{c^{-1}} = c^1$

$\therefore \dfrac{a^{-3}b^2}{c^{-1}} = \dfrac{b^2c}{a^3}$

Example 7

Self Tutor

Write $\dfrac{1}{2^{1-n}}$ in non-fractional form.

$$\begin{aligned}\frac{1}{2^{1-n}} &= 2^{-(1-n)}\\ &= 2^{-1+n}\\ &= 2^{n-1}\end{aligned}$$

EXERCISE 3C

1 Simplify using the index laws:

a $5^4 \times 5^7$
b $d^2 \times d^6$
c $\dfrac{k^8}{k^3}$
d $\dfrac{7^5}{7^6}$
e $\left(x^2\right)^5$
f $\left(3^4\right)^4$
g $\dfrac{p^3}{p^7}$
h $n^3 \times n^9$
i $\left(5^t\right)^3$
j $7^x \times 7^2$
k $\dfrac{10^3}{10^q}$
l $\left(c^4\right)^m$

2 Write as powers of 2:

a 4
b $\frac{1}{4}$
c 8
d $\frac{1}{8}$
e 32
f $\frac{1}{32}$
g 2
h $\frac{1}{2}$
i 64
j $\frac{1}{64}$
k 128
l $\frac{1}{128}$

3 Write as powers of 3:

a 9
b $\frac{1}{9}$
c 27
d $\frac{1}{27}$
e 3
f $\frac{1}{3}$
g 81
h $\frac{1}{81}$
i 1
j 243
k $\frac{1}{243}$

4 Write as a single power of 2:

a 2×2^a
b 4×2^b
c 8×2^t
d $(2^{x+1})^2$
e $(2^{1-n})^{-1}$
f $\dfrac{2^c}{4}$
g $\dfrac{2^m}{2^{-m}}$
h $\dfrac{4}{2^{1-n}}$
i $\dfrac{2^{x+1}}{2^x}$
j $\dfrac{4^x}{2^{1-x}}$

5 Write as a single power of 3:

a 9×3^p
b 27^a
c 3×9^n
d 27×3^d
e 9×27^t
f $\dfrac{3^y}{3}$
g $\dfrac{3}{3^y}$
h $\dfrac{9}{27^t}$
i $\dfrac{9^a}{3^{1-a}}$
j $\dfrac{9^{n+1}}{3^{2n-1}}$

6 Write without brackets:

a $(2a)^2$
b $(3b)^3$
c $(ab)^4$
d $(pq)^3$
e $\left(\dfrac{m}{n}\right)^2$
f $\left(\dfrac{a}{3}\right)^3$
g $\left(\dfrac{b}{c}\right)^4$
h $\left(\dfrac{2a}{b}\right)^0$
i $\left(\dfrac{m}{3n}\right)^4$
j $\left(\dfrac{xy}{2}\right)^3$

7 Write the following in simplest form, without brackets:

a $(-2a)^2$ **b** $(-6b^2)^2$ **c** $(-2a)^3$ **d** $(-3m^2n^2)^3$

e $(-2ab^4)^4$ **f** $\left(\dfrac{-2a^2}{b^2}\right)^3$ **g** $\left(\dfrac{-4a^3}{b}\right)^2$ **h** $\left(\dfrac{-3p^2}{q^3}\right)^2$

8 Write without negative indices:

a ab^{-2} **b** $(ab)^{-2}$ **c** $(2ab^{-1})^2$ **d** $(3a^{-2}b)^2$ **e** $\dfrac{a^2b^{-1}}{c^2}$

f $\dfrac{a^2b^{-1}}{c^{-2}}$ **g** $\dfrac{1}{a^{-3}}$ **h** $\dfrac{a^{-2}}{b^{-3}}$ **i** $\dfrac{2a^{-1}}{d^2}$ **j** $\dfrac{12a}{m^{-3}}$

9 Write in non-fractional form:

a $\dfrac{1}{a^n}$ **b** $\dfrac{1}{b^{-n}}$ **c** $\dfrac{1}{3^{2-n}}$ **d** $\dfrac{a^n}{b^{-m}}$ **e** $\dfrac{a^{-n}}{a^{2+n}}$

10 Simplify, giving answers in simplest rational form:

a $\left(\frac{5}{3}\right)^0$ **b** $\left(\frac{7}{4}\right)^{-1}$ **c** $\left(\frac{1}{6}\right)^{-1}$ **d** $\dfrac{3^3}{3^0}$

e $\left(\frac{4}{3}\right)^{-2}$ **f** $2^1 + 2^{-1}$ **g** $\left(1\frac{2}{3}\right)^{-3}$ **h** $5^2 + 5^1 + 5^{-1}$

11 Write as powers of 2, 3 and/or 5:

a $\frac{1}{9}$ **b** $\frac{1}{16}$ **c** $\frac{1}{125}$ **d** $\frac{3}{5}$

e $\frac{4}{27}$ **f** $\dfrac{2^c}{8 \times 9}$ **g** $\dfrac{9^k}{10}$ **h** $\dfrac{6^p}{75}$

12 Read about Nicomachus' pattern on page **95** and find the series of odd numbers for:

a 5^3 **b** 7^3 **c** 12^3

D RATIONAL INDICES

The index laws used previously can also be applied to **rational indices**, or indices which are written as a fraction.

Notice that $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$ {index laws}

and $\sqrt{a} \times \sqrt{a} = a$ also.

So, $\boxed{a^{\frac{1}{2}} = \sqrt{a}}$ {by direct comparison}

Likewise $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^1 = a$

and $\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a$

suggests $\boxed{a^{\frac{1}{3}} = \sqrt[3]{a}}$

In general, $\boxed{a^{\frac{1}{n}} = \sqrt[n]{a}}$ where $\sqrt[n]{a}$ reads 'the nth root of a'.

Notice also that
$$a^{\frac{2}{3}} \times a^{\frac{2}{3}} \times a^{\frac{2}{3}} = a^2$$
$$\left(a^{\frac{2}{3}}\right)^3 = a^2$$
$$\therefore \quad a^{\frac{2}{3}} = \sqrt[3]{a^2}$$

In general, $\boxed{a^{\frac{m}{n}} = \sqrt[n]{a^m}}$

Example 8 **Self Tutor**

Write as a single power of 2: **a** $\sqrt[3]{2}$ **b** $\dfrac{1}{\sqrt{2}}$ **c** $\sqrt[5]{4}$

a $\sqrt[3]{2}$
$= 2^{\frac{1}{3}}$

b $\dfrac{1}{\sqrt{2}}$
$= \dfrac{1}{2^{\frac{1}{2}}}$
$= 2^{-\frac{1}{2}}$

c $\sqrt[5]{4}$
$= (2^2)^{\frac{1}{5}}$
$= 2^{2\times\frac{1}{5}}$
$= 2^{\frac{2}{5}}$

Example 9 **Self Tutor**

Use your calculator to evaluate: **a** $2^{\frac{7}{5}}$ **b** $\dfrac{1}{\sqrt[3]{4}}$.

		Calculator:	*Answer:*
a	For $2^{\frac{7}{5}}$ press:	2 ^ (7 ÷ 5) ENTER	$\approx 2.639\,015$
b	$\dfrac{1}{\sqrt[3]{4}} = 4^{-\frac{1}{3}}$	4 ^ ((−) 1 ÷ 3) ENTER	$\approx 0.629\,961$

Example 10 **Self Tutor**

Without using a calculator, write in simplest rational form: **a** $8^{\frac{4}{3}}$ **b** $27^{-\frac{2}{3}}$

a $8^{\frac{4}{3}}$
$= (2^3)^{\frac{4}{3}}$
$= 2^{3\times\frac{4}{3}}$ $\{(a^m)^n = a^{mn}\}$
$= 2^4$
$= 16$

b $27^{-\frac{2}{3}}$
$= (3^3)^{-\frac{2}{3}}$
$= 3^{3\times-\frac{2}{3}}$
$= 3^{-2}$
$= \frac{1}{9}$

EXERCISE 3D

1 Write as a single power of 2:

a $\sqrt[5]{2}$ **b** $\dfrac{1}{\sqrt[5]{2}}$ **c** $2\sqrt{2}$ **d** $4\sqrt{2}$ **e** $\dfrac{1}{\sqrt[3]{2}}$

f $2 \times \sqrt[3]{2}$ **g** $\dfrac{4}{\sqrt{2}}$ **h** $(\sqrt{2})^3$ **i** $\dfrac{1}{\sqrt[3]{16}}$ **j** $\dfrac{1}{\sqrt{8}}$

2 Write as a single power of 3:

a $\sqrt[3]{3}$ **b** $\dfrac{1}{\sqrt[3]{3}}$ **c** $\sqrt[4]{3}$ **d** $3\sqrt{3}$ **e** $\dfrac{1}{9\sqrt{3}}$

3 Write the following in the form a^x where a is a prime number and x is rational:

a $\sqrt[3]{7}$ **b** $\sqrt[4]{27}$ **c** $\sqrt[5]{16}$ **d** $\sqrt[3]{32}$ **e** $\sqrt[7]{49}$

f $\dfrac{1}{\sqrt[3]{7}}$ **g** $\dfrac{1}{\sqrt[4]{27}}$ **h** $\dfrac{1}{\sqrt[5]{16}}$ **i** $\dfrac{1}{\sqrt[3]{32}}$ **j** $\dfrac{1}{\sqrt[7]{49}}$

4 Use your calculator to find:

a $3^{\frac{3}{4}}$ **b** $2^{\frac{7}{8}}$ **c** $2^{-\frac{1}{3}}$ **d** $4^{-\frac{3}{5}}$ **e** $\sqrt[4]{8}$

f $\sqrt[5]{27}$ **g** $\dfrac{1}{\sqrt[3]{7}}$

5 Without using a calculator, write in simplest rational form:

a $4^{\frac{3}{2}}$ **b** $8^{\frac{5}{3}}$ **c** $16^{\frac{3}{4}}$ **d** $25^{\frac{3}{2}}$ **e** $32^{\frac{2}{5}}$

f $4^{-\frac{1}{2}}$ **g** $9^{-\frac{3}{2}}$ **h** $8^{-\frac{4}{3}}$ **i** $27^{-\frac{4}{3}}$ **j** $125^{-\frac{2}{3}}$

E ALGEBRAIC EXPANSION AND FACTORISATION

We can use the usual expansion laws to simplify expressions containing indices:

$$\begin{aligned}
(a+b)(c+d) &= ac+ad+bc+bd\\
(a+b)(a-b) &= a^2-b^2\\
(a+b)^2 &= a^2+2ab+b^2\\
(a-b)^2 &= a^2-2ab+b^2
\end{aligned}$$

Example 11 **Self Tutor**

Expand and simplify: $x^{-\frac{1}{2}}(x^{\frac{3}{2}}+2x^{\frac{1}{2}}-3x^{-\frac{1}{2}})$

$x^{-\frac{1}{2}}(x^{\frac{3}{2}}+2x^{\frac{1}{2}}-3x^{-\frac{1}{2}})$

$= x^{-\frac{1}{2}}\times x^{\frac{3}{2}}+x^{-\frac{1}{2}}\times 2x^{\frac{1}{2}}-x^{-\frac{1}{2}}\times 3x^{-\frac{1}{2}}$ {each term is $\times$ by $x^{-\frac{1}{2}}$}

$= x^1+2x^0-3x^{-1}$ {adding indices}

$= x+2-\dfrac{3}{x}$

Example 12 **Self Tutor**

Expand and simplify: **a** $(2^x+3)(2^x+1)$ **b** $(7^x+7^{-x})^2$

a
$$\begin{aligned}&(2^x+3)(2^x+1)\\&=2^x\times2^x+2^x+3\times2^x+3\\&=2^{2x}+4\times2^x+3\\&=4^x+2^{2+x}+3\end{aligned}$$

b
$$\begin{aligned}&(7^x+7^{-x})^2\\&=(7^x)^2+2\times7^x\times7^{-x}+(7^{-x})^2\\&=7^{2x}+2\times7^0+7^{-2x}\\&=7^{2x}+2+7^{-2x}\end{aligned}$$

EXERCISE 3E.1

1 Expand and simplify:

- **a** $x^2(x^3+2x^2+1)$
- **b** $2^x(2^x+1)$
- **c** $x^{\frac{1}{2}}(x^{\frac{1}{2}}+x^{-\frac{1}{2}})$
- **d** $7^x(7^x+2)$
- **e** $3^x(2-3^{-x})$
- **f** $x^{\frac{1}{2}}(x^{\frac{3}{2}}+2x^{\frac{1}{2}}+3x^{-\frac{1}{2}})$
- **g** $2^{-x}(2^x+5)$
- **h** $5^{-x}(5^{2x}+5^x)$
- **i** $x^{-\frac{1}{2}}(x^2+x+x^{\frac{1}{2}})$

2 Expand and simplify:

- **a** $(2^x+1)(2^x+3)$
- **b** $(3^x+2)(3^x+5)$
- **c** $(5^x-2)(5^x-4)$
- **d** $(2^x+3)^2$
- **e** $(3^x-1)^2$
- **f** $(4^x+7)^2$
- **g** $(x^{\frac{1}{2}}+2)(x^{\frac{1}{2}}-2)$
- **h** $(2^x+3)(2^x-3)$
- **i** $(x^{\frac{1}{2}}+x^{-\frac{1}{2}})(x^{\frac{1}{2}}-x^{-\frac{1}{2}})$
- **j** $(x+\frac{2}{x})^2$
- **k** $(7^x-7^{-x})^2$
- **l** $(5-2^{-x})^2$

FACTORISATION AND SIMPLIFICATION

Example 13 **Self Tutor**

Factorise: **a** $2^{n+3}+2^n$ **b** $2^{n+3}+8$ **c** $2^{3n}+2^{2n}$

a
$$\begin{aligned}&2^{n+3}+2^n\\&=2^n2^3+2^n\\&=2^n(2^3+1)\\&=2^n\times9\end{aligned}$$

b
$$\begin{aligned}&2^{n+3}+8\\&=2^n2^3+8\\&=8(2^n)+8\\&=8(2^n+1)\end{aligned}$$

c
$$\begin{aligned}&2^{3n}+2^{2n}\\&=2^{2n}2^n+2^{2n}\\&=2^{2n}(2^n+1)\end{aligned}$$

Example 14 **Self Tutor**

Factorise: **a** 4^x-9 **b** $9^x+4(3^x)+4$

a
$$\begin{aligned}&4^x-9\\&=(2^x)^2-3^2 &&\{\text{difference of two squares}\}\\&=(2^x+3)(2^x-3)\end{aligned}$$

b
$$\begin{aligned}&9^x+4(3^x)+4\\&=(3^x)^2+4(3^x)+4 &&\{\text{compare } a^2+4a+4\}\\&=(3^x+2)^2 &&\{\text{as } a^2+4a+4=(a+2)^2\}\end{aligned}$$

Example 15

Self Tutor

Simplify: **a** $\dfrac{6^n}{3^n}$ **b** $\dfrac{4^n}{6^n}$

a $\dfrac{6^n}{3^n}$ **or** $\dfrac{6^n}{3^n}$

$$= \frac{2^n\cancel{3^n}}{\cancel{3^n}_1} \qquad = \left(\frac{6}{3}\right)^n$$

$$= 2^n \qquad = 2^n$$

b $\dfrac{4^n}{6^n}$ **or** $\dfrac{4^n}{6^n}$

$$= \frac{\cancel{2^n}2^n}{\cancel{2^n}3^n} \qquad = \left(\frac{4}{6}\right)^n$$

$$= \frac{2^n}{3^n} \qquad = \left(\frac{2}{3}\right)^n$$

Example 16

Self Tutor

Simplify: **a** $\dfrac{3^n+6^n}{3^n}$ **b** $\dfrac{2^{m+2}-2^m}{2^m}$ **c** $\dfrac{2^{m+3}+2^m}{9}$

a

$$\frac{3^n+6^n}{3^n}$$
$$= \frac{3^n+2^n3^n}{3^n}$$
$$= \frac{\cancel{3^n}(1+2^n)}{\cancel{3^n}_1}$$
$$= 1+2^n$$

b

$$\frac{2^{m+2}-2^m}{2^m}$$
$$= \frac{2^m2^2-2^m}{2^m}$$
$$= \frac{\cancel{2^m}(4-1)}{\cancel{2^m}_1}$$
$$= 3$$

c

$$\frac{2^{m+3}+2^m}{9}$$
$$= \frac{2^m2^3+2^m}{9}$$
$$= \frac{2^m\cancel{(8+1)}}{\cancel{9}_1}$$
$$= 2^m$$

EXERCISE 3E.2

1 Factorise:

a $5^{2x}+5^x$ **b** $3^{n+2}+3^n$ **c** 7^n+7^{3n}

d $5^{n+1}-5$ **e** $6^{n+2}-6$ **f** $4^{n+2}-16$

2 Factorise:

a 9^x-4 **b** 4^x-25 **c** $16-9^x$

d $25-4^x$ **e** 9^x-4^x **f** $4^x+6(2^x)+9$

g $9^x+10(3^x)+25$ **h** $4^x-14(2^x)+49$ **i** $25^x-4(5^x)+4$

3 Factorise:

a $4^x+9(2^x)+18$ **b** 4^x-2^x-20 **c** $9^x+9(3^x)+14$

d $9^x+4(3^x)-5$ **e** 25^x+5^x-2 **f** $49^x-7^{x+1}+12$

4 Simplify:

a $\dfrac{12^n}{6^n}$ **b** $\dfrac{20^a}{2^a}$ **c** $\dfrac{6^b}{2^b}$ **d** $\dfrac{4^n}{20^n}$

e $\dfrac{35^x}{7^x}$ **f** $\dfrac{6^a}{8^a}$ **g** $\dfrac{5^{n+1}}{5^n}$ **h** $\dfrac{5^{n+1}}{5}$

5 Simplify:

a $\dfrac{6^m + 2^m}{2^m}$ **b** $\dfrac{2^n + 12^n}{2^n}$ **c** $\dfrac{8^n + 4^n}{2^n}$

d $\dfrac{12^x - 3^x}{3^x}$ **e** $\dfrac{6^n + 12^n}{1 + 2^n}$ **f** $\dfrac{5^{n+1} - 5^n}{4}$

g $\dfrac{5^{n+1} - 5^n}{5^n}$ **h** $\dfrac{4^n - 2^n}{2^n}$ **i** $\dfrac{2^n - 2^{n-1}}{2^n}$

6 Simplify:

a $2^n(n+1) + 2^n(n-1)$ **b** $3^n\left(\dfrac{n-1}{6}\right) - 3^n\left(\dfrac{n+1}{6}\right)$

F EXPONENTIAL EQUATIONS

An **exponential equation** is an equation in which the unknown occurs as part of the index or exponent.

For example: $2^x = 8$ and $30 \times 3^x = 7$ are both exponential equations.

If $2^x = 8$ then $2^x = 2^3$. Thus $x = 3$, and this is the only solution.

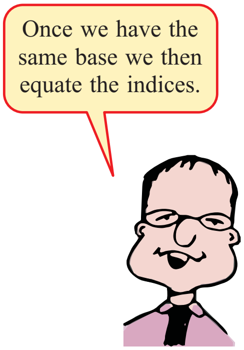
> If $a^x = a^k$ then $x = k$.
> So, if the base numbers are the same, we can **equate indices**.

Example 17 **Self Tutor**

Solve for x: **a** $2^x = 16$ **b** $3^{x+2} = \frac{1}{27}$

a
$$\begin{aligned} 2^x &= 16 \\ \therefore\ 2^x &= 2^4 \\ \therefore\ x &= 4 \end{aligned}$$

b
$$\begin{aligned} 3^{x+2} &= \tfrac{1}{27} \\ \therefore\ 3^{x+2} &= 3^{-3} \\ \therefore\ x+2 &= -3 \\ \therefore\ x &= -5 \end{aligned}$$



Example 18 **Self Tutor**

Solve for x:

a $4^x = 8$

b $9^{x-2} = \frac{1}{3}$

a
$$\begin{aligned} 4^x &= 8 \\ \therefore\ (2^2)^x &= 2^3 \\ \therefore\ 2^{2x} &= 2^3 \\ \therefore\ 2x &= 3 \\ \therefore\ x &= \tfrac{3}{2} \end{aligned}$$

b
$$\begin{aligned} 9^{x-2} &= \tfrac{1}{3} \\ \therefore\ (3^2)^{x-2} &= 3^{-1} \\ \therefore\ 3^{2(x-2)} &= 3^{-1} \\ \therefore\ 2x-4 &= -1 \\ \therefore\ 2x &= 3 \\ \therefore\ x &= \tfrac{3}{2} \end{aligned}$$

Example 19 **Self Tutor**

Solve for x: $4^x + 2^x - 20 = 0$

$$\begin{aligned} 4^x + 2^x - 20 &= 0 \\ \therefore\ (2^x)^2 + 2^x - 20 &= 0 && \{\text{compare } a^2 + a - 20 = 0\} \\ \therefore\ (2^x - 4)(2^x + 5) &= 0 && \{\text{as } a^2 + a - 20 = (a-4)(a+5)\} \\ \therefore\ 2^x = 4 \text{ or } 2^x &= -5 \\ \therefore\ 2^x &= 2^2 && \{2^x \text{ cannot be negative}\} \\ \therefore\ x &= 2 \end{aligned}$$

EXERCISE 3F

1 Solve for x:

a $2^x = 8$ **b** $5^x = 25$ **c** $3^x = 81$ **d** $7^x = 1$

e $3^x = \frac{1}{3}$ **f** $4^x = \frac{1}{16}$ **g** $5^x = \frac{1}{125}$ **h** $4^{x+1} = 64$

i $2^{x-2} = \frac{1}{32}$ **j** $3^{x+1} = \frac{1}{27}$ **k** $7^{x+1} = 343$ **l** $5^{1-2x} = \frac{1}{5}$

2 Solve for x:

a $8^x = 32$ **b** $4^x = \frac{1}{8}$ **c** $9^x = \frac{1}{27}$ **d** $25^x = \frac{1}{5}$

e $27^x = \frac{1}{9}$ **f** $16^x = \frac{1}{32}$ **g** $4^{x+2} = 128$ **h** $25^{1-x} = \frac{1}{125}$

i $4^{4x-1} = \frac{1}{2}$ **j** $9^{x-3} = 27$ **k** $(\frac{1}{2})^{x+1} = 8$ **l** $(\frac{1}{3})^{x+2} = 9$

m $81^x = 27^{-x}$ **n** $(\frac{1}{4})^{1-x} = 32$ **o** $(\frac{1}{7})^x = 49$ **p** $(\frac{1}{3})^{x+1} = 243$

3 Solve for x:

a $4^{2x+1} = 8^{1-x}$ **b** $9^{2-x} = (\frac{1}{3})^{2x+1}$ **c** $2^x \times 8^{1-x} = \frac{1}{4}$

4 Solve for x:

a $3 \times 2^x = 24$ **b** $7 \times 2^x = 56$ **c** $3 \times 2^{x+1} = 24$

d $12 \times 3^{-x} = \frac{4}{3}$ **e** $4 \times (\frac{1}{3})^x = 36$ **f** $5 \times (\frac{1}{2})^x = 20$

5 Solve for x:

a $4^x - 6(2^x) + 8 = 0$ **b** $4^x - 2^x - 2 = 0$ **c** $9^x - 12(3^x) + 27 = 0$

d $9^x = 3^x + 6$ **e** $25^x - 23(5^x) - 50 = 0$ **f** $49^x + 1 = 2(7^x)$

G GRAPHS OF EXPONENTIAL FUNCTIONS

We have learned to deal with b^n where $n \in \mathbb{Q}$, or in other words n is any rational number.

But what about b^n where $n \in \mathbb{R}$, so n is real but not necessarily rational?

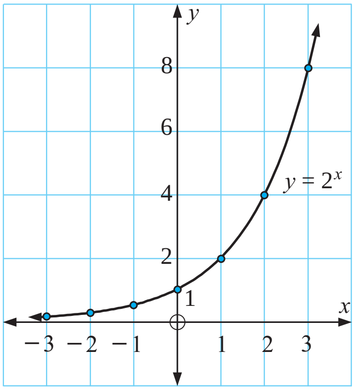
To answer this question, we can look at graphs of exponential functions.

The most simple general **exponential function** has the form $y = b^x$ where $b > 0$, $b \neq 1$.

For example, $y = 2^x$ is an exponential function.

We construct a table of values from which we graph the function:

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



We notice that when $x = -10$, $y = 2^{-10} \approx 0.001$ and when $x = -50$, $y = 2^{-50} \approx 8.88 \times 10^{-16}$.

As x becomes large and negative, the graph of $y = 2^x$ approaches the x-axis from above it.

We say that $y = 2^x$ is '**asymptotic** to the x-axis' or '$y = 0$ is a **horizontal asymptote**'.

INVESTIGATION 1 EXPONENTIAL GRAPHS

The object of this investigation is to examine the graphs of various families of exponential functions.

GRAPHING PACKAGE

What to do:

1 **a** On the same set of axes, use a **graphing package** or **graphics calculator** to graph the following functions: $y = 2^x$, $y = 3^x$, $y = 10^x$, $y = (1.3)^x$.

b The functions in **a** are all members of the family $y = b^x$.

- **i** What effect does changing b have on the shape of the graph?
- **ii** What is the y-intercept of each graph?
- **iii** What is the horizontal asymptote of each graph?

2 **a** On the same set of axes, use a **graphing package** or **graphics calculator** to graph the following functions: $y = 2^x$, $y = 2^x + 1$, $y = 2^x - 2$.

b The functions in **a** are all members of the family $y = 2^x + d$ where d is a constant.

- **i** What effect does changing d have on the position of the graph?
- **ii** What effect does changing d have on the shape of the graph?
- **iii** What is the horizontal asymptote of each graph?
- **iv** What is the horizontal asymptote of $y = 2^x + d$?

c To graph $y = 2^x + d$ from $y = 2^x$ what transformation is used?

3 **a** On the same set of axes, use a **graphing package** or **graphics calculator** to graph the following functions: $y = 2^x$, $y = 2^{x-1}$, $y = 2^{x+2}$, $y = 2^{x-3}$.

b The functions in **a** are all members of the family $y = 2^{x-c}$.

- **i** What effect does changing c have on the position of the graph?
- **ii** What effect does changing c have on the shape of the graph?

iii What is the horizontal asymptote of each graph?

c To graph $y = 2^{x-c}$ from $y = 2^x$ what transformation is used?

4 **a** On the same set of axes, use a **graphing package** or **graphics calculator** to graph the functions $y = 2^x$ and $y = 2^{-x}$.

b **i** What is the y-intercept of each graph?

ii What is the horizontal asymptote of each graph?

iii What transformation moves $y = 2^x$ to $y = 2^{-x}$?

5 **a** On the same set of axes, use a **graphing package** or **graphics calculator** to graph the following functions:

i $y = 2^x$, $y = 3 \times 2^x$, $y = \frac{1}{2} \times 2^x$

ii $y = -2^x$, $y = -3 \times 2^x$, $y = -\frac{1}{2} \times 2^x$

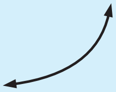
b The functions in **a** are all members of the family $y = a \times 2^x$ where a is a constant. Comment on the effect on the graph when **i** $a > 0$ **ii** $a < 0$.

c What is the horizontal asymptote of each graph? Explain your answer.

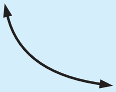
From your investigation you should have discovered that:

For the general exponential function $\boldsymbol{y = a \times b^{x-c} + d}$

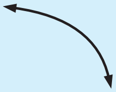
- b controls how steeply the graph increases or decreases
- c controls horizontal translation
- d controls vertical translation and $y = d$ is the equation of the horizontal asymptote.
-
  - if $a > 0$, $b > 1$ the function is increasing.



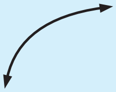
  - if $a > 0$, $0 < b < 1$ the function is decreasing.



  - if $a < 0$, $b > 1$ the function is decreasing.

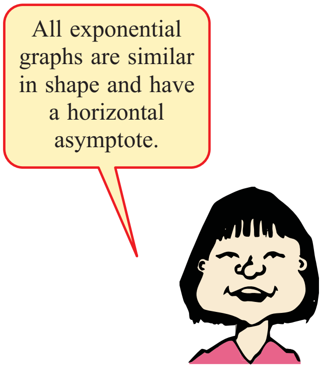


  - if $a < 0$, $0 < b < 1$ the function is increasing.



We can sketch reasonably accurate graphs of exponential functions using:

- the horizontal asymptote
- the y-intercept
- two other points, for example, when $x = 2$, $x = -2$



Example 20 **Self Tutor**

Sketch the graph of $y = 2^{-x} - 3$.

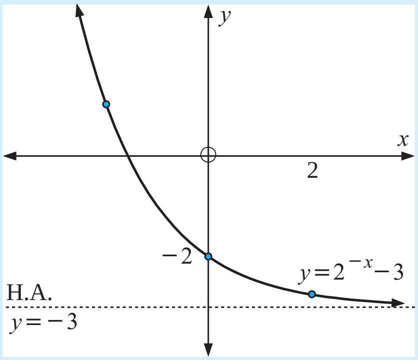
For $y = 2^{-x} - 3$
the horizontal asymptote is $y = -3$

When $x = 0$, $y = 2^0 - 3$
$= 1 - 3$
$= -2$

$\therefore$ the y-intercept is -2

When $x = 2$, $y = 2^{-2} - 3$
$= \frac{1}{4} - 3$
$= -2\frac{3}{4}$

When $x = -2$, $y = 2^2 - 3 = 1$



We now have a well-defined meaning for b^n where $b, n \in \mathbb{R}$ because simple exponential functions have smooth increasing or decreasing graphs.

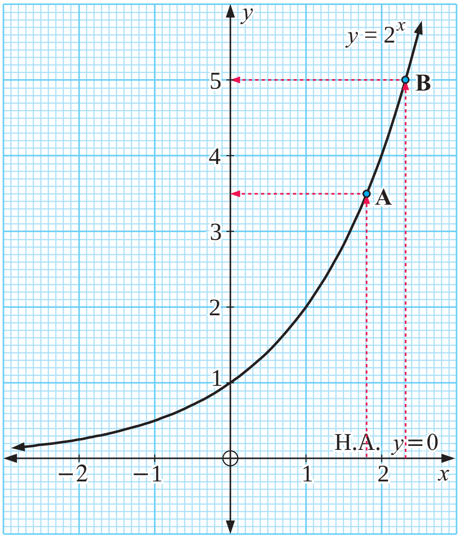
EXERCISE 3G

1 Given the graph of $y = 2^x$ we can estimate values of 2^x for various values of x.
For example:

- $2^{1.8} \approx 3.5$ (point A)
- $2^{2.3} \approx 5$ (point B)

Use the graph to determine approximate values of:

a $2^{\frac{1}{2}}$ or $\sqrt{2}$ **b** $2^{0.8}$
c $2^{1.5}$ **d** $2^{-1.6}$
e $2^{\sqrt{2}}$ **f** $2^{-\sqrt{2}}$



2 Draw freehand sketches of the following pairs of graphs using your observations from the previous investigation:

GRAPHING PACKAGE

a $y = 2^x$ and $y = 2^x - 2$ **b** $y = 2^x$ and $y = 2^{-x}$
c $y = 2^x$ and $y = 2^{x-2}$ **d** $y = 2^x$ and $y = 2(2^x)$

3 Draw freehand sketches of the following pairs of graphs:

a $y = 3^x$ and $y = 3^{-x}$ **b** $y = 3^x$ and $y = 3^x + 1$
c $y = 3^x$ and $y = -3^x$ **d** $y = 3^x$ and $y = 3^{x-1}$

4 Sketch the graphs of:

a $y = 2^x + 1$ **b** $y = 2 - 2^x$ **c** $y = 2^{-x} + 3$ **d** $y = 3 - 2^{-x}$

5 Use your graphics calculator to graph the functions in question **4**. In each case find the value of y when $x = \sqrt{2}$.

6 For the graphs of the functions in question **4**, discuss the behaviour of y as $x \to \pm\infty$. Hence determine the horizontal asymptotes for each graph.

H GROWTH AND DECAY

In this section we will examine situations where quantities are either increasing or decreasing exponentially. These situations are known as **growth** and **decay**, and occur frequently in the world around us.

For example, populations of animals, people, and bacteria usually *grow* in an exponential way. Radioactive substances, and items that depreciate in value, usually *decay* exponentially.

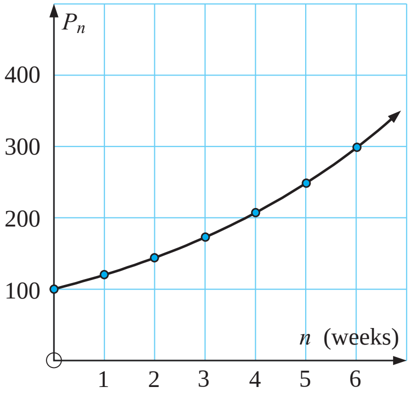
GROWTH

Consider a population of 100 mice which under favourable conditions is increasing by 20% each week. To increase a quantity by 20%, we multiply it by 120% or 1.2.

If P_n is the population after n weeks, then

$P_0 = 100$ {the *original* population}
$P_1 = P_0 \times 1.2 = 100 \times 1.2$
$P_2 = P_1 \times 1.2 = 100 \times (1.2)^2$
$P_3 = P_2 \times 1.2 = 100 \times (1.2)^3$, and so on.

From this pattern we see that $P_n = 100 \times (1.2)^n$.



Alternatively:

This is an example of a *geometric sequence* and we could find the rule to generate it.

Clearly $P_0 = 100$ and $r = 1.2$, and so $P_n = P_0 r^n = 100 \times (1.2)^n$ for $n = 0, 1, 2, 3, ...$

Example 21 **Self Tutor**

An entomologist monitoring a grasshopper plague notices that the area affected by the grasshoppers is given by $A_n = 1000 \times 2^{0.2n}$ hectares, where n is the number of weeks after the initial observation.

a Find the original affected area.
b Find the affected area after: **i** 5 weeks **ii** 10 weeks.
c Find the affected area after 12 weeks.
d Draw the graph of A_n against n.

a $A_0 = 1000 \times 2^0$
$= 1000 \times 1$
$= 1000$ $\quad\therefore$ the original affected area was 1000 ha.

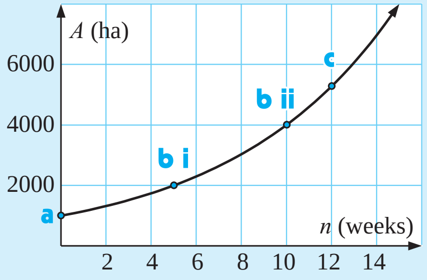
b **i** $A_5 = 1000 \times 2^1$
$= 2000$
The affected area is 2000 ha.

ii $A_{10} = 1000 \times 2^2$
$= 4000$
The affected area is 4000 ha.

c $A_{12} = 1000 \times 2^{0.2 \times 12}$
$= 1000 \times 2^{2.4}$ $\quad$ {*Press:* 1000 $\boxed{\times}$ 2 $\boxed{\wedge}$ 2.4 $\boxed{\text{ENTER}}$ }
≈ 5280

$\therefore$ after 12 weeks, the area affected is about 5280 ha.

d



EXERCISE 3H.1

GRAPHING PACKAGE

1 The weight W_t of bacteria in a culture t hours after establishment is given by $W_t = 100 \times 2^{0.1t}$ grams.

a Find the initial weight.

b Find the weight after: **i** 4 hours **ii** 10 hours **iii** 24 hours.

c Sketch the graph of W_t against t using the results of **a** and **b** only.

d Use technology to graph $\text{Y}_1 = 100 \times 2^{0.1\text{X}}$ and check your answers to **a**, **b** and **c**.

2 A breeding program to ensure the survival of pygmy possums was established with an initial population of 50 (25 pairs). From a previous program, the expected population P_n in n years' time is given by $P_n = P_0 \times 2^{0.3n}$.

a What is the value of P_0?

b What is the expected population after: **i** 2 years **ii** 5 years **iii** 10 years?

c Sketch the graph of P_n against n using **a** and **b** only.

d Use technology to graph $\text{Y}_1 = 50 \times 2^{0.3\text{X}}$ and check your answers in **b**.

3 The speed V_t of a chemical reaction is given by $V_t = V_0 \times 2^{0.05t}$ where t is the temperature in ^{o}C. Find:

a the speed at 0^oC **b** the speed at 20^oC

c the percentage increase in speed at 20^oC compared with the speed at 0^oC.

d Find $\left(\dfrac{V_{50} - V_{20}}{V_{20}}\right) \times 100\%$. What does this calculation represent?

4 A species of bear is introduced to a large island off Alaska where previously there were no bears. 6 pairs of bears were introduced in 1998. It is expected that the population will increase according to $B_t = B_0 \times 2^{0.18t}$ where t is the time since the introduction.

a Find B_0. **b** Find the expected bear population in 2018.

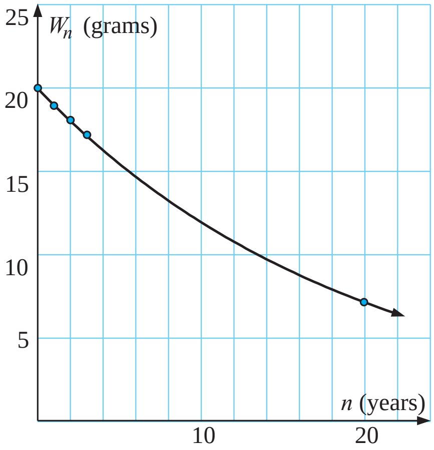
c Find the expected percentage increase from 2008 to 2018.

DECAY

Consider a radioactive substance with original weight 20 grams. It *decays* or reduces by 5% each year. The multiplier is now 95% or 0.95.

If W_n is the weight after n years, then:

$$\begin{aligned}
W_0 &= 20 \text{ grams}\\
W_1 &= W_0 \times 0.95 = 20 \times 0.95 \text{ grams}\\
W_2 &= W_1 \times 0.95 = 20 \times (0.95)^2 \text{ grams}\\
W_3 &= W_2 \times 0.95 = 20 \times (0.95)^3 \text{ grams}\\
&\vdots\\
W_{20} &= 20 \times (0.95)^{20} \approx 7.2 \text{ grams}\\
&\vdots\\
W_{100} &= 20 \times (0.95)^{100} \approx 0.1 \text{ grams}
\end{aligned}$$



and from this pattern we see that $W_n = 20 \times (0.95)^n$.

Alternatively:

Once again we have a *geometric sequence*. In this case $W_0 = 20$ and $r = 0.95$, and so $W_n = 20 \times (0.95)^n$ for $n = 0, 1, 2, 3, ...$

Example 22 **Self Tutor**

When a diesel-electric generator is switched off, the current dies away according to the formula $I(t) = 24 \times (0.25)^t$ amps, where t is the time in seconds.

a Find $I(t)$ when $t = 0, 1, 2$ and 3.

b What current flowed in the generator at the instant when it was switched off?

c Plot the graph of $I(t)$ for $t \geqslant 0$ using the information above.

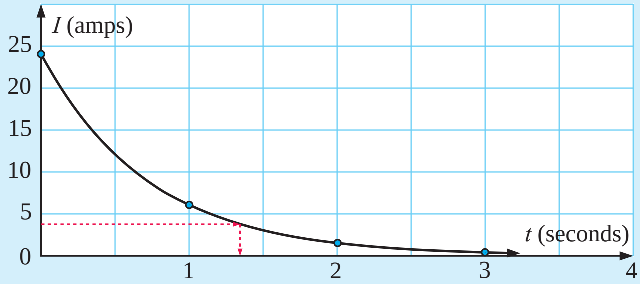
d Use your graph or technology to find how long it takes for the current to reach 4 amps.

a $I(t) = 24 \times (0.25)^t$ amps

$I(0)$	$I(1)$	$I(2)$	$I(3)$
$= 24 \times (0.25)^0$	$= 24 \times (0.25)^1$	$= 24 \times (0.25)^2$	$= 24 \times (0.25)^3$
$= 24$ amps	$= 6$ amps	$= 1.5$ amps	$= 0.375$ amps

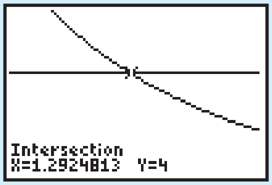
b When $t = 0$, $I(0) = 24$ $\therefore$ 24 amps of current flowed.

c



d From the graph above, the time to reach 4 amps is about 1.3 seconds. *or*

By finding the **point of intersection** of $Y_1 = 24 \times (0.25)\hat{}X$ and $Y_2 = 4$ on a graphics calculator, the solution is ≈ 1.29 seconds.



```
Intersection
X=1.2924813  Y=4
```

Example 23 **Self Tutor**

The weight of radioactive material remaining after t years is given by $W_t = W_0 \times 2^{-0.001t}$ grams.

a Find the original weight.

b Find the percentage remaining after 200 years.

a When $t = 0$, $W_0 = W_0 \times 2^0 = W_0$

$\therefore$ W_0 is the original weight.

b When $t = 200$,

$$\begin{aligned} W_{200} &= W_0 \times 2^{-0.001\times 200} \\ &= W_0 \times 2^{-0.2} \\ &\approx W_0 \times 0.8706 \\ &\approx 87.06\% \text{ of } W_0 \qquad \therefore \quad 87.1\% \text{ remains.} \end{aligned}$$

EXERCISE 3H.2

1 The weight of a radioactive substance t years after being set aside is given by $W(t) = 250 \times (0.998)^t$ grams.

- **a** How much radioactive substance was initially set aside?
- **b** Determine the weight of the substance after:
  - **i** 400 years **ii** 800 years **iii** 1200 years.
- **c** Sketch the graph of $W(t)$ for $t \geqslant 0$, using the above information.
- **d** Use your graph or graphics calculator to find how long it takes for the substance to decay to 125 grams.

2 The temperature T of a liquid which has been placed in a refrigerator is given by $T(t) = 100 \times 2^{-0.02t}$ ^{o}C where t is the time in minutes. Find:

- **a** the initial temperature
- **b** the temperature after: **i** 15 minutes **ii** 20 minutes **iii** 78 minutes.
- **c** Sketch the graph of $T(t)$ for $t \geqslant 0$ using **a** and **b** only.

3 The weight W_t grams of radioactive substance remaining after t years is given by $W_t = 1000 \times 2^{-0.03t}$ grams. Find:

- **a** the initial weight
- **b** the weight after:
  - **i** 10 years
  - **ii** 100 years
  - **iii** 1000 years.
- **c** Graph W_t against t using **a** and **b** only.

4 The weight W_t of radioactive uranium remaining after t years is given by the formula $W_t = W_0 \times 2^{-0.0002t}$ grams, $t \geqslant 0$. Find:

- **a** the original weight
- **b** the percentage weight loss after 1000 years.

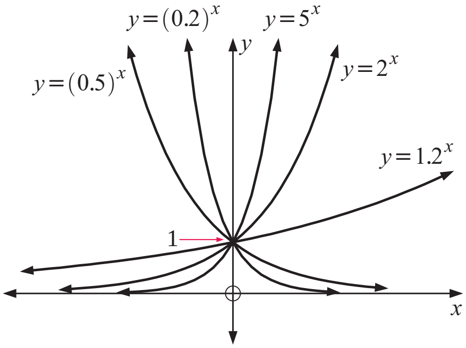
I THE NATURAL EXPONENTIAL 'e'

We have seen that the simplest exponential functions are of the form $f(x) = b^x$ where $b > 0$, $b \neq 1$.

Graphs of some of these functions are shown alongside.

We can see that for all positive values of the base b, the graph is always positive.

Hence $\boxed{b^x > 0 \text{ for all } b > 0.}$



There is an infinite number of possible choices for the base number.

However, where exponential data is examined in science, engineering, and other areas, the base $e \approx 2.7183$ is commonly used.

e is a special number in mathematics. It is irrational like π, and just as π is the ratio of a circle's area to its diameter, e also has a physical meaning. We explore this meaning in the following investigation.

INVESTIGATION 2 CONTINUOUS COMPOUND INTEREST

A formula for calculating the amount to which an investment grows is $u_n = u_0(1 + i)^n$ where:

u_n is the **final amount**, u_0 is the **initial amount**,
i is the **interest rate per compounding period**,
n is the **number of periods** or number of times the interest is compounded.

We will investigate the final value of an investment for various values of n, and allow n to get extremely large.

What to do:

1 Suppose \$1000 is invested for one year at a fixed rate of 6% per annum. Use your calculator to find the final amount or *maturing value* if the interest is paid:

  a annually $(n = 1,\ i = 6\% = 0.06)$ **b** quarterly $(n = 4,\ i = \frac{6\%}{4} = 0.015)$
  c monthly **d** daily **e** by the second **f** by the millisecond.

2 Comment on your answers obtained in **1**.

3 If r is the percentage rate per year, t is the number of years, and N is the number of interest payments per year, then $i = \dfrac{r}{N}$ and $n = Nt$.

This means that the growth formula becomes $u_n = u_0\left(1 + \dfrac{r}{N}\right)^{Nt}$.

If we let $a = \dfrac{N}{r}$, show that $u_n = u_0\left[\left(1 + \dfrac{1}{a}\right)^{a}\right]^{rt}$.

4 For continuous compound growth, the number of interest payments per year N gets very large.

  a Explain why a gets very large as N gets very large.
  b Copy and complete the table:
  Give answers as accurately as technology permits.

a	$\left(1 + \dfrac{1}{a}\right)^{a}$
10	
100	
1000	
10 000	
100 000	
⋮	

5 You should have found that for very large values of a,

$$\left(1 + \frac{1}{a}\right)^{a} \approx 2.718\,281\,828\,459......$$

6 Now use the $\boxed{e^x}$ key of your calculator to find the value of e^1.
What do you notice?

7 For continuous growth, $u_n = u_0 e^{rt}$ where u_0 is the initial amount
r is the annual percentage rate
t is the number of years

Use this formula to find the final value if \$1000 is invested for 4 years at a fixed rate of 6% per annum, where the interest is calculated continuously.

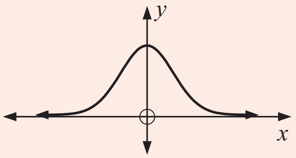
From **Investigation 2** we observe that:

If interest is paid *continuously* or *instantaneously* then the formula for calculating a compounding amount $u_n = u_0(1 + i)^n$ can be replaced by $u_n = u_0 e^{rt}$, where r is the percentage rate per annum and t is the number of years.

RESEARCH RESEARCHING e

What to do:

1 The 'bell curve' which models statistical distributions is shown alongside. Research the equation of this curve.



2 $e^{i\pi} + 1 = 0$ is called **Euler's equation** where i is the *imaginary* number $\sqrt{-1}$. Research the significance of this equation.

3 The series $f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2\times3}x^3 + \frac{1}{2\times3\times4}x^4 + ...$ has infinitely many terms. It has been shown that $f(x) = e^x$.
Check this statement by finding an approximation for $f(1)$ using its first 20 terms.

EXERCISE 3I

1 Use the $\boxed{e^x}$ key on a calculator to find the value of e to as many digits as possible.

2 Sketch, on the same set of axes, the graphs of $y = 2^x$, $y = e^x$ and $y = 3^x$. Comment on any observations.

GRAPHING PACKAGE

3 Sketch, on the same set of axes, the graphs of $y = e^x$ and $y = e^{-x}$. What is the geometric connection between these two graphs?

4 For the general exponential function $y = ae^{kx}$, what is the y-intercept?

5 Consider $y = 2e^x$.

a Explain why y can never be < 0.
b Find y if: **i** $x = -20$ **ii** $x = 20$.

6 Find, to 3 significant figures, the value of:

a e^2 **b** e^3 **c** $e^{0.7}$ **d** $\sqrt{e}$ **e** e^{-1}

7 Write the following as powers of e:

a $\sqrt{e}$ **b** $e\sqrt{e}$ **c** $\dfrac{1}{\sqrt{e}}$ **d** $\dfrac{1}{e^2}$

8 Simplify:

a $(e^{0.36})^{\frac{t}{2}}$ **b** $(e^{0.064})^{\frac{t}{16}}$ **c** $(e^{-0.04})^{\frac{t}{8}}$ **d** $(e^{-0.836})^{\frac{t}{5}}$

9 Find, to five significant figures, the values of:

a $e^{2.31}$ **b** $e^{-2.31}$ **c** $e^{4.829}$ **d** $e^{-4.829}$
e $50e^{-0.1764}$ **f** $80e^{-0.6342}$ **g** $1000e^{1.2642}$ **h** $0.25e^{-3.6742}$

10 On the same set of axes, sketch and clearly label the graphs of:
$f : x \mapsto e^x$, $g : x \mapsto e^{x-2}$, $h : x \mapsto e^x + 3$
State the domain and range of each function.

11 On the same set of axes, sketch and clearly label the graphs of:
$f : x \mapsto e^x$, $g : x \mapsto -e^x$, $h : x \mapsto 10 - e^x$
State the domain and range of each function.

12 The weight of bacteria in a culture is given by $W(t) = 2e^{\frac{t}{2}}$ grams where t is the time in hours after the culture was set to grow.

a Find the weight of the culture when:
i $t = 0$ **ii** $t = 30$ min **iii** $t = 1\frac{1}{2}$ hours **iv** $t = 6$ hours.

b Use **a** to sketch the graph of $W(t) = 2e^{\frac{t}{2}}$.

13 The current flowing in an electrical circuit t seconds after it is switched off is given by $I(t) = 75e^{-0.15t}$ amps.

a What current is still flowing in the circuit after:

i $t = 1$ sec **ii** $t = 10$ sec?

b Use your graphics calculator to sketch $I(t) = 75e^{-0.15t}$ and $I = 1$.

c Find how long it will take for the current to fall to 1 amp.

14 Consider the function $f(x) = e^x$. On the same set of axes sketch $y = f(x)$, $y = x$ and $y = f^{-1}(x)$.

REVIEW SET 3A NON-CALCULATOR

1 Simplify:

a $-(-1)^{10}$ **b** $-(-3)^3$ **c** $3^0 - 3^{-1}$

2 Simplify using the index laws:

a $a^4b^5 \times a^2b^2$ **b** $6xy^5 \div 9x^2y^5$ **c** $\dfrac{5(x^2y)^2}{(5x^2)^2}$

3 Write as powers of 2:

a $\frac{1}{16}$ **b** $2^x \times 4$ **c** $4^x \div 8$

4 Write without brackets or negative indices:

a $x^{-2} \times x^{-3}$ **b** $2(ab)^{-2}$ **c** $2ab^{-2}$

5 Write as a single power of 3:

a $\dfrac{27}{9^a}$ **b** $(\sqrt{3})^{1-x} \times 9^{1-2x}$

6 Evaluate:

a $8^{\frac{2}{3}}$ **b** $27^{-\frac{2}{3}}$

7 Write without negative indices:

a mn^{-2} **b** $(mn)^{-3}$ **c** $\dfrac{m^2n^{-1}}{p^{-2}}$ **d** $(4m^{-1}n)^2$

8 Expand and simplify:

a $(3 - 2^a)^2$ **b** $(\sqrt{x} + 2)(\sqrt{x} - 2)$ **c** $2^{-x}(2^{2x} + 2^x)$

9 Find the value of x:

a $2^{x-3} = \frac{1}{32}$ **b** $9^x = 27^{2-2x}$

10 Solve for x:

a $27^x = 3$ **b** $9^{1-x} = 27^{x+2}$

REVIEW SET 3B CALCULATOR

1 **a** Write 4×2^n as a power of 2. **b** Evaluate $7^{-1} - 7^0$.

c Write $(\frac{2}{3})^{-3}$ in simplest fractional form.

d Simplify $\left(\dfrac{2a^{-1}}{b^2}\right)^2$. Do not have negative indices or brackets in your answer.

2 Evaluate, correct to 3 significant figures:

a $3^{\frac{3}{4}}$ **b** $27^{-\frac{1}{5}}$ **c** $\sqrt[4]{100}$

3 If $f(x) = 3 \times 2^x$, find the value of:

a $f(0)$ **b** $f(3)$ **c** $f(-2)$

4 Write as powers of 3:

a 81 **b** 1 **c** $\frac{1}{27}$ **d** $\frac{1}{243}$

5 On the same set of axes draw the graphs of: **a** $y = 2^x$ **b** $y = 2^x - 4$.
In each case state the y-intercept and the equation of the horizontal asymptote.

6 The temperature of a liquid t minutes after it was heated is given by
$T = 80 \times (0.913)^t$ ^{o}C.

a Find the initial temperature of the liquid.

b Find the temperature after: **i** $t = 12$ **ii** $t = 24$ **iii** $t = 36$ minutes.

c Draw the graph of T against t for $t \geqslant 0$, using the above or technology.

d Hence, find the time taken for the temperature to reach 25^oC.

7 Consider $y = 3^x - 5$.

a Find y when $x = 0, \pm 1, \pm 2$. **b** Discuss y as $x \to \pm\infty$.

c Sketch the graph of $y = 3^x - 5$. **d** State the equation of any asymptote.

8 On the same set of axes, sketch and clearly label the graphs of:

$f : x \mapsto e^x$, $g : x \mapsto e^{x-1}$, $h : x \mapsto 3 - e^x$

State the domain and range of each function.

9 For $y = 3 - 2^{-x}$:

a find y when $x = 0, \pm 1, \pm 2$

b discuss y as $x \to \infty$ and as $x \to -\infty$

c sketch the graph of $y = 3 - 2^{-x}$

d state the equation of any asymptote.

10 The weight of a radioactive substance after t years is given by
$W = 1500 \times (0.993)^t$ grams.

a Find the original amount of radioactive material.

b Find the amount of radioactive material remaining after:

i 400 years **ii** 800 years.

c Sketch the graph of W against t, $t \geqslant 0$, using the above or technology.

d Hence, find the time taken for the weight to reduce to 100 grams.

REVIEW SET 3C

1 Simplify:

a $-(-2)^3$ **b** $5^{-1} - 5^0$

2 Simplify using the index laws:

a $(a^7)^3$ **b** $pq^2 \times p^3q^4$ **c** $\dfrac{8ab^5}{2a^4b^4}$

3 Write the following as a power of 2:

a 2×2^{-4} **b** $16 \div 2^{-3}$ **c** 8^4

4 Write without brackets or negative indices:

a b^{-3} **b** $(ab)^{-1}$ **c** ab^{-1}

5 Simplify $\dfrac{2^{x+1}}{2^{1-x}}$.

6 Write as powers of 5 in simplest form:

a 1 **b** $5\sqrt{5}$ **c** $\dfrac{1}{\sqrt[4]{5}}$ **d** 25^{a+3}

7 Expand and simplify:

a $e^x(e^{-x} + e^x)$ **b** $(2^x + 5)^2$ **c** $(x^{\frac{1}{2}} - 7)(x^{\frac{1}{2}} + 7)$

8 Solve for x:

a $6 \times 2^x = 192$ **b** $4 \times (\frac{1}{3})^x = 324$

9 Solve for x without using a calculator:

a $2^{x+1} = 32$ **b** $4^{x+1} = \left(\frac{1}{8}\right)^x$

10 Consider $y = 2e^{-x} + 1$.

a Find y when $x = 0, \pm1, \pm2$.

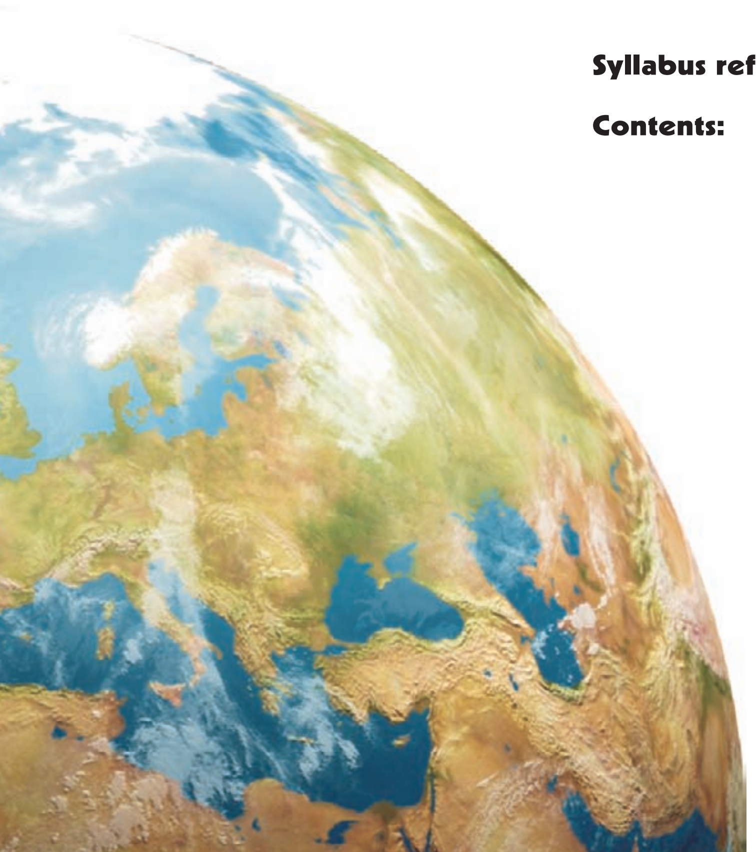
b Discuss y as $x \to \pm\infty$.

c Sketch the graph of $y = 2e^{-x} + 1$.

d State the equation of any asymptote.

Chapter 4

Logarithms



Syllabus reference: 1.2, 2.2, 2.7, 2.8

Contents:

OPENING PROBLEM

Paulo knows that when he invests €12 000 for n years at an interest rate of 8.35% p.a. compounded annually, the value of the investment at the end of this period is given by $A_{n+1} = 12\,000 \times (1.0835)^n$ euros.

Things to think about:

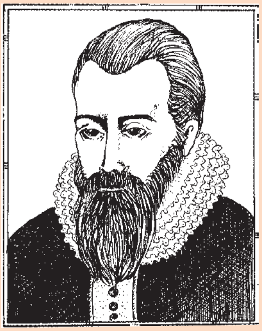
- **a** What is the value of A_1? What does it mean?
- **b** How would we find the value of the investment after 5 years?
- **c** If we let $n = 2.25$, $A_{3.25} = 12\,000 \times (1.0835)^{2.25}$. Does the power 2.25 have a meaning? What is the meaning of the value of $A_{3.25}$?
- **d** How long would it take for the investment to double in value?
- **e** What would the graph of A_{n+1} against n look like?

A LOGARITHMS

HISTORICAL NOTE

In the late 16th century, astronomers spent a large part of their working lives doing the complex and tedious calculations of spherical trigonometry needed to understand the movement of celestial bodies.

A Scotsman, **John Napier**, discovered a method of simplifying these calculations using logarithms. So effective was Napier's method that it was said he effectively doubled the life of an astronomer by reducing the time required to do these calculations.

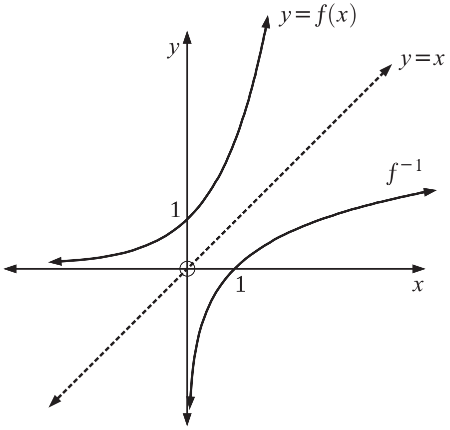


Consider the function $f : x \mapsto 10^x$ or $f(x) = 10^x$.

The graph of $y = f(x)$ is shown alongside, along with its inverse function f^{-1}.

How can we write f^{-1} in functional form?

As f is defined by $y = 10^x$,
f^{-1} is defined by $x = 10^y$.
{interchanging x and y}



If $x = 10^y$ then y is the exponent to which the base 10 is raised in order to get x.

We write this as $y = \log_{10} x$ and say that 'y is the logarithm of x in base 10.'

So,
- if $f(x) = 10^x$, then $f^{-1}(x) = \log_{10} x$
- if $f(x) = a^x$, then $f^{-1}(x) = \log_a x$.

LOGARITHMS IN BASE a

If $b = a^x$, $a \neq 1$, $a > 0$, we say that x is the logarithm of b in base a, and that $b = a^x \Leftrightarrow x = \log_a b$, $b > 0$.

$b = a^x \Leftrightarrow x = \log_a b$ is a short way of writing:
'if $b = a^x$ then $x = \log_a b$, and if $x = \log_a b$ then $b = a^x$'.

We say that $b = a^x$ and $x = \log_a b$ are *equivalent* or *interchangeable* statements.

For example:
- $8 = 2^3$ means that $3 = \log_2 8$ and vice versa.
- $\log_5 25 = 2$ means that $25 = 5^2$ and vice versa.

If $y = a^x$ then $x = \log_a y$, and so $\boldsymbol{x = \log_a a^x}$.

If $x = a^y$ then $y = \log_a x$, and so $\boldsymbol{x = a^{\log_a x}}$ provided $x > 0$.

Example 1 **Self Tutor**

a Write an equivalent exponential statement for $\log_{10} 1000 = 3$.

b Write an equivalent logarithmic statement for $3^4 = 81$.

a From $\log_{10} 1000 = 3$ we deduce that $10^3 = 1000$.

b From $3^4 = 81$ we deduce that $\log_3 81 = 4$.

Example 2 **Self Tutor**

Find: **a** $\log_{10} 100$ **b** $\log_2 32$ **c** $\log_5(0.2)$

a To find $\log_{10} 100$ we ask "What power must 10 be raised to, to get 100?"
$10^2 = 100$ so $\log_{10} 100 = 2$.

b $2^5 = 32$ so $\log_2 32 = 5$.

c $5^{-1} = \frac{1}{5} = 0.2$ so $\log_5(0.2) = -1$.

EXERCISE 4A

1 Write an equivalent exponential statement for:

a $\log_{10} 10\,000 = 4$ **b** $\log_{10}(0.1) = -1$ **c** $\log_{10} \sqrt{10} = \frac{1}{2}$

d $\log_2 8 = 3$ **e** $\log_2(\frac{1}{4}) = -2$ **f** $\log_3 \sqrt{27} = 1.5$

2 Write an equivalent logarithmic statement for:

a $2^2 = 4$ **b** $2^{-3} = \frac{1}{8}$ **c** $10^{-2} = 0.01$

d $7^2 = 49$ **e** $2^6 = 64$ **f** $3^{-3} = \frac{1}{27}$

3 Find:

a $\log_{10} 100\,000$ **b** $\log_{10}(0.01)$ **c** $\log_3 \sqrt{3}$ **d** $\log_2 8$

e $\log_2 64$ **f** $\log_2 128$ **g** $\log_5 25$ **h** $\log_5 125$

i $\log_2(0.125)$ **j** $\log_9 3$ **k** $\log_4 16$ **l** $\log_{36} 6$

m $\log_3 243$ **n** $\log_2 \sqrt[3]{2}$ **o** $\log_a a^n$ **p** $\log_8 2$

q $\log_t \left(\frac{1}{t}\right)$ **r** $\log_6 6\sqrt{6}$ **s** $\log_4 1$ **t** $\log_9 9$

4 Use your calculator to find:

a $\log_{10} 152$ **b** $\log_{10} 25$ **c** $\log_{10} 74$ **d** $\log_{10} 0.8$

5 Solve for x:

a $\log_2 x = 3$ **b** $\log_4 x = \frac{1}{2}$ **c** $\log_x 81 = 4$ **d** $\log_2(x - 6) = 3$

Example 3 **Self Tutor**

Use $\log_a a^x = x$ to find:

a $\log_2 16$ **b** $\log_{10} \sqrt[5]{100}$ **c** $\log_2 \left(\frac{1}{\sqrt{2}}\right)$

a
$$\begin{aligned} &\log_2 16 \\ &= \log_2 2^4 \\ &= 4 \end{aligned}$$

b
$$\begin{aligned} &\log_{10} \sqrt[5]{100} \\ &= \log_{10} (10^2)^{\frac{1}{5}} \\ &= \log_{10} 10^{\frac{2}{5}} \\ &= \frac{2}{5} \end{aligned}$$

c
$$\begin{aligned} &\log_2 \left(\frac{1}{\sqrt{2}}\right) \\ &= \log_2 2^{-\frac{1}{2}} \\ &= -\frac{1}{2} \end{aligned}$$

6 Use $\log_a a^x = x$ to find:

a $\log_4 16$ **b** $\log_2 4$ **c** $\log_3 \left(\frac{1}{3}\right)$ **d** $\log_{10} \sqrt[4]{1000}$

e $\log_7 \left(\frac{1}{\sqrt{7}}\right)$ **f** $\log_5(25\sqrt{5})$ **g** $\log_3 \left(\frac{1}{\sqrt{27}}\right)$ **h** $\log_4 \left(\frac{1}{2\sqrt{2}}\right)$

B LOGARITHMS IN BASE 10

Many positive numbers can be easily written in the form 10^x.

For example:

$$\begin{aligned} 10\,000 &= 10^4 \\ 1000 &= 10^3 \\ 100 &= 10^2 \\ 10 &= 10^1 \\ 1 &= 10^0 \\ 0.1 &= 10^{-1} \\ 0.01 &= 10^{-2} \\ 0.001 &= 10^{-3} \end{aligned}$$

Numbers like $\sqrt{10}$, $10\sqrt{10}$ and $\dfrac{1}{\sqrt[5]{10}}$ can also be written in the form 10^x as follows:

$$\begin{aligned}&\sqrt{10}\\ &= 10^{\frac{1}{2}}\\ &= 10^{0.5}\end{aligned} \qquad \begin{aligned}&10\sqrt{10}\\ &= 10^1 \times 10^{0.5}\\ &= 10^{1.5}\end{aligned} \qquad \begin{aligned}&\frac{1}{\sqrt[5]{10}}\\ &= 10^{-\frac{1}{5}}\\ &= 10^{-0.2}\end{aligned}$$

In fact, all positive numbers can be written in the form 10^x by using logarithms in base 10.

The **logarithm in base 10** of a positive number is its power of 10.

For example:

- Since $1000 = 10^3$, we write $\log_{10} 1000 = 3$ or $\log 1000 = 3$.
- Since $0.01 = 10^{-2}$, we write $\log_{10}(0.01) = -2$ or $\log(0.01) = -2$.

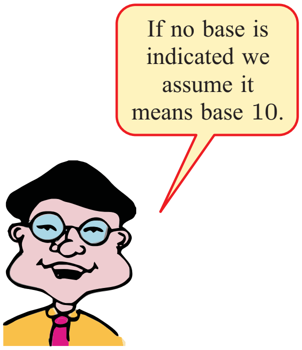
$$\boldsymbol{a = 10^{\log a}} \quad \text{for any } a > 0.$$

Notice that a must be positive since $10^x > 0$ for all $x \in \mathbb{R}$.

Notice also that $\log 1000 = \log 10^3 = 3$

and $\log 0.01 = \log 10^{-2} = -2$

give us the useful alternative $\boldsymbol{\log 10^x = x}$



Example 4

Self Tutor

a Without using a calculator, find: **i** $\log 100$ **ii** $\log(\sqrt[4]{10})$.

b Check your answers using technology.

a **i** $\log 100 = \log 10^2 = 2$ **ii** $\log(\sqrt[4]{10}) = \log(10^{\frac{1}{4}}) = \frac{1}{4}$

b **i** press $\boxed{\text{log}}$ 100 $\boxed{)}$ $\boxed{\text{ENTER}}$ *Answer:* 2

ii press $\boxed{\text{log}}$ 10 $\boxed{\wedge}$ 0.25 $\boxed{)}$ $\boxed{\text{ENTER}}$ *Answer:* 0.25

Example 5

Self Tutor

Use your calculator to write the following in the form 10^x where x is correct to 4 decimal places:

a 8 **b** 800 **c** 0.08

a
$$\begin{aligned}&8\\ &= 10^{\log 8}\\ &\approx 10^{0.9031}\end{aligned}$$

b
$$\begin{aligned}&800\\ &= 10^{\log 800}\\ &\approx 10^{2.9031}\end{aligned}$$

c
$$\begin{aligned}&0.08\\ &= 10^{\log 0.08}\\ &\approx 10^{-1.0969}\end{aligned}$$

Example 6 Self Tutor

a Use your calculator to find: **i** $\log 2$ **ii** $\log 20$

b Explain why $\log 20 = \log 2 + 1$.

a **i** $\log 2 \approx 0.3010$

ii $\log 20 \approx 1.3010$ {calculator}

b
$$\begin{aligned} \log 20 &= \log(2 \times 10) \\ &\approx \log(10^{0.3010} \times 10^1) \\ &\approx \log 10^{1.3010} \qquad \{\text{adding indices}\} \\ &\approx 1.3010 \\ &\approx \log 2 + 1 \end{aligned}$$

Example 7 Self Tutor

Find x if:

a $\log x = 3$

b $\log x \approx -0.271$

a
$$\begin{aligned} x &= 10^{\log x} \\ \therefore \quad x &= 10^3 \\ \therefore \quad x &= 1000 \end{aligned}$$

b
$$\begin{aligned} x &= 10^{\log x} \\ \therefore \quad x &\approx 10^{-0.271} \\ \therefore \quad x &\approx 0.536 \end{aligned}$$

EXERCISE 4B

1 Without using a calculator, find:

a $\log 10\,000$
b $\log 0.001$
c $\log 10$
d $\log 1$
e $\log \sqrt{10}$
f $\log(\sqrt[3]{10})$
g $\log\left(\frac{1}{\sqrt[4]{10}}\right)$
h $\log 10\sqrt{10}$
i $\log \sqrt[3]{100}$
j $\log\left(\frac{100}{\sqrt{10}}\right)$
k $\log(10 \times \sqrt[3]{10})$
l $\log 1000\sqrt{10}$
m $\log 10^n$
n $\log(10^a \times 100)$
o $\log\left(\frac{10}{10^m}\right)$
p $\log\left(\frac{10^a}{10^b}\right)$

2 Find using a calculator:

a $\log 10\,000$
b $\log 0.001$
c $\log \sqrt{10}$
d $\log \sqrt[3]{10}$
e $\log \sqrt[3]{100}$
f $\log 10\sqrt{10}$
g $\log\left(\frac{1}{\sqrt{10}}\right)$
h $\log\left(\frac{1}{\sqrt[4]{10}}\right)$

3 Use your calculator to write these in the form 10^x where x is correct to 4 decimal places:

a 6
b 60
c 6000
d 0.6
e 0.006
f 15
g 1500
h 1.5
i 0.15
j 0.000 15

4 **a** Use your calculator to find: **i** $\log 3$ **ii** $\log 300$

b Explain why $\log 300 = \log 3 + 2$.

5 **a** Use your calculator to find: **i** $\log 5$ **ii** $\log 0.05$

b Explain why $\log 0.05 = \log 5 - 2$.

6 Find x if:

a $\log x = 2$ **b** $\log x = 1$ **c** $\log x = 0$ **d** $\log x = -1$

e $\log x = \frac{1}{2}$ **f** $\log x = -\frac{1}{2}$ **g** $\log x = 4$ **h** $\log x = -5$

i $\log x \approx 0.8351$ **j** $\log x \approx 2.1457$ **k** $\log x \approx -1.378$ **l** $\log x \approx -3.1997$

C LAWS OF LOGARITHMS

INVESTIGATION DISCOVERING THE LAWS OF LOGARITHMS

What to do:

1 Use your calculator to find:

a $\log 2 + \log 3$ **b** $\log 3 + \log 7$ **c** $\log 4 + \log 20$

d $\log 6$ **e** $\log 21$ **f** $\log 80$

From your answers, suggest a possible simplification for $\log a + \log b$.

2 Use your calculator to find:

a $\log 6 - \log 2$ **b** $\log 12 - \log 3$ **c** $\log 3 - \log 5$

d $\log 3$ **e** $\log 4$ **f** $\log(0.6)$

From your answers, suggest a possible simplification for $\log a - \log b$.

3 Use your calculator to find:

a $3\log 2$ **b** $2\log 5$ **c** $-4\log 3$

d $\log(2^3)$ **e** $\log(5^2)$ **f** $\log(3^{-4})$

From your answers, suggest a possible simplification for $n\log a$.

From the investigation, you should have discovered the three important **laws of logarithms**:

If A and B are both positive then:

- $\log A + \log B = \log(AB)$
- $\log A - \log B = \log\left(\frac{A}{B}\right)$
- $n\log A = \log(A^n)$

More generally, in any base c we have these **laws of logarithms**:

If A and B are both positive then:

- $\log_c A + \log_c B = \log_c(AB)$
- $\log_c A - \log_c B = \log_c\left(\frac{A}{B}\right)$
- $n\log_c A = \log_c(A^n)$

These laws are easily established using the first three index laws.

Example 8 **Self Tutor**

Use the laws of logarithms to write the following as a single logarithm:

a $\log 5 + \log 3$ **b** $\log 24 - \log 8$ **c** $\log 5 - 1$

a
$$\begin{aligned} &\log 5 + \log 3 \\ &= \log(5 \times 3) \\ &= \log 15 \end{aligned}$$

b
$$\begin{aligned} &\log 24 - \log 8 \\ &= \log\left(\tfrac{24}{8}\right) \\ &= \log 3 \end{aligned}$$

c
$$\begin{aligned} &\log 5 - 1 \\ &= \log 5 - \log 10^1 \\ &= \log\left(\tfrac{5}{10}\right) \\ &= \log\left(\tfrac{1}{2}\right) \end{aligned}$$

Example 9 **Self Tutor**

Write as a single logarithm in the form $\log a,\ a \in \mathbb{Q}$.

a $2\log 7 - 3\log 2$

b $2\log 3 + 3$

a
$$\begin{aligned} &2\log 7 - 3\log 2 \\ &= \log(7^2) - \log(2^3) \\ &= \log 49 - \log 8 \\ &= \log\left(\tfrac{49}{8}\right) \end{aligned}$$

b
$$\begin{aligned} &2\log 3 + 3 \\ &= \log(3^2) + \log(10^3) \\ &= \log 9 + \log 1000 \\ &= \log(9000) \end{aligned}$$

Example 10 **Self Tutor**

Simplify $\dfrac{\log 8}{\log 4}$ without using a calculator.

$$\begin{aligned} \frac{\log 8}{\log 4} &= \frac{\log 2^3}{\log 2^2} \\ &= \frac{3\cancel{\log 2}}{2\cancel{\log 2}} \\ &= \frac{3}{2} \end{aligned}$$

Example 11 **Self Tutor**

Show that:

a $\log\left(\frac{1}{9}\right) = -2\log 3$

b $\log 500 = 3 - \log 2$

a
$$\begin{aligned} &\log\left(\tfrac{1}{9}\right) \\ &= \log(3^{-2}) \\ &= -2\log 3 \end{aligned}$$

b
$$\begin{aligned} &\log 500 \\ &= \log\left(\tfrac{1000}{2}\right) \\ &= \log 1000 - \log 2 \\ &= \log 10^3 - \log 2 \\ &= 3 - \log 2 \end{aligned}$$

EXERCISE 4C.1

1 Write as a single logarithm:

a $\log 8 + \log 2$ **b** $\log 8 - \log 2$ **c** $\log 40 - \log 5$

d $\log 4 + \log 5$ **e** $\log 5 + \log(0.4)$ **f** $\log 2 + \log 3 + \log 4$

g $1 + \log 3$ **h** $\log 4 - 1$ **i** $\log 5 + \log 4 - \log 2$

j $2 + \log 2$ **k** $\log 40 - 2$ **l** $\log 6 - \log 2 - \log 3$

m $\log 50 - 4$ **n** $3 - \log 50$ **o** $\log\left(\frac{4}{3}\right) + \log 3 + \log 7$

2 Write as a single logarithm or integer:

a $5\log 2 + \log 3$ **b** $2\log 3 + 3\log 2$ **c** $3\log 4 - \log 8$

d $2\log 5 - 3\log 2$ **e** $\frac{1}{2}\log 4 + \log 3$ **f** $\frac{1}{3}\log\left(\frac{1}{8}\right)$

g $3 - \log 2 - 2\log 5$ **h** $1 - 3\log 2 + \log 20$ **i** $2 - \frac{1}{2}\log 4 - \log 5$

3 Simplify without using a calculator:

a $\dfrac{\log 4}{\log 2}$ **b** $\dfrac{\log 27}{\log 9}$ **c** $\dfrac{\log 8}{\log 2}$ **d** $\dfrac{\log 3}{\log 9}$ **e** $\dfrac{\log 25}{\log(0.2)}$ **f** $\dfrac{\log 8}{\log(0.25)}$

Check your answers using a calculator.

4 Show that:

a $\log 9 = 2\log 3$ **b** $\log\sqrt{2} = \frac{1}{2}\log 2$ **c** $\log\left(\frac{1}{8}\right) = -3\log 2$

d $\log\left(\frac{1}{5}\right) = -\log 5$ **e** $\log 5 = 1 - \log 2$ **f** $\log 5000 = 4 - \log 2$

5 If $p = \log_b 2$, $q = \log_b 3$ and $r = \log_b 5$ write in terms of p, q and r:

a $\log_b 6$ **b** $\log_b 108$ **c** $\log_b 45$

d $\log_b\left(\frac{5\sqrt{3}}{2}\right)$ **e** $\log_b\left(\frac{5}{32}\right)$ **f** $\log_b(0.\overline{2})$

6 If $\log_2 P = x$, $\log_2 Q = y$ and $\log_2 R = z$ write in terms of x, y and z:

a $\log_2(PR)$ **b** $\log_2(RQ^2)$ **c** $\log_2\left(\dfrac{PR}{Q}\right)$

d $\log_2(P^2\sqrt{Q})$ **e** $\log_2\left(\dfrac{Q^3}{\sqrt{R}}\right)$ **f** $\log_2\left(\dfrac{R^2\sqrt{Q}}{P^3}\right)$

7 If $\log_t M = 1.29$ and $\log_t N^2 = 1.72$ find:

a $\log_t N$ **b** $\log_t(MN)$ **c** $\log_t\left(\dfrac{N^2}{\sqrt{M}}\right)$

LOGARITHMIC EQUATIONS

Example 12 **Self Tutor**

Write these as logarithmic equations (in base 10):

a $y = a^2 b$ **b** $y = \dfrac{a}{b^3}$ **c** $P = \dfrac{20}{\sqrt{n}}$

a
$$\begin{aligned} y &= a^2 b \\ \therefore\ \log y &= \log(a^2 b) \\ \therefore\ \log y &= \log a^2 + \log b \\ \therefore\ \log y &= 2\log a + \log b \end{aligned}$$

b
$$\begin{aligned} y &= \frac{a}{b^3} \\ \therefore\ \log y &= \log\left(\frac{a}{b^3}\right) \\ \therefore\ \log y &= \log a - \log b^3 \\ \therefore\ \log y &= \log a - 3\log b \end{aligned}$$

c $P = \left(\dfrac{20}{\sqrt{n}}\right)$ $\therefore$ $\log P = \log\left(\dfrac{20}{n^{\frac{1}{2}}}\right)$ and so $\log P = \log 20 - \frac{1}{2}\log n.$

Example 13 **Self Tutor**

Write the following equations without logarithms:

a $\log A = \log b + 2\log c$ **b** $\log M = 3\log a - 2$

a
$$\begin{aligned} \log A &= \log b + 2\log c \\ \therefore\ \log A &= \log b + \log c^2 \\ \therefore\ \log A &= \log(bc^2) \\ \therefore\ A &= bc^2 \end{aligned}$$

b
$$\begin{aligned} \log M &= 3\log a - 2 \\ \therefore\ \log M &= \log a^3 - \log 10^2 \\ \therefore\ \log M &= \log\left(\frac{a^3}{100}\right) \\ \therefore\ M &= \frac{a^3}{100} \end{aligned}$$

EXERCISE 4C.2

1 Write the following as logarithmic equations (in base 10):

a $y = 2^x$ **b** $y = 20b^3$ **c** $M = ad^4$ **d** $T = 5\sqrt{d}$

e $R = b\sqrt{l}$ **f** $Q = \dfrac{a}{b^n}$ **g** $y = ab^x$ **h** $F = \dfrac{20}{\sqrt{n}}$

i $L = \dfrac{ab}{c}$ **j** $N = \sqrt{\dfrac{a}{b}}$ **k** $S = 200 \times 2^t$ **l** $y = \dfrac{a^m}{b^n}$

2 Write the following equations without logarithms:

a $\log D = \log e + \log 2$ **b** $\log F = \log 5 - \log t$

c $\log P = \frac{1}{2}\log x$ **d** $\log M = 2\log b + \log c$

e $\log B = 3\log m - 2\log n$ **f** $\log N = -\frac{1}{3}\log p$

g $\log P = 3\log x + 1$ **h** $\log Q = 2 - \log x$

3 Solve for x:

a $\log_3 27 + \log_3(\frac{1}{3}) = \log_3 x$ **b** $\log_5 x = \log_5 8 - \log_5(6 - x)$

c $\log_5 125 - \log_5 \sqrt{5} = \log_5 x$ **d** $\log_{20} x = 1 + \log_{20} 10$

e $\log x + \log(x + 1) = \log 30$ **f** $\log(x + 2) - \log(x - 2) = \log 5$

D NATURAL LOGARITHMS

In **Chapter 3** we came across the **natural exponential** $e \approx 2.718\,28$.

If f is the exponential function $f(x) = e^x$ or $y = e^x$ then its inverse function is $x = e^y$ or $f^{-1}(x) = \log_e x$.

So,

$y = \log_e x$ is the reflection of $y = e^x$ in the mirror line $y = x$.

$\ln x$ is used to represent $\log_e x$.

$\ln x$ is called the **natural logarithm** of x.

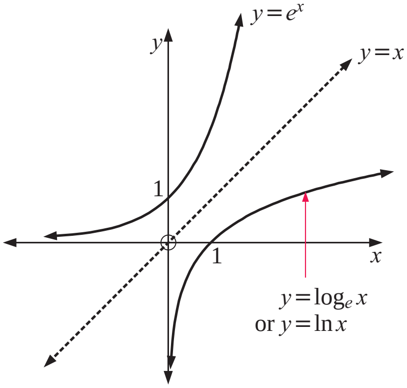
Notice that: $\ln 1 = \ln e^0 = 0$ $\ln e = \ln e^1 = 1$

$\ln e^2 = 2$ $\ln \sqrt{e} = \ln e^{\frac{1}{2}} = \frac{1}{2}$

and $\ln\left(\frac{1}{e}\right) = \ln e^{-1} = -1$

In general, $\boldsymbol{\ln e^x = x}$ **and** $\boldsymbol{e^{\ln x} = x}$.

Also, since $a^x = \left(e^{\ln a}\right)^x = e^{x \ln a}$, $\boldsymbol{a^x = e^{x \ln a}}$.



Example 14 **Self Tutor**

Use your calculator to write the following in the form e^k where k is correct to 4 decimal places: **a** 50 **b** 0.005

a 50
$= e^{\ln 50}$ $\{$using $x = e^{\ln x}\}$
$\approx e^{3.9120}$

b 0.005
$= e^{\ln 0.005}$
$\approx e^{-5.2983}$

Example 15 **Self Tutor**

Find x if:

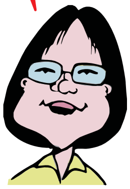
a $\ln x = 2.17$

b $\ln x = -0.384$

a $\ln x = 2.17$
$\therefore\ x = e^{2.17}$
$\therefore\ x \approx 8.76$

b $\ln x = -0.384$
$\therefore\ x = e^{-0.384}$
$\therefore\ x \approx 0.681$

If $\ln x = a$ then $x = e^a$.



EXERCISE 4D.1

1 Without using a calculator find:

a $\ln e^3$ **b** $\ln 1$ **c** $\ln \sqrt[3]{e}$ **d** $\ln\left(\frac{1}{e^2}\right)$

2 Check your answers to question **1** using a calculator.

3 Explain why $\ln(-2)$ and $\ln 0$ cannot be found.

4 Simplify:

a $\ln e^a$ **b** $\ln(e \times e^a)$ **c** $\ln\left(e^a \times e^b\right)$ **d** $\ln(e^a)^b$ **e** $\ln\left(\frac{e^a}{e^b}\right)$

5 Use your calculator to write these in the form e^x where x is correct to 4 decimal places:

a 6 **b** 60 **c** 6000 **d** 0.6 **e** 0.006
f 15 **g** 1500 **h** 1.5 **i** 0.15 **j** 0.000 15

6 Find x if:

a $\ln x = 3$ **b** $\ln x = 1$ **c** $\ln x = 0$ **d** $\ln x = -1$
e $\ln x = -5$ **f** $\ln x \approx 0.835$ **g** $\ln x \approx 2.145$ **h** $\ln x \approx -3.2971$

LAWS OF NATURAL LOGARITHMS

The laws for natural logarithms are the laws for logarithms written in base e:

For positive A and B:

- $\ln A + \ln B = \ln(AB)$
- $\ln A - \ln B = \ln\left(\dfrac{A}{B}\right)$
- $n \ln A = \ln(A^n)$

Example 16

Self Tutor

Use the laws of logarithms to write the following as a single logarithm:

a $\ln 5 + \ln 3$ **b** $\ln 24 - \ln 8$ **c** $\ln 5 - 1$

a
$$\begin{aligned} &\ln 5 + \ln 3 \\ &= \ln(5 \times 3) \\ &= \ln 15 \end{aligned}$$

b
$$\begin{aligned} &\ln 24 - \ln 8 \\ &= \ln\left(\tfrac{24}{8}\right) \\ &= \ln 3 \end{aligned}$$

c
$$\begin{aligned} &\ln 5 - 1 \\ &= \ln 5 - \ln\ e^1 \\ &= \ln\left(\tfrac{5}{e}\right) \end{aligned}$$

Example 17

Self Tutor

Use the laws of logarithms to simplify:

a $2\ln 7 - 3\ln 2$

b $2\ln 3 + 3$

a
$$\begin{aligned} &2\ln 7 - 3\ln 2 \\ &= \ln(7^2) - \ln(2^3) \\ &= \ln 49 - \ln 8 \\ &= \ln\left(\tfrac{49}{8}\right) \end{aligned}$$

b
$$\begin{aligned} &2\ln 3 + 3 \\ &= \ln(3^2) + \ln e^3 \\ &= \ln 9 + \ln e^3 \\ &= \ln\left(9e^3\right) \end{aligned}$$

Example 18

Self Tutor

Show that:

a $\ln\left(\frac{1}{9}\right) = -2\ln 3$

b $\ln 500 \approx 6.9078 - \ln 2$

a
$$\begin{aligned} &\ln\left(\tfrac{1}{9}\right) \\ &= \ln(3^{-2}) \\ &= -2\ln 3 \end{aligned}$$

b
$$\begin{aligned} \ln 500 &= \ln\left(\tfrac{1000}{2}\right) \\ &= \ln 1000 - \ln 2 \\ &\approx 6.9078 - \ln 2 \end{aligned}$$

Example 19

Self Tutor

Write the following equations without logarithms:

a $\ln A = 2\ln c + 3$ **b** $\ln M = 3\ln a - 2$

a
$$\begin{aligned} \ln A &= 2\ln c + 3 \\ \therefore\ \ln A - 2\ln c &= 3 \\ \therefore\ \ln A - \ln c^2 &= 3 \\ \therefore\ \ln\left(\frac{A}{c^2}\right) &= 3 \\ \therefore\ \frac{A}{c^2} &= e^3 \\ \therefore\ A &= c^2e^3 \end{aligned}$$

b
$$\begin{aligned} \ln M &= 3\ln a - 2 \\ \therefore\ \ln M - 3\ln a &= -2 \\ \therefore\ \ln M - \ln a^3 &= -2 \\ \therefore\ \ln\left(\frac{M}{a^3}\right) &= -2 \\ \therefore\ \frac{M}{a^3} &= e^{-2} \\ \therefore\ M &= a^3e^{-2} \ \text{ or }\ M = \frac{a^3}{e^2} \end{aligned}$$

EXERCISE 4D.2

1 Write as a single logarithm:

a $\ln 15 + \ln 3$ **b** $\ln 15 - \ln 3$ **c** $\ln 20 - \ln 5$

d $\ln 4 + \ln 6$ **e** $\ln 5 + \ln(0.2)$ **f** $\ln 2 + \ln 3 + \ln 5$

g $1 + \ln 4$ **h** $\ln 6 - 1$ **i** $\ln 5 + \ln 8 - \ln 2$

j $2 + \ln 4$ **k** $\ln 20 - 2$ **l** $\ln 12 - \ln 4 - \ln 3$

2 Write in the form $\ln a,\ a \in \mathbb{Q}$:

a $5\ln 3 + \ln 4$ **b** $3\ln 2 + 2\ln 5$ **c** $3\ln 2 - \ln 8$

d $3\ln 4 - 2\ln 2$ **e** $\frac{1}{3}\ln 8 + \ln 3$ **f** $\frac{1}{3}\ln(\frac{1}{27})$

g $-\ln 2$ **h** $-\ln(\frac{1}{2})$ **i** $-2\ln(\frac{1}{4})$

3 Show that:

a $\ln 27 = 3\ln 3$ **b** $\ln\sqrt{3} = \frac{1}{2}\ln 3$ **c** $\ln(\frac{1}{16}) = -4\ln 2$

d $\ln(\frac{1}{6}) = -\ln 6$ **e** $\ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\ln 2$ **f** $\ln\left(\frac{e}{5}\right) = 1 - \ln 5$

g $\ln\sqrt[3]{5} = \frac{1}{3}\ln 5$ **h** $\ln(\frac{1}{32}) = -5\ln 2$ **i** $\ln\left(\frac{1}{\sqrt[5]{2}}\right) = -\frac{1}{5}\ln 2$

4 Show that $\ln\left(\frac{e^2}{8}\right) = 2 - 3\ln 2$.

5 Write the following equations without logarithms:

a $\ln D = \ln x + 1$ **b** $\ln F = -\ln p + 2$ **c** $\ln P = \frac{1}{2}\ln x$

d $\ln M = 2\ln y + 3$ **e** $\ln B = 3\ln t - 1$ **f** $\ln N = -\frac{1}{3}\ln g$

g $\ln Q \approx 3\ln x + 2.159$ **h** $\ln D \approx 0.4\ln n - 0.6582$

E EXPONENTIAL EQUATIONS USING LOGARITHMS

In **Chapter 3** we found solutions to simple exponental equations by creating equal bases and then equating indices. However, it is not always easy to make the bases the same. In these situations we use **logarithms** to find the solution.

Example 20 **Self Tutor**

Solve for x, giving an exact answer: $2^x = 30$.

$$\begin{aligned} 2^x &= 30 \\ \therefore\quad \log 2^x &= \log 30 \quad \{\text{find the logarithm of each side}\} \\ \therefore\quad x\log 2 &= \log 30 \quad \{\log(A^n) = n\log A\} \\ \therefore\quad x &= \frac{\log 30}{\log 2} \end{aligned}$$

Example 21 **Self Tutor**

Solve for t to 3 significant figures: $200 \times 2^{0.04t} = 6$.

$$\begin{aligned} 200 \times 2^{0.04t} &= 6 \\ \therefore\quad 2^{0.04t} &= \tfrac{6}{200} && \{\text{dividing both sides by } 200\} \\ \therefore\quad 2^{0.04t} &= 0.03 \\ \therefore\quad \log 2^{0.04t} &= \log 0.03 && \{\text{find the logarithm of each side}\} \\ \therefore\quad 0.04t \times \log 2 &= \log 0.03 && \{\log(A)^n = n \log A\} \\ \therefore\quad t &= \frac{\log 0.03}{0.04 \times \log 2} \approx -126 \end{aligned}$$

For help using your calculator to evaluate logarithms, see the graphics calculator instructions chapter.

To solve exponential equations of the form $e^x = a$ we simply use the property:

If $e^x = a$ then $x = \ln a$.

This rule is clearly true, because if $e^x = a$

$$\begin{aligned} \text{then}\quad \ln e^x &= \ln a && \{\text{finding } \ln \text{ of both sides}\} \\ \therefore\quad x &= \ln a && \{\ln e^x = x\} \end{aligned}$$

Example 22 **Self Tutor**

Find x to 4 significant figures if:

a $e^x = 30$ **b** $e^{\frac{x}{3}} = 21.879$ **c** $20e^{4x} = 0.0382$

a
$$\begin{aligned} e^x &= 30 \\ \therefore\quad x &= \ln 30 \\ \therefore\quad x &\approx 3.401 \end{aligned}$$

b
$$\begin{aligned} e^{\frac{x}{3}} &= 21.879 \\ \therefore\quad \frac{x}{3} &= \ln 21.879 \\ \therefore\quad x &\approx 9.257 \end{aligned}$$

c
$$\begin{aligned} 20e^{4x} &= 0.0382 \\ \therefore\quad e^{4x} &= 0.001\,91 \\ \therefore\quad 4x &= \ln 0.001\,91 \\ \therefore\quad 4x &\approx -6.2607 \\ \therefore\quad x &\approx -1.565 \end{aligned}$$

EXERCISE 4E

1 Solve for x, giving your answer correct to 3 significant figures:

a $2^x = 10$ **b** $3^x = 20$ **c** $4^x = 100$

d $(1.2)^x = 1000$ **e** $2^x = 0.08$ **f** $3^x = 0.000\,25$

g $(\frac{1}{2})^x = 0.005$ **h** $(\frac{3}{4})^x = 10^{-4}$ **i** $(0.99)^x = 0.000\,01$

2 Find the solution to the following correct to 4 significant figures:

a $200 \times 2^{0.25t} = 600$ **b** $20 \times 2^{0.06t} = 450$ **c** $30 \times 3^{-0.25t} = 3$

d $12 \times 2^{-0.05t} = 0.12$ **e** $50 \times 5^{-0.02t} = 1$ **f** $300 \times 2^{0.005t} = 1000$

3 Solve for x, giving answers correct to 4 significant figures:

a $e^x = 10$ **b** $e^x = 1000$ **c** $e^x = 0.008\,62$

d $e^{\frac{x}{2}} = 5$ **e** $e^{\frac{x}{3}} = 157.8$ **f** $e^{\frac{x}{10}} = 0.016\,82$

g $20 \times e^{0.06x} = 8.312$ **h** $50 \times e^{-0.03x} = 0.816$ **i** $41.83e^{0.652x} = 1000$

F THE CHANGE OF BASE RULE

If $\log_b a = x$, then $b^x = a$

$\therefore\ \log_c b^x = \log_c a$ {taking logarithms in base c}

$\therefore\ x\log_c b = \log_c a$ {power law of logarithms}

$$\therefore\ x = \frac{\log_c a}{\log_c b}$$

So, $$\boldsymbol{\log_b a = \frac{\log_c a}{\log_c b}}$$

Example 23 **Self Tutor**

Find $\log_2 9$ by: **a** letting $\log_2 9 = x$

b using the rule $\log_b a = \dfrac{\log_c a}{\log_c b}$ with: **i** $c = 10$ **ii** $c = e$

a Let $\log_2 9 = x$

$$\begin{aligned} \therefore\ 9 &= 2^x \\ \therefore\ \log 2^x &= \log 9 \\ \therefore\ x\log 2 &= \log 9 \\ \therefore\ x &= \frac{\log 9}{\log 2} \approx 3.17 \end{aligned}$$

b **i** $$\log_2 9 = \frac{\log_{10} 9}{\log_{10} 2} \approx 3.17$$

ii $$\log_2 9 = \frac{\ln 9}{\ln 2} \approx 3.17$$

Example 24 **Self Tutor**

Solve for x: $8^x - 5(4^x) = 0$

$$\begin{aligned} 8^x - 5(4^x) &= 0 \\ \therefore\ 2^{3x} - 5(2^{2x}) &= 0 \\ \therefore\ 2^{2x}(2^x - 5) &= 0 \\ \therefore\ 2^x &= 5 \quad \{\text{as } 2^{2x} > 0 \text{ for all } x\} \\ \therefore\ x &= \log_2 5 \\ \therefore\ x &= \frac{\log 5}{\log 2} \approx 2.32 \end{aligned}$$

EXERCISE 4F

1 Use the rule $\log_b a = \dfrac{\log_{10} a}{\log_{10} b}$ to find, correct to 3 significant figures:

a $\log_3 12$ **b** $\log_{\frac{1}{2}} 1250$ **c** $\log_3(0.067)$ **d** $\log_{0.4}(0.006\,984)$

2 Use the rule $\log_b a = \dfrac{\ln a}{\ln b}$ to solve, correct to 3 significant figures:

a $2^x = 0.051$ **b** $4^x = 213.8$ **c** $3^{2x+1} = 4.069$

Hint: In **2a** $2^x = 0.051$ implies that $x = \log_2(0.051)$.

3 Solve for x:

a $25^x - 3(5^x) = 0$

b $8(9^x) - 3^x = 0$

4 Solve for x:

a $\log_4 x^3 + \log_2 \sqrt{x} = 8$

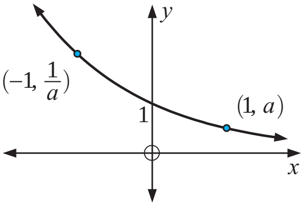
b $\log_{16} x^5 = \log_{64} 125 - \log_4 \sqrt{x}$

5 Find the exact value of x for which $4^x \times 5^{4x+3} = 10^{2x+3}$.

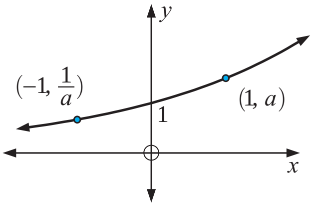
G GRAPHS OF LOGARITHMIC FUNCTIONS

Consider the general exponential function $f(x) = a^x$, $a > 0$, $a \neq 1$. The graph of $y = a^x$ is:

For $0 < a < 1$:



For $a > 1$:



These functions have the **horizontal asymptote** $y = 0$ (the x-axis). They have domain $\mathbb{R}$ (all real numbers) and range $\{y \mid y > 0\}$.

f^{-1} is given by $x = a^y$, so $y = \log_a x$.

$$\textbf{If } \boldsymbol{f(x) = a^x} \textbf{ then } \boldsymbol{f^{-1}(x) = \log_a x}.$$

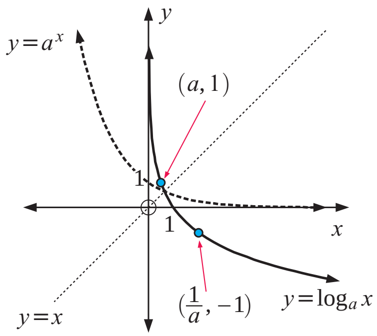
Note:

- The domain of f^{-1} is $\{x \mid x > 0\}$.
  The range of f^{-1} is $y \in \mathbb{R}$.
- The domain of f^{-1} = the range of f.
  The range of f^{-1} = the domain of f.

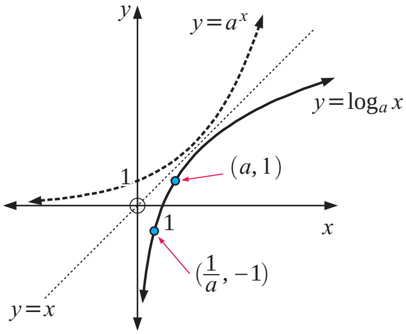
LOGARITHMIC GRAPHS

The graph of $y = \log_a x$ is:

For $0 < a < 1$:



For $a > 1$:



We can see that:

- both graphs are reflections of $y = a^x$ in the line $y = x$
- both functions have domain $\{x \mid x > 0\}$
- we can only find logarithms of positive numbers
- both graphs have the vertical asymptote $x = 0$ (the y-axis)
- for $0 < a < 1$, as $x \to \infty$, $y \to -\infty$ and as $x \to 0$ (from right), $y \to \infty$
- for $a > 1$, as $x \to \infty$, $y \to \infty$ and as $x \to 0$ (from right), $y \to -\infty$
- to find the domain of $\log_a g(x)$, we find the solutions of $g(x) > 0$.

Example 25 **Self Tutor**

Consider the function $f(x) = \log_2(x-1) + 1$.

a Find the domain and range of f.

b Find any asymptotes and axes intercepts.

c Sketch the graph of f showing all important features.

d Find f^{-1} and explain how to verify your answer.

a $x - 1 > 0$ when $x > 1$

So, the domain is $\{x \mid x > 1\}$ and the range is $y \in \mathbb{R}$.

b As $x \to 1$ from the right, $y \to -\infty$, so $x = 1$ is the vertical asymptote.

As $x \to \infty$, $y \to \infty$.

When $x = 0$, y is undefined, so there is no y-intercept.

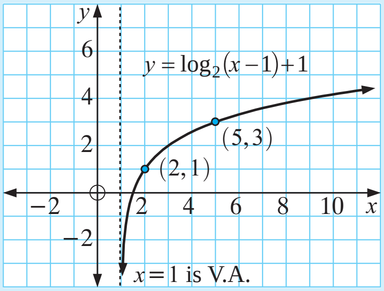
When $y = 0$, $\log_2(x-1) = -1$

$\therefore \quad x - 1 = 2^{-1}$

$\therefore \quad x = 1\frac{1}{2}$ So, the x-intercept is $1\frac{1}{2}$.

c To graph the function using your calculator we need to change the base to base 10.

So, we graph $y = \dfrac{\log(x-1)}{\log 2} + 1$



d f is defined by $y = \log_2(x-1) + 1$

$\therefore$ f^{-1} is defined by $x = \log_2(y-1) + 1$

$\therefore \quad x - 1 = \log_2(y-1)$

$\therefore \quad y - 1 = 2^{x-1}$

$\therefore \quad y = 2^{x-1} + 1$

$\therefore \quad f^{-1}(x) = 2^{x-1} + 1$ which has a H.A. of $y = 1$ ✓

Its domain is $x \in \mathbb{R}$, range is $\{y \mid y > 1\}$.

Graphics calculator tip:

When graphing f, f^{-1} and $y = x$ on the same axes, it is best to set the scale so that $y = x$ makes a 45^o angle with both axes.

To ensure the graphs are not distorted, use a square window.

Example 26 **Self Tutor**

Consider the function $f : x \mapsto e^{x-3}$.

a Find the defining equation of f^{-1}.

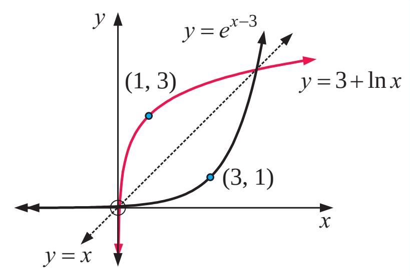
b Sketch the graphs of f and f^{-1} on the same set of axes.

c State the domain and range of f and f^{-1}.

d Find any asymptotes and intercepts.

a
$$f(x) = e^{x-3}$$
$$\therefore\ f^{-1} \text{ is } x = e^{y-3}$$
$$\therefore\ y - 3 = \ln x$$
$$\therefore\ y = 3 + \ln x$$
So, $f^{-1}(x) = 3 + \ln x$

b



c

	f	f^{-1}
domain	$x \in \mathbb{R}$	$x > 0$
range	$y > 0$	$y \in \mathbb{R}$

d For f: H.A. is $y = 0$, y-int is $(0,\ e^{-3})$.

For f^{-1}: V.A. is $x = 0$, x-int is $(e^{-3},\ 0)$.

EXERCISE 4G

1 For the following functions f:

- **i** Find the domain and range.
- **ii** Find any asymptotes and axes intercepts.
- **iii** Sketch the graph of $y = f(x)$ showing all important features.
- **iv** Solve $f(x) = -1$ algebraically and check the solution on your graph.
- **v** Find f^{-1} and explain how to verify your answer.

a $f : x \mapsto \log_3(x+1)$

b $f : x \mapsto 1 - \log_3(x+1)$

c $f : x \mapsto \log_5(x-2) - 2$

d $f : x \mapsto 1 - \log_5(x-2)$

e $f : x \mapsto 1 - \log_2 x^2$

2 For the following functions f:

- **i** Find the defining equation of f^{-1}.
- **ii** Sketch the graphs of f and f^{-1} on the same set of axes.
- **iii** State the domain and range of f and f^{-1}.
- **iv** Find any asymptotes.

a $f(x) \mapsto e^x + 5$

b $f(x) \mapsto e^{x+1} - 3$

c $f(x) \mapsto \ln x - 4$ where $x > 0$

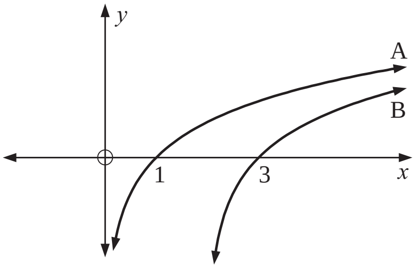
d $f(x) \mapsto \ln(x-1) + 2$ where $x > 1$

3 Given $f : x \mapsto e^{2x}$ and $g : x \mapsto 2x - 1$, find the defining equations of:

a $(f^{-1} \circ g)(x)$

b $(g \circ f)^{-1}(x)$

4 Consider the graphs A and B. One of them is the graph of $y = \ln x$ and the other is the graph of $y = \ln(x-2)$.

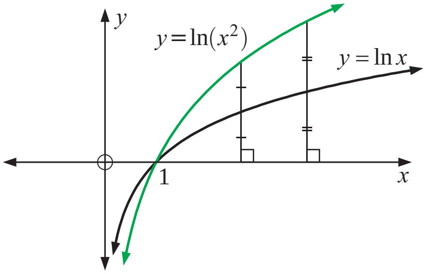


a Identify which is which. Give evidence for your answer.

b Redraw the graphs on a new set of axes and add to them the graph of $y = \ln(x+2)$.

c Find the equation of the vertical asymptote for each graph.

5 Kelly said that in order to graph $y = \ln(x^2)$, you could first graph $y = \ln x$ and then double the distances away from the x-axis. Connecting these points will give the graph of $y = \ln x^2$. Is she correct? Give evidence.



6 For the function $f : x \mapsto e^{x+3} + 2$

a Find the defining equation for f^{-1}.

b Find the values of x for which:

i $f(x) < 2.1$ **ii** $f(x) < 2.01$ **iii** $f(x) < 2.001$ **iv** $f(x) < 2.0001$

and hence conjecture the horizontal asymptote for the graph of f.

c Determine the horizontal asymptote of $f(x)$ by discussing the behaviour of $f(x)$ as $x \to \pm\infty$.

d Hence, determine the vertical asymptote and the domain of f^{-1}.

H GROWTH AND DECAY

In **Chapter 3** we showed how exponential functions can be used to model a variety of growth and decay situations. These included the growth of populations and the decay of radioactive substances. In this section we consider more growth and decay problems, focussing particularly on how logarithms can be used in their solution.

POPULATION GROWTH

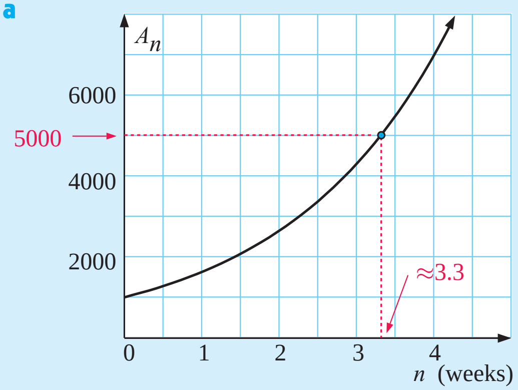
Example 27 **Self Tutor**

A farmer monitoring an insect plague notices that the area affected by the insects is given by $A_n = 1000 \times 2^{0.7n}$ hectares, where n is the number of weeks after the initial observation.

a Draw an accurate graph of A_n against n and use your graph to estimate the time taken for the affected area to reach 5000 ha.

b Check your answer to **a** using logarithms and using suitable technology.

a



b

$$\begin{aligned} \text{When} \quad A_n &= 5000, \\ 1000 \times 2^{0.7n} &= 5000 \\ \therefore \quad 2^{0.7n} &= 5 \\ \therefore \quad \log 2^{0.7n} &= \log 5 \\ \therefore \quad 0.7n \log 2 &= \log 5 \\ \therefore \quad n &= \frac{\log 5}{0.7 \times \log 2} \\ \therefore \quad n &\approx 3.32 \end{aligned}$$

$\therefore$ it takes about 3 weeks and 2 days.

Using technology we find the intersection of $y = 1000 \times 2^{0.7x}$ and $y = 5000$. This confirms $n \approx 3.32$.

FINANCIAL GROWTH

Suppose an amount u_1 is invested at a rate of $r\%$ each compounding period. In this case the value of the investment after n periods is given by $u_{n+1} = u_1 \times r^n$. In order to find n, the **period** of the investment, we need to use **logarithms**.

Example 28

Self Tutor

Iryna has €5000 to invest in an account that pays 5.2% p.a. interest compounded annually. How long will it take for her investment to reach €20 000?

$$\begin{aligned} u_{n+1} &= 20\,000 \quad \text{after } n \text{ years} \\ u_1 &= 5000 \\ r &= 105.2\% = 1.052 \end{aligned}$$

$$\begin{aligned} \text{Now} \quad u_{n+1} &= u_1 \times r^n \\ \therefore \quad 20\,000 &= 5000 \times (1.052)^n \\ \therefore \quad (1.052)^n &= 4 \\ \therefore \quad \log(1.052)^n &= \log 4 \\ \therefore \quad n \times \log 1.052 &= \log 4 \\ \therefore \quad n &= \frac{\log 4}{\log 1.052} \approx 27.3 \text{ years} \end{aligned}$$

$\therefore$ it will take 28 years.

EXERCISE 4H

1 The weight W_t of bacteria in a culture t hours after establishment is given by $W_t = 20 \times 2^{0.15t}$ grams. Find the time for the weight of the culture to reach:

a 30 grams **b** 100 grams.

2 The mass M_t of bacteria in a culture t hours after establishment is given by $M_t = 25 \times e^{0.1t}$ grams. Find the time for the mass of the culture to reach:

a 50 grams **b** 100 grams.

3 A biologist monitoring a fire ant infestation notices that the area affected by the ants is given by $A_n = 2000 \times e^{0.57n}$ hectares, where n is the number of weeks after the initial observation.

- **a** Draw an accurate graph of A_n against n and use your graph to estimate the time taken for the infested area to reach 10 000 ha.
- **b** Find the answer to **a** using logarithms.
- **c** Check your answer to **b** using suitable technology.

4 A house is expected to increase in value at an average rate of 7.5% p.a. If the house is worth £160 000 now, when would you expect it to reach £250 000?

5 Thabo has $10 000 to invest in an account that pays 4.8% p.a. compounded annually. How long will it take for his investment to grow to $15 000?

6 Dien invests $15 000 at 8.4% p.a. compounded *monthly*. He will withdraw his money when it reaches $25 000, at which time he plans to travel. The formula $u_{n+1} = u_1 \times r^n$ can be used to calculate the time needed, where n is the time in months.

- **a** Explain why $r = 1.007$.
- **b** After how many months can he withdraw the money?

7 Revisit the **Opening Problem** on page **120** and answer the questions posed.

8 The mass M_t of radioactive substance remaining after t years is given by $M_t = 1000 \times e^{-0.04t}$ grams. Find the time taken for the mass to:

- **a** halve
- **b** reach 25 grams
- **c** reach 1% of its original value.

9 A man jumps from an aeroplane. His speed of descent is given by $V = 50(1 - e^{-0.2t})$ m s^{-1} where t is the time in seconds. Find the time taken for his speed to reach 40 m s^{-1}.

10 The temperature T of a liquid which has been placed in a refrigerator is given by $T = 4 + 96 \times e^{-0.03t}$ ^{o}C, where t is the time in minutes. Find the time required for the temperature to reach:

- **a** 25^{o}C
- **b** 5^{o}C.

11 The weight W_t of radioactive substance remaining after t years is given by $W_t = 1000 \times 2^{-0.04t}$ grams. Find the time taken for the weight to:

- **a** halve
- **b** reach 20 grams
- **c** reach 1% of its original value.

12 The weight $W(t)$ of radioactive uranium remaining after t years is given by the formula $W(t) = W_0 \times 2^{-0.0002t}$ grams, $t \geqslant 0$. Find the time taken for the original weight to fall to:

- **a** 25% of its original value
- **b** 0.1% of its original value.

13 The current I flowing in a transistor radio t seconds after it is switched off is given by $I = I_0 \times 2^{-0.02t}$ amps. Find the time taken for the current to drop to 10% of its original value.

14 A parachutist jumps from the basket of a stationary hot air balloon. His speed of descent is given by $V = 60(1 - 2^{-0.2t})$ m s^{-1} where t is the time in seconds. Find the time taken for his speed to reach 50 m s^{-1}.

REVIEW SET 4A NON-CALCULATOR

1 Find the following, showing all working.

a $\log_4 64$ **b** $\log_2 256$ **c** $\log_2(0.25)$ **d** $\log_{25} 5$ **e** $\log_8 1$

f $\log_6 6$ **g** $\log_{81} 3$ **h** $\log_9(0.\overline{1})$ **i** $\log_{27} 3$ **j** $\log_k \sqrt{k}$

2 Find:

a $\log \sqrt{10}$ **b** $\log \dfrac{1}{\sqrt[3]{10}}$ **c** $\log(10^a \times 10^{b+1})$

3 Simplify:

a $4\ln 2 + 2\ln 3$ **b** $\frac{1}{2}\ln 9 - \ln 2$ **c** $2\ln 5 - 1$ **d** $\frac{1}{4}\ln 81$

4 Find:

a $\ln(e\sqrt{e})$ **b** $\ln\left(\dfrac{1}{e^3}\right)$ **c** $\ln\left(\dfrac{e}{\sqrt{e^5}}\right)$

5 Write as a single logarithm:

a $\log 16 + 2\log 3$ **b** $\log_2 16 - 2\log_2 3$ **c** $2 + \log_4 5$

6 Write as logarithmic equations:

a $P = 3 \times b^x$ **b** $m = \dfrac{n^3}{p^2}$

7 Simplify:

a $\ln(e^{2x})$ **b** $\ln(e^2 e^x)$ **c** $\ln\left(\dfrac{e}{e^x}\right)$

8 Write the following equations without logarithms:

a $\log T = 2\log x - \log y$ **b** $\log_2 K = \log_2 n + \frac{1}{2}\log_2 t$

9 Write in the form $a\ln k$ where a and k are positive whole numbers and k is prime:

a $\ln 32$ **b** $\ln 125$ **c** $\ln 729$

10 If $A = \log_5 2$ and $B = \log_5 3$, write in terms of A and B:

a $\log_5 36$ **b** $\log_5 54$ **c** $\log_5(8\sqrt{3})$ **d** $\log_5(20.25)$ **e** $\log_5(0.\overline{8})$

REVIEW SET 4B CALCULATOR

1 Write in the form 10^x:

a 32 **b** 0.0013 **c** 8.963×10^{-5}

2 Find x if:

a $\log_2 x = -3$ **b** $\log_5 x \approx 2.743$ **c** $\log_3 x \approx -3.145$

3 Write the following equations without logarithms:

a $\log_2 k \approx 1.699 + x$ **b** $\log_a Q = 3\log_a P + \log_a R$

c $\log A \approx 5\log B - 2.602$

4 Solve for x, giving your answer correct to 4 significant figures:

a $5^x = 7$ **b** $20 \times 2^{2x+1} = 500$

5 The weight of radioactive substance after t years is

$W_t = 2500 \times 3^{-\frac{t}{3000}}$ grams.

a Find the initial weight.

b Find the time taken for the weight to reduce to 30% of its original value.

c Find the percentage weight loss after 1500 years.

d Sketch the graph of W_t against t.

6 Solve for x: $16^x - 5 \times 8^x = 0$

7 Solve for x, giving your answers correct to 3 significant figures:

a $\ln x = 5$ **b** $3 \ln x + 2 = 0$

8 Solve for x, giving your answers correct to 3 significant figures:

a $e^x = 400$ **b** $e^{2x+1} = 11$ **c** $25e^{\frac{x}{2}} = 750$ **d** $e^{2x} = 7e^x - 12$

9 A population of seals is given by $P_t = P_0 2^{\frac{t}{3}}$ where t is the time in years, $t \geqslant 0$. Find:

a the time required for the population to double in size

b the percentage increase in population during the first 4 years.

10 Consider $g : x \mapsto 2e^x - 5$.

a Find the defining equation of g^{-1}.

b Sketch the graphs of g and g^{-1} on the same set of axes.

c State the domain and range of g and g^{-1}.

REVIEW SET 4C

1 Without using a calculator, find the base 10 logarithms of:

a $\sqrt{1000}$ **b** $\dfrac{10}{\sqrt[3]{10}}$ **c** $\dfrac{10^a}{10^{-b}}$

2 Without using a calculator, find:

a $\ln(e^5)$ **b** $\ln(\sqrt{e})$ **c** $\ln\left(\dfrac{1}{e}\right)$

3 Write in the form e^x:

a 20 **b** 3000 **c** 0.075

4 Solve for x:

a $\log x = 3$ **b** $\log_3(x+2) = 1.732$ **c** $\log_2\left(\dfrac{x}{10}\right) = -0.671$

5 Write as a single logarithm:

a $\ln 6 + \ln 4$ **b** $\ln 60 - \ln 20$ **c** $\ln 4 + \ln 1$

d $\ln 200 - \ln 8 + \ln 5$

6 Write as logarithmic equations:

a $M = ab^n$ **b** $T = \dfrac{5}{\sqrt{l}}$ **c** $G = \dfrac{a^2 b}{c}$

7 Solve for x:

a $3^x = 300$ **b** $30 \times 5^{1-x} = 0.15$ **c** $3^{x+2} = 2^{1-x}$

8 Write the following equations without logarithms:

a $\ln P = 1.5 \ln Q + \ln T$ **b** $\ln M = 1.2 - 0.5 \ln N$

9 For the function $g : x \mapsto \log_3(x+2) - 2$:

a Find the domain and range.

b Find any asymptotes and axes intercepts for the graph of the function.

c Sketch the graph of $y = g(x)$.

d Find g^{-1}. Explain how to verify your answer for g^{-1}.

e Sketch the graphs of g, g^{-1} and $y = x$ on the same axes.

10 The weight W_t grams of radioactive substance remaining after t weeks is given by $W_t = 8000 \times e^{-\frac{t}{20}}$ grams. Find the time for the weight to:

a halve **b** reach 1000 g **c** reach 0.1% of its original value.

Chapter 5

Graphing and transforming functions

Syllabus reference: 2.2, 2.3

Contents:

A FAMILIES OF FUNCTIONS

There are several families of functions that you are already familiar with. These include:

Name	*General form*	*Function notation*
Linear	$f(x) = ax + b, \quad a \neq 0$	$f : x \mapsto ax + b, \quad a \neq 0$
Quadratic	$f(x) = ax^2 + bx + c, \quad a \neq 0$	$f : x \mapsto ax^2 + bx + c, \quad a \neq 0$
Cubic	$f(x) = ax^3 + bx^2 + cx + d, \ a \neq 0$	$f : x \mapsto ax^3 + bx^2 + cx + d, \ a \neq 0$
Exponential	$f(x) = a^x, \quad a > 0, \quad a \neq 1$	$f : x \mapsto a^x, \quad a > 0, \quad a \neq 1$
Logarithmic	$f(x) = \log_e x \quad \text{or} \quad f(x) = \ln x$	$f : x \mapsto \ln x$
Reciprocal	$f(x) = \dfrac{k}{x}, \quad x \neq 0$	$f : x \mapsto \dfrac{k}{x}, \quad x \neq 0$

These families of functions have different and distinctive graphs. We can compare them by considering important graphical features such as:

- the **axes intercepts** where the graph cuts the x and y-axes
- **gradients**
- **turning points** which are maxima and minima
- values of x where the function does not exist
- the presence of **asymptotes**, which are lines or curves that the graph approaches.

INVESTIGATION FUNCTION FAMILIES

In this investigation you are encouraged to use the graphing package supplied. Click on the icon to access this package.

GRAPHING PACKAGE

What to do:

1 From the menu, graph on the same set of axes:
$y = 2x + 1, \quad y = 2x + 3, \quad y = 2x - 1$
Comment on all lines of the form $y = 2x + b$.

2 From the menu, graph on the same set of axes:
$y = x + 2, \quad y = 2x + 2, \quad y = 4x + 2, \quad y = -x + 2, \quad y = -\frac{1}{2}x + 2$
Comment on all lines of the form $y = ax + 2$.

3 On the same set of axes graph:
$y = x^2, \quad y = 2x^2, \quad y = \frac{1}{2}x^2, \quad y = -x^2, \quad y = -3x^2, \quad y = -\frac{1}{5}x^2$
Comment on all functions of the form $y = ax^2, \quad a \neq 0$.

4 On the same set of axes graph:
$y = x^2, \quad y = (x-1)^2 + 2, \quad y = (x+1)^2 - 3, \quad y = (x-2)^2 - 1$
and other functions of the form $y = (x-h)^2 + k$ of your choice.
Comment on the functions of this form.

5 On the same set of axes, graph these functions:

a $y = \frac{1}{x}$, $y = \frac{3}{x}$, $y = \frac{10}{x}$

b $y = \frac{-1}{x}$, $y = \frac{-2}{x}$, $y = \frac{-5}{x}$

c $y = \frac{1}{x}$, $y = \frac{1}{x-2}$, $y = \frac{1}{x+3}$

d $y = \frac{1}{x}$, $y = \frac{1}{x} + 2$, $y = \frac{1}{x} - 2$

e $y = \frac{2}{x}$, $y = \frac{2}{x-1} + 2$, $y = \frac{2}{x+2} - 1$

Write a brief report on your discoveries.

Example 1 **Self Tutor**

If $f(x) = x^2$, find in simplest form:

a $f(2x)$ **b** $f\left(\frac{x}{3}\right)$ **c** $2f(x) + 1$ **d** $f(x+3) - 4$

a
$$\begin{aligned} &f(2x) \\ &= (2x)^2 \\ &= 4x^2 \end{aligned}$$

b
$$\begin{aligned} &f\left(\frac{x}{3}\right) \\ &= \left(\frac{x}{3}\right)^2 \\ &= \frac{x^2}{9} \end{aligned}$$

c
$$\begin{aligned} &2\,f(x) + 1 \\ &= 2x^2 + 1 \end{aligned}$$

d
$$\begin{aligned} &f(x+3) - 4 \\ &= (x+3)^2 - 4 \\ &= x^2 + 6x + 9 - 4 \\ &= x^2 + 6x + 5 \end{aligned}$$

EXERCISE 5A

1 If $f(x) = x$, find in simplest form:

a $f(2x)$ **b** $f(x) + 2$ **c** $\frac{1}{2}f(x)$ **d** $2f(x) + 3$

2 If $f(x) = x^2$, find in simplest form:

a $f(3x)$ **b** $f\left(\frac{x}{2}\right)$ **c** $3f(x)$ **d** $2f(x-1) + 5$

3 If $f(x) = x^3$, find in simplest form:

a $f(4x)$ **b** $\frac{1}{2}f(2x)$ **c** $f(x+1)$ **d** $2f(x+1) - 3$

Hint: $(x+1)^3 = x^3 + 3x^2 + 3x + 1$. See the binomial theorem in **Chapter 7**.

4 If $f(x) = 2^x$, find in simplest form:

a $f(2x)$ **b** $f(-x) + 1$ **c** $f(x-2) + 3$ **d** $2f(x) + 3$

5 If $f(x) = \frac{1}{x}$, find in simplest form:

a $f(-x)$ **b** $f(\frac{1}{2}x)$ **c** $2f(x) + 3$ **d** $3f(x-1) + 2$

For the following questions, use the graphing package or your graphics calculator to graph and find the key features of the functions.

GRAPHING PACKAGE

6 Consider $f : x \mapsto 2x + 3$ or $y = 2x + 3$.

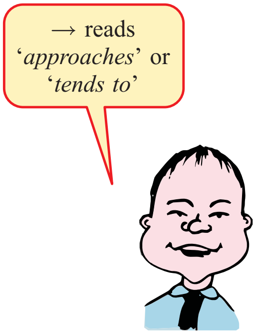
- **a** Graph the function.
- **b** Find algebraically the: **i** x-axis intercept **ii** y-axis intercept **iii** gradient.
- **c** Use technology to check the axes intercepts found in **b**.

7 Consider $f : x \mapsto (x - 2)^2 - 9$.

- **a** Graph the function.
- **b** Find algebraically the axes intercepts.
- **c** Use technology to check that:
  - **i** the x-axis intercepts are -1 and 5
  - **ii** the y-intercept is -5
  - **iii** the vertex is $(2, -9)$.

8 Consider $f : x \mapsto 2x^3 - 9x^2 + 12x - 5$.

- **a** Graph the function.
- **b** Check that:
  - **i** the x-intercepts are 1 and $2\frac{1}{2}$
  - **ii** the y-intercept is -5
  - **iii** the minimum turning point is at $(2, -1)$
  - **iv** the maximum turning point is at $(1, 0)$.



9 Consider $f : x \mapsto 2^x$. Graph the function and check these key features:

- **a** as $x \to \infty$, $2^x \to \infty$
- **b** as $x \to -\infty$, $2^x \to 0$ (from above)
- **c** the y-intercept is 1
- **d** $2^x > 0$ for all x.

10 Consider $f : x \mapsto \ln x$.
Graph the function and then check that:

- **a** as $x \to \infty$, $\ln x \to \infty$
- **b** as $x \to 0$ (from the right), $\ln x \to -\infty$
- **c** $\ln x$ only exists if $x > 0$
- **d** the x-intercept is 1
- **e** the y-axis is an asymptote.

B TRANSFORMATION OF GRAPHS

In the next exercise you should discover the graphical connection between $y = f(x)$ and functions of the form:

- $y = f(x) + b$, b is a constant
- $y = f(x - a)$, a is a constant
- $y = pf(x)$, p is a positive constant
- $y = f\left(\dfrac{x}{q}\right)$, q is a positive constant
- $y = -f(x)$
- $y = f(-x)$

We will see that these forms correspond to **translations**, **stretches**, and **reflections** of the original function $y = f(x)$.

TRANSLATIONS $y = f(x) + b$ AND $y = f(x - a)$

EXERCISE 5B.1

GRAPHING PACKAGE

1 **a** Sketch the graph of $f(x) = x^2$.

b On the same set of axes sketch the graphs of:

i $y = f(x) + 2$ or $y = x^2 + 2$ **ii** $y = f(x) - 3$ or $y = x^2 - 3$.

c What is the connection between the graphs of $y = f(x)$ and $y = f(x) + b$ if:

i $b > 0$ **ii** $b < 0$?

2 For each of the following functions f, sketch on the same set of axes $y = f(x)$, $y = f(x) + 1$ and $y = f(x) - 2$.

a $f(x) = 2^x$ **b** $f(x) = x^3$ **c** $f(x) = \dfrac{1}{x}$ **d** $f(x) = (x - 1)^2$

Summarise your observations by describing the graphical transformation of $y = f(x)$ as it becomes $y = f(x) + b$.

3 **a** On the same set of axes, graph: $f(x) = x^2$, $y = f(x - 3)$ and $y = f(x + 2)$.

b What is the connection between the graphs of $y = f(x)$ and $y = f(x - a)$ if:

i $a > 0$ **ii** $a < 0$?

4 For each of the following functions f, sketch on the same set of axes the graphs of $y = f(x)$, $y = f(x - 1)$ and $y = f(x + 2)$.

a $f(x) = x^3$ **b** $f(x) = \ln x$ **c** $f(x) = \dfrac{1}{x}$ **d** $f(x) = (x + 1)^2 + 2$

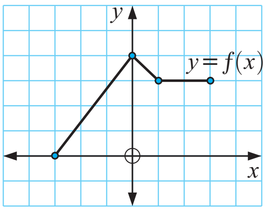
Summarise your observations by describing the geometrical transformation of $y = f(x)$ as it becomes $y = f(x - a)$.

5 For each of the following functions sketch, on the same set of axes:
$y = f(x)$, $y = f(x - 2) + 3$ and $y = f(x + 1) - 4$.

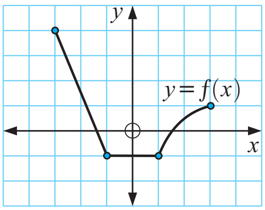
a $f(x) = x^2$ **b** $f(x) = e^x$ **c** $f(x) = \dfrac{1}{x}$

6 Copy these functions and then draw the graph of $y = f(x - 2) - 3$.

a



b



7 Suppose $f(x) = x^2$ is transformed to $g(x) = (x - 3)^2 + 2$.

a Find the images of the following points on $f(x)$:

i $(0, 0)$ **ii** $(-3, 9)$ **iii** where $x = 2$

b Find the points on $f(x)$ which correspond to the following points on $g(x)$:

i $(1, 6)$ **ii** $(-2, 27)$ **iii** $(1\frac{1}{2}, 4\frac{1}{4})$

- For $y = f(x) + b$, the effect of b is to **translate** the graph **vertically** through b units.
  - If $b > 0$ it moves **upwards**.
  - If $b < 0$ it moves **downwards**.
- For $y = f(x - a)$, the effect of a is to **translate** the graph **horizontally** through a units.
  - If $a > 0$ it moves to the **right.**
  - If $a < 0$ it moves to the **left**.
- For $y = f(x - a) + b$, the graph is translated horizontally a units and vertically b units. We say it is **translated by the vector** $\binom{a}{b}$.

STRETCHES $y = pf(x)$, $p > 0$ AND $y = f\left(\frac{x}{q}\right)$, $q > 0$

Stretches are also known as **dilations**.

EXERCISE 5B.2

1 Sketch, on the same set of axes, the graphs of $y = f(x)$, $y = 2f(x)$ and $y = 3f(x)$ for each of:

a $f(x) = x^2$ **b** $f(x) = x^3$ **c** $f(x) = e^x$

d $f(x) = \ln x$ **e** $f(x) = \dfrac{1}{x}$

2 Sketch, on the same set of axes, the graphs of $y = f(x)$, $y = \frac{1}{2}f(x)$ and $y = \frac{1}{4}f(x)$ for each of:

a $f(x) = x^2$ **b** $f(x) = x^3$ **c** $f(x) = e^x$

3 Using **1** and **2**, summarise your observations by describing the graphical transformation of $y = f(x)$ to $y = pf(x)$ for $p > 0$.

4 Sketch, on the same set of axes, the graphs of $y = f(x)$ and $y = f\left(\frac{x}{2}\right)$ for each of:

a $y = x^2$ **b** $y = 2x$ **c** $y = (x + 2)^2$

5 Sketch, on the same set of axes, the graphs of $y = f(x)$ and $y = f(2x)$ for each of:

a $y = x^2$ **b** $y = (x - 1)^2$ **c** $y = (x + 3)^2$

6 Sketch, on the same set of axes, the graphs of $y = f(x)$ and $y = f(3x)$ for each of:

a $y = x$ **b** $y = x^2$ **c** $y = e^x$

7 Using **4**, **5** and **6**, summarise your observations by describing the graphical transformation of $y = f(x)$ to $y = f\left(\dfrac{x}{q}\right)$ for $q > 0$.

8 Consider the function $f : x \mapsto x^2$.
On the same set of axes sketch the graphs of:

a $y = f(x)$, $y = 3f(x - 2) + 1$ and $y = 2f(x + 1) - 3$

b $y = f(x)$, $y = f(x - 3)$, $y = f(\frac{x}{2} - 3)$, $y = 2f(\frac{x}{2} - 3)$ and $y = 2f(\frac{x}{2} - 3) + 4$

c $y = f(x)$ and $y = \frac{1}{4}f(2x + 5) + 1$.

9 **a** Given that the following points lie on $y = f(x)$, find the corresponding points on the image function $y = 3f(2x)$:

i $(3, -5)$ **ii** $(1, 2)$ **iii** $(-2, 1)$

b Find the points on $y = f(x)$ which are moved to the following points under the transformation $y = 3f(2x)$:

i $(2, 1)$ **ii** $(-3, 2)$ **iii** $(-7, 3)$

- For $y = pf(x)$, $p > 0$, the effect of p is to **vertically stretch** the graph by the factor p. p is called the **dilation factor**.
  - If $p > 1$ it moves points of $y = f(x)$ **further away** from the x-axis.
  - If $0 < p < 1$ it moves points of $y = f(x)$ **closer** to the x-axis.
- For $y = f\left(\dfrac{x}{q}\right)$, $q > 0$, the effect of q is to **horizontally stretch** the graph by a factor of q. q is called the **dilation factor**.
  - If $q > 1$ it moves points of $y = f(x)$ **further away** from the y-axis.
  - If $0 < q < 1$ it moves points of $y = f(x)$ **closer** to the y-axis.

REFLECTIONS $y = -f(x)$ AND $y = f(-x)$

EXERCISE 5B.3

1 On the same set of axes, sketch the graphs of:

a $y = 3x$ and $y = -3x$ **b** $y = e^x$ and $y = -e^x$

c $y = x^2$ and $y = -x^2$ **d** $y = \ln x$ and $y = -\ln x$

e $y = x^3 - 2$ and $y = -x^3 + 2$ **f** $y = 2(x+1)^2$ and $y = -2(x+1)^2$.

2 Based on question **1**, what transformation moves $y = f(x)$ to $y = -f(x)$?

3 **a** Find $f(-x)$ for:

i $f(x) = 2x + 1$ **ii** $f(x) = x^2 + 2x + 1$ **iii** $f(x) = x^3$

b Graph $y = f(x)$ and $y = f(-x)$ for:

i $f(x) = 2x + 1$ **ii** $f(x) = x^2 + 2x + 1$ **iii** $f(x) = x^3$

4 Based on question **3**, what transformation moves $y = f(x)$ to $y = f(-x)$?

5 The function $y = f(x)$ is transformed to $g(x) = -f(x)$.

a Find the points on $g(x)$ corresponding to the following points on $f(x)$:

i $(3, 0)$ **ii** $(2, -1)$ **iii** $(-3, 2)$

b Find the points on $f(x)$ that have been transformed to the following points on $g(x)$:

i $(7, -1)$ **ii** $(-5, 0)$ **iii** $(-3, -2)$

6 The function $y = f(x)$ is transformed to $h(x) = f(-x)$.

a Find the image points on $h(x)$ for the following points on $f(x)$:

i $(2, -1)$ **ii** $(0, 3)$ **iii** $(-1, 2)$

b Find the points on $f(x)$ corresponding to the following points on $h(x)$:

i $(5, -4)$ **ii** $(0, 3)$ **iii** $(2, 3)$

7 A function $y = f(x)$ is transformed to the function $y = -f(-x) = g(x)$.

a Describe the nature of the transformation.

b If $(3, -7)$ lies on $y = f(x)$, find the transformed point on $g(x)$.

c Find the point on $f(x)$ that transforms to the point $(-5, -1)$.

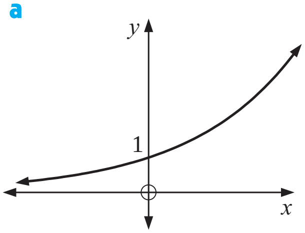
- For $y = -f(x)$, we **reflect** $y = f(x)$ **in the x-axis**.
- For $y = f(-x)$, we **reflect** $y = f(x)$ **in the y-axis**.

MISCELLANEOUS TRANSFORMATIONS

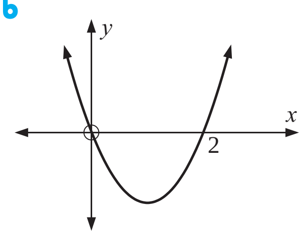
EXERCISE 5B.4

1 Copy the following graphs for $y = f(x)$ and sketch the graphs of $y = -f(x)$ on the same axes.

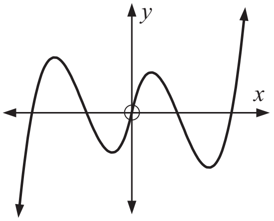
a



b

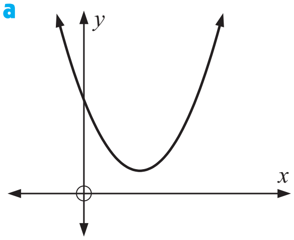


c

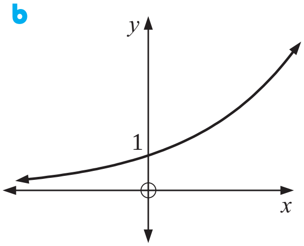


2 Given the following graphs of $y = f(x)$, sketch graphs of $y = f(-x)$:

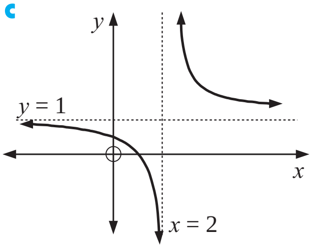
a



b



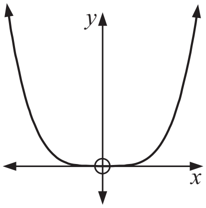
c



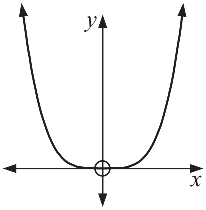
3 The scales on the graphs below are the same. Match each equation to its graph.

A $y = x^4$ **B** $y = 2x^4$ **C** $y = \frac{1}{2}x^4$ **D** $y = 6x^4$

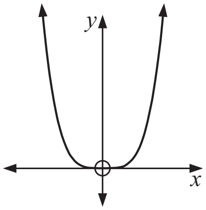
a



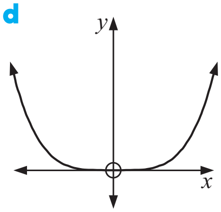
b



c



d

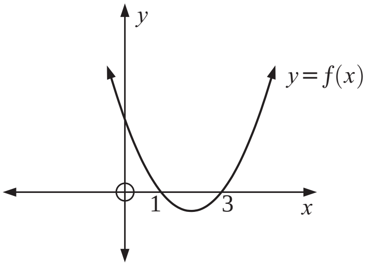


4 For the graph of $y = f(x)$ given, sketch the graph of:

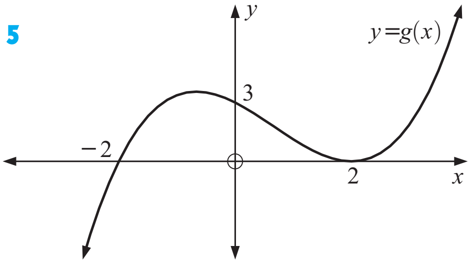
a $y = 2f(x)$ **b** $y = \frac{1}{2}f(x)$

c $y = f(x+2)$ **d** $y = f(2x)$

e $y = f(\frac{1}{2}x)$



5



For the graph of $y = g(x)$ given, sketch the graph of:

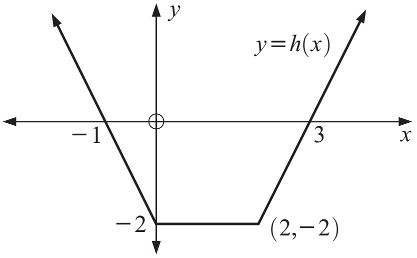
a $y = g(x) + 2$ **b** $y = -g(x)$

c $y = g(-x)$ **d** $y = g(x+1)$

6 For the graph of $y = h(x)$ given, sketch the graph of:

a $y = h(x) + 1$ **b** $y = \frac{1}{2}h(x)$

c $y = h(-x)$ **d** $y = h\left(\frac{x}{2}\right)$



REVIEW SET 5A NON-CALCULATOR

1 If $f(x) = x^2 - 2x$, find in simplest form:

a $f(3)$ **b** $f(-2)$ **c** $f(2x)$ **d** $f(-x)$ **e** $3f(x) - 2$

2 If $f(x) = 5 - x - x^2$, find in simplest form:

a $f(4)$ **b** $f(-1)$ **c** $f(x-1)$ **d** $f\left(\frac{x}{2}\right)$ **e** $2f(x) - f(-x)$

3 Consider $f : x \mapsto 3x - 2$.

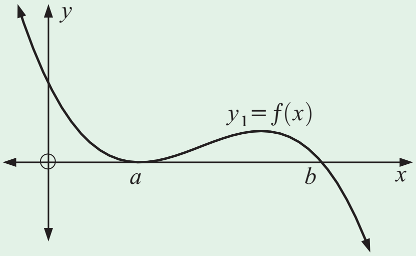
a Sketch the function f.

b Find algebraically the **i** x-intercept **ii** y-intercept **iii** gradient of the line.

c **i** Find y when $x = 0.3$. **ii** Find x when $y = 0.7$.

4 The graph of $f(x) = 3x^3 - 2x^2 + x + 2$ is translated to its image $g(x)$ by the vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Write the equation of $g(x)$ in the form $g(x) = ax^3 + bx^2 + cx + d$.

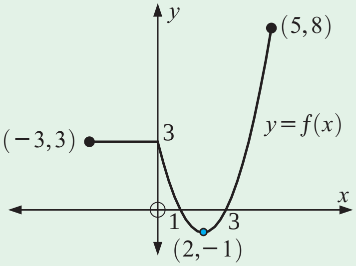
5 The graph of $y_1 = f(x)$ is shown alongside. The x-axis is a tangent to $f(x)$ at $x = a$ and $f(x)$ cuts the x-axis at $x = b$. On the same diagram sketch the graph of $y_2 = f(x - c)$ where $0 < c < b - a$. Indicate the coordinates of the points of intersection of y_2 with the x-axis.



6 For the graph of $y = f(x)$, sketch graphs of:

a $y = f(-x)$ **b** $y = -f(x)$

c $y = f(x + 2)$ **d** $y = f(x) + 2$

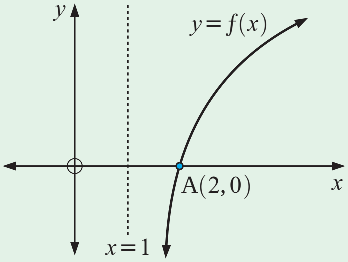


7 The graph of $y = f(x)$ is shown alongside.

a Sketch the graph of $y = g(x)$ where $g(x) = f(x + 3) - 1$.

b State the equation of the vertical asymptote of $y = g(x)$.

c Identify the point A$'$ on the graph of $y = g(x)$ which corresponds to point A.



8 Consider the function $f : x \mapsto x^2$.
On the same set of axes graph:

a $y = f(x)$ **b** $y = f(x - 1)$ **c** $y = 3f(x - 1)$

d $y = 3f(x - 1) + 2$

REVIEW SET 5B CALCULATOR

1 If $f(x) = x^2 - 2$, find in simplest form:

a $f(-3)$ **b** $f(x + 4)$ **c** $3f(x) + 5$

2 Consider $f(x) = (x + 1)^2 - 4$.

a Use your calculator to help graph the function.

b Find: **i** the x-intercepts **ii** the y-intercept.

c What are the coordinates of the vertex of the function?

3 Consider the function $f : x \mapsto x^2$. On the same set of axes graph:

a $y = f(x)$ **b** $y = f(x + 2)$ **c** $y = 2f(x + 2)$ **d** $y = 2f(x + 2) - 3$

4 Consider $f : x \mapsto 2^{-x}$.

- **a** Use your calculator to help graph the function.
- **b** True or false?
  - **i** As $x \to \infty$, $2^{-x} \to 0$.
  - **ii** As $x \to -\infty$, $2^{-x} \to 0$.
  - **iii** The y-intercept is $\frac{1}{2}$.
  - **iv** $2^{-x} > 0$ for all x.

5 The graph of the function $f(x) = (x+1)^2 + 4$ is translated 2 units to the right and 4 units up.

- **a** Find the function $g(x)$ corresponding to the translated graph.
- **b** State the range of $f(x)$.
- **c** State the range of $g(x)$.

6 For each of the following functions:

- **i** Find $y = f(x)$, the result of a translation $\binom{1}{-2}$.
- **ii** Sketch the original function and its translated function on the same set of axes.
- **iii** Clearly state any asymptotes of each function.
- **iv** State the domain and range of each function.

a $y = \dfrac{1}{x}$ **b** $y = 2^x$ **c** $y = \log_4 x$

7 Consider the function $g(x) = (x+3)^2 - 4$.

- **a** Use technology to help sketch a graph of the function.
- **b** Find the axes intercepts.
- **c** Find the coordinates of the vertex of $g(x)$.

8 Sketch the graph of $f(x) = x^2 + 1$, and on the same set of axes sketch the graphs of:

a $-f(x)$ **b** $f(2x)$ **c** $f(x) + 3$

REVIEW SET 5C

1 If $f(x) = \dfrac{4}{x}$, find in simplest form:

a $f(-4)$ **b** $f(2x)$ **c** $f\left(\dfrac{x}{2}\right)$ **d** $4f(x+2) - 3$

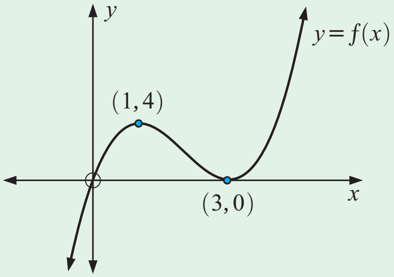
2 Sketch the graph of $f(x) = -x^2$, and on the same set of axes sketch the graph of:

a $y = f(-x)$ **b** $y = -f(x)$ **c** $y = f(2x)$ **d** $y = f(x-2)$

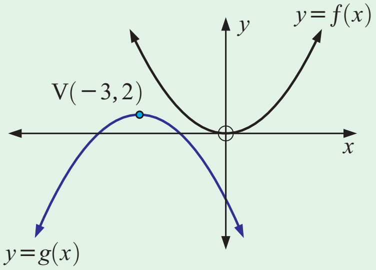
3 The graph of a cubic function $y = f(x)$ is shown alongside.

a Sketch the graph of $g(x) = -f(x-1)$.

b State the coordinates of the turning points of $y = g(x)$.



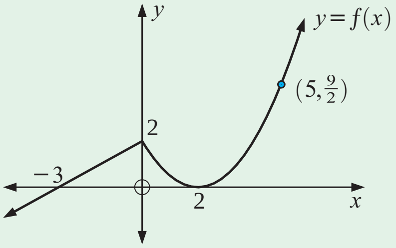
4 The graph of $f(x) = x^2$ is transformed to the graph of $g(x)$ by a reflection and a translation as illustrated in the diagram alongside.
Find the formula for $g(x)$ in the form $g(x) = ax^2 + bx + c$.



5 Given the graph of $y = f(x)$, sketch graphs of:

a $f(-x)$ **b** $f(x+1)$

c $f(x) - 3$.



6 The graph of $f(x) = x^3 + 3x^2 - x + 4$ is translated to its image, $y = g(x)$, by the vector $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.
Write the equation of $g(x)$ in the form $g(x) = ax^3 + bx^2 + cx + d$.

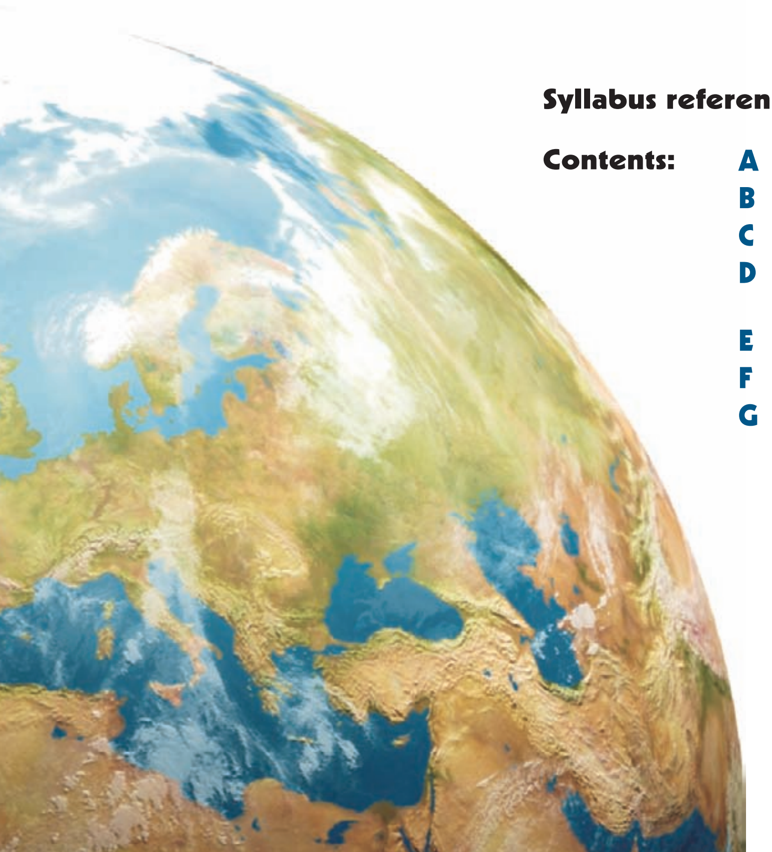
7 **a** Find the equation of the line that results when the line $f(x) = 3x + 2$ is translated:

i 2 units to the left **ii** 6 units upwards.

b Show that when the linear function $f(x) = ax + b$ is translated k units to the left, the resulting line is the same as when $f(x)$ is translated ka units upwards.

Chapter 6

Quadratic equations and functions



Syllabus reference: 2.2, 2.5, 2.6

Contents:

QUADRATICS

A **quadratic equation** is an equation of the form $ax^2 + bx + c = 0$ where a, b and c are constants, $a \neq 0$.

A **quadratic function** is a function of the form $f(x) = ax^2 + bx + c$, $a \neq 0$.

Alternatively, it can be written as $f : x \mapsto ax^2 + bx + c$, $a \neq 0$.

Quadratic functions are members of the family of **polynomials**.

$f : x \mapsto ax^3 + bx^2 + cx + d$, $a \neq 0$ is a *cubic polynomial.*
$f : x \mapsto ax^4 + bx^3 + cx^2 + dx + e$, $a \neq 0$ is a *quartic polynomial.*

HISTORICAL NOTE

Galileo Galilei (1564 - 1642) was born in Pisa, Tuscany. He was a philosopher who played a significant role in the scientific revolution of that time.

Within his research he conducted a series of experiments on the paths of projectiles, attempting to find a mathematical description of falling bodies.

Galileo

Two of Galileo's experiments consisted of rolling a ball down a grooved ramp that was placed at a fixed height above the floor and inclined at a fixed angle to the horizontal. In one experiment the ball left the end of the ramp and descended to the floor. In the second, a horizontal shelf was placed at the end of the ramp, and the ball travelled along this shelf before descending to the floor.

In each experiment Galileo altered the release height h of the ball and measured the distance d the ball travelled before landing. The units of measurement were called 'punti' (points).

In both experiments Galileo found that once the ball left the ramp or shelf, its path was parabolic and could therefore be modelled by a quadratic function.

OPENING PROBLEM

A tennis ball is thrown directly upwards. Its height H above the ground is given at one second intervals in the following table:

t (s)	0	1	2	3	4	5
H (m)	1.8	16.2	24.2	25.8	21	9.8

Things to think about:

- What would the flight of the ball look like from a distance of 50 m away?

- What function $H(t)$ gives the height H in terms of time t? What would its graph look like when H is plotted against t?
- What is the maximum height reached by the ball and when does this occur?
- When is the ball 6 m above the ground?

SIMULATION

A QUADRATIC EQUATIONS

Acme Leather Jacket Co. makes and sells x leather jackets each week and their profit function is given by $P = -12.5x^2 + 550x - 2125$ dollars.

How many jackets must be made and sold each week in order to obtain a weekly profit of \$3000?

Clearly we need to solve the equation:

$$-12.5x^2 + 550x - 2125 = 3000$$

We can rearrange the equation to give

$$12.5x^2 - 550x + 5125 = 0,$$

which is of the form $ax^2 + bx + c = 0$ and is thus a quadratic equation.

To solve quadratic equations we can:

- **factorise** the quadratic and use the **Null Factor law**:

  If $ab = 0$ then $a = 0$ or $b = 0$.
- **complete the square**
- use the **quadratic formula**
- use **technology**.

The **roots** or **solutions** of $ax^2 + bx + c = 0$ are the values of x which satisfy the equation, or make it true.

For example, $x = 2$ is a root of $x^2 - 3x + 2 = 0$ since, when $x = 2$,
$x^2 - 3x + 2 = (2)^2 - 3(2) + 2 = 4 - 6 + 2 = 0$ ✓

SOLVING BY FACTORISATION

Step 1: Make one side of the equation 0 by transferring all terms to one side.
Step 2: Fully factorise the other side.
Step 3: Use the Null Factor law: If $ab = 0$ then $a = 0$ or $b = 0$.
Step 4: Solve the resulting linear equations.

Example 1 Self Tutor

Solve for x: **a** $3x^2+5x=0$ **b** $x^2=5x+6$

a
$$\begin{aligned} 3x^2+5x&=0\\ \therefore\ x(3x+5)&=0\\ \therefore\ x=0 \text{ or } 3x+5&=0\\ \therefore\ x=0 \text{ or } x&=-\tfrac{5}{3} \end{aligned}$$

b
$$\begin{aligned} x^2&=5x+6\\ \therefore\ x^2-5x-6&=0\\ \therefore\ (x-6)(x+1)&=0\\ \therefore\ x&=6 \text{ or } -1 \end{aligned}$$

Example 2 Self Tutor

Solve for x: **a** $4x^2+1=4x$ **b** $6x^2=11x+10$

a
$$\begin{aligned} 4x^2+1&=4x\\ \therefore\ 4x^2-4x+1&=0\\ \therefore\ (2x-1)^2&=0\\ \therefore\ x&=\tfrac{1}{2} \end{aligned}$$

b
$$\begin{aligned} 6x^2&=11x+10\\ \therefore\ 6x^2-11x-10&=0\\ \therefore\ (2x-5)(3x+2)&=0\\ \therefore\ x&=\tfrac{5}{2} \text{ or } -\tfrac{2}{3} \end{aligned}$$

Example 3 Self Tutor

Solve for x:

$3x+\dfrac{2}{x}=-7$

$$\begin{aligned} 3x+\frac{2}{x}&=-7\\ \therefore\ x\left(3x+\frac{2}{x}\right)&=-7x && \{\text{multiplying both sides by } x\}\\ \therefore\ 3x^2+2&=-7x && \{\text{expanding the brackets}\}\\ \therefore\ 3x^2+7x+2&=0 && \{\text{making the RHS } 0\}\\ \therefore\ (x+2)(3x+1)&=0 && \{\text{factorising}\}\\ \therefore\ x&=-2 \text{ or } -\tfrac{1}{3} \end{aligned}$$

EXERCISE 6A.1

1 Solve the following by factorisation:

a $4x^2+7x=0$
b $6x^2+2x=0$
c $3x^2-7x=0$
d $2x^2-11x=0$
e $3x^2=8x$
f $9x=6x^2$
g $x^2-5x+6=0$
h $x^2=2x+8$
i $x^2+21=10x$
j $9+x^2=6x$
k $x^2+x=12$
l $x^2+8x=33$

2 Solve the following by factorisation:

a $9x^2-12x+4=0$
b $2x^2-13x-7=0$
c $3x^2=16x+12$
d $3x^2+5x=2$
e $2x^2+3=5x$
f $3x^2+8x+4=0$
g $3x^2=10x+8$
h $4x^2+4x=3$
i $4x^2=11x+3$
j $12x^2=11x+15$
k $7x^2+6x=1$
l $15x^2+2x=56$

3 Solve for x:

a $(x+1)^2 = 2x^2 - 5x + 11$

b $(x+2)(1-x) = -4$

c $5 - 4x^2 = 3(2x+1) + 2$

d $x + \frac{2}{x} = 3$

e $2x - \frac{1}{x} = -1$

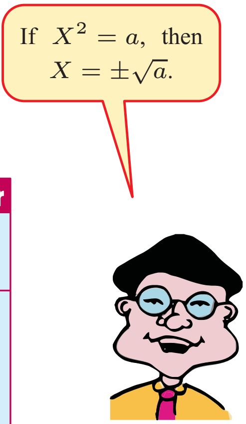
f $\frac{x+3}{1-x} = -\frac{9}{x}$

SOLVING BY 'COMPLETING THE SQUARE'

As you would be aware by now, not all quadratics factorise easily. For example, x^2+4x+1 cannot be factorised by using a simple factorisation approach. This means that we need a different approach in order to solve $x^2+4x+1=0$.

One way is to use the 'completing the square' technique.

Equations of the form $ax^2+bx+c=0$ can be converted to the form $(x+p)^2 = q$ from which the solutions are easy to obtain.



Example 4 **Self Tutor**

Solve exactly for x: **a** $(x+2)^2 = 7$ **b** $(x-1)^2 = -5$

a $(x+2)^2 = 7$

$\therefore \quad x+2 = \pm\sqrt{7}$

$\therefore \quad x = -2 \pm \sqrt{7}$

b $(x-1)^2 = -5$
has no real solutions since the perfect square $(x-1)^2$ cannot be negative.

Example 5 **Self Tutor**

Solve for exact values of x: $x^2+4x+1=0$

$x^2+4x+1=0$

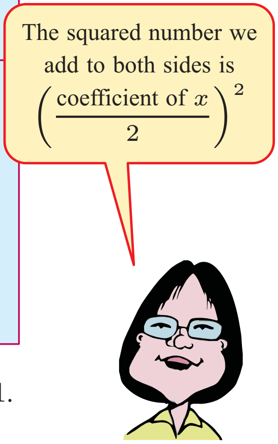
$\therefore \quad x^2+4x = -1$ {put the constant on the RHS}

$\therefore \quad x^2+4x+2^2 = -1+2^2$ {completing the square}

$\therefore \quad (x+2)^2 = 3$ {factorising}

$\therefore \quad x+2 = \pm\sqrt{3}$

$\therefore \quad x = -2 \pm \sqrt{3}$



If the coefficient of x^2 is not 1, we first divide throughout to make it 1.

For example, $2x^2+10x+3=0$ becomes $x^2+5x+\frac{3}{2}=0$

$-3x^2+12x+5=0$ becomes $x^2-4x-\frac{5}{3}=0$

EXERCISE 6A.2

1 Solve for exact values of x:

a $(x+5)^2 = 2$

b $(x+6)^2 = -11$

c $(x-4)^2 = 8$

d $(x-8)^2=7$ **e** $2(x+3)^2=10$ **f** $3(x-2)^2=18$
g $(x+1)^2+1=11$ **h** $(2x+1)^2=3$ **i** $(1-3x)^2-7=0$

2 Solve exactly by completing the square:

a $x^2-4x+1=0$ **b** $x^2+6x+2=0$ **c** $x^2-14x+46=0$
d $x^2=4x+3$ **e** $x^2+6x+7=0$ **f** $x^2=2x+6$
g $x^2+6x=2$ **h** $x^2+10=8x$ **i** $x^2+6x=-11$

3 Solve exactly by completing the square:

a $2x^2+4x+1=0$ **b** $2x^2-10x+3=0$ **c** $3x^2+12x+5=0$
d $3x^2=6x+4$ **e** $5x^2-15x+2=0$ **f** $4x^2+4x=5$

THE QUADRATIC FORMULA

Many quadratic equations cannot be solved by factorising, and completing the square can be rather tedious. Consequently, the **quadratic formula** has been developed. This formula is:

$$\text{If}\quad \boldsymbol{ax^2+bx+c=0},\quad \text{then}\quad \boldsymbol{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}.$$

Proof: If $ax^2+bx+c=0$,

$$\text{then}\quad x^2+\frac{b}{a}x+\frac{c}{a}=0 \qquad \{\text{dividing each term by } a, \text{ as } a\neq 0\}$$

$$\therefore\quad x^2+\frac{b}{a}x=-\frac{c}{a}$$

$$\therefore\quad x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2$$

$$\therefore\quad \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2} \qquad \{\text{completing the square on LHS}\}$$

$$\therefore\quad x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$\therefore\quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

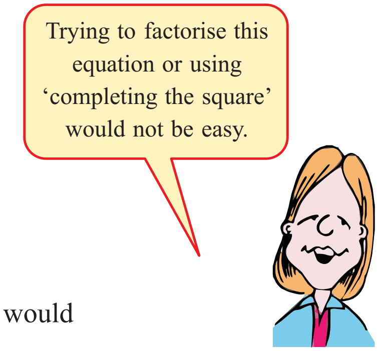
For example, consider the Acme Leather Jacket Co. equation from page **157**.

We need to solve: $12.5x^2-550x+5125=0$
so in this case $a=12.5,\quad b=-550,\quad c=5125$

$$\therefore\quad x=\frac{550\pm\sqrt{(-550)^2-4(12.5)(5125)}}{2(12.5)}$$

$$=\frac{550\pm\sqrt{46\,250}}{25}$$

$$\approx 30.60 \quad\text{or}\quad 13.40$$



However, x needs to be a whole number, so $x=13$ or 31 would produce a profit of around \$3000 each week.

Example 6 **Self Tutor**

Solve for x: **a** $x^2 - 2x - 6 = 0$ **b** $2x^2 + 3x - 6 = 0$

a $x^2 - 2x - 6 = 0$ has
$a = 1,\ b = -2,\ c = -6$

$$\therefore\ x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-6)}}{2(1)}$$

$$\therefore\ x = \frac{2 \pm \sqrt{4 + 24}}{2}$$

$$\therefore\ x = \frac{2 \pm \sqrt{28}}{2}$$

$$\therefore\ x = \frac{2 \pm 2\sqrt{7}}{2}$$

$$\therefore\ x = 1 \pm \sqrt{7}$$

b $2x^2 + 3x - 6 = 0$ has
$a = 2,\ b = 3,\ c = -6$

$$\therefore\ x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-6)}}{2(2)}$$

$$\therefore\ x = \frac{-3 \pm \sqrt{9 + 48}}{4}$$

$$\therefore\ x = \frac{-3 \pm \sqrt{57}}{4}$$

EXERCISE 6A.3

1 Use the quadratic formula to solve exactly for x:

a $x^2 - 4x - 3 = 0$ **b** $x^2 + 6x + 7 = 0$ **c** $x^2 + 1 = 4x$
d $x^2 + 4x = 1$ **e** $x^2 - 4x + 2 = 0$ **f** $2x^2 - 2x - 3 = 0$
g $(3x + 1)^2 = -2x$ **h** $(x + 3)(2x + 1) = 9$

2 Use the quadratic formula to solve exactly for x:

a $(x + 2)(x - 1) = 2 - 3x$ **b** $(2x + 1)^2 = 3 - x$ **c** $(x - 2)^2 = 1 + x$
d $\dfrac{x - 1}{2 - x} = 2x + 1$ **e** $x - \dfrac{1}{x} = 1$ **f** $2x - \dfrac{1}{x} = 3$

SOLVING USING TECHNOLOGY

You can use your graphics calculator to solve quadratic equations.

If the right hand side is zero, you can graph the expression on the left hand side. The x-intercepts of the graph will be the solutions to the quadratic.

If the right hand side is non-zero, you can either:

- rearrange the equation so the right hand side is zero, then graph the expression and find the x-intercepts, or
- graph the expressions on the left and right hand sides on the same set of axes, then find the x-coordinates of the point where they meet.

Instructions for these tasks can be found in the graphics calculator chapter at the front of the book.

Use technology to check your answers to **Exercise 6A.3** above.

B THE DISCRIMINANT OF A QUADRATIC

In the quadratic formula, the quantity $b^2 - 4ac$ under the square root sign is called the **discriminant**.

The symbol **delta** Δ is used to represent the discriminant, so $\Delta = b^2 - 4ac$.

The quadratic formula becomes $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$ where Δ replaces $b^2 - 4ac$.

- If $\Delta = 0$, $x = \dfrac{-b}{2a}$ is the **only solution** (a **repeated** or **double root**)
- If $\Delta > 0$, $\sqrt{\Delta}$ is a positive real number, so there are **two distinct real roots**:
  $$\frac{-b+\sqrt{\Delta}}{2a} \quad \text{and} \quad \frac{-b-\sqrt{\Delta}}{2a}$$
- If $\Delta < 0$, $\sqrt{\Delta}$ is not a real number and so there are **no real roots**.
- If a, b and c are rational and Δ is a **perfect square** then the equation has two rational roots which can be found by factorisation.

Example 7 **Self Tutor**

Use the discriminant to determine the nature of the roots of:

a $2x^2 - 2x + 3 = 0$

b $3x^2 - 4x - 2 = 0$

a
$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (-2)^2 - 4(2)(3)\\ &= -20 \text{ which is } < 0\end{aligned}$$
$\therefore$ there are no real roots

b
$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (-4)^2 - 4(3)(-2)\\ &= 40 \text{ which is } > 0\end{aligned}$$
40 is not a perfect square so there are 2 distinct irrational roots.

Example 8 **Self Tutor**

For $x^2 - 2x + m = 0$, find Δ and hence find the values of m for which the equation has:

a a repeated root

b 2 distinct real roots

c no real roots.

$x^2 - 2x + m = 0$ has $a = 1$, $b = -2$ and $c = m$

$\therefore \quad \Delta = b^2 - 4ac = (-2)^2 - 4(1)(m) = 4 - 4m$

a For a repeated root
$$\begin{aligned}\Delta &= 0\\ \therefore \quad 4 - 4m &= 0\\ \therefore \quad 4 &= 4m\\ \therefore \quad m &= 1\end{aligned}$$

b For 2 distinct real roots
$$\begin{aligned}\Delta &> 0\\ \therefore \quad 4 - 4m &> 0\\ \therefore \quad -4m &> -4\\ \therefore \quad m &< 1\end{aligned}$$

c For no real roots
$$\begin{aligned}\Delta &< 0\\ \therefore \quad 4 - 4m &< 0\\ \therefore \quad -4m &< -4\\ \therefore \quad m &> 1\end{aligned}$$

Summary:

Roots of quadratic	*Discriminant value*
two real distinct roots	$\Delta > 0$
two identical real roots (repeated)	$\Delta = 0$
no real roots	$\Delta < 0$

These outcomes correspond to a quadratic factorising into 2 distinct linear factors, factorising into 2 identical linear factors, or not factorising at all.

Example 9

Self Tutor

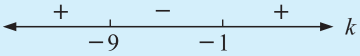
For the equation $kx^2 + (k+3)x = 1$ find the discriminant Δ and draw a sign diagram for it. Hence, find the value of k for which the equation has:

a two distinct real roots
b two real roots
c a repeated root
d no real roots.

For $kx^2 + (k+3)x - 1 = 0$, $\quad a = k,\ b = (k+3),\ c = -1$

So,
$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (k+3)^2 - 4(k)(-1)\\ &= k^2 + 6k + 9 + 4k\\ &= k^2 + 10k + 9\\ &= (k+9)(k+1)\end{aligned}$$

and has sign diagram:



a For two distinct real roots, $\Delta > 0$ $\therefore$ $k < -9$ or $k > -1$.
b For two real roots, $\Delta \geqslant 0$ $\therefore$ $k \leqslant -9$ or $k \geqslant -1$.
c For a repeated root, $\Delta = 0$ $\therefore$ $k = -9$ or $k = -1$.
d For no real roots, $\Delta < 0$ $\therefore$ $-9 < k < -1$.

EXERCISE 6B

1 By using the discriminant only, state the nature of the solutions of:

a $x^2 + 7x - 3 = 0$
b $3x^2 + 2x - 1 = 0$
c $5x^2 + 4x - 3 = 0$
d $x^2 + x + 5 = 0$
e $16x^2 - 8x + 1 = 0$

2 By using the discriminant only, determine which of the following quadratic equations have rational roots which can be found by factorisation.

a $6x^2 - 5x - 6 = 0$
b $2x^2 - 7x - 5 = 0$
c $3x^2 + 4x + 1 = 0$
d $6x^2 - 47x - 8 = 0$
e $4x^2 - 3x + 2 = 0$
f $8x^2 + 2x - 3 = 0$

3 For the following quadratic equations, determine Δ in simplest form and draw a sign diagram for it. Hence find the value of m for which the equation has:

i a repeated root
ii two distinct real roots
iii no real roots.

a $x^2 + 4x + m = 0$
b $mx^2 + 3x + 2 = 0$
c $mx^2 - 3x + 1 = 0$

4 For the following quadratic equations, find the discriminant Δ and hence draw a sign diagram for it. Find all k values for which the equation has:

i two distinct real roots
ii two real roots
iii a repeated root
iv no real roots.

a $2x^2 + kx - k = 0$
b $kx^2 - 2x + k = 0$
c $x^2 + (k+2)x + 4 = 0$
d $2x^2 + (k-2)x + 2 = 0$
e $x^2 + (3k-1)x + (2k+10) = 0$
f $(k+1)x^2 + kx + k = 0$

C GRAPHING QUADRATIC FUNCTIONS

REVIEW OF TERMINOLOGY

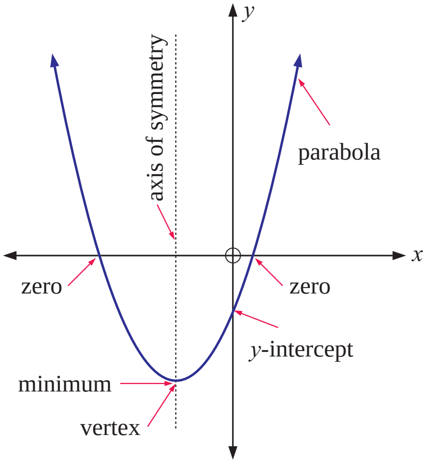
The equation of a **quadratic function** is given by $y = ax^2 + bx + c$, where $a \neq 0$.

The graph of a quadratic function is called a **parabola**. The point where the graph 'turns' is called the **vertex**.

If the graph opens upward, the y-coordinate of the vertex is the **minimum** and the graph is concave upwards.

If the graph opens downward, the y-coordinate of the vertex is the **maximum** and the graph is concave downwards.

The vertical line that passes through the vertex is called the **axis of symmetry**. All parabolas are symmetrical about the axis of symmetry.



The point where the graph crosses the y-axis is the ***y*-intercept**.

The points (if they exist) where the graph crosses the x-axis are called the ***x*-intercepts**, and correspond to the **zeros** of the function.

INVESTIGATION 1 GRAPHING $y = a(x-p)(x-q)$

This investigation is best done using a **graphing package** or **graphics calculator**.

GRAPHING PACKAGE

What to do:

1 **a** Use technology to help you to sketch:
$y = (x-1)(x-3)$, $y = 2(x-1)(x-3)$, $y = -(x-1)(x-3)$,
$y = -3(x-1)(x-3)$ and $y = -\frac{1}{2}(x-1)(x-3)$

b Find the x-intercepts for each function in **a**.

c What is the geometrical significance of a in $y = a(x-1)(x-3)$?

2 **a** Use technology to help you to sketch:
$y = 2(x-1)(x-4)$, $y = 2(x-3)(x-5)$, $y = 2(x+1)(x-2)$,
$y = 2x(x+5)$ and $y = 2(x+2)(x+4)$

b Find the x-intercepts for each function in **a**.

c What is the geometrical significance of p and q in $y = 2(x-p)(x-q)$?

3 **a** Use technology to help you to sketch:
$y = 2(x-1)^2$, $y = 2(x-3)^2$, $y = 2(x+2)^2$, $y = 2x^2$

b Find the x-intercepts for each function in **a**.

c What is the geometrical significance of p in $y = 2(x-p)^2$?

4 Copy and complete:

- If a quadratic has the form $y = a(x-p)(x-q)$ it ...... the x-axis at ......
- If a quadratic has the form $y = a(x-p)^2$ it ...... the x-axis at ......

INVESTIGATION 2 GRAPHING $y = a(x-h)^2+k$

This investigation is also best done using technology.

GRAPHING PACKAGE

What to do:

1 **a** Use technology to help you to sketch:
$y = (x-3)^2+2$, $y = 2(x-3)^2+2$, $y = -2(x-3)^2+2$,
$y = -(x-3)^2+2$ and $y = -\frac{1}{3}(x-3)^2+2$

b Find the coordinates of the vertex for each function in **a**.

c What is the geometrical significance of a in $y = a(x-3)^2+2$?

2 **a** Use technology to help you to sketch:
$y = 2(x-1)^2+3$, $y = 2(x-2)^2+4$, $y = 2(x-3)^2+1$,
$y = 2(x+1)^2+4$, $y = 2(x+2)^2-5$ and $y = 2(x+3)^2-2$

b Find the coordinates of the vertex for each function in **a**.

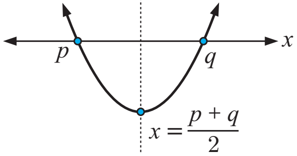
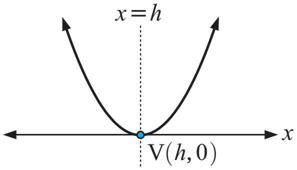
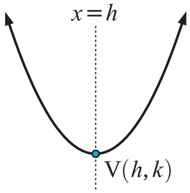
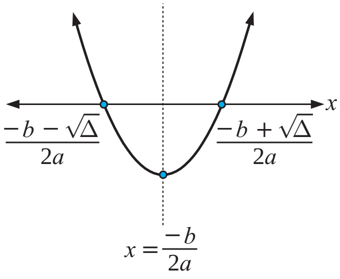
c What is the geometrical significance of h and k in $y = 2(x-h)^2+k$?

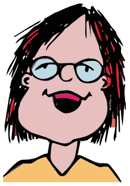
3 Copy and complete:
If a quadratic has the form $y = a(x-h)^2+k$ then its vertex has coordinates ......
The graph of $y = a(x-h)^2+k$ is a ...... of the graph of $y = ax^2$ with vector ....

From **Investigations 1** and **2** you should have discovered that the coefficient of x^2 (which is a) controls the width of the graph and whether it opens upwards or downwards.

- $a > 0$ produces ∪ or concave up. $a < 0$ produces ∩ or concave down.
- If $-1 < a < 1$, $a \neq 0$ the graph is wider than $y = x^2$.
If $a < -1$ or $a > 1$ the graph is narrower than $y = x^2$.

For different quadratic forms we can summarise the properties of their graphs as follows:

Quadratic form, $a \neq 0$	*Graph*	*Facts*
• $y = a(x-p)(x-q)$ p, q are real		x-intercepts are p and q axis of symmetry is $x = \frac{p+q}{2}$ vertex is $\left(\frac{p+q}{2}, f\left(\frac{p+q}{2}\right)\right)$
• $y = a(x-h)^2$ h is real		touches x-axis at h axis of symmetry is $x = h$ vertex is $(h, 0)$
• $y = a(x-h)^2 + k$		axis of symmetry is $x = h$ vertex is (h, k) translation of $y = ax^2$ with vector $\begin{pmatrix} h \\ k \end{pmatrix}$
• $y = ax^2 + bx + c$		y-intercept c axis of symmetry is $x = \frac{-b}{2a}$ x-intercepts for $\Delta \geqslant 0$ are $\frac{-b \pm \sqrt{\Delta}}{2a}$ where $\Delta = b^2 - 4ac$



$-\frac{b}{2a}$ is the **average** of

$\frac{-b-\sqrt{\Delta}}{2a}$ and $\frac{-b+\sqrt{\Delta}}{2a}$

irrespective of the sign of Δ.

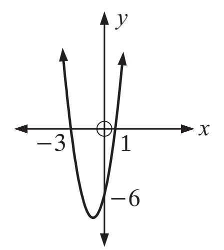
SKETCHING GRAPHS USING KEY FACTS

Example 10 **Self Tutor**

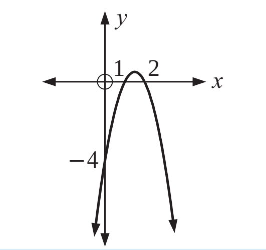
Using axes intercepts only, sketch the graphs of:

a $y = 2(x+3)(x-1)$ **b** $y = -2(x-1)(x-2)$ **c** $y = \frac{1}{2}(x+2)^2$

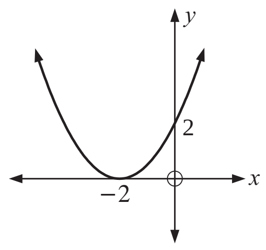
a $y = 2(x+3)(x-1)$
has x-intercepts -3, 1
When $x = 0$,
$y = 2(3)(-1)$
$= -6$
$\therefore$ y-intercept is -6



b $y = -2(x-1)(x-2)$
has x-intercepts 1, 2
When $x = 0$,
$y = -2(-1)(-2)$
$= -4$
$\therefore$ y-intercept is -4



c $y = \frac{1}{2}(x+2)^2$
touches x-axis at -2
When $x = 0$,
$y = \frac{1}{2}(2)^2$
$= 2$
$\therefore$ y-intercept is 2



Example 11

Self Tutor

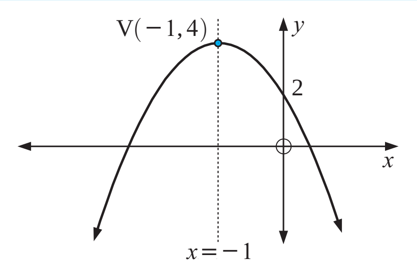
Use the vertex, axis of symmetry and y-intercept to graph $y = -2(x+1)^2 + 4$.

The vertex is $(-1, 4)$.

The axis of symmetry is $x = -1$.

When $x = 0$, $y = -2(1)^2 + 4$
$= 2$

$a < 0$ so the shape is 



Example 12

Self Tutor

Determine the coordinates of the vertex of $y = 2x^2 - 8x + 1$.

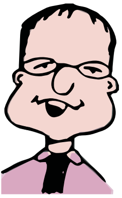
$y = 2x^2 - 8x + 1$ has $a = 2$, $b = -8$, $c = 1$

$$\text{and so } \frac{-b}{2a} = \frac{-(-8)}{2 \times 2} = 2$$

$\therefore$ equation of axis of symmetry is $x = 2$
and when $x = 2$, $y = 2(2)^2 - 8(2) + 1$
$= -7$

$\therefore$ the vertex has coordinates $(2, -7)$.

The vertex is called the **maximum turning point** or the **minimum turning point** depending on whether the graph is concave down or concave up.



Example 13 **Self Tutor**

For the quadratic $y = 2x^2 + 6x - 3$, find:

a the equation of the axis of symmetry
b the coordinates of the vertex
c the axes intercepts.
d Hence, sketch the graph.

For $y = 2x^2 + 6x - 3$, $a = 2$, $b = 6$, $c = -3$.

$a > 0$ so the shape is ∪

a $\dfrac{-b}{2a} = \frac{-6}{4} = -\frac{3}{2}$

$\therefore$ axis of symmetry is $x = -\frac{3}{2}$.

b When $x = -\frac{3}{2}$,

$$y = 2(-\tfrac{3}{2})^2 + 6(-\tfrac{3}{2}) - 3$$
$$= -7\tfrac{1}{2} \quad \{\text{simplifying}\}$$

$\therefore$ vertex is $(-\frac{3}{2}, -7\frac{1}{2})$.

c When $x = 0$, $y = -3$

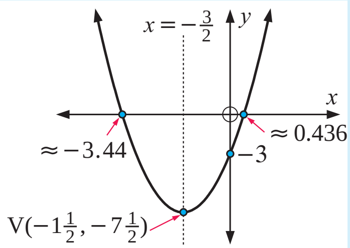
$\therefore$ y-intercept is -3.

When $y = 0$, $2x^2 + 6x - 3 = 0$

$$\therefore \quad x = \frac{-6 \pm \sqrt{36 - 4(2)(-3)}}{4}$$

$$\therefore \quad x \approx -3.44 \text{ or } 0.436$$

d



EXERCISE 6C.1

1 Using axes intercepts only, sketch the graphs of:

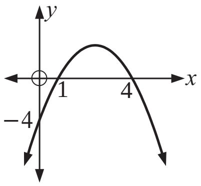
a $y = (x - 4)(x + 2)$
b $y = -(x - 4)(x + 2)$
c $y = 2(x + 3)(x + 5)$
d $y = -3x(x + 4)$
e $y = 2(x + 3)^2$
f $y = -\frac{1}{4}(x + 2)^2$

2 State the equation of the axis of symmetry for each graph in question **1**.

3 Match the given graphs with the possible formulae stated:

a $y = 2(x - 1)(x - 4)$
b $y = -(x + 1)(x - 4)$
c $y = (x - 1)(x - 4)$
d $y = (x + 1)(x - 4)$
e $y = 2(x + 4)(x - 1)$
f $y = -3(x + 4)(x - 1)$
g $y = -(x - 1)(x - 4)$
h $y = -3(x - 1)(x - 4)$

A



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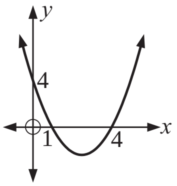
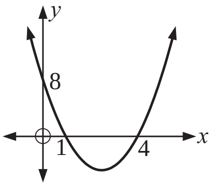


C



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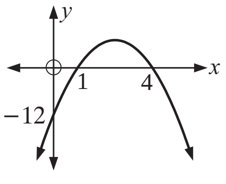

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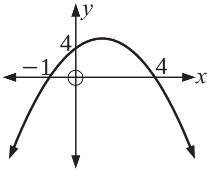

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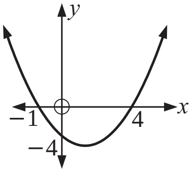
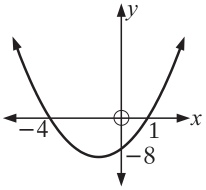
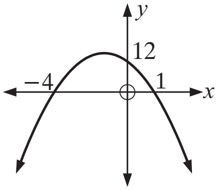


G



H



4 Use the vertex, axis of symmetry and y-intercept to graph:

a $y=(x-1)^2+3$ **b** $y=2(x+2)^2+1$ **c** $y=-2(x-1)^2-3$

d $y=\frac{1}{2}(x-3)^2+2$ **e** $y=-\frac{1}{3}(x-1)^2+4$ **f** $y=-\frac{1}{10}(x+2)^2-3$

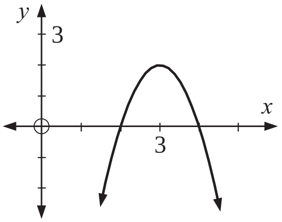
5 Match each quadratic function with its corresponding graph:

a $y=-(x+1)^2+3$ **b** $y=-2(x-3)^2+2$ **c** $y=x^2+2$

d $y=-(x-1)^2+1$ **e** $y=(x-2)^2-2$ **f** $y=\frac{1}{3}(x+3)^2-3$

g $y=-x^2$ **h** $y=-\frac{1}{2}(x-1)^2+1$ **i** $y=2(x+2)^2-1$

A



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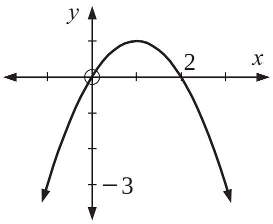

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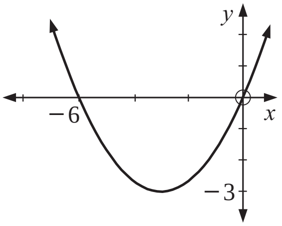

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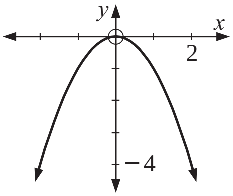

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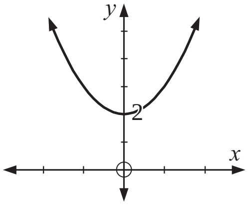

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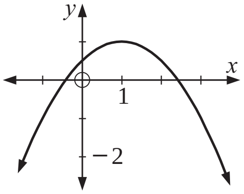

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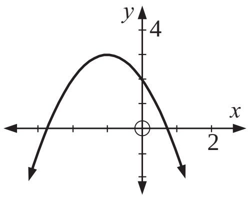

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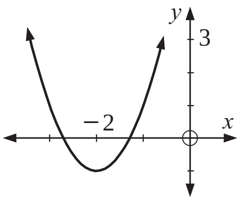
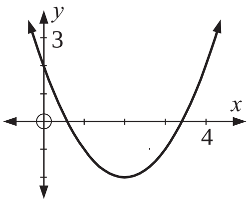


I



6 Find the turning point or vertex for the following quadratic functions:

a $y=x^2-4x+2$ **b** $y=x^2+2x-3$ **c** $y=2x^2+4$

d $y=-3x^2+1$ **e** $y=2x^2+8x-7$ **f** $y=-x^2-4x-9$

g $y=2x^2+6x-1$ **h** $y=2x^2-10x+3$ **i** $y=-\frac{1}{2}x^2+x-5$

7 Find the x-intercepts for:

a $y = x^2 - 9$
b $y = 2x^2 - 6$
c $y = x^2 + 7x + 10$
d $y = x^2 + x - 12$
e $y = 4x - x^2$
f $y = -x^2 - 6x - 8$
g $y = -2x^2 - 4x - 2$
h $y = 4x^2 - 24x + 36$
i $y = x^2 - 4x + 1$
j $y = x^2 + 4x - 3$
k $y = x^2 - 6x - 2$
l $y = x^2 + 8x + 11$

8 For the following quadratics, find:

i the equation of the axis of symmetry
ii the coordinates of the vertex
iii the axes intercepts, if they exist.
iv Hence, sketch the graph.

a $y = x^2 - 2x + 5$
b $y = x^2 + 4x - 1$
c $y = 2x^2 - 5x + 2$
d $y = -x^2 + 3x - 2$
e $y = -3x^2 + 4x - 1$
f $y = -2x^2 + x + 1$
g $y = 6x - x^2$
h $y = -x^2 - 6x - 8$
i $y = -\frac{1}{4}x^2 + 2x + 1$

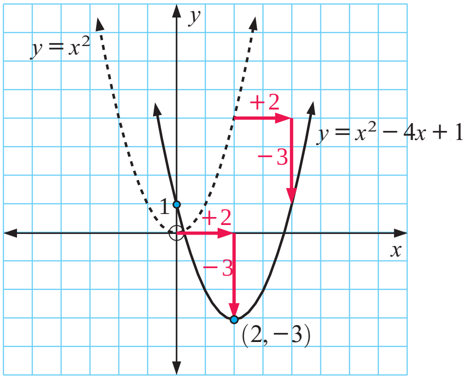
SKETCHING GRAPHS BY 'COMPLETING THE SQUARE'

If we wish to find the vertex of a quadratic given in general form $y = ax^2 + bx + c$ then one approach is to convert it to the form $y = a(x-h)^2 + k$ where we can read off the vertex (h, k). To do this we may choose to '**complete the square**'.

Consider the simple case $y = x^2 - 4x + 1$, for which $a = 1$.

$$y = x^2 - 4x + 1$$
$$\therefore\ y = \underbrace{x^2 - 4x + 2^2}\ \underbrace{+1 - 2^2}$$
$$\therefore\ y = \quad (x-2)^2 \qquad -3$$

The graph of $y = x^2 - 4x + 1$ is the graph of $y = x^2$ translated through $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

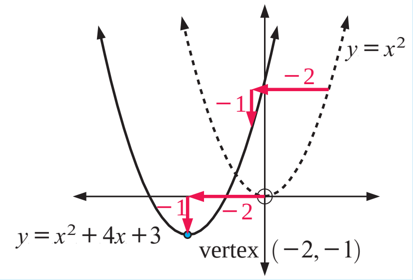


Example 14 Self Tutor

Write $y = x^2 + 4x + 3$ in the form $y = (x-h)^2 + k$ by 'completing the square'.
Hence sketch $y = x^2 + 4x + 3$, stating the coordinates of the vertex.

$$y = x^2 + 4x + 3$$
$$\therefore\ y = x^2 + 4x + 2^2 + 3 - 2^2$$
$$\therefore\ y = \underset{\substack{\downarrow \\ \text{shift 2} \\ \text{units left}}}{(x+2)^2}\ \underset{\substack{\downarrow \\ \text{shift 1} \\ \text{unit down}}}{-\,1}$$

Vertex is $(-2, -1)$ and y-intercept is 3.



Example 15 **Self Tutor**

Convert $y = 3x^2 - 4x + 1$ into the form $y = a(x-h)^2 + k$ by 'completing the square'. Hence, write down the coordinates of its vertex and sketch the graph of the function.

$y = 3x^2 - 4x + 1$

$= 3[x^2 - \frac{4}{3}x + \frac{1}{3}]$ {taking out a factor of 3}

$= 3[x^2 - 2(\frac{2}{3})x + (\frac{2}{3})^2 - (\frac{2}{3})^2 + \frac{1}{3}]$ {completing the square}

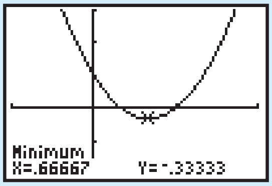
$= 3[(x - \frac{2}{3})^2 - \frac{4}{9} + \frac{3}{9}]$ {writing as a perfect square}

$= 3[(x - \frac{2}{3})^2 - \frac{1}{9}]$

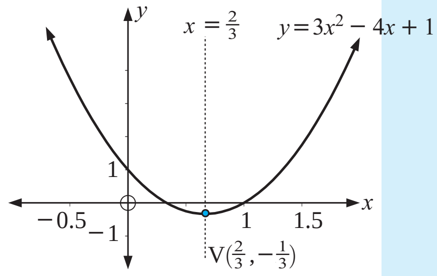
$= 3(x - \frac{2}{3})^2 - \frac{1}{3}$

The vertex is $(\frac{2}{3}, -\frac{1}{3})$.

The y-intercept is 1.



Minimum
X=.66667 Y=-.33333



EXERCISE 6C.2

1 Write the following quadratics in the form $y = (x-h)^2 + k$ by 'completing the square'. Hence sketch each function, stating the vertex:

a $y = x^2 - 2x + 3$ **b** $y = x^2 + 4x - 2$ **c** $y = x^2 - 4x$

d $y = x^2 + 3x$ **e** $y = x^2 + 5x - 2$ **f** $y = x^2 - 3x + 2$

g $y = x^2 - 6x + 5$ **h** $y = x^2 + 8x - 2$ **i** $y = x^2 - 5x + 1$

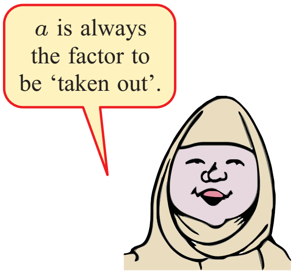
2 For each of the following quadratics:

- **i** convert into the form $y = a(x-h)^2 + k$ by 'completing the square'
- **ii** state the coordinates of the vertex
- **iii** find the y-intercept
- **iv** sketch the graph of the quadratic
- **v** use technology to check your answer.

a $y = 2x^2 + 4x + 5$ **b** $y = 2x^2 - 8x + 3$

c $y = 2x^2 - 6x + 1$ **d** $y = 3x^2 - 6x + 5$

e $y = -x^2 + 4x + 2$ **f** $y = -2x^2 - 5x + 3$



a is always the factor to be 'taken out'.

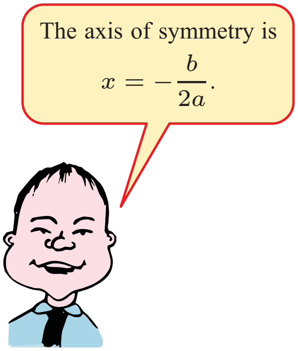
3 Use your **graphing package** or **graphics calculator** to graph each of the following functions. *Hence* write each function in the form $y = a(x-h)^2 + k$.

GRAPHING PACKAGE

a $y = x^2 - 4x + 7$ **b** $y = x^2 + 6x + 3$ **c** $y = -x^2 + 4x + 5$

d $y = 2x^2 + 6x - 4$ **e** $y = -2x^2 - 10x + 1$ **f** $y = 3x^2 - 9x - 5$

THE DISCRIMINANT AND THE QUADRATIC GRAPH

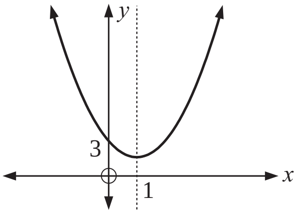
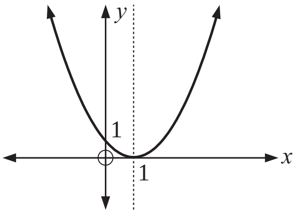
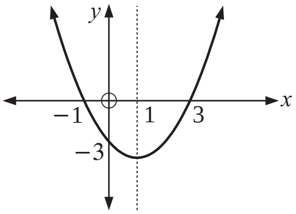


The axis of symmetry is $x = -\dfrac{b}{2a}$.

Consider the graphs of:

$$y = x^2 - 2x + 3, \quad y = x^2 - 2x + 1, \quad y = x^2 - 2x - 3.$$

All of these curves have the same axis of symmetry: $x = 1$.

$y = x^2 - 2x + 3$	$y = x^2 - 2x + 1$	$y = x^2 - 2x - 3$
		
$\Delta = b^2 - 4ac$ $= (-2)^2 - 4(1)(3)$ $= -8$	$\Delta = b^2 - 4ac$ $= (-2)^2 - 4(1)(1)$ $= 0$	$\Delta = b^2 - 4ac$ $= (-2)^2 - 4(1)(-3)$ $= 16$
$\Delta < 0$	$\Delta = 0$	$\Delta > 0$
does not cut the x-axis	touches the x-axis	cuts the x-axis twice

The **discriminant** Δ determines if the graph:

- does not cut the x-axis $(\Delta < 0)$
- touches the x-axis $(\Delta = 0)$
- cuts the x-axis twice $(\Delta > 0)$.

Example 16 **Self Tutor**

Use the discriminant to determine the relationship between the graph and the x-axis for: **a** $y = x^2 + 3x + 4$ **b** $y = -2x^2 + 5x + 1$

a $a = 1, \quad b = 3, \quad c = 4$

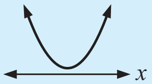
$\therefore \quad \Delta = b^2 - 4ac$

$= 9 - 4(1)(4)$

$= -7$ which is < 0

The graph does not cut the x-axis.

$a > 0 \quad \therefore$ concave up



It lies entirely above the x-axis.

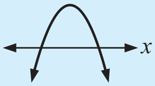
b $a = -2, \quad b = 5, \quad c = 1$

$\therefore \quad \Delta = b^2 - 4ac$

$= 25 - 4(-2)(1)$

$= 33$ which is > 0

$a < 0 \quad \therefore$ concave down

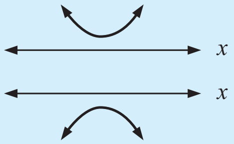


$\therefore$ the graph cuts the x-axis twice.

POSITIVE DEFINITE AND NEGATIVE DEFINITE QUADRATICS

Positive definite quadratics are quadratics which are positive for all values of x. So, $ax^2+bx+c>0$ for all $x \in \mathbb{R}$.

Negative definite quadratics are quadratics which are negative for all values of x. So, $ax^2+bx+c<0$ for all $x \in \mathbb{R}$.



Tests:

- A quadratic is **positive definite** if $a>0$ and $\Delta<0$.
- A quadratic is **negative definite** if $a<0$ and $\Delta<0$.

EXERCISE 6C.3

1 Use the discriminant to determine the relationship between the graph and x-axis for:

a $y=x^2+7x-2$ **b** $y=x^2+8x+16$ **c** $y=-2x^2+3x+1$

d $y=6x^2+5x-4$ **e** $y=-x^2+x+6$ **f** $y=9x^2+6x+1$

2 Show that:

a $x^2-3x+6>0$ for all x **b** $4x-x^2-6<0$ for all x

c $2x^2-4x+7$ is positive definite **d** $-2x^2+3x-4$ is negative definite.

3 Explain why $3x^2+kx-1$ is never positive definite for any value of k.

4 Under what conditions is $2x^2+kx+2$ positive definite?

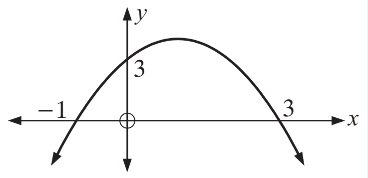
D FINDING A QUADRATIC FROM ITS GRAPH

If we are given sufficient information on or about a graph we can determine the quadratic function in whatever form is required.

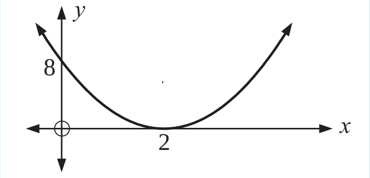
Example 17 **Self Tutor**

Find the equation of the quadratic with graph:

a



b

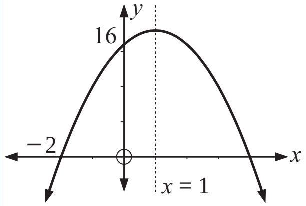


a Since the x-intercepts are -1 and 3,
$y=a(x+1)(x-3)$, $a<0$.
But when $x=0$, $y=3$
$\therefore\ 3=a(1)(-3)$
$\therefore\ a=-1$
So, $y=-(x+1)(x-3)$.

b Since it touches the x-axis at 2,
$y=a(x-2)^2$, $a>0$.
But when $x=0$, $y=8$
$\therefore\ 8=a(-2)^2$
$\therefore\ a=2$
So, $y=2(x-2)^2$.

Example 18 Self Tutor

Find the equation of the quadratic with graph:



The axis of symmetry is $x = 1$, so the other x-intercept is 4

$\therefore \quad y = a(x+2)(x-4)$

But when $x = 0$, $y = 16$

$\therefore \quad 16 = a(2)(-4)$

$\therefore \quad a = -2$

$\therefore$ the quadratic is $y = -2(x+2)(x-4)$.

Example 19 Self Tutor

Find, in the form $y = ax^2 + bx + c$, the equation of the quadratic whose graph cuts the x-axis at 4 and -3 and passes through the point $(2, -20)$.

Since the x-intercepts are 4 and -3, the equation is

$$y = a(x-4)(x+3) \quad \text{where} \quad a \neq 0.$$

But when $x = 2$, $y = -20$ $\quad \therefore \quad -20 = a(2-4)(2+3)$

$\therefore \quad -20 = a(-2)(5)$

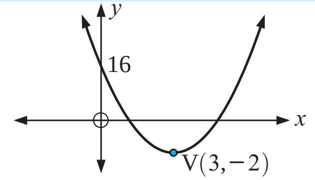
$\therefore \quad a = 2$

$\therefore$ the equation is $y = 2(x-4)(x+3)$ or $y = 2x^2 - 2x - 24$.

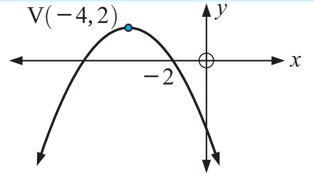
Example 20 Self Tutor

Find the equation of the quadratic given its graph is:

a



b



a For a vertex $(3, -2)$ the quadratic has the form

$y = a(x-3)^2 - 2$

But when $x = 0$, $y = 16$

$\therefore \quad 16 = a(-3)^2 - 2$

$\therefore \quad 16 = 9a - 2$

$\therefore \quad 18 = 9a$

$\therefore \quad a = 2$

So, $y = 2(x-3)^2 - 2$.

b For a vertex $(-4, 2)$ the quadratic has the form

$y = a(x+4)^2 + 2$

But when $x = -2$, $y = 0$

$\therefore \quad 0 = a(2)^2 + 2$

$\therefore \quad 4a = -2$

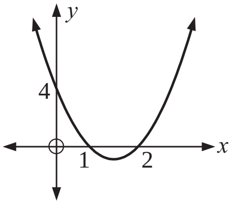
$\therefore \quad a = -\frac{1}{2}$

So, $y = -\frac{1}{2}(x+4)^2 + 2$.

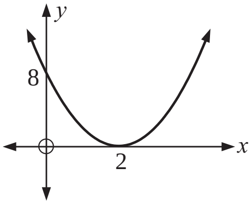
EXERCISE 6D

1 Find the equation of the quadratic with graph:

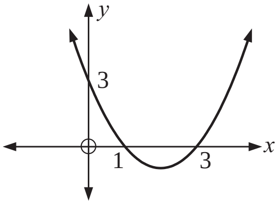
a



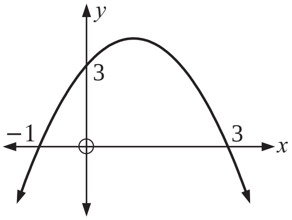
b



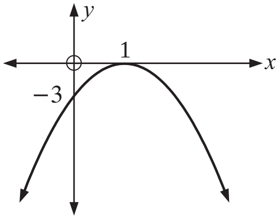
c



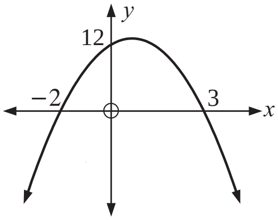
d



e

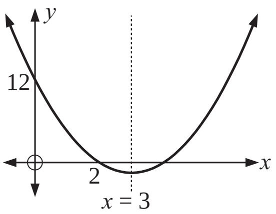


f

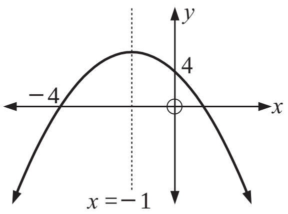


2 Find the quadratic with graph:

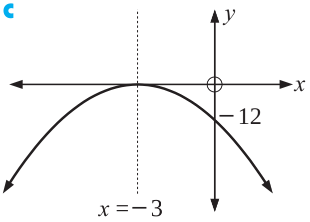
a



b



c

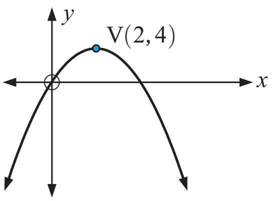


3 Find, in the form $y = ax^2 + bx + c$, the equation of the quadratic whose graph:

- **a** cuts the x-axis at 5 and 1, and passes through $(2, -9)$
- **b** cuts the x-axis at 2 and $-\frac{1}{2}$, and passes through $(3, -14)$
- **c** touches the x-axis at 3 and passes through $(-2, -25)$
- **d** touches the x-axis at -2 and passes through $(-1, 4)$
- **e** cuts the x-axis at 3, passes through $(5, 12)$ and has axis of symmetry $x = 2$
- **f** cuts the x-axis at 5, passes through $(2, 5)$ and has axis of symmetry $x = 1$.

4 If V is the vertex, find the equation of the quadratic given its graph is:

a



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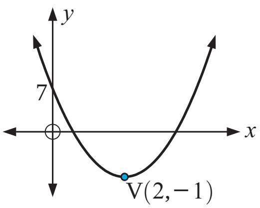
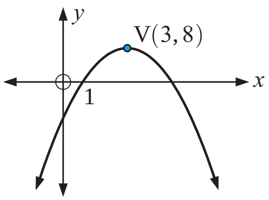
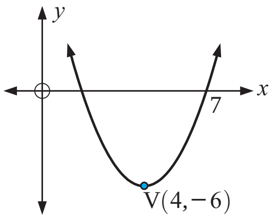


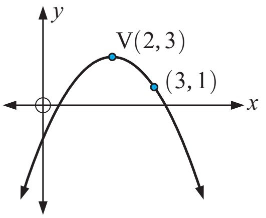
c



d



e



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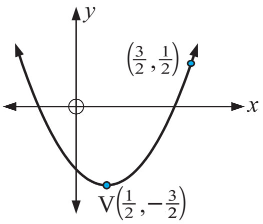


INVESTIGATION 3 FINDING QUADRATIC FUNCTIONS

$y = 2x^2 + 3x + 7$ is a quadratic function from which the following table of values is obtained:

x	0	1	2	3	4	5
y	7	12	21	34	51	72

Consider adding two further rows to this table:

- a row called Δ_1 which gives differences between successive y-values, and
- a row called Δ_2 which gives differences between successive Δ_1-values.

So, we have:

x	0	1	2	3	4	5
y	7	12	21	34	51	72
Δ_1		5	9	13	17	21
Δ_2		4	4	4	4	

$9 - 5$ $34 - 21$ $72 - 51$

This table is known as a **difference table**.

What to do:

1 Construct difference tables for $x = 0, 1, 2, 3, 4, 5$ for each of the following quadratic functions:

a $y = x^2 + 4x + 3$
b $y = 3x^2 - 4x$
c $y = 5x - x^2$
d $y = 4x^2 - 5x + 2$

2 What do you notice about the Δ_2 row for the quadratic functions in **1**?

3 Consider the general quadratic $y = ax^2 + bx + c$, $a \neq 0$.

a Copy and complete the following difference table:

x	0	1	2	3	4	5
y	ⓒ	$a + b + c$	$4a + 2b + c$	......	......	......
Δ_1		○	......	......	......	......
Δ_2		○	......	......	......	

b Comment on the Δ_2 row.

c What can the encircled numbers be used for?

4 Use your observations in **3** to determine, if possible, the quadratic functions with the following tables of values:

a

x	0	1	2	3	4
y	6	5	8	15	26

b

x	0	1	2	3	4
y	8	10	18	32	52

c

x	0	1	2	3	4
y	1	2	-1	-8	-18

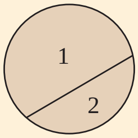
d

x	0	1	2	3	4
y	5	3	-1	-7	-15

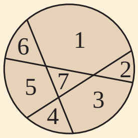
5 **Cutting up Pizzas**

We wish to determine the **maximum** number of pieces into which a pizza can be cut using n cuts across it.

For example, for $n = 1$ we have which has 2 pieces



for $n = 3$ we have which has 7 pieces.



a Copy and complete:

Number of cuts, n	0	1	2	3	4	5
Maximum number of pieces, P_n						

b Complete the Δ_1 and Δ_2 rows. Hence determine a quadratic formula for P_n.

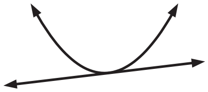
c For a huge pizza with 12 cuts across it, find the maximum number of pieces which can result.

E WHERE FUNCTIONS MEET

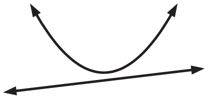
Consider the graphs of a quadratic function and a linear function on the same set of axes.

Notice that we could have:

cutting
(2 points of intersection)



touching
(1 point of intersection)



missing
(no points of intersection)

If the graphs meet, the coordinates of the points of intersection of the graphs of the two functions can be found by *solving the two equations simultaneously*.

Example 21 **Self Tutor**

Find the coordinates of the points of intersection of the graphs with equations $y = x^2 - x - 18$ and $y = x - 3$.

$y = x^2 - x - 18$ meets $y = x - 3$ where

$$x^2 - x - 18 = x - 3$$
$$\therefore \quad x^2 - 2x - 15 = 0 \qquad \{\text{RHS} = 0\}$$
$$\therefore \quad (x - 5)(x + 3) = 0 \qquad \{\text{factorising}\}$$
$$\therefore \quad x = 5 \text{ or } -3$$

Substituting into $y = x - 3$, when $x = 5$, $y = 2$ and when $x = -3$, $y = -6$.

$\therefore$ the graphs meet at (5, 2) and (−3, −6).

Example 22 **Self Tutor**

$y = 2x + k$ is a tangent to $y = 2x^2 - 3x + 4$. Find k.

$y = 2x + k$ meets $y = 2x^2 - 3x + 4$ where

$$2x^2 - 3x + 4 = 2x + k$$

$$\therefore \quad 2x^2 - 5x + (4 - k) = 0$$

Now this quadratic has $\Delta = 0$ since the graphs touch.

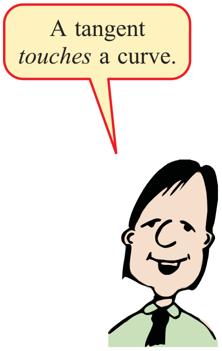
$$\therefore \quad (-5)^2 - 4(2)(4 - k) = 0$$

$$\therefore \quad 25 - 8(4 - k) = 0$$

$$\therefore \quad 25 - 32 + 8k = 0$$

$$\therefore \quad 8k = 7$$

$$\therefore \quad k = \tfrac{7}{8}$$



EXERCISE 6E

1 Find the coordinates of the point(s) of intersection of the graphs with equations:

a $y = x^2 - 2x + 8$ and $y = x + 6$ **b** $y = -x^2 + 3x + 9$ and $y = 2x - 3$

c $y = x^2 - 4x + 3$ and $y = 2x - 6$ **d** $y = -x^2 + 4x - 7$ and $y = 5x - 4$

2 Use a **graphing package** or a **graphics calculator** to find the coordinates of the points of intersection (to 2 decimal places) of the graphs with equations:

a $y = x^2 - 3x + 7$ and $y = x + 5$

b $y = x^2 - 5x + 2$ and $y = x - 7$

c $y = -x^2 - 2x + 4$ and $y = x + 8$

d $y = -x^2 + 4x - 2$ and $y = 5x - 6$

GRAPHING PACKAGE

3 Find, by algebraic means, the points of intersection of the graphs with equations:

a $y = x^2$ and $y = x + 2$ **b** $y = x^2 + 2x - 3$ and $y = x - 1$

c $y = 2x^2 - x + 3$ and $y = 2 + x + x^2$ **d** $xy = 4$ and $y = x + 3$

4 Use technology to check your solutions to the questions in **3**.

5 Find possible values of c for which the lines $y = 3x + c$ are tangents to the parabola with equation $y = x^2 - 5x + 7$.

6 Find the values of m for which the lines $y = mx - 2$ are tangents to the curve with equation $y = x^2 - 4x + 2$.

7 Find the gradients of the lines with y-intercept $(0, 1)$ that are tangents to the curve $y = 3x^2 + 5x + 4$.

8 **a** For what values of c do the lines $y = x + c$ never meet the parabola with equation $y = 2x^2 - 3x - 7$?

b Choose one of the values of c found in part **a** above. Sketch the graphs using technology to illustrate that these curves never meet.

F PROBLEM SOLVING WITH QUADRATICS

When solving some problems algebraically, a quadratic equation results. We are generally only interested in any **real solutions** which result. If the resulting quadratic equation has no real roots then the problem has no real solution.

Any answer we obtain must be checked to see if it is reasonable. For example:

- if we are finding a length then it must be positive and we reject any negative solutions
- if we are finding 'how many people are present' then clearly the answer must be an integer.

We employ the following general problem solving method:

Step 1: If the information is given in words, translate it into algebra using a pronumeral such as x for the unknown. Write down the resulting equation.

Step 2: Solve the equation by a suitable method.

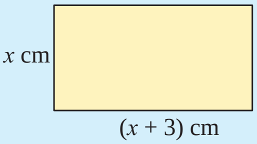
Step 3: Examine the solutions carefully to see if they are acceptable.

Step 4: Give your answer in a sentence.

Example 23 **Self Tutor**

A rectangle has length 3 cm longer than its width. Its area is 42 cm^2. Find its width.

If the width is x cm then the length is $(x+3)$ cm.



$$\therefore \quad x(x+3) = 42 \quad \{\text{equating areas}\}$$
$$\therefore \quad x^2 + 3x - 42 = 0$$
$$\therefore \quad x \approx -8.15 \text{ or } 5.15 \quad \{\text{using technology}\}$$

We reject the negative solution as lengths are positive.

So, the width ≈ 5.15 cm.

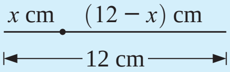
Example 24 **Self Tutor**

Is it possible to bend a 12 cm length of wire to form the legs of a right angled triangle with area 20 cm^2?

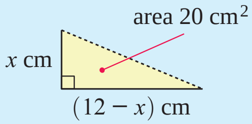
Suppose the wire is bent x cm from one end.

The area $A = \frac{1}{2}x(12-x)$

$$\therefore \quad \tfrac{1}{2}x(12-x) = 20$$
$$\therefore \quad x(12-x) = 40$$



becomes



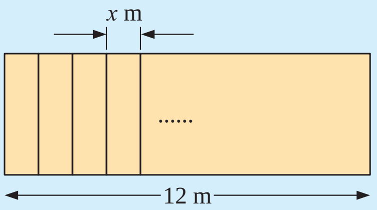
$$\therefore \quad 12x - x^2 - 40 = 0$$
$$\therefore \quad x^2 - 12x + 40 = 0 \quad \text{which has} \quad \Delta = (-12)^2 - 4(1)(40)$$
$$= -16 \quad \text{which is} \ < 0$$

There are no real solutions, indicating this situation is **impossible**.

Example 25 **Self Tutor**

A wall is 12 m long. It is panelled using vertical sheets of timber which have equal width. If the sheets had been 0.2 m wider, 2 less sheets would have been required.

What is the width of the timber panelling used?



Let x m be the width of each sheet.

$\therefore \ \dfrac{12}{x}$ is the number of sheets needed.

Now if the sheets had been $\left(x + \frac{1}{5}\right)$ m in width,

$\left(\dfrac{12}{x} - 2\right)$ sheets would have been needed.

$$\begin{aligned}
\text{So,}\quad \left(x+\frac{1}{5}\right)\left(\frac{12}{x}-2\right) &= 12 && \{\text{length of wall}\}\\
\therefore\quad 12 - 2x + \frac{12}{5x} - \tfrac{2}{5} &= 12 && \{\text{expanding LHS}\}\\
\therefore\quad -2x + \frac{12}{5x} - \tfrac{2}{5} &= 0\\
\therefore\quad -10x^2 + 12 - 2x &= 0 && \{\times\ \text{each term by } 5x\}\\
\therefore\quad 5x^2 + x - 6 &= 0 && \{\div\ \text{each term by } -2\}\\
\therefore\quad (5x+6)(x-1) &= 0\\
\therefore\quad x &= -\tfrac{6}{5} \text{ or } 1 && \text{where}\quad x > 0
\end{aligned}$$

So, each sheet is 1 m wide.

EXERCISE 6F

1 Two integers differ by 12 and the sum of their squares is 74. Find the integers.

2 The sum of a number and its reciprocal is $5\frac{1}{5}$. Find the number.

3 The sum of a natural number and its square is 210. Find the number.

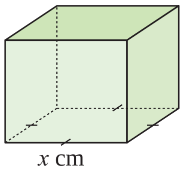
4 The product of two consecutive even numbers is 360. Find the numbers.

5 The product of two consecutive odd numbers is 255. Find the numbers.

6 The number of diagonals of an n-sided polygon is given by the formula $D = \dfrac{n}{2}(n-3)$.
A polygon has 90 diagonals. How many sides does it have?

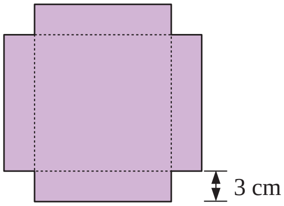
7 The length of a rectangle is 4 cm longer than its width. Find its width given that its area is 26 cm^2.

8 A rectangular box has a square base, and its height is 1 cm longer than the length of each side of its base.



- **a** If each side of its base has length x cm, show that its total surface area is given by $A = 6x^2 + 4x$ cm^2.
- **b** If the total surface area is 240 cm^2, find the dimensions of the box.

9

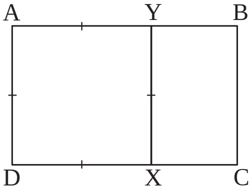


An open box can hold 80 cm^3. It is made from a square piece of tinplate with 3 cm squares cut from each of its 4 corners. Find the dimensions of the original piece of tinplate.

10 Is it possible to bend a 20 cm length of wire into the shape of a rectangle which has an area of 30 cm^2?

11 The **golden rectangle** is the rectangle which can be divided into a square and a smaller rectangle by a line which is parallel to its shorter sides. The smaller rectangle is **similar** to the original rectangle.

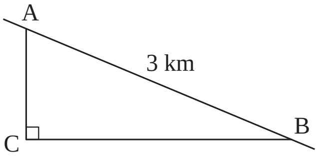
Thus, if ABCD is the golden rectangle, ADXY is a square and BCXY is similar to ABCD.



The ratio of $\frac{AB}{AD}$ for the golden rectangle is called the **golden ratio**.

Show that the golden ratio is $\frac{1+\sqrt{5}}{2}$. **Hint:** Let AB $= x$ units and AD $= 1$ unit.

12



A triangular paddock has a road (AB) forming its longest side. [AB] is 3 km long. The fences [AC] and [CB] are at right angles. If [BC] is 400 m longer than [AC], find the area of the paddock in hectares.

13 A uniform concrete path is paved around a 30 m by 40 m rectangular lawn. The concrete has area one quarter of the lawn. Find the width of the path.

14 Chuong and Hassan both drive 40 km from home to work each day. One day Chuong said to Hassan, "If you drive home at your usual speed, I will average 40 km h^{-1} faster than you and arrive home in 20 minutes less time." Find Hassan's speed.

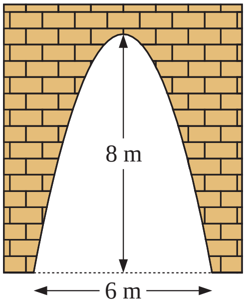
15 If the average speed of a small aeroplane had been 120 km h^{-1} less, it would have taken a half an hour longer to fly 1000 km. Find the speed of the plane.

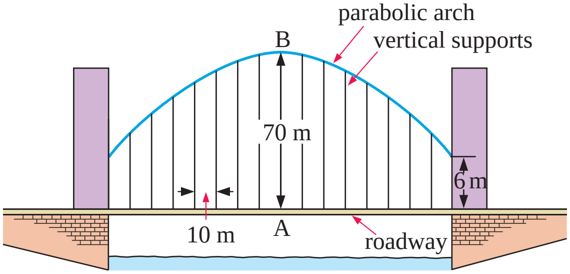
16 Two trains travel a 160 km track each day. The express travels 10 km h^{-1} faster and takes 30 minutes less time than the normal train. Find the speed of the express.

17 

A group of elderly citizens chartered a bus for \$160. However, at the last minute 8 of them fell ill and had to miss the trip.
As a consequence, the other citizens had to pay an extra \$1 each. How many elderly citizens went on the trip?

18 A tunnel is parabolic in shape with dimensions shown.
A truck carrying a wide load needs to pass through the tunnel.
The truck is 4.8 m high and 3.9 m wide.
Determine whether the truck will fit.



19 

[AB] is the longest vertical support of a bridge which contains a parabolic arch. The vertical supports are 10 m apart. The arch meets the vertical end supports 6 m above the road.

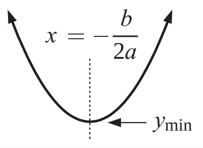
a If axes are drawn on the diagram of the bridge above, with x-axis the road and y-axis (AB), find the equation of the parabolic arch in the form $y = ax^2 + c$.

b Hence, determine the lengths of all other vertical supports.

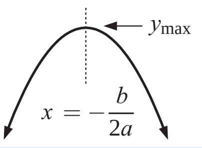
G QUADRATIC OPTIMISATION

For a quadratic function $y = ax^2 + bx + c$, we have already seen that:

- if $a > 0$,
  the **minimum**
  value of y occurs
  at $x = -\dfrac{b}{2a}$



- if $a < 0$,
  the **maximum**
  value of y occurs
  at $x = -\dfrac{b}{2a}$.

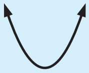


The process of finding the maximum or minimum value of a function is called **optimisation**.

Optimisation is a very useful tool when looking at such issues as maximising profits and minimising costs.

Example 26 **Self Tutor**

Find the maximum or minimum value of the following quadratics, and the corresponding value of x: **a** $y = x^2 + x - 3$ **b** $y = 3 + 3x - 2x^2$

a For $y = x^2 + x - 3$
$a = 1, \quad b = 1, \quad c = -3.$
As $a > 0$, the shape is 

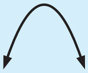
$\therefore$ the minimum value occurs

when $x = \dfrac{-b}{2a} = -\frac{1}{2}$

and $y = (-\frac{1}{2})^2 + (-\frac{1}{2}) - 3$

$= -3\frac{1}{4}$

The minimum value of y is $-3\frac{1}{4}$, occurring when $x = -\frac{1}{2}$.

b For $y = -2x^2 + 3x + 3$
$a = -2, \quad b = 3, \quad c = 3.$
As $a < 0$, the shape is 

$\therefore$ the maximum value occurs

when $x = \dfrac{-b}{2a} = \frac{-3}{-4} = \frac{3}{4}$

and $y = -2(\frac{3}{4})^2 + 3(\frac{3}{4}) + 3$

$= 4\frac{1}{8}$

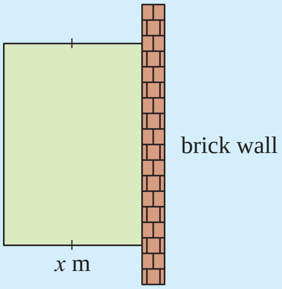
The maximum value of y is $4\frac{1}{8}$, occurring when $x = \frac{3}{4}$.

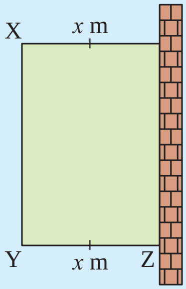
Example 27 **Self Tutor**

A vegetable gardener has 40 m of fencing to enclose a rectangular garden plot, where one side is an existing brick wall. If the two equal sides are x m long:

a show that the area enclosed is given by $A = x(40 - 2x)$ m^2

b find the dimensions of the vegetable garden of maximum area.





a Side [XY] has length $(40 - 2x)$ m.
Now area = length × width
$\therefore \quad A = x(40 - 2x)$ m^2.

b $A = 40x - 2x^2 = -2x^2 + 40x$
is a quadratic in x, with $a = -2$, $b = 40$, $c = 0$.

As $a < 0$, the shape is 

The maximum area occurs when $x = \dfrac{-b}{2a} = \frac{-40}{-4} = 10$

$\therefore$ the area is maximised when YZ = 10 m and XY = 20 m.

EXERCISE 6G

1 Find the maximum or minimum values of the following quadratics, and the corresponding values of x:

a $y = x^2 - 2x$
b $y = 7 - 2x - x^2$
c $y = 8 + 2x - 3x^2$
d $y = 2x^2 + x - 1$
e $y = 4x^2 - x + 5$
f $y = 7x - 2x^2$

2 The profit in manufacturing x refrigerators per day, is given by the profit relation $P = -3x^2 + 240x - 800$ dollars. How many refrigerators should be made each day to maximise the total profit? What is the maximum profit?

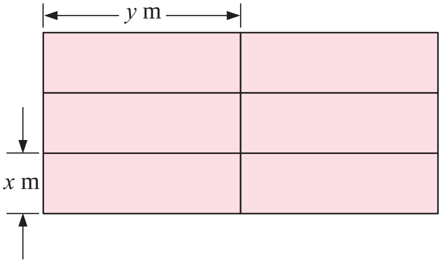
3 A rectangular plot is enclosed by 200 m of fencing and has an area of A square metres. Show that:

- **a** $A = 100x - x^2$ where x m is the length of one of its sides
- **b** the area is maximised when the rectangle is a square.

4 A rectangular paddock to enclose horses is to be made, with one side being a straight water drain. If 1000 m of fencing is available for the other 3 sides, what dimensions should be used for the paddock so that it encloses the maximum possible area?

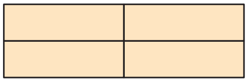
5 1800 m of fencing is available to fence six identical pens as shown in the diagram.

- **a** Explain why $9x + 8y = 1800$.
- **b** Show that the total area of each pen is given by $A = -\frac{9}{8}x^2 + 225x$ m^2.
- **c** If the area enclosed is to be maximised, what is the shape of each pen?

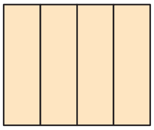


6 500 m of fencing is available to make 4 rectangular pens of identical shape. Find the dimensions that maximise the area of each pen if the plan is:

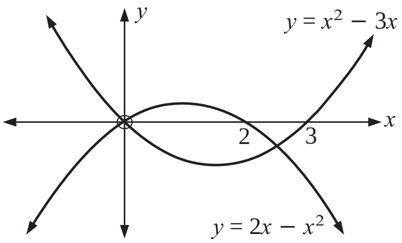
a



b



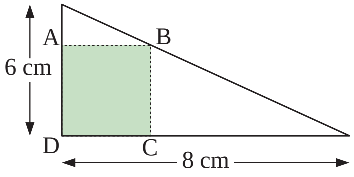
7 The graphs of $y = x^2 - 3x$ and $y = 2x - x^2$ are illustrated.



- **a** Prove that the graphs meet where $x = 0$ and $x = 2\frac{1}{2}$.
- **b** Find the maximum vertical separation between the curves for $0 \leqslant x \leqslant 2\frac{1}{2}$.

8 Infinitely many rectangles may be inscribed within the right angled triangle shown alongside. One of them is illustrated.

- **a** Let AB $= x$ cm and BC $= y$ cm. Use similar triangles to find y in terms of x.
- **b** Find the dimensions of rectangle ABCD of maximum area.



9 A manufacturer of pot-belly stoves has the following situation to consider.
If x are made per week, each one will cost $(50 + \frac{400}{x})$ dollars and the total receipts per week for selling them will be $(550x - 2x^2)$ dollars.
How many pot-belly stoves should be made per week in order to maximise profits?

10 The total cost of producing x toasters per day is given by $C = \left(\frac{1}{10}x^2 + 20x + 25\right)$ euros, and the selling price of each toaster is $\left(44 - \frac{1}{5}x\right)$ euros. How many toasters should be produced each day in order to maximise the total profit?

11 A manufacturer of barbeques knows that if x of them are made each week then each one will cost $\left(60 + \frac{800}{x}\right)$ pounds and the total receipts per week will be $(1000x - 3x^2)$ pounds. How many barbeques should be made per week to maximise profits?

INVESTIGATION 4 SUM AND PRODUCT OF ROOTS

Answer the following questions:

1 $ax^2 + bx + c = 0$ has roots p and q.

Prove that $p + q = \dfrac{-b}{a}$ and $pq = \dfrac{c}{a}$.

2 $2x^2 - 5x + 1 = 0$ has roots p and q. Without finding the values of p and q, find:

a $p + q$ **b** pq **c** $p^2 + q^2$ **d** $\dfrac{1}{p} + \dfrac{1}{q}$

3 Find *all* quadratic equations with roots which are:

a one more than the roots of $2x^2 - 5x + 1 = 0$

b the squares of the roots of $2x^2 - 5x + 1 = 0$

c the reciprocals of the roots of $2x^2 - 5x + 1 = 0$.

REVIEW SET 6A NON-CALCULATOR

1 Consider the quadratic function $y = -2(x + 2)(x - 1)$.

a State the x-intercepts. **b** State the equation of the axis of symmetry.

c Find the y-intercept. **d** Find the coordinates of the vertex.

e Sketch the graph of the function.

2 Solve the following equations, giving exact answers:

a $3x^2 - 12x = 0$ **b** $3x^2 - x - 10 = 0$ **c** $x^2 - 11x = 60$

3 Solve using the quadratic formula:

a $x^2 + 5x + 3 = 0$ **b** $3x^2 + 11x - 2 = 0$

4 Solve the following equation by ‘completing the square’: $x^2 + 7x - 4 = 0$

5 Use the vertex, axis of symmetry and y-intercept to graph:

a $y = (x - 2)^2 - 4$ **b** $y = -\frac{1}{2}(x + 4)^2 + 6$

6 Find, in the form $y = ax^2 + bx + c$, the equation of the quadratic whose graph:

a touches the x-axis at 4 and passes through $(2, 12)$.

b has vertex $(-4, 1)$ and passes through $(1, 11)$.

7 Find the maximum or minimum value of the relation $y = -2x^2 + 4x + 3$ and the value of x for which the maximum or minimum occurs.

8 Find the points of intersection of $y = x^2 - 3x$ and $y = 3x^2 - 5x - 24$.

9 For what values of k does the graph of $y = -2x^2 + 5x + k$ not cut the x-axis?

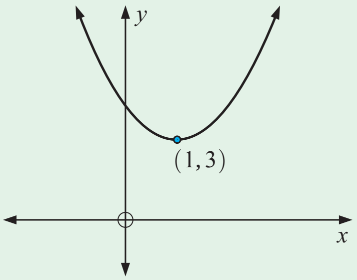
10 Find the values of m for which $2x^2 - 3x + m = 0$ has:

a a repeated root **b** two distinct real roots **c** no real roots.

11 The sum of a number and its reciprocal is $2\frac{1}{30}$. Find the number.

12 Show that no line with a y-intercept of $(0, 10)$ will ever be tangential to the curve with equation $y = 3x^2 + 7x - 2$.

13 The diagram shows a quadratic $f(x) = x^2 + mx + n$.



a Determine the values of m and n.

b Find k given that the graph passes through the point $(3, k)$.

c State the vertex of $y = g(x)$ given $g(x) = f(x - 1) + 2$.

d Find the domain and range of $f(x)$ and $g(x)$.

REVIEW SET 6B CALCULATOR

1 Consider the quadratic function $y = 2x^2 + 6x - 3$.

a Convert it into the form $y = a(x - h)^2 + k$ by 'completing the square'.

b State the coordinates of the vertex. **c** Find the y-intercept.

d Sketch the graph of the function. **e** Use technology to check your answers.

2 Use technology to solve:

a $(x - 2)(x + 1) = 3x - 4$ **b** $2x - \frac{1}{x} = 5$

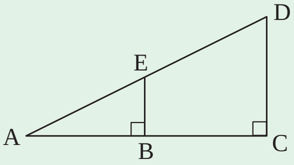
3 Draw the graph of $y = -x^2 + 2x$.

4 Find the equation of the axis of symmetry and the vertex of $y = -3x^2 + 8x + 7$.

5 Using the discriminant only, determine the nature of the solutions of:

a $2x^2 - 5x - 7 = 0$ **b** $3x^2 - 24x + 48 = 0$

6 If [AB] has the same length as [CD], [BC] is 2 cm shorter than [AB], and [BE] is 7 cm in length, find the length of [AB].



7 **a** For what values of c do the lines with equations $y = 3x + c$ intersect the parabola $y = x^2 + x - 5$ in two distinct points?

b Choose one such value of c from part **a** and find the points of intersection.

8 For the quadratic $y = 2x^2 + 4x - 1$, find:

a the equation of the axis of symmetry **b** the coordinates of the vertex

c the axes intercepts. **d** Hence sketch the graph.

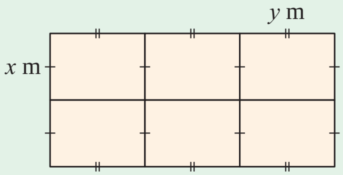
9 An open square container is made by cutting 4 cm square pieces out of a piece of tinplate. If the capacity is 120 cm^3, find the size of the original piece of tinplate.

10 Find the points where $y = -x^2 - 5x + 3$ and $y = x^2 + 3x + 11$ meet.

11 Find the maximum or minimum value of the following quadratics, and the corresponding value of x:

a $y = 3x^2 + 4x + 7$ **b** $y = -2x^2 - 5x + 2$

12 600 m of fencing is used to construct 6 rectangular animal pens as shown.



a Show that $y = \dfrac{600 - 8x}{9}$.

b Find the area A of each pen in terms of x.

c Find the dimensions of each pen if each pen is to have maximum area.

d What is the maximum area of each pen?

13 Two different quadratic functions of the form $f(x) = 9x^2 - kx + 4$ each touch the x-axis.

a Find the two values of k.

b Find the point of intersection of the two quadratic functions.

c Describe the transformation which maps one function onto the other.

REVIEW SET 6C

1 Consider the quadratic function $y = \frac{1}{2}(x-2)^2 - 4$.

a State the equation of the axis of symmetry.

b Find the coordinates of the vertex. **c** Find the y-intercept.

d Sketch the graph of the function. **e** Use technology to check your answers.

2 Solve the following equations:

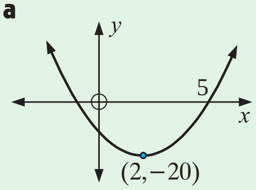
a $x^2 - 5x - 3 = 0$ **b** $2x^2 - 7x - 3 = 0$

3 Solve the following using the quadratic formula:

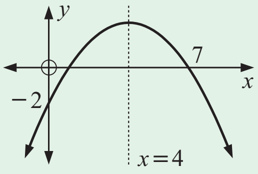
a $x^2 - 7x + 3 = 0$ **b** $2x^2 - 5x + 4 = 0$

4 Find the equation of the quadratic relation with graph:

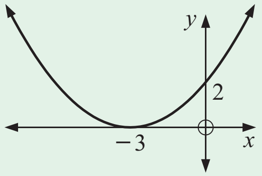
a



b



c



5 Use the discriminant only to find the relationship between the graph and the x-axis for:

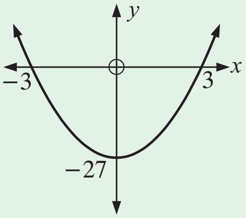
a $y = 2x^2 + 3x - 7$ **b** $y = -3x^2 - 7x + 4$

6 Determine if the quadratic functions are positive definite, negative definite or neither:

a $y = -2x^2 + 3x + 2$ **b** $y = 3x^2 + x + 11$

7 Find the equation of the quadratic relation with graph:

a



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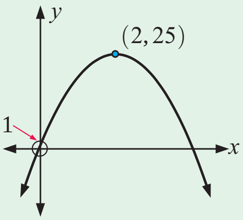


8 In a right angled triangle, one leg is 7 cm longer than the other, and the hypotenuse is 2 cm longer than the longer leg. Find the length of the hypotenuse.

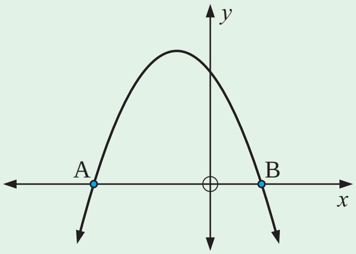
9 Find the y-intercept of the line with gradient -3 that is tangential to the parabola $y = 2x^2 - 5x + 1$.

10 For what values of k would the graph of $y = x^2 - 2x + k$ cut the x-axis twice?

11 Find an expression for the quadratic which cuts the x-axis at 3 and -2 and has y-intercept 24. Give your answer in the form $y = ax^2 + bx + c$.

12 For what values of m are the lines $y = mx - 10$ tangents to the parabola $y = 3x^2 + 7x + 2$?

13 The diagram shows a parabola $y = a(x + m)(x + n)$ where $m > n$.



a Find, in terms of m and n, the:
  - **i** coordinates of the x-intercepts A and B
  - **ii** equation of the axis of symmetry.

b State the sign of:
  - **i** the discriminant Δ
  - **ii** a.

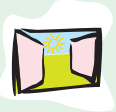
Chapter 7

The binomial expansion

Syllabus reference: **1.3**

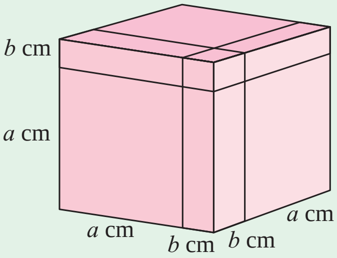
Contents:

OPENING PROBLEM



Consider the cube alongside, which has sides of length $(a+b)$ cm.

The cube has been subdivided into 8 blocks by making 3 cuts parallel to the cube's surfaces as shown.



We know that the total volume of the cube is $(a+b)^3$ cm^3. However, we can also find an expression for the cube's volume by adding the volumes of the individual blocks.

Things to think about:

- How many of the blocks that make up the cube have dimensions:
  - a by a by a
  - a by a by b
  - a by b by b
  - b by b by b?
- By adding the volumes of the blocks, can you find an expression which is equivalent to $(a+b)^3$?

The **Opening Problem** is an example of a **counting** problem.

The following exercises will help us to solve counting problems without having to list and count the possibilities one by one. To do this we will examine:

- the product principle
- counting permutations
- counting combinations

A BINOMIAL EXPANSIONS

The sum $\boldsymbol{a+b}$ is called a **binomial** as it contains two terms.
Any expression of the form $\boldsymbol{(a+b)^n}$ is called a **power of a binomial**.

All binomials raised to a power can be expanded using the same general principles. In this chapter, therefore, we consider the expansion of the general expression $(a+b)^n$ where $n \geqslant 2$.

Consider the following algebraic expansions:

$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2+2ab+b^2$$

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)^2\\ &= (a+b)(a^2+2ab+b^2)\\ &= a^3+2a^2b+ab^2+a^2b+2ab^2+b^3\\ &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

We say that:

$\boldsymbol{a^2+2ab+b^2}$ is the **binomial expansion** of $\boldsymbol{(a+b)^2}$
$\boldsymbol{a^3+3a^2b+3ab^2+b^3}$ is the **binomial expansion** of $\boldsymbol{(a+b)^3}$

INVESTIGATION 1 THE BINOMIAL EXPANSION OF $(a+b)^n$, $n \geqslant 4$

What to do:

1 Expand $(a+b)^4$ in the same way as for $(a+b)^3$ above.

2 Similarly, expand algebraically $(a+b)^5$ using your expansion for $(a+b)^4$ from **1**.

3 Expand $(a+b)^6$ using your expansion for $(a+b)^5$ from **2**.

4 The $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ expansion contains 4 terms: a^3, $3a^2b$, $3ab^2$ and b^3. The coefficients of these terms are: 1 3 3 1

- **a** What happens to the powers of a and b in each term of the expansion of $(a+b)^3$?
- **b** Does the pattern in **a** continue for the expansions of $(a+b)^4$, $(a+b)^5$, and so on?
- **c** Write down the triangle of coefficients to row 6:

$n = 0$ 1
$n = 1$ 1 1
$n = 2$ 1 2 1
$n = 3$ 1 3 3 1 ← row 3
⋮

5 The triangle of coefficients in **c** above is called **Pascal's triangle**. Investigate:

- **a** the predictability of each row from the previous one
- **b** a formula for finding the sum of the numbers in the nth row of Pascal's triangle.

6 Use your results from **5** to predict the elements of the 7th row of Pascal's triangle.
Hence write down the binomial expansion of $(a+b)^7$.
Check your result algebraically by using $(a+b)^7 = (a+b)(a+b)^6$ and your results from **3**.

From the investigation we obtained
$$\begin{aligned}(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\\ &= a^4 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + b^4\end{aligned}$$

Notice that:

- As we look from left to right across the expansion, the powers of a decrease by 1 and the powers of b increase by 1.
- The sum of the powers of a and b in each term of the expansion is 4.
- The number of terms in the expansion is $4 + 1 = 5$.

For the expansion of $(a+b)^n$ where $n = 1, 2, 3, 4, 5, ...$:

- As we look from left to right across the expansion, the powers of a *decrease* by 1, while the powers of b *increase* by 1.
- The sum of the powers of a and b in each term of the expansion is n.
- The number of terms in the expansion is $n + 1$.

In the following examples we see how the general binomial expansion $(a+b)^n$ may be put to use.

Example 1 **Self Tutor**

Using $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, find the binomial expansion of:

a $(2x+3)^3$ **b** $(x-5)^3$

a In the expansion of $(a+b)^3$ we substitute $a = (2x)$ and $b = (3)$

$$\therefore \quad (2x+3)^3 = (2x)^3 + 3(2x)^2(3) + 3(2x)^1(3)^2 + (3)^3$$
$$= 8x^3 + 36x^2 + 54x + 27 \quad \text{on simplifying.}$$

b This time we substitute $a = (x)$ and $b = (-5)$

$$\therefore \quad (x-5)^3 = (x)^3 + 3(x)^2(-5) + 3(x)(-5)^2 + (-5)^3$$
$$= x^3 - 15x^2 + 75x - 125$$

Example 2 **Self Tutor**

Find the: **a** 5th row of Pascal's triangle **b** binomial expansion of $\left(x - \frac{2}{x}\right)^5$.

a

1 ← the 0th row, for $(a+b)^0$
1 1 ← the 1st row, for $(a+b)^1$
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1 ← the 5th row, for $(a+b)^5$

b Using the coefficients obtained in **a**,

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Letting $a = (x)$ and $b = \left(\frac{-2}{x}\right)$, we find

$$\left(x - \frac{2}{x}\right)^5 = (x)^5 + 5(x)^4\left(\frac{-2}{x}\right) + 10(x)^3\left(\frac{-2}{x}\right)^2 + 10(x)^2\left(\frac{-2}{x}\right)^3$$
$$+ 5(x)\left(\frac{-2}{x}\right)^4 + \left(\frac{-2}{x}\right)^5$$
$$= x^5 - 10x^3 + 40x - \frac{80}{x} + \frac{80}{x^3} - \frac{32}{x^5}$$

EXERCISE 7A

1 Use the binomial expansion of $(a+b)^3$ to expand and simplify:

a $(p+q)^3$ **b** $(x+1)^3$ **c** $(x-3)^3$ **d** $(2+x)^3$

e $(3x-1)^3$ **f** $(2x+5)^3$ **g** $\left(3x - \frac{1}{3}\right)^3$ **h** $\left(2x + \frac{1}{x}\right)^3$

2 Use $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ to expand and simplify:

a $(1+x)^4$ **b** $(p-q)^4$ **c** $(x-2)^4$ **d** $(3-x)^4$

e $(1+2x)^4$ **f** $(2x+3)^4$ **g** $\left(x + \frac{1}{x}\right)^4$ **h** $\left(2x - \frac{1}{x}\right)^4$

3 Expand and simplify:

a $(x+2)^5$ **b** $(x-2y)^5$ **c** $(1+2x)^5$ **d** $\left(x-\frac{1}{x}\right)^5$

4 **a** Write down the 6th row of Pascal's triangle.

b Find the binomial expansion of:

i $(x+2)^6$ **ii** $(2x-1)^6$ **iii** $\left(x+\frac{1}{x}\right)^6$

5 Expand and simplify:

a $(1+\sqrt{2})^3$ **b** $(\sqrt{5}+2)^4$ **c** $(2-\sqrt{2})^5$

6 **a** Expand $(2+x)^6$.

b Use the expansion of **a** to find the value of $(2.01)^6$.

7 Expand and simplify $(2x+3)(x+1)^4$.

8 Find the coefficient of:

a a^3b^2 in the expansion of $(3a+b)^5$ **b** a^3b^3 in the expansion of $(2a+3b)^6$.

B THE BINOMIAL THEOREM

INVESTIGATION 2 THE BINOMIAL COEFFICIENT

You can use your calculator to find the values in Pascal's triangle.

TI-$n$spire
TI-84
Casio

What to do:

1 Write down the first 6 rows of Pascal's triangle.

2 Use a graphics calculator to find the **binomial coefficient** $\binom{n}{r}$ for $n=3$ and $r=0$, 1, 2, and 3. Click on the appropriate icon for instructions. What do you notice about these numbers?

3 Use your calculator to find $\binom{4}{0}$, $\binom{4}{1}$, $\binom{4}{2}$, $\binom{4}{3}$ and $\binom{4}{4}$. What do you notice about these numbers?

4 Use your calculator to help you write down the expansion for $(a+b)^5$.

5 Copy and complete: In the expansion of $(a+b)^5$, $\binom{5}{r}$ is the coefficient of $a^{\cdots}b^{\cdots}$.

6 Copy and complete: In the expansion of $(a+b)^n$, the binomial coefficient $\binom{n}{r}$ is the coefficient of $a^{\cdots}b^{\cdots}$.

From the investigation above you should have found that $\binom{n}{0}=\binom{n}{n}=1$ for all n. You should also have discovered the **binomial theorem**:

$$\boldsymbol{(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \dots + \binom{n}{r}a^{n-r}b^r + \dots + b^n}$$

where $\binom{n}{r}$ is the **binomial coefficient** of $a^{n-r}b^r$ and $r=0, 1, 2, 3, \dots, n$.

The binomial coefficient is sometimes written ${}^nC_r$ or C_r^n.

The **general term** or $(r+1)$th term in the binomial expansion is $\boldsymbol{T_{r+1} = \binom{n}{r}a^{n-r}b^r}$.

So, $$\boldsymbol{(a+b)^n = \sum_{r=0}^{n} \binom{n}{r}a^{n-r}b^r}.$$

Example 3 **Self Tutor**

Write down the first 3 and last 2 terms of the expansion of $\left(2x+\frac{1}{x}\right)^{12}$.

$$\left(2x+\tfrac{1}{x}\right)^{12} = (2x)^{12} + \binom{12}{1}(2x)^{11}\left(\tfrac{1}{x}\right) + \binom{12}{2}(2x)^{10}\left(\tfrac{1}{x}\right)^2 + \ldots\ldots$$
$$\ldots\ldots + \binom{12}{11}(2x)\left(\tfrac{1}{x}\right)^{11} + \left(\tfrac{1}{x}\right)^{12}$$

Example 4 **Self Tutor**

Find the 7th term of $\left(3x - \dfrac{4}{x^2}\right)^{14}$. Do not simplify your answer.

$a = (3x)$, $b = \left(\dfrac{-4}{x^2}\right)$ and $n = 14$

So, as $T_{r+1} = \binom{n}{r}a^{n-r}b^r$, we let $r = 6$

$$\therefore \quad T_7 = \binom{14}{6}(3x)^8\left(\frac{-4}{x^2}\right)^6$$

Example 5 **Self Tutor**

In the expansion of $\left(x^2+\frac{4}{x}\right)^{12}$, find:

a the coefficient of x^6

b the constant term

$a = (x^2)$, $b = \left(\frac{4}{x}\right)$ and $n = 12$

$$\therefore \quad T_{r+1} = \binom{12}{r}(x^2)^{12-r}\left(\tfrac{4}{x}\right)^r$$
$$= \binom{12}{r}x^{24-2r}\frac{4^r}{x^r}$$
$$= \binom{12}{r}4^r x^{24-3r}$$

a If $24 - 3r = 6$

then $3r = 18$

$\therefore \quad r = 6$

$\therefore \quad T_7 = \binom{12}{6}4^6x^6$

$\therefore$ the coefficient of x^6 is

$\binom{12}{6}4^6$ or 3 784 704.

b If $24 - 3r = 0$

then $3r = 24$

$\therefore \quad r = 8$

$\therefore \quad T_9 = \binom{12}{8}4^8x^0$

$\therefore$ the constant term is

$\binom{12}{8}4^8$ or 32 440 320.

Example 6 **Self Tutor**

Find the coefficient of x^5 in the expansion of $(x+3)(2x-1)^6$.

$$\begin{aligned} &(x+3)(2x-1)^6 \\ &= (x+3)[(2x)^6 + \tbinom{6}{1}(2x)^5(-1) + \tbinom{6}{2}(2x)^4(-1)^2 + ...] \\ &= (\underset{\uparrow (1)}{x} + \underset{\uparrow (2)}{3})(2^6x^6 - \underset{\uparrow (2)}{\tbinom{6}{1}2^5x^5} + \underset{\uparrow (1)}{\tbinom{6}{2}2^4x^4} - ....) \end{aligned}$$

So, the terms containing x^5 are $\binom{6}{2}2^4x^5$ from (1)

and $-3\binom{6}{1}2^5x^5$ from (2)

$\therefore$ the coefficient of x^5 is $\binom{6}{2}2^4 - 3\binom{6}{1}2^5 = -336$

EXERCISE 7B

1 Write down the first three and last two terms of the binomial expansion of:

a $(1+2x)^{11}$ **b** $\left(3x+\frac{2}{x}\right)^{15}$ **c** $\left(2x-\frac{3}{x}\right)^{20}$

2 Without simplifying, write down:

a the 6th term of $(2x+5)^{15}$ **b** the 4th term of $(x^2+y)^9$

c the 10th term of $\left(x-\frac{2}{x}\right)^{17}$ **d** the 9th term of $\left(2x^2-\frac{1}{x}\right)^{21}$.

3 Find the coefficient of:

a x^{10} in the expansion of $(3+2x^2)^{10}$ **b** x^3 in the expansion of $\left(2x^2-\frac{3}{x}\right)^6$

c x^6y^3 in the expansion of $(2x^2-3y)^6$ **d** x^{12} in the expansion of $\left(2x^2-\frac{1}{x}\right)^{12}$.

4 Find the constant term in:

a the expansion of $\left(x+\frac{2}{x^2}\right)^{15}$ **b** the expansion of $\left(x-\frac{3}{x^2}\right)^9$.

5 **a** Write down the first 6 rows of Pascal's triangle.

b Find the sum of the numbers in:

i row 1 **ii** row 2 **iii** row 3 **iv** row 4 **v** row 5.

c Copy and complete:

The sum of the numbers in row n of Pascal's triangle is ......

d Show that $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + ... + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$.

Hence deduce that $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n-1} + \binom{n}{n} = 2^n$.

6 **a** Find the coefficient of x^5 in the expansion of $(x+2)(x^2+1)^8$.

b Find the coefficient of x^6 in the expansion of $(2-x)(3x+1)^9$.

7 Consider the expression $(x^2y-2y^2)^6$. Find the term in which x and y are raised to the same power.

8 **a** The third term of $(1+x)^n$ is $36x^2$. Find the fourth term.

b If $(1+kx)^n = 1 - 12x + 60x^2 - \ldots\ldots$, find the values of k and n.

9 Find a if the coefficient of x^{11} in the expansion of $(x^2 + \frac{1}{ax})^{10}$ is 15.

REVIEW SET 7

1 Use the binomial expansion to find: **a** $(x-2y)^3$ **b** $(3x+2)^4$

2 Find the coefficient of x^3 in the expansion of $(2x+5)^6$.

3 Find the constant term in the expansion of $\left(2x^2 - \frac{1}{x}\right)^6$.

4 Use Pascal's triangle to expand $(a+b)^6$.

Hence, find the binomial expansion of: **a** $(x-3)^6$ **b** $\left(1+\frac{1}{x}\right)^6$

5 Expand and simplify $(\sqrt{3}+2)^5$. Give your answer in the form $a + b\sqrt{3}$ where $a, b \in \mathbb{Z}$.

6 Use the expansion of $(4+x)^3$ to find the exact value of $(4.02)^3$.

7 Find the coefficient of x^{-6} in the expansion of $\left(2x - \frac{3}{x^2}\right)^{12}$.

8 Find the coefficient of x^5 in the expansion of $(2x+3)(x-2)^6$.

9 Find k in the expansion $(m-2n)^{10} = m^{10} - 20m^9n + km^8n^2 - \ldots. + 1024n^{10}$.

10 Find c given that the expansion $(1+cx)(1+x)^4$ includes the term $22x^3$.

11 Consider the expansion of $\left(3x^2 + \frac{1}{x}\right)^6$.

- **a** How many terms does the expansion include?
- **b** Find the constant term.
- **c** Show that the expansion has no terms involving x^5.

Chapter 8

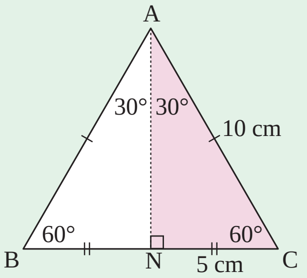
The unit circle and radian measure

Syllabus reference: 3.1, 3.2

Contents:

OPENING PROBLEM

Consider an equilateral triangle with sides 10 cm long. Altitude [AN] bisects side [BC] and the vertical angle BAC.



- Can you see from this figure that $\sin 30^o = \frac{1}{2}$?
- Use your calculator to find the values of $\sin 30^o$, $\sin 150^o$, $\sin 390^o$, $\sin 1110^o$ and $\sin(-330^o)$. What do you notice? Can you explain why this result occurs even though the angles are not between 0^o and 90^o?

A RADIAN MEASURE

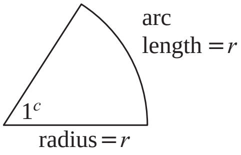
DEGREE MEASUREMENT OF ANGLES

We have seen previously that one full revolution makes an angle of 360^o, and the angle on a straight line is 180^o. Hence, one **degree**, 1^o, can be defined as $\frac{1}{360}$th of one full revolution. This measure of angle is probably most useful for surveyors and architects, and is the one you have probably used in earlier years.

For greater accuracy we define one **minute**, $1'$, as $\frac{1}{60}$th of one degree and one **second**, $1''$, as $\frac{1}{60}$th of one minute. Obviously a minute and a second are very small angles.

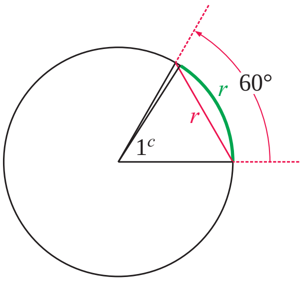
Most graphics calculators have the capacity to convert fractions of angles measured in degrees into minutes and seconds. This is also useful for converting fractions of hours into minutes and seconds for time measurement, as one minute is $\frac{1}{60}$th of one hour, and one second is $\frac{1}{60}$th of one minute.

RADIAN MEASUREMENT OF ANGLES



An angle is said to have a measure of 1 **radian** (1^c) if it is subtended at the centre of a circle by an arc equal in length to the radius.

The symbol 'c' is used for radian measure but is usually omitted. By contrast, the degree symbol is *always* used when the measure of an angle is given in degrees.



From the diagram to the right, it can be seen that 1^c is slightly smaller than 60^o. In fact, $1^c \approx 57.3^o$.

The word 'radian' is an abbreviation of 'radial angle'.

DEGREE-RADIAN CONVERSIONS

If the radius of a circle is r, then an arc of length $2r$ will subtend an angle of 2 radians at the centre. An arc of length πr, or half the circumference, will subtend an angle of π radians.

Therefore, π radians $\equiv 180^o$.

So, $1^c = \left(\frac{180}{\pi}\right)^o \approx 57.3^o$ and $1^o = \left(\frac{\pi}{180}\right)^c \approx 0.0175^c$.

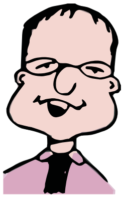
To convert from degrees to radians, we multiply by $\frac{\pi}{180}$.

To convert from radians to degrees, we multiply by $\frac{180}{\pi}$.

We can summarise these results in the conversion diagram:

$$\begin{tikzcd}
\text{Degrees} \arrow[r, bend left, "\times\frac{\pi}{180}"] & \text{Radians} \arrow[l, bend left, "\times\frac{180}{\pi}"]
\end{tikzcd}$$

We indicate degrees with a small ^{o}. To indicate radians we use a small ^{c} or else use no symbol at all.



Example 1 Self Tutor

Convert 45^o to radians in terms of π.

$45^o = (45 \times \frac{\pi}{180})$ radians
$= \frac{\pi}{4}$ radians

or

$180^o = \pi$ radians
$\therefore \left(\frac{180}{4}\right)^o = \frac{\pi}{4}$ radians
$\therefore 45^o = \frac{\pi}{4}$ radians

Example 2 Self Tutor

Convert 126.5^o to radians.

126.5^o
$= (126.5 \times \frac{\pi}{180})$ radians
≈ 2.21 radians

Example 3 Self Tutor

Convert $\frac{5\pi}{6}$ to degrees.

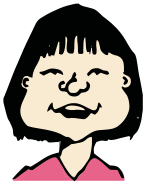
$\frac{5\pi}{6}$
$= \left(\frac{5\pi}{6} \times \frac{180}{\pi}\right)^o$
$= 150^o$

Example 4 Self Tutor

Convert 0.638 radians to degrees.

0.638 radians
$= (0.638 \times \frac{180}{\pi})^o$
$\approx 36.6^o$

Angles in radians are expressed either in terms of π or as decimals.



EXERCISE 8A

1 Convert to radians, in terms of π:

a 90^o **b** 60^o **c** 30^o **d** 18^o **e** 9^o
f 135^o **g** 225^o **h** 270^o **i** 360^o **j** 720^o
k 315^o **l** 540^o **m** 36^o **n** 80^o **o** 230^o

2 Convert to radians, correct to 3 significant figures:

a 36.7^o **b** 137.2^o **c** 317.9^o **d** 219.6^o **e** 396.7^o

3 Convert the following radian measure to degrees:

a $\frac{\pi}{5}$ **b** $\frac{3\pi}{5}$ **c** $\frac{3\pi}{4}$ **d** $\frac{\pi}{18}$ **e** $\frac{\pi}{9}$
f $\frac{7\pi}{9}$ **g** $\frac{\pi}{10}$ **h** $\frac{3\pi}{20}$ **i** $\frac{5\pi}{6}$ **j** $\frac{\pi}{8}$

4 Convert the following radians to degrees. Give your answers correct to 2 decimal places.

a 2 **b** 1.53 **c** 0.867 **d** 3.179 **e** 5.267

5 Copy and complete:

a

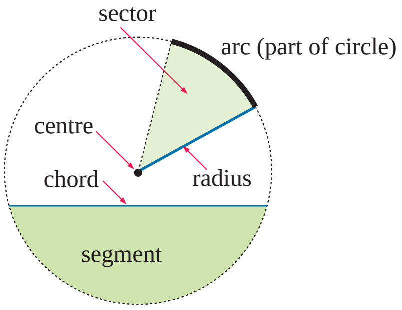
Degrees	0	45	90	135	180	225	270	315	360
Radians									

b

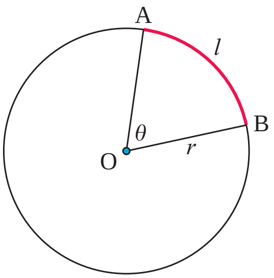
Degrees	0	30	60	90	120	150	180	210	240	270	300	330	360
Radians													

B ARC LENGTH AND SECTOR AREA

You should be familiar with these terms relating to the parts of a circle:



ARC LENGTH



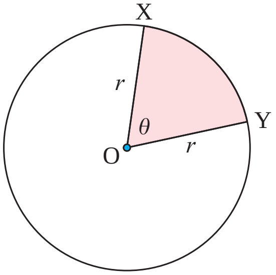
In the diagram, the **arc length** AB is l. θ is measured in **radians**.

$$\frac{\text{arc length}}{\text{circumference}} = \frac{\theta}{2\pi}$$

$$\therefore \quad \frac{l}{2\pi r} = \frac{\theta}{2\pi}$$

$$\boldsymbol{l = \theta r}$$

AREA OF SECTOR



In the diagram, the area of minor sector XOY is shaded. θ is measured in **radians**.

$$\frac{\text{area of minor sector XOY}}{\text{area of circle}} = \frac{\theta}{2\pi}$$

$$\therefore \quad \frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

$$\boldsymbol{A = \tfrac{1}{2}\theta r^2}$$

If θ is in **degrees**, $\boldsymbol{l = \dfrac{\theta}{360} \times 2\pi r}$ and $\boldsymbol{A = \dfrac{\theta}{360} \times \pi r^2}$.

Example 5 **Self Tutor**

A sector has radius 12 cm and angle 3 radians. Use radians to find its:

a arc length **b** area

a arc length $= \theta r$
$= 3 \times 12$
$= 36$ cm

b area $= \frac{1}{2}\theta r^2$
$= \frac{1}{2} \times 3 \times 12^2$
$= 216 \text{ cm}^2$

Example 6 **Self Tutor**

A sector has radius 8.2 cm and arc length 13.3 cm.

Find the area of this sector.

$l = \theta r \quad \{\theta \text{ in radians}\}$

$\therefore \quad \theta = \dfrac{l}{r} = \dfrac{13.3}{8.2}$

$\therefore \quad \text{area} = \frac{1}{2}\theta r^2$

$= \frac{1}{2} \times \dfrac{13.3}{8.2} \times 8.2^2$

$\approx 54.5 \text{ cm}^2$

EXERCISE 8B

1 Use radians to find **i** the arc length **ii** the area of a sector of a circle of:

a radius 9 cm and angle $\frac{7\pi}{4}$ **b** radius 4.93 cm and angle 4.67 radians.

2 A sector has an angle of 107.9^o and an arc length of 5.92 m. Find:

a its radius **b** its area.

3 A sector has an angle of 1.19 radians and an area of 20.8 cm^2. Find:

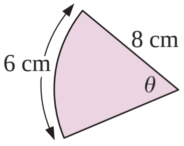
a its radius **b** its perimeter.

4 Find, in radians, the angle of a sector of:

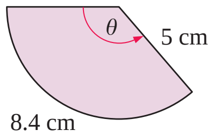
- **a** radius 4.3 m and arc length 2.95 m
- **b** radius 10 cm and area 30 cm^2.

5 Find θ (in radians) for each of the following, and hence find the area of each figure:

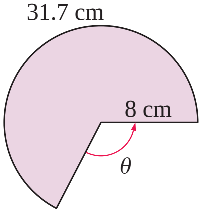
a



b



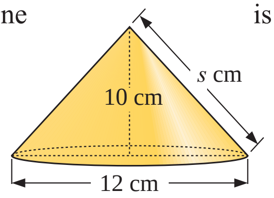
c



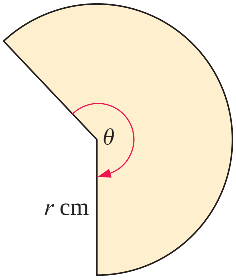
6 Find the arc length and area of a sector of radius 5 cm and angle 2 radians.

7 If a sector has radius $2x$ cm and arc length x cm, show that its area is x^2 cm^2.

8 This cone



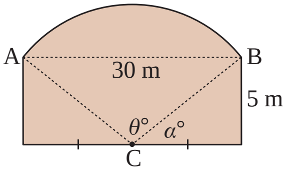
is made from this sector:



Find correct to 3 significant figures:

- **a** the slant length s cm
- **b** the value of r
- **c** the arc length of the sector
- **d** the sector angle in radians.

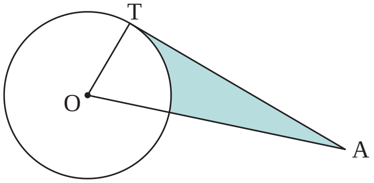
9



The end wall of a building has the shape illustrated, where the centre of arc AB is at C. Find:

- **a** α to 4 significant figures
- **b** θ to 4 significant figures
- **c** the area of the wall.

10

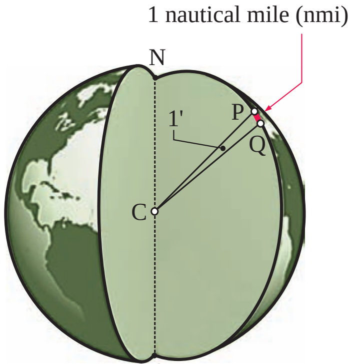


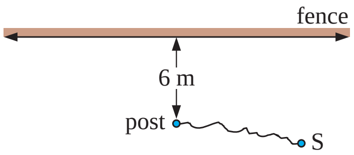
[AT] is a tangent to the given circle. If OA = 13 cm and the circle has radius 5 cm, find the perimeter of the shaded region.

11 A **nautical mile** (nmi) is the distance on the Earth's surface that subtends an angle of 1 minute (or $\frac{1}{60}$ of a degree) of the Great Circle arc measured from the centre of the Earth.

A **knot** is a speed of 1 nautical mile per hour.

- **a** Given that the radius of the Earth is 6370 km, show that 1 nmi is approximately equal to 1.853 km.
- **b** Calculate how long it would take a plane to fly from Perth to Adelaide (a distance of 2130 km) if the plane can fly at 480 knots.

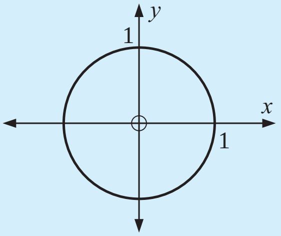


12 

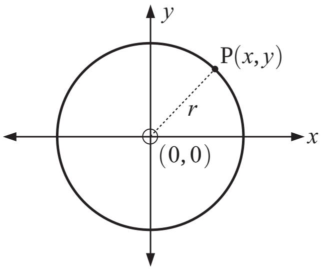
A sheep is tethered to a post which is 6 m from a long fence. The length of rope is 9 m. Find the area which is available for the sheep to feed on.

C THE UNIT CIRCLE AND THE BASIC TRIGONOMETRIC RATIOS

The **unit circle** is the circle with centre $(0, 0)$ and radius 1 unit.



CIRCLES WITH CENTRE $(0, 0)$



Consider a circle with centre $(0, 0)$ and radius r units, and suppose $P(x, y)$ is any point on this circle.

Since $OP = r$, then

$$\sqrt{(x-0)^2+(y-0)^2}=r \qquad \{\text{distance formula}\}$$

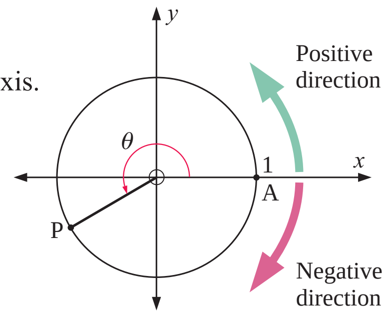
$$\therefore \quad x^2+y^2=r^2$$

$x^2+y^2=r^2$ is the equation of a circle with centre $(0, 0)$ and radius r.
The equation of the **unit circle** is $x^2+y^2=1$.

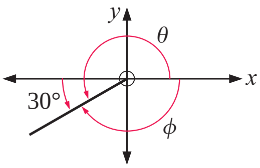
ANGLE MEASUREMENT

Suppose P lies anywhere on the unit circle and A is $(1, 0)$.
Let θ be the angle measured from [OA] on the positive x-axis.

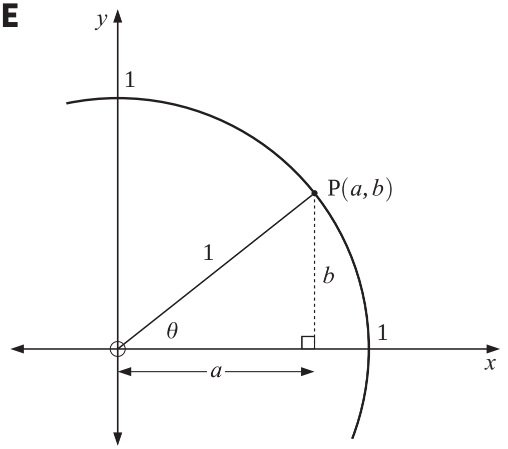
θ is **positive** for anticlockwise rotations and **negative** for clockwise rotations.



For example, $\theta = 210^o$ and $\phi = -150^o$.



DEFINITION OF SINE AND COSINE



Consider a point P(a, b) which lies on the unit circle in the first quadrant. [OP] makes an angle θ with the x-axis as shown.

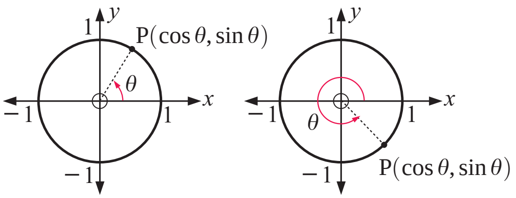
Using right angled triangle trigonometry:

$$\cos\theta = \frac{\text{ADJ}}{\text{HYP}} = \frac{a}{1} = a$$

$$\sin\theta = \frac{\text{OPP}}{\text{HYP}} = \frac{b}{1} = b$$

$$\tan\theta = \frac{\text{OPP}}{\text{ADJ}} = \frac{b}{a} = \frac{\sin\theta}{\cos\theta}$$

In general, for a point P anywhere on the unit circle,

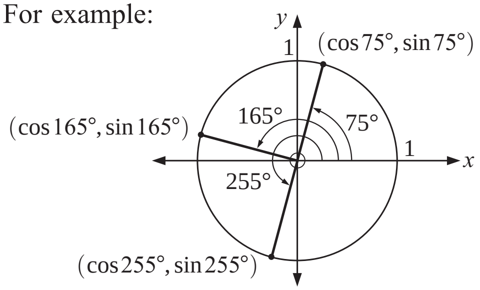


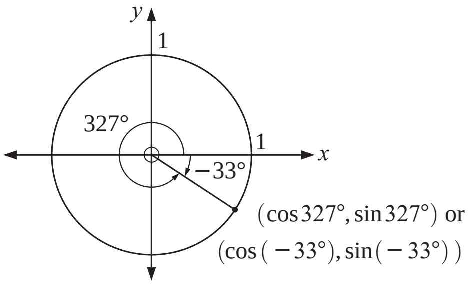
$\cos\theta$ is the x-coordinate of P.

$\sin\theta$ is the y-coordinate of P.

We can hence find the coordinates of any point on the unit circle with given angle θ measured from the positive x-axis.

For example:





Since the unit circle has equation $x^2 + y^2 = 1$,

$$(\cos\theta)^2 + (\sin\theta)^2 = 1 \quad \text{for all } \theta$$

We commonly write this as $\boldsymbol{\cos^2\theta + \sin^2\theta = 1}$.

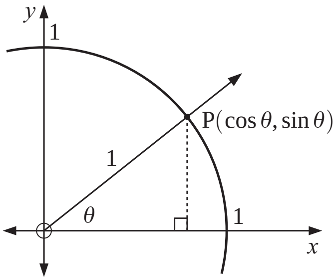
For all points on the unit circle: $-1 \leqslant x \leqslant 1$ and $-1 \leqslant y \leqslant 1$

So, $\boldsymbol{-1 \leqslant \cos\theta \leqslant 1}$ and $\boldsymbol{-1 \leqslant \sin\theta \leqslant 1}$ for all θ.

DEFINITION OF TANGENT

The **tangent ratio** is defined as $\boldsymbol{\tan\theta = \dfrac{\sin\theta}{\cos\theta}}$.

Consider the straight line passing through the origin and the point P$(\cos\theta, \sin\theta)$ on the unit circle.



The gradient of this line is
$$m = \frac{y\text{-step}}{x\text{-step}} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

and since the y-intercept is 0 the line has equation $y = x\tan\theta$.

By considering the first quadrant, we can easily see that the right angled triangle definitions of sine, cosine and tangent are consistent with the unit circle definition, but are restricted to acute angles only. The unit circle definitions apply to angles of any value.

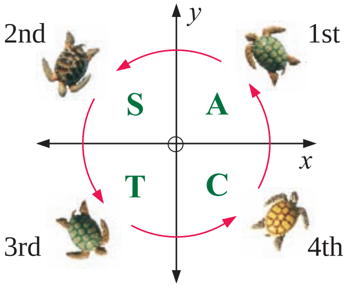
Helpful hint:

We have seen previously how the quadrants of the Cartesian Plane are labelled in anticlockwise order from 1st to 4th.

We can use a letter to show which trigonometric ratios are positive in each quadrant. The A stands for *all* of the ratios.

You might like to remember them using

All **S**illy **T**urtles **C**rawl.



PERIODICITY OF TRIGONOMETRIC RATIOS

Since there are 2π radians in a full revolution, if we add any integer multiple of 2π to θ then the position of P on the unit circle is unchanged.

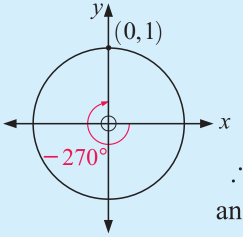
So, for all $k \in \mathbb{Z}$ and angles θ, $\boldsymbol{\cos(\theta + 2k\pi) = \cos\theta}$ **and** $\boldsymbol{\sin(\theta + 2k\pi) = \sin\theta}$.

This **periodic** feature is an important property of the trigonometric functions.

Example 7

Self Tutor

Use a unit circle diagram to find the values of $\cos(-270^o)$ and $\sin(-270^o)$.



$\therefore$ $\cos(-270^o) = 0$ {the x-coordinate}

and $\sin(-270^o) = 1$ {the y-coordinate}

EXERCISE 8C.1

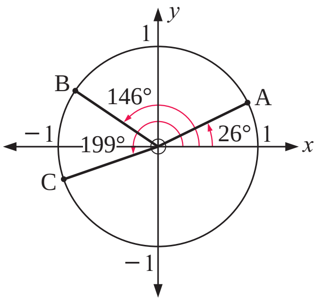
1 Sketch the graph of the curve with equation:

a $x^2 + y^2 = 1$ **b** $x^2 + y^2 = 4$ **c** $x^2 + y^2 = 1, \quad y \geqslant 0$

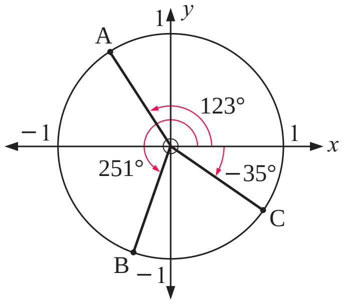
2 For each unit circle illustrated:

- **i** state the exact coordinates of points A, B and C in terms of sine or cosine
- **ii** use your calculator to give the coordinates of A, B and C correct to 3 significant figures.

a



b



3 With the aid of a unit circle, complete the following table:

θ (degrees)	0^o	90^o	180^o	270^o	360^o	450^o
θ (radians)						
sine						
cosine						
tangent						

4 **a** Use your calculator to evaluate: **i** $\frac{1}{\sqrt{2}}$ **ii** $\frac{\sqrt{3}}{2}$

b Copy and complete the following table. Use your calculator to evaluate the trigonometric ratios, then **a** to write them exactly.

θ (degrees)	30^o	45^o	60^o	135^o	150^o	240^o	315^o
θ (radians)							
sine							
cosine							
tangent							

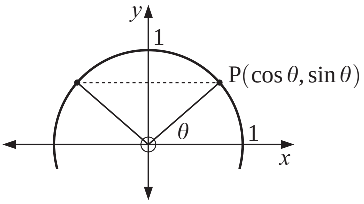
5 **a** Use your calculator to evaluate:

i $\sin 100^o$	**ii** $\sin 80^o$	**iii** $\sin 120^o$	**iv** $\sin 60^o$
v $\sin 150^o$	**vi** $\sin 30^o$	**vii** $\sin 45^o$	**viii** $\sin 135^o$

b Use the results from **a** to copy and complete:

$\sin(180^o - \theta) = \ldots\ldots$

c Justify your answer using the diagram alongside:



d Find the obtuse angle with the same sine as:

i 45^o **ii** 51^o **iii** $\frac{\pi}{3}$ **iv** $\frac{\pi}{6}$

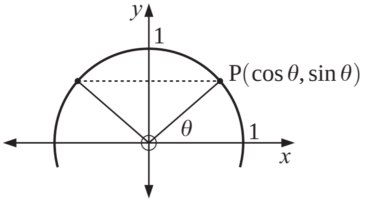
6 **a** Use your calculator to evaluate:

- **i** $\cos 70^o$ **ii** $\cos 110^o$ **iii** $\cos 60^o$ **iv** $\cos 120^o$
- **v** $\cos 25^o$ **vi** $\cos 155^o$ **vii** $\cos 80^o$ **viii** $\cos 100^o$

b Use the results from **a** to copy and complete:

$\cos(180^o - \theta) = ...$

c Justify your answer using the diagram alongside:



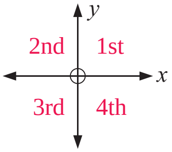
d Find the obtuse angle which has the negative cosine of:

- **i** 40^o **ii** 19^o **iii** $\frac{\pi}{5}$ **iv** $\frac{2\pi}{5}$

7 Without using your calculator find:

- **a** $\sin 137^o$ if $\sin 43^o \approx 0.6820$
- **b** $\sin 59^o$ if $\sin 121^o \approx 0.8572$
- **c** $\cos 143^o$ if $\cos 37^o \approx 0.7986$
- **d** $\cos 24^o$ if $\cos 156^o \approx -0.9135$
- **e** $\sin 115^o$ if $\sin 65^o \approx 0.9063$
- **f** $\cos 132^o$ if $\cos 48^o \approx 0.6691$

8 The diagram alongside shows the 4 quadrants. They are numbered anticlockwise.



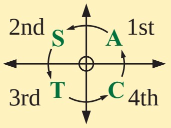
a Copy and complete:

Quadrant	*Degree measure*	*Radian measure*	$\cos\theta$	$\sin\theta$	$\tan\theta$
1	$0^o < \theta < 90^o$	$0 < \theta < \frac{\pi}{2}$	positive	positive	
2					
3					
4					

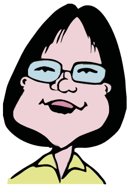
b In which quadrants are the following true?

- **i** $\cos\theta$ is positive.
- **ii** $\cos\theta$ is negative.
- **iii** $\cos\theta$ and $\sin\theta$ are both negative.
- **iv** $\cos\theta$ is negative and $\sin\theta$ is positive.

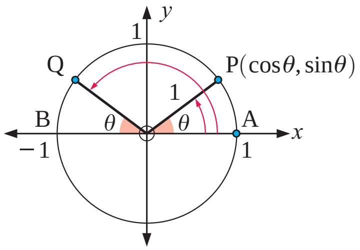
Remember:



All **S**illy **T**urtles **C**rawl



9 **a** If angle AOP $= \theta$ and angle BOQ $= \theta$ also, what is the measure of angle AOQ?



b Copy and complete:

[OQ] is a reflection of [OP] in the .....
and so Q has coordinates .....

c What trigonometric formulae can be deduced from **a** and **b**?

From the previous exercise you should have discovered that:

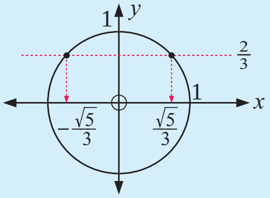
$$\sin(180^o - \theta) = \sin\theta \qquad \text{and} \qquad \cos(180^o - \theta) = -\cos\theta$$

APPLICATIONS OF THE UNIT CIRCLE

Example 8 **Self Tutor**

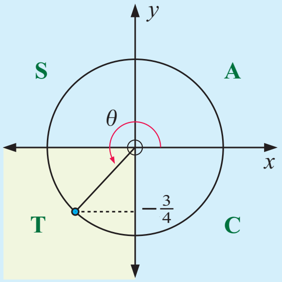
Find the possible values of $\cos\theta$ for $\sin\theta = \frac{2}{3}$. Illustrate your answers.

$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \therefore \quad \cos^2\theta + \left(\tfrac{2}{3}\right)^2 &= 1 \\ \therefore \quad \cos^2\theta &= \tfrac{5}{9} \\ \therefore \quad \cos\theta &= \pm\tfrac{\sqrt{5}}{3} \end{aligned}$$



Example 9 **Self Tutor**

If $\sin\theta = -\frac{3}{4}$ and $\pi < \theta < \frac{3\pi}{2}$, find $\cos\theta$ and $\tan\theta$ without using a calculator.



$$\begin{aligned} \text{Now} \quad \cos^2\theta + \sin^2\theta &= 1 \\ \therefore \quad \cos^2\theta + \tfrac{9}{16} &= 1 \\ \therefore \quad \cos^2\theta &= \tfrac{7}{16} \\ \therefore \quad \cos\theta &= \pm\tfrac{\sqrt{7}}{4} \end{aligned}$$

But $\pi < \theta < \frac{3\pi}{2}$, so θ is a quadrant 3 angle

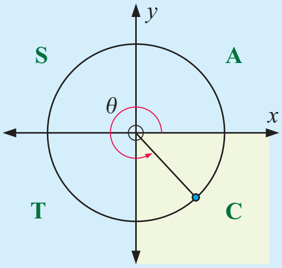
$\therefore$ $\cos\theta$ is negative.

$$\therefore \quad \cos\theta = -\tfrac{\sqrt{7}}{4}$$

$$\text{and} \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{3}{4}}{-\frac{\sqrt{7}}{4}} = \frac{3}{\sqrt{7}}$$

Example 10 **Self Tutor**

If $\tan\theta = -2$ and $\frac{3\pi}{2} < \theta < 2\pi$, find $\sin\theta$ and $\cos\theta$.



$$\begin{aligned} \frac{\sin\theta}{\cos\theta} &= -2 \\ \therefore \quad \sin\theta &= -2\cos\theta \\ \text{Now} \quad \sin^2\theta + \cos^2\theta &= 1 \\ \therefore \quad 4\cos^2\theta + \cos^2\theta &= 1 \\ \therefore \quad 5\cos^2\theta &= 1 \\ \therefore \quad \cos\theta &= \pm\tfrac{1}{\sqrt{5}} \end{aligned}$$

But $\frac{3\pi}{2} \leqslant \theta \leqslant 2\pi$, so θ is a quadrant 4 angle.

$\therefore$ $\cos\theta$ is positive and $\sin\theta$ is negative.

$\therefore$ $\cos\theta = \frac{1}{\sqrt{5}}$ and $\sin\theta = -\frac{2}{\sqrt{5}}$.

EXERCISE 8C.2

1 Find the possible exact values of $\cos\theta$ for:

a $\sin\theta = \frac{1}{2}$ **b** $\sin\theta = -\frac{1}{3}$ **c** $\sin\theta = 0$ **d** $\sin\theta = -1$

2 Find the possible exact values of $\sin\theta$ for:

a $\cos\theta = \frac{4}{5}$ **b** $\cos\theta = -\frac{3}{4}$ **c** $\cos\theta = 1$ **d** $\cos\theta = 0$

3 Without using a calculator, find:

a $\sin\theta$ if $\cos\theta = \frac{2}{3}$, $0 < \theta < \frac{\pi}{2}$ **b** $\cos\theta$ if $\sin\theta = \frac{2}{5}$, $\frac{\pi}{2} < \theta < \pi$

c $\cos\theta$ if $\sin\theta = -\frac{3}{5}$, $\frac{3\pi}{2} < \theta < 2\pi$ **d** $\sin\theta$ if $\cos\theta = -\frac{5}{13}$, $\pi < \theta < \frac{3\pi}{2}$

4 **a** If $\sin x = \frac{1}{3}$ and $\frac{\pi}{2} < x < \pi$, find $\tan x$ in radical form.

b If $\cos x = \frac{1}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\tan x$ in radical form.

c If $\sin x = -\frac{1}{\sqrt{3}}$ and $\pi < x < \frac{3\pi}{2}$, find $\tan x$ in radical form.

d If $\cos x = -\frac{3}{4}$ and $\frac{\pi}{2} < x < \pi$, find $\tan x$ in radical form.

5 Find $\sin x$ and $\cos x$ given that:

a $\tan x = \frac{2}{3}$ and $0 < x < \frac{\pi}{2}$ **b** $\tan x = -\frac{4}{3}$ and $\frac{\pi}{2} < x < \pi$

c $\tan x = \frac{\sqrt{5}}{3}$ and $\pi < x < \frac{3\pi}{2}$ **d** $\tan x = -\frac{12}{5}$ and $\frac{3\pi}{2} < x < 2\pi$.

INVESTIGATION PARAMETRIC EQUATIONS

Usually we write functions in the form $y = f(x)$.

For example: $y = 3x + 7$, $y = x^2 - 6x + 8$, $y = \sin x$

However, sometimes it is useful to express **both** x and y in terms of another variable t, called the **parameter**. In this case we say we have **parametric equations**.

GRAPHING PACKAGE

TI-$n$spire
TI-84
Casio

What to do:

1 Either click on the icon or use your graphics calculator (with the same scale on both axes) to plot $\{(x, y) \mid x = \cos t,\ y = \sin t,\ 0^o \leqslant t \leqslant 360^o\}$.

Note: Your calculator will need to be set to degrees.

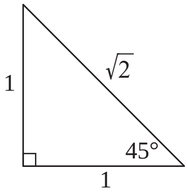
2 Describe the resulting graph.

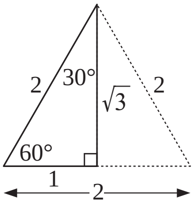
3 What is the equation of this graph? There are two possible answers.

4 If using a graphics calculator, use the *trace* key to move along the curve. What do you notice?

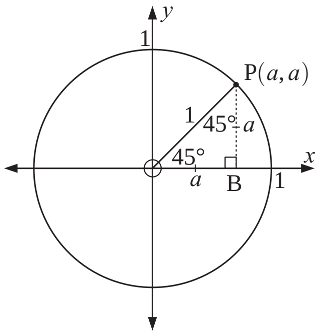
MULTIPLES OF $\frac{\pi}{6}$ AND $\frac{\pi}{4}$

The following diagrams may be helpful when finding exact trigonometric ratios.





MULTIPLES OF $\frac{\pi}{4}$



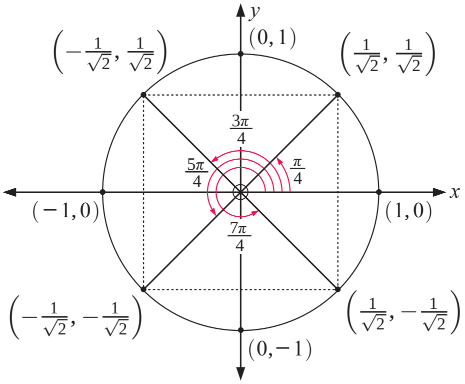
Consider $\theta = \frac{\pi}{4} = 45^o$:

Triangle OBP is isosceles as angle OPB also measures 45^o.

Letting OB = BP = a,

$$a^2 + a^2 = 1^2 \qquad \{\text{Pythagoras}\}$$
$$\therefore \quad 2a^2 = 1$$
$$\therefore \quad a^2 = \tfrac{1}{2}$$
$$\therefore \quad a = \tfrac{1}{\sqrt{2}} \quad \text{as} \quad a > 0$$

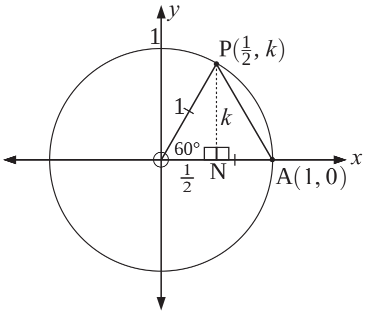
Hence, P is $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ where $\frac{1}{\sqrt{2}} \approx 0.7$.



We can find the coordinates of all points corresponding to angles of $\frac{3\pi}{4}$, $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ using suitable rotations and reflections.

MULTIPLES OF $\frac{\pi}{6}$

Consider $\theta = \frac{\pi}{3} = 60^o$:



Triangle OAP is isosceles with $\mathrm{A\widehat{O}P} = 60^o$. The remaining angles are therefore 60^o and so triangle AOP is equilateral. The altitude [PN] bisects base [OA],

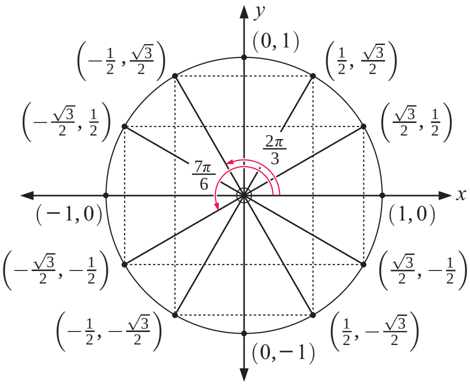
$$\therefore \quad \text{ON} = \tfrac{1}{2}$$

If P is $(\frac{1}{2}, k)$, then $(\frac{1}{2})^2 + k^2 = 1$

$$\therefore \quad k^2 = \tfrac{3}{4}$$
$$\therefore \quad k = \tfrac{\sqrt{3}}{2} \quad \{\text{as} \quad k > 0\}$$

$\therefore$ P is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ where $\frac{\sqrt{3}}{2} \approx 0.9$.

We can find the coordinates of all points on the unit circle corresponding to multiples of $\frac{\pi}{6}$ using rotations and reflections.



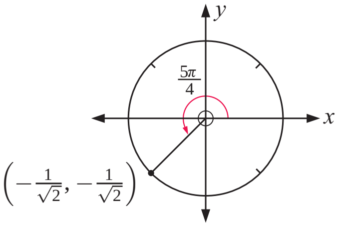
Summary:

- If θ is a **multiple of $\frac{\pi}{2}$**, the coordinates of the points on the unit circle involve 0 and ±1.
- If θ is a **multiple of $\frac{\pi}{4}$**, but not a multiple of $\frac{\pi}{2}$, the coordinates involve $\pm\frac{1}{\sqrt{2}}$.
- If θ is a **multiple of $\frac{\pi}{6}$**, but not a multiple of $\frac{\pi}{2}$, the coordinates involve $\pm\frac{1}{2}$ and $\pm\frac{\sqrt{3}}{2}$.

You should not try to memorise the coordinates on the above circles for every multiple of $\frac{\pi}{6}$ and $\frac{\pi}{4}$, but rather use the summary to work out the correct result.

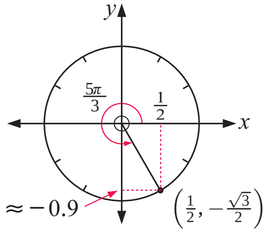
For example:

Consider $225^o = \frac{5\pi}{4}$.



$\frac{5\pi}{4}$ is in quad 3, so signs are both negative and both have $\frac{1}{\sqrt{2}}$ size.

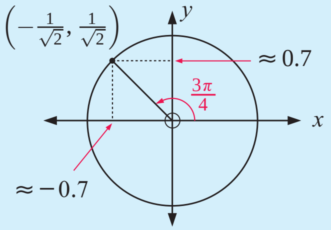
Consider $300^o = \frac{5\pi}{3}$ which is a multiple of $\frac{\pi}{6}$.



$\frac{5\pi}{4}$ is in quad 4, so signs are $(+, -)$ and from the diagram the x-value is $\frac{1}{2}$.

Example 11 **Self Tutor**

Use a unit circle to find the exact values of $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$ for $\alpha = \frac{3\pi}{4}$.

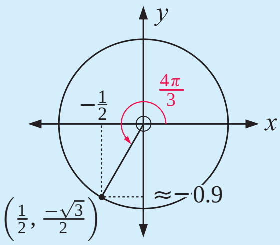


$$\therefore \quad \cos\left(\tfrac{3\pi}{4}\right) = -\tfrac{1}{\sqrt{2}}, \quad \sin\left(\tfrac{3\pi}{4}\right) = \tfrac{1}{\sqrt{2}}$$

$$\tan\left(\tfrac{3\pi}{4}\right) = -1$$

Example 12 **Self Tutor**

Use a unit circle diagram to find the exact values of $\sin A$, $\cos A$ and $\tan A$ for $A = \frac{4\pi}{3}$.



$\therefore \quad \cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$

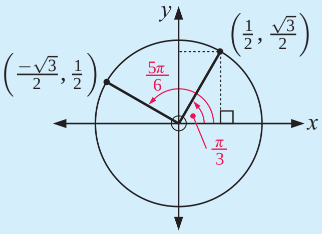
and $\quad \sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$\therefore \quad \tan\left(\frac{4\pi}{3}\right) = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}}$

$\therefore \quad \tan\left(\frac{4\pi}{3}\right) = \sqrt{3}$

Example 13 **Self Tutor**

Without using a calculator, find the value of $8\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{5\pi}{6}\right)$.

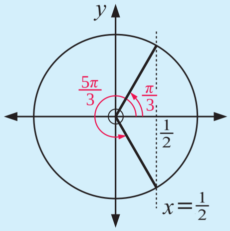


$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

$$\begin{aligned} \therefore \quad 8\sin\left(\tfrac{\pi}{3}\right)\cos\left(\tfrac{5\pi}{6}\right) &= 8\left(\tfrac{\sqrt{3}}{2}\right)\left(-\tfrac{\sqrt{3}}{2}\right) \\ &= 2(-3) \\ &= -6 \end{aligned}$$

Example 14 **Self Tutor**

Use a unit circle diagram to find all angles $0 \leqslant \theta \leqslant 2\pi$ with a cosine of $\frac{1}{2}$.



As the cosine is $\frac{1}{2}$, we draw the vertical line $x = \frac{1}{2}$.

Because $\frac{1}{2}$ is involved we know the required angles are multiples of $\frac{\pi}{6}$.

They are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.

EXERCISE 8C.3

1 Use a unit circle diagram to find $\sin\theta$, $\cos\theta$ and $\tan\theta$ for θ equal to:

a $\frac{\pi}{4}$ **b** $\frac{5\pi}{4}$ **c** $\frac{7\pi}{4}$ **d** π **e** $\frac{-3\pi}{4}$

2 Use a unit circle diagram to find $\sin\beta$, $\cos\beta$ and $\tan\beta$ for β equal to:

a $\frac{\pi}{6}$ **b** $\frac{2\pi}{3}$ **c** $\frac{7\pi}{6}$ **d** $\frac{5\pi}{3}$ **e** $\frac{11\pi}{6}$

3 Without using a calculator, evaluate:

a $\sin^2 60^o$ **b** $\sin 30^o \cos 60^o$ **c** $4\sin 60^o \cos 30^o$

d $1 - \cos^2\left(\frac{\pi}{6}\right)$ **e** $\sin^2\left(\frac{2\pi}{3}\right) - 1$ **f** $\cos^2\left(\frac{\pi}{4}\right) - \sin\left(\frac{7\pi}{6}\right)$

g $\sin(\frac{3\pi}{4}) - \cos(\frac{5\pi}{4})$ **h** $1 - 2\sin^2(\frac{7\pi}{6})$ **i** $\cos^2(\frac{5\pi}{6}) - \sin^2(\frac{5\pi}{6})$

j $\tan^2(\frac{\pi}{3}) - 2\sin^2(\frac{\pi}{4})$ **k** $2\tan(-\frac{5\pi}{4}) - \sin(\frac{3\pi}{2})$ **l** $\dfrac{2\tan 150^o}{1 - \tan^2 150^o}$

Check all answers using your calculator.

4 Use a unit circle diagram to find all angles between 0^o and 360^o with:

a a sine of $\frac{1}{2}$ **b** a sine of $\frac{\sqrt{3}}{2}$ **c** a cosine of $\frac{1}{\sqrt{2}}$

d a cosine of $-\frac{1}{2}$ **e** a cosine of $-\frac{1}{\sqrt{2}}$ **f** a sine of $-\frac{\sqrt{3}}{2}$

5 Use a unit circle diagram to find all angles between 0 and 2π (inclusive) which have:

a a tangent of 1 **b** a tangent of -1 **c** a tangent of $\sqrt{3}$

d a tangent of 0 **e** a tangent of $\frac{1}{\sqrt{3}}$ **f** a tangent of $-\sqrt{3}$

6 Use a unit circle diagram to find all angles between 0 and 4π with:

a a cosine of $\frac{\sqrt{3}}{2}$ **b** a sine of $-\frac{1}{2}$ **c** a sine of -1

7 Find θ in radians if $0 \leqslant \theta \leqslant 2\pi$ and:

a $\cos\theta = \frac{1}{2}$ **b** $\sin\theta = \frac{\sqrt{3}}{2}$ **c** $\cos\theta = -1$ **d** $\sin\theta = 1$

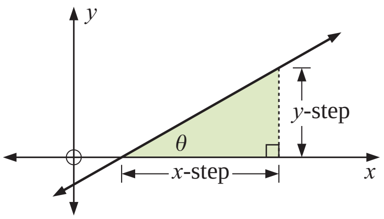
e $\cos\theta = -\frac{1}{\sqrt{2}}$ **f** $\sin^2\theta = 1$ **g** $\cos^2\theta = 1$ **h** $\cos^2\theta = \frac{1}{2}$

i $\tan\theta = -\frac{1}{\sqrt{3}}$ **j** $\tan^2\theta = 3$

D THE EQUATION OF A STRAIGHT LINE

If a straight line makes an angle of θ with the positive x-axis then its gradient is $m = \tan\theta$.

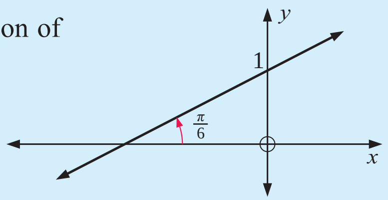
Proof:



$$\text{Gradient } m = \frac{y\text{-step}}{x\text{-step}} = \tan\theta$$

Example 15 **Self Tutor**

Find the equation of the given line:



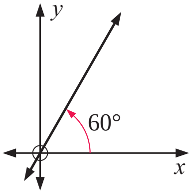
The line has gradient $m = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$
and its y-intercept is 1.

$\therefore$ the line has equation $y = \frac{1}{\sqrt{3}}x + 1$.

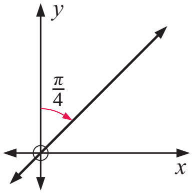
EXERCISE 8D

1 Find the equations of the following lines:

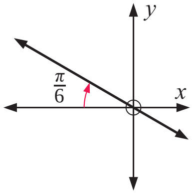
a



b

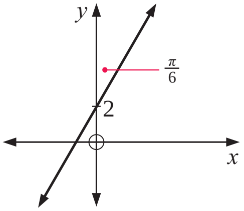


c



2 Find the equations of the following lines:

a



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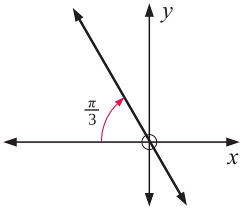

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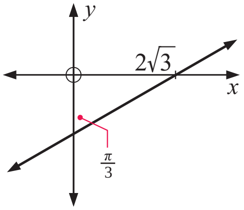


REVIEW SET 8A NON-CALCULATOR

1 Convert these to radians in terms of π: **a** 120^o **b** 225^o **c** 150^o **d** 540^o

2 Find the acute angles that would have the same:

a sine as $\frac{2\pi}{3}$ **b** sine as 165^o **c** cosine as 276^o.

3 Find:

a $\sin 159^o$ if $\sin 21^o \approx 0.358$ **b** $\cos 92^o$ if $\cos 88^o \approx 0.035$

c $\cos 75^o$ if $\cos 105^o \approx -0.259$ **d** $\sin 227^o$ if $\sin 47^o \approx 0.731$.

4 Use a unit circle diagram to find:

a $\cos 360^o$ and $\sin 360^o$ **b** $\cos(-\pi)$ and $\sin(-\pi)$.

5 Explain how to use the unit circle to find θ when $\cos\theta = -\sin\theta$.

6 Use a unit circle diagram to find exact values for $\sin\theta$ and $\cos\theta$ for θ equal to:

a $\frac{2\pi}{3}$ **b** $\frac{8\pi}{3}$

7 If $\sin x = -\frac{1}{4}$ and $\pi < x < \frac{3\pi}{2}$, find $\tan x$ in radical form.

8 If $\cos 42^o \approx 0.743$, find the value of:

a $\cos 138^o$ **b** $\cos 222^o$ **c** $\cos 318^o$ **d** $\cos(-222^o)$

9 If $\cos\theta = \frac{3}{4}$ find the possible values of $\sin\theta$.

10 Evaluate:

a $2\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)$ **b** $\tan^2\left(\frac{\pi}{4}\right)-1$ **c** $\cos^2\left(\frac{\pi}{6}\right)-\sin^2\left(\frac{\pi}{6}\right)$

11 Given $\tan x = -\frac{3}{2}$ and $\frac{3\pi}{2} < x < 2\pi$, find: **a** $\sin x$ **b** $\cos x$.

REVIEW SET 8B CALCULATOR

1 Determine the coordinates of the point on the unit circle corresponding to an angle of:

a 320^o **b** 163^o

2 Convert to radians to 4 significant figures:

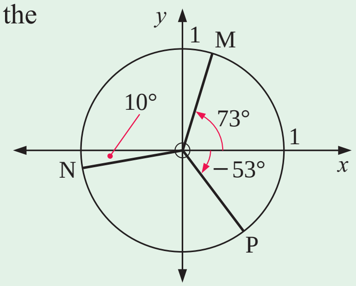
a 71^o **b** 124.6^o **c** -142^o

3 Convert these radian measurements to degrees, to 2 decimal places:

a 3 **b** 1.46 **c** 0.435 **d** -5.271

4 Determine the area of a sector of angle $\frac{5\pi}{12}$ and radius 13 cm.

5 Find the coordinates of the points M, N and P on the unit circle.



6 Find the angle [OA] makes with the positive x-axis if the x-coordinate of the point A on the unit circle is -0.222.

7 Use a unit circle diagram to find all angles between 0^o and 360^o which have:

a a cosine of $-\frac{\sqrt{3}}{2}$ **b** a sine of $\frac{1}{\sqrt{2}}$ **c** a tangent of $-\sqrt{3}$

8 Find θ for $0 \leqslant \theta \leqslant 2\pi$ if:

a $\cos\theta = -1$ **b** $\sin^2\theta = \frac{3}{4}$

9 Find the obtuse angles which have the same:

a sine as 47^o **b** sine as $\frac{\pi}{15}$ **c** cosine as 186^o.

10 Find the perimeter and area of a sector of radius 11 cm and angle 63^o.

11 Find the radius and area of a sector of perimeter 36 cm with an angle of $\frac{2\pi}{3}$.

REVIEW SET 8C

1 Convert these radian measurements to degrees: **a** $\frac{2\pi}{5}$ **b** $\frac{5\pi}{4}$ **c** $\frac{7\pi}{9}$ **d** $\frac{11\pi}{6}$

2 Illustrate the regions where $\sin\theta$ and $\cos\theta$ have the same sign.

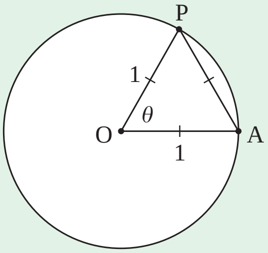
3 Use a unit circle diagram to find:

a $\cos\left(\frac{3\pi}{2}\right)$ and $\sin\left(\frac{3\pi}{2}\right)$ **b** $\cos\left(-\frac{\pi}{2}\right)$ and $\sin\left(-\frac{\pi}{2}\right)$

4 If $\sin 74^o \approx 0.961$, find without using a calculator the value of:

a $\sin 106^o$ **b** $\sin 254^o$ **c** $\sin 286^o$ **d** $\sin 646^o$

5



a State the value of θ in:

i degrees **ii** radians.

b State the arc length AP.

6 Without a calculator, evaluate $\tan^2\left(\frac{2\pi}{3}\right)$.

7 Suppose $f(x) = \cos(x) - \sin(x)$. Show that $f\left(\frac{3\pi}{4}\right) = -\sqrt{2}$.

8 If $\cos\theta = -\frac{3}{4}$, $\frac{\pi}{2} < \theta < \pi$ find the exact value of:

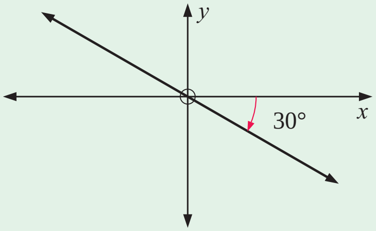
a $\sin\theta$ **b** $\tan\theta$ **c** $\sin(\theta + \pi)$

9 Without using a calculator, evaluate:

a $\tan^2 60^o - \sin^2 45^o$ **b** $\cos^2(\frac{\pi}{4}) + \sin(\frac{\pi}{2})$ **c** $\cos(\frac{5\pi}{3}) - \tan(\frac{5\pi}{4})$

10 Simplify: **a** $\sin(\pi - \theta) - \sin\theta$ **b** $\cos\theta\tan\theta$

11



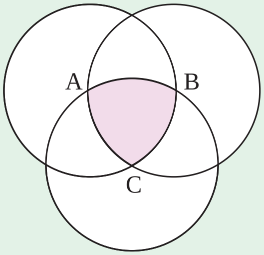
a Find the equation of the line drawn.

b Find the exact value of k given the point $(k, 2)$ lies on the line.

12 Three circles with radius r are drawn as shown, each with its centre on the circumference of the other two circles. A, B and C are the centres of the three circles.

Prove that an expression for the area of the shaded region is:

$$A = \frac{r^2}{2}(\pi - \sqrt{3})$$



Chapter 9

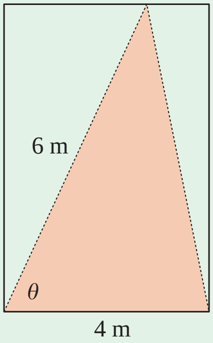
Non-right angled triangle trigonometry

Syllabus reference: 3.6

Contents:

OPENING PROBLEM

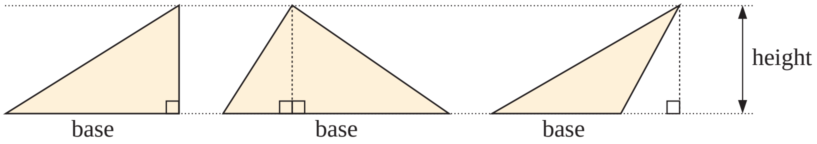
A triangular sail is to be cut from a section of cloth. Two of the sides must have lengths 4 m and 6 m as illustrated. The total area for the sail must be 11.6 m^2, the maximum allowed for the boat to race in its particular class.



Things to think about:

- **a** Can you find the size of the angle θ between the two sides of given length?
- **b** Can you find the length of the third side?

A AREAS OF TRIANGLES



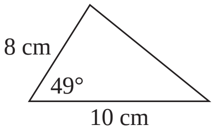
If we know the base and height measurements of a triangle we can calculate the area using **area** $= \frac{1}{2}$ **base** $\times$ **height**.

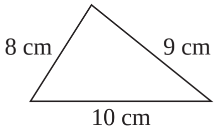
However, cases arise where we do not know the height but we can use trigonometry to calculate the area.

These cases are:

- knowing two sides and the **included angle** between them
- knowing all three sides

For example:

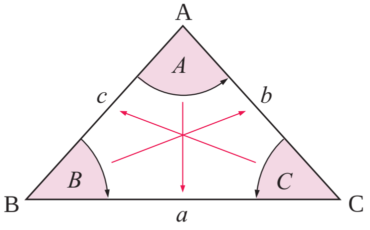




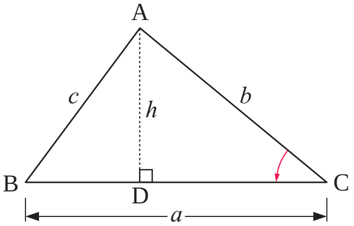
USING THE INCLUDED ANGLE

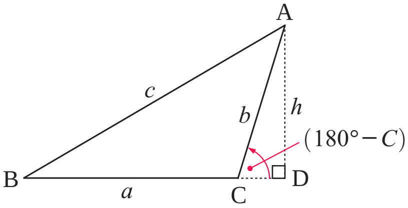
If triangle ABC has angles of size A, B and C, the sides opposite these angles are labelled a, b and c respectively.

Using trigonometry, we can develop an alternative area formula that does not depend on a perpendicular height.



Any triangle that is not right angled must be either acute or obtuse. We will consider both cases:





In both triangles a perpendicular is constructed from A to D on [BC] (extended if necessary).

$$\sin C = \frac{h}{b} \qquad\qquad \sin(180^o - C) = \frac{h}{b}$$

$$\therefore \quad h = b\sin C \qquad\qquad \therefore \quad h = b\sin(180^o - C)$$

$$\text{but} \quad \sin(180^o - C) = \sin C$$

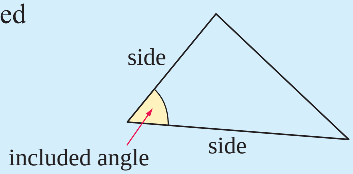
$$\therefore \quad h = b\sin C$$

So, area $= \frac{1}{2}ah$ gives $\boldsymbol{A = \frac{1}{2}ab\sin C}$.

Using different altitudes we can show that the area is also $\frac{1}{2}bc\sin A$ or $\frac{1}{2}ac\sin B$.

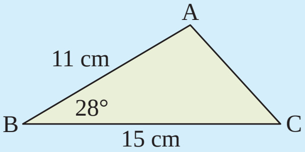
Given the lengths of two sides of a triangle and the included angle between them, the area of the triangle is

a half of the product of two sides and the sine of the included angle.



Example 1 **Self Tutor**

Find the area of triangle ABC:



Area $= \frac{1}{2}ac\sin B$

$= \frac{1}{2} \times 15 \times 11 \times \sin 28^o$

$\approx 38.7\ \text{cm}^2$

When we use the area formula to find the included angle between two sides, we need to use the **inverse** sine ratio. This is denoted $\sin^{-1}$ or arcsin. For help with this and the other inverse trigonometric ratios you should consult the Background Knowledge chapter on the CD.

Example 2 **Self Tutor**

A triangle has sides of length 10 cm and 11 cm and an area of 50 cm^2. Show that the included angle may have two possible sizes.

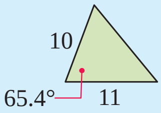
If the included angle is θ, then $\frac{1}{2} \times 10 \times 11 \times \sin\theta = 50$

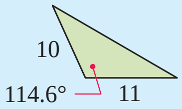
$\therefore \quad \sin\theta = \frac{50}{55}$

Now $\sin^{-1}(\frac{50}{55}) \approx 65.4^o$

$\therefore \quad \theta \approx 65.4^o$ or $180^o - 65.4^o$

So, $\theta \approx 65.4^o$ or 114.6^o



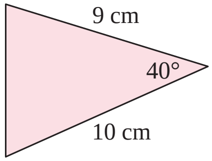


The two different possible angles are 65.4^o and 114.6^o.

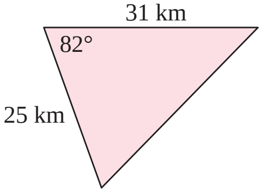
EXERCISE 9A

1 Find the area of:

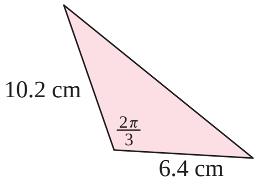
a



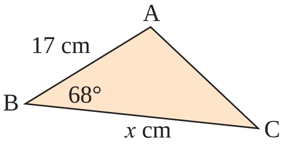
b



c



2 If triangle ABC has area 150 cm^2, find the value of x:



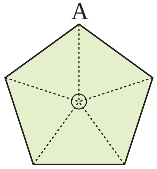
3 A parallelogram has two adjacent sides of length 4 cm and 6 cm respectively. If the included angle measures 52^o, find the area of the parallelogram.

4 A rhombus has sides of length 12 cm and an angle of 72^o. Find its area.

5 Find the area of a regular hexagon with sides of length 12 cm.

6 A rhombus has an area of 50 cm^2 and an internal angle of size 63^o. Find the length of its sides.

7 A regular pentagonal garden plot has centre of symmetry O and an area of 338 m^2. Find the distance OA.



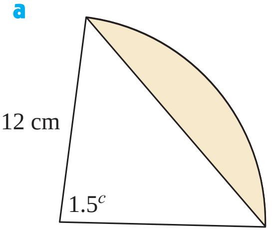
8 Find the possible values of the included angle of a triangle with:

- **a** sides of length 5 cm and 8 cm, and area 15 cm^2
- **b** sides of length 45 km and 53 km, and area 800 km^2.

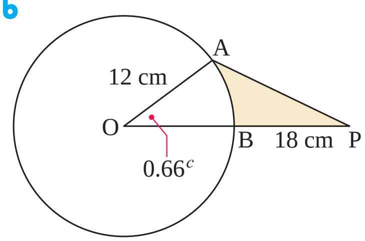
9 The Australian 50 cent coin has the shape of a regular dodecagon, which is a polygon with 12 sides. Eight of these 50 cent coins will fit exactly on an Australian \$10 note as shown. What fraction of the \$10 note is *not* covered?

10 Find the shaded area in:

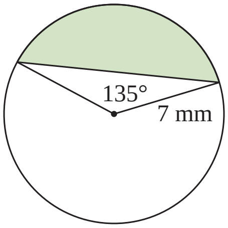
a



b

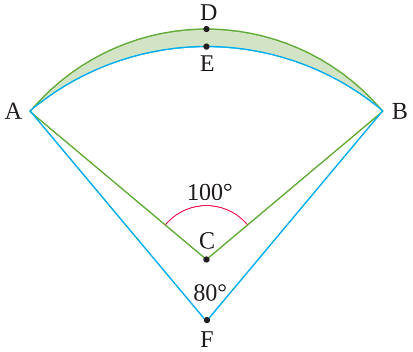


c



11 ADB is an arc of the circle with centre C and radius 7.3 cm. AEB is an arc of the circle with centre F and radius 8.7 cm.

Find the shaded area.

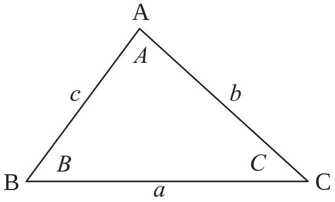


B THE COSINE RULE

The **cosine rule** involves the sides and angles of a triangle.

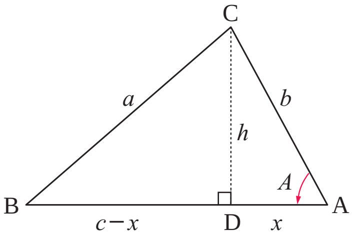
In any ΔABC:

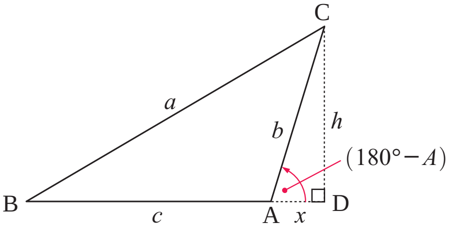
$$a^2 = b^2 + c^2 - 2bc\cos A$$
or $$b^2 = a^2 + c^2 - 2ac\cos B$$
or $$c^2 = a^2 + b^2 - 2ab\cos C$$



We will develop the first formula for both an acute and an obtuse triangle.

Proof:





In both triangles drop a perpendicular from C to meet [AB] (extended if necessary) at D.

Let AD $= x$ and let CD $= h$.

Apply the theorem of Pythagoras in ΔBCD:

$$a^2 = h^2 + (c - x)^2$$
$$\therefore \quad a^2 = h^2 + c^2 - 2cx + x^2$$

$$a^2 = h^2 + (c + x)^2$$
$$\therefore \quad a^2 = h^2 + c^2 + 2cx + x^2$$

In both cases, applying Pythagoras to ΔADC gives $h^2 + x^2 = b^2$.

$\therefore$ $h^2 = b^2 - x^2$, and we substitute this into the equations above.

$$\therefore \quad a^2 = b^2 + c^2 - 2cx$$

In ΔADC: $\cos A = \dfrac{x}{b}$

$$\therefore \quad b\cos A = x$$

$$\therefore \quad a^2 = b^2 + c^2 - 2bc\cos A$$

$$\therefore \quad a^2 = b^2 + c^2 + 2cx$$

$$\cos(180^o - A) = \frac{x}{b}$$

$$\therefore \quad b\cos(180^o - A) = x$$

But $\cos(180^o - A) = -\cos A$

$$\therefore \quad -b\cos A = x$$

$$\therefore \quad a^2 = b^2 + c^2 - 2bc\cos A$$

The other variations of the cosine rule could be developed by rearranging the vertices of ΔABC.

Note that if $A = 90^o$ then $\cos A = 0$ and $a^2 = b^2 + c^2 - 2bc\cos A$ reduces to $a^2 = b^2 + c^2$, which is the Pythagorean Rule.

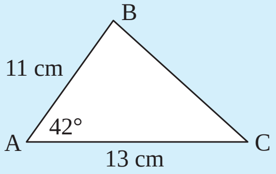
The **cosine rule** can be used to solve triangles given:

- **two sides** and an **included angle**
- **three sides**.

If we are given **two sides** and a **non-included angle**, then when we try to find the third side we will end up with a quadratic equation. This is an *ambiguous* case where there may be two plausible solutions.

Example 3 **Self Tutor**

Find, correct to 2 decimal places, the length of [BC].



By the cosine rule:

$$\mathrm{BC}^2 = 11^2 + 13^2 - 2 \times 11 \times 13 \times \cos 42^o$$

$$\therefore \quad \mathrm{BC} \approx \sqrt{(11^2 + 13^2 - 2 \times 11 \times 13 \times \cos 42^o)}$$

$$\therefore \quad \mathrm{BC} \approx 8.801$$

$\therefore$ [BC] is 8.80 cm in length.

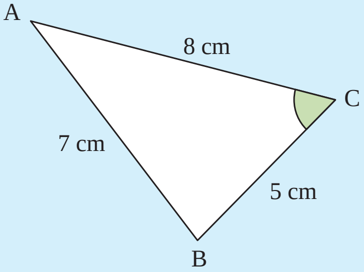
Rearrangement of the original cosine rule formulae can be used for finding angles if we know all three sides. The formulae for finding the angles are:

$$\boldsymbol{\cos A = \frac{b^2 + c^2 - a^2}{2bc}} \qquad \boldsymbol{\cos B = \frac{c^2 + a^2 - b^2}{2ca}} \qquad \boldsymbol{\cos C = \frac{a^2 + b^2 - c^2}{2ab}}$$

We then need to use the **inverse** cosine ratio $\cos^{-1}$ or arccos to evaluate the angle.

Example 4 **Self Tutor**

In triangle ABC, if AB = 7 cm, BC = 5 cm and CA = 8 cm, find the measure of angle BCA.



By the cosine rule:

$$\cos C = \frac{(5^2 + 8^2 - 7^2)}{(2 \times 5 \times 8)}$$

$$\therefore \quad C = \cos^{-1}\left(\frac{5^2 + 8^2 - 7^2}{2 \times 5 \times 8}\right)$$

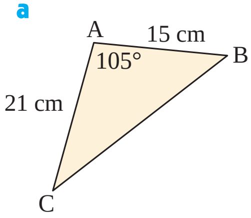
$$\therefore \quad C = 60^o$$

So, angle BCA measures 60^o.

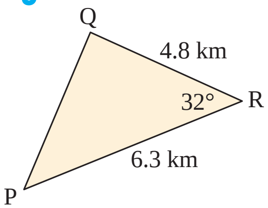
EXERCISE 9B

1 Find the length of the remaining side in the given triangle:

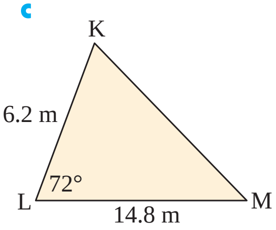
a



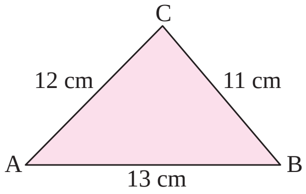
b



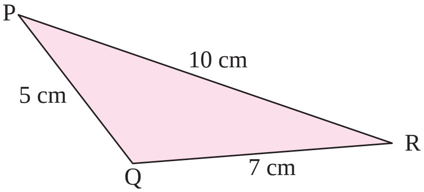
c



2 Find the measure of all angles of:

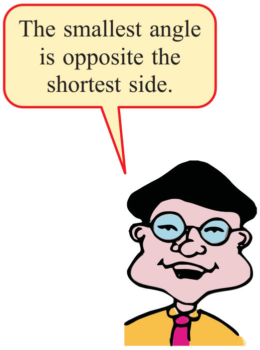


3 Find the measure of obtuse angle PQR.



4 **a** Find the smallest angle of a triangle with sides 11 cm, 13 cm and 17 cm.

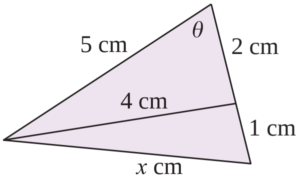
b Find the largest angle of a triangle with sides 4 cm, 7 cm and 9 cm.



The smallest angle is opposite the shortest side.

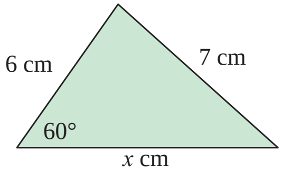
5 **a** Find $\cos\theta$ but not θ.

b Find the value of x.

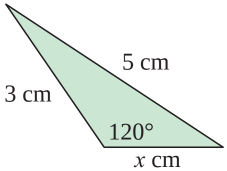


6 Find the exact value of x in each of the following diagrams:

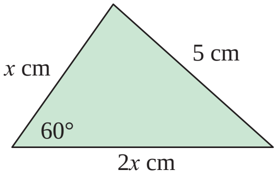
a



b

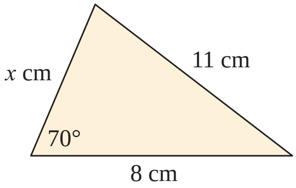


c

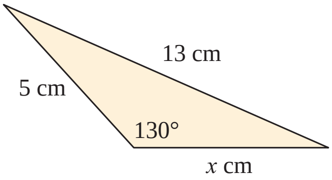


7 Find x in each of the following diagrams:

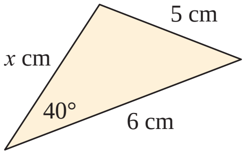
a



b



c

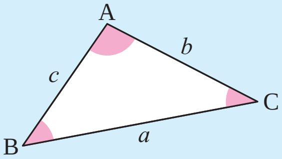


C THE SINE RULE

The **sine rule** is a set of equations which connects the lengths of the sides of any triangle with the sines of the angles of the triangle. The triangle does not have to be right angled for the sine rule to be used.

In any triangle ABC with sides a, b and c units in length, and opposite angles A, B and C respectively,

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c} \quad \text{or} \quad \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$



Proof: The area of any triangle ABC is given by $\frac{1}{2}bc\sin A=\frac{1}{2}ac\sin B=\frac{1}{2}ab\sin C$.

Dividing each expression by $\frac{1}{2}abc$ gives $\dfrac{\sin A}{a}=\dfrac{\sin B}{b}=\dfrac{\sin C}{c}$.

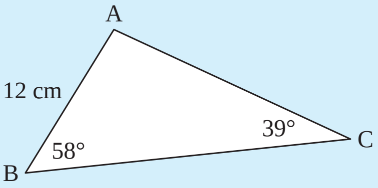
The sine rule is used to solve problems involving triangles, given:

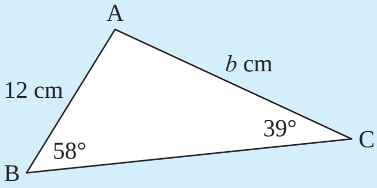
- **two angles** and **one side**
- **two sides** and a **non-included** angle.

FINDING SIDES

Example 5 **Self Tutor**

Find the length of [AC] correct to two decimal places.





Using the sine rule, $\dfrac{b}{\sin 58^o}=\dfrac{12}{\sin 39^o}$

$\therefore \quad b=\dfrac{12\times\sin 58^o}{\sin 39^o}$

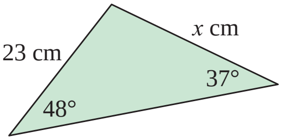
$\therefore \quad b\approx 16.170\,74$

$\therefore$ [AC] is about 16.17 cm long.

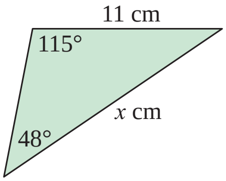
EXERCISE 9C.1

1 Find the value of x:

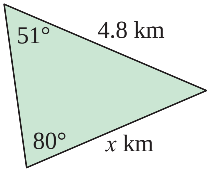
a



b



c



2 In triangle ABC find:

- **a** a if $A = 63^o$, $B = 49^o$ and $b = 18$ cm
- **b** b if $A = 82^o$, $C = 25^o$ and $c = 34$ cm
- **c** c if $B = 21^o$, $C = 48^o$ and $a = 6.4$ cm.

FINDING ANGLES

The problem of finding angles using the sine rule is more complicated because there may be two possible answers. We call this situation the **ambiguous case**.

You can click on the icon to obtain an interactive demonstration of the ambiguous case, or else you can work through the following investigation.

DEMO

INVESTIGATION THE AMBIGUOUS CASE

You will need a blank sheet of paper, a ruler, a protractor, and a compass for the tasks that follow. In each task you will be required to construct triangles from given information. You could also do this using a computer package such as 'The Geometer's Sketchpad'.

Task 1: Draw AB = 10 cm. At A construct an angle of 30^o. Using B as the centre, draw an arc of a circle of radius 6 cm. Let the arc intersect the ray from A at C. How many different positions may C have, and therefore how many different triangles ABC may be constructed?

Task 2: As before, draw AB = 10 cm and construct a 30^o angle at A. This time draw an arc of radius 5 cm centred at B. How many different triangles are possible?

Task 3: Repeat, but this time draw an arc of radius 3 cm centred at B. How many different triangles are possible?

Task 4: Repeat with an arc of radius 12 cm from B. How many triangles are possible now?

You should have discovered that when you are given two sides and a non-included angle there are a number of different possibilities. You could get two triangles, one triangle, or it may be impossible to draw any triangles at all from the given data.

Now consider the calculations involved in each of the cases of the investigation.

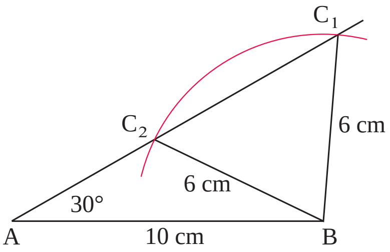
Task 1: Given: $c = 10$ cm, $a = 6$ cm, $A = 30^o$

Finding C:

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\therefore \quad \sin C = \frac{c \sin A}{a}$$

$$\therefore \quad \sin C = \frac{10 \times \sin 30^o}{6} \approx 0.8333$$



Because $\sin\theta = \sin(180^o - \theta)$ there are two possible angles:
$C \approx 56.44^o$ *or* $180^o - 56.44^o = 123.56^o$

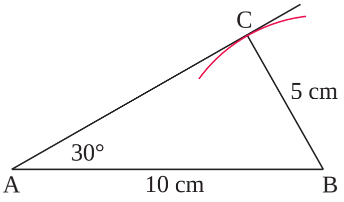
Task 2: Given: $c = 10$ cm, $a = 5$ cm, $A = 30^o$

Finding C:

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\therefore \quad \sin C = \frac{c \sin A}{a}$$

$$\therefore \quad \sin C = \frac{10 \times \sin 30^o}{5} = 1$$



There is only one possible solution for C in the range from 0^o to 180^o, and that is $C = 90^o$. Only one triangle is therefore possible. Complete the solution of the triangle yourself.

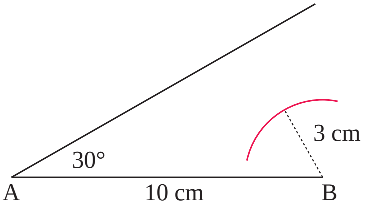
Task 3: Given: $c = 10$ cm, $a = 3$ cm, $A = 30^o$

Finding C:

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\therefore \quad \sin C = \frac{c \sin A}{a}$$

$$\therefore \quad \sin C = \frac{10 \times \sin 30^o}{3} \approx 1.6667$$



There is no angle that has a sine ratio > 1. Therefore there is *no solution* for this given data, and no triangles can be drawn to match the information given.

Task 4: Given: $c = 10$ cm, $a = 12$ cm, $A = 30^o$

Finding C:

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

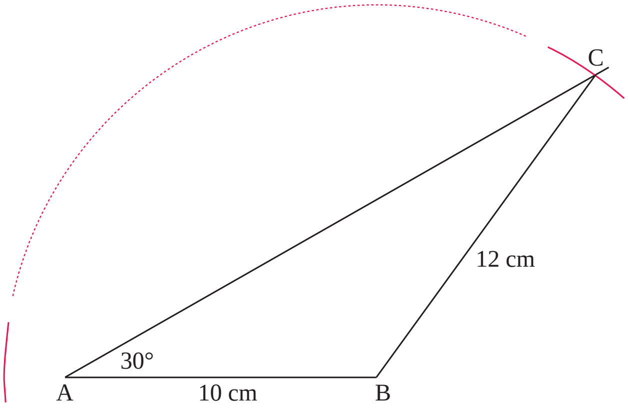
$$\therefore \quad \sin C = \frac{c \sin A}{a}$$

$$\therefore \quad \sin C = \frac{10 \times \sin 30^o}{12}$$

$$\therefore \quad \sin C = 0.4167$$

Two angles have a sine ratio of 0.4167:

$C \approx 24.62^o$ *or*
$180^o - 24.62^o = 155.38^o$



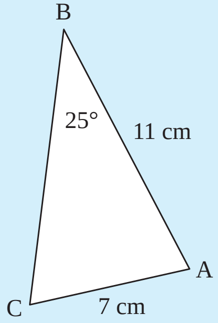
However, in this case only one of these two angles is valid. If $A = 30^o$ then C cannot possibly equal 155.38^o because $30^o + 155.38^o > 180^o$.

Therefore, there is only one possible solution, $C \approx 24.62^o$.

Conclusion: Each situation using the sine rule with two sides and a non-included angle must be examined very carefully.

Example 6 **Self Tutor**

Find the measure of angle C in triangle ABC if AC $= 7$ cm, AB $= 11$ cm, and angle B measures 25^o.



$$\frac{\sin C}{c} = \frac{\sin B}{b} \quad \text{\{by the sine rule\}}$$

$$\therefore \quad \frac{\sin C}{11} = \frac{\sin 25^o}{7}$$

$$\therefore \quad \sin C = \frac{11 \times \sin 25^o}{7}$$

$$\therefore \quad C = \sin^{-1}\left(\frac{11 \times \sin 25^o}{7}\right) \text{ or its supplement}$$

$$\therefore \quad C \approx 41.6^o \text{ or } 180^o - 41.6^o$$

{as C may be obtuse}

$$\therefore \quad C \approx 41.6^o \text{ or } 138.4^o$$

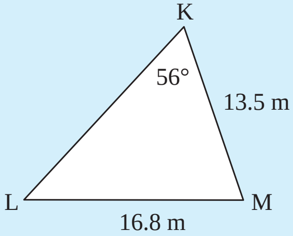
$\therefore$ C measures 41.6^o if angle C is acute, or 138.4^o if angle C is obtuse.

In this case there is insufficient information to determine the actual shape of the triangle.

Sometimes there is information in the question which enables us to **reject** one of the answers.

Example 7 **Self Tutor**

Find the measure of angle L in triangle KLM given that angle LKM measures 56^o, LM $= 16.8$ m, and KM $= 13.5$ m.



$$\frac{\sin L}{13.5} = \frac{\sin 56^o}{16.8} \quad \text{\{by the sine rule\}}$$

$$\therefore \quad \sin L = \frac{13.5 \times \sin 56^o}{16.8}$$

$$\therefore \quad L = \sin^{-1}\left(\frac{13.5 \times \sin 56^o}{16.8}\right) \text{ or its supplement}$$

$$\therefore \quad L \approx 41.8^o \text{ or } 180^o - 41.8^o$$

$$\therefore \quad L \approx 41.8^o \text{ or } 138.2^o$$

We reject $L \approx 138.2^o$, since $138.2^o + 56^o > 180^o$ which is impossible.

$\therefore$ $L \approx 41.8^o$.

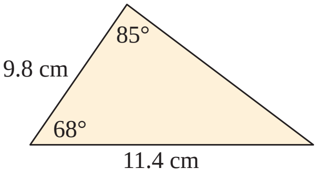
EXERCISE 9C.2

1 Triangle ABC has angle $B = 40^o$, $b = 8$ cm, and $c = 11$ cm. Find the two possible values for angle C.

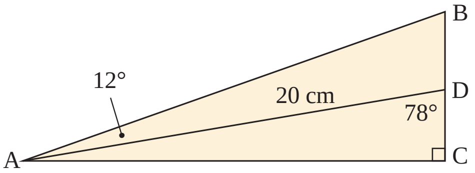
2 In triangle ABC, find the measure of:

- **a** angle A if $a = 14.6$ cm, $b = 17.4$ cm and $\mathrm{A\widehat{B}C} = 65^o$
- **b** angle B if $b = 43.8$ cm, $c = 31.4$ cm and $\mathrm{A\widehat{C}B} = 43^o$
- **c** angle C if $a = 6.5$ km, $c = 4.8$ km and $\mathrm{B\widehat{A}C} = 71^o$.

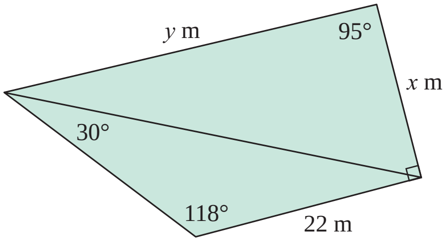
3 Is it possible to have a triangle with the measurements shown? Explain.



4 Find the magnitude of the angle ABC and hence the length BD.



5 Find x and y in the given figure.

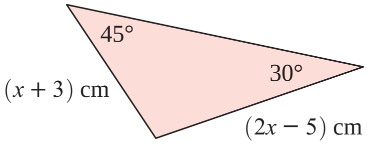


6 Triangle ABC has $\widehat{A} = 58^o$, AB = 10 cm and AC = 5.1 cm.

- **a** Find $\widehat{C}$ correct to the nearest tenth of a degree using the sine rule.
- **b** Find $\widehat{C}$ correct to the nearest tenth of a degree using the cosine rule.
- **c** Copy and complete: "When faced with using either the sine rule or the cosine rule it is better to use the .... as it avoids ...."

7 In triangle ABC, $\mathrm{A\widehat{B}C} = 30^o$, AC = 9 cm and AB = 7 cm. Find the area of the triangle.

8 In the diagram alongside, find the exact value of x. Express your answer in the form $a + b\sqrt{2}$ where $a, b \in \mathbb{Q}$.



D USING THE SINE AND COSINE RULES

If we are given a problem involving a triangle, we must first decide which rule to use.

If the triangle is right angled then the trigonometric ratios or Pythagoras' Theorem can be used. For some problems we can add an extra line or two to the diagram to create a right angled triangle.

However, if we do not have a right angled triangle and we have to choose between the sine and cosine rules, the following checklist may be helpful:

Use the **cosine rule** when given:

- three sides
- two sides and an included angle.

Use the **sine rule** when given:

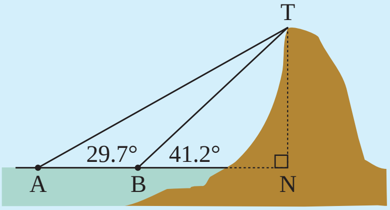
- one side and two angles
- two sides and a non-included angle, but beware of the *ambiguous case* which can occur when the smaller of the two given sides is opposite the given angle.

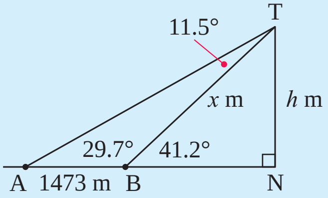
Example 8 **Self Tutor**

The angles of elevation to the top of a mountain are measured from two beacons A and B at sea.

These angles are as shown on the diagram.

If the beacons are 1473 m apart, how high is the mountain?





$$A\widehat{T}B = 41.2^o - 29.7^o \quad \{\text{exterior angle of } \Delta\}$$
$$= 11.5^o$$

We find x in ΔABT using the sine rule:

$$\frac{x}{\sin 29.7^o} = \frac{1473}{\sin 11.5^o}$$
$$\therefore \quad x = \frac{1473}{\sin 11.5^o} \times \sin 29.7^o$$
$$\approx 3660.62$$

Now, in ΔBNT, $\sin 41.2^o = \frac{h}{x} \approx \frac{h}{3660.62}$

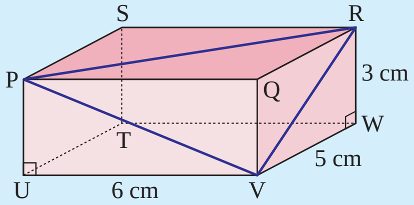
$\therefore \quad h \approx \sin 41.2^o \times 3660.62$

$\therefore \quad h \approx 2410$

So, the mountain is about 2410 m high.

Example 9 **Self Tutor**

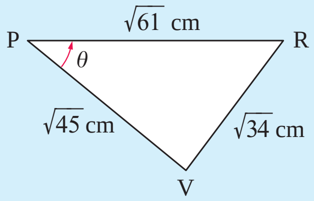
Find the measure of angle RPV.



In ΔRVW, $\text{RV} = \sqrt{5^2+3^2} = \sqrt{34}$ cm. {Pythagoras}

In ΔPUV, $\text{PV} = \sqrt{6^2+3^2} = \sqrt{45}$ cm. {Pythagoras}

In ΔPQR, $\text{PR} = \sqrt{6^2+5^2} = \sqrt{61}$ cm. {Pythagoras}



By rearrangement of the cosine rule,

$$\cos\theta = \frac{(\sqrt{61})^2+(\sqrt{45})^2-(\sqrt{34})^2}{2\sqrt{61}\sqrt{45}}$$

$$= \frac{61+45-34}{2\sqrt{61}\sqrt{45}}$$

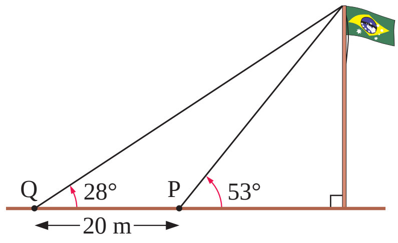
$$= \frac{72}{2\sqrt{61}\sqrt{45}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{36}{\sqrt{61}\sqrt{45}}\right) \approx 46.6^o$$

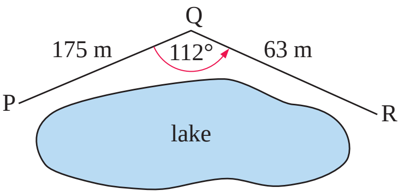
$\therefore$ angle RPV measures about 46.6^o.

EXERCISE 9D

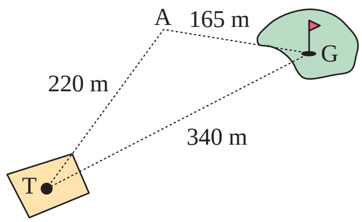
1 Rodrigo wishes to determine the height of a flagpole. He takes a sighting to the top of the flagpole from point P. He then moves further away from the flagpole by 20 metres to point Q and takes a second sighting. The information is shown in the diagram alongside. How high is the flagpole?



2 To get from P to R, a park ranger had to walk along a path to Q and then to R as shown. What is the distance in a straight line from P to R?



3 A golfer played his tee shot a distance of 220 m to point A. He then played a 165 m six iron to the green. If the distance from tee to green is 340 m, determine the number of degrees the golfer was off line with his tee shot.



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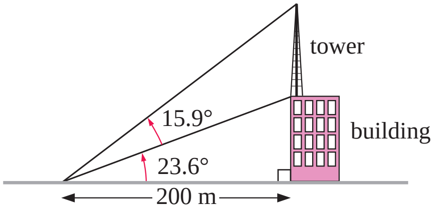
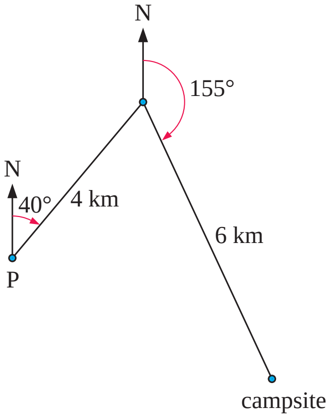


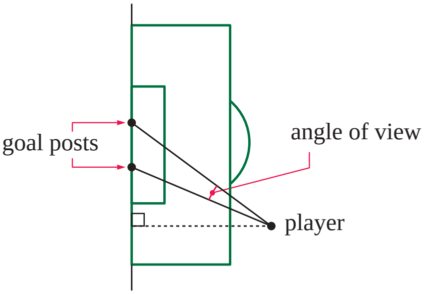
A communications tower is constructed on top of a building as shown. Find the height of the tower.

5 Hikers Ritva and Esko leave point P at the same time. Ritva walks 4 km on a bearing of 040^o and then walks a further 6 km on a bearing of 155^o. Esko hikes directly from P to the camp site.

- **a** **i** How far does Esko hike?
  - **ii** In which direction does Esko hike?
- **b** Ritva hikes at 10 km h^{-1} and Esko hikes at 6 km h^{-1}.
  - **i** Who will arrive at the camp site first?
  - **ii** How long will this person need to wait before the other person arrives?
- **c** On what bearing should they walk from the camp site to return to P?



6 A football goal is 5 metres wide. When a player is 26 metres from one goal post and 23 metres from the other, he shoots for goal. What is the angle of view of the goals that the player sees?

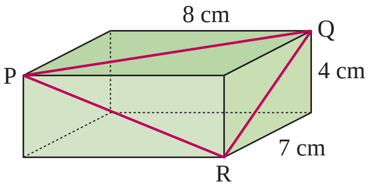


7 A tower 42 metres high stands on top of a hill. From a point some distance from the base of the hill, the angle of elevation to the top of the tower is 13.2^o and the angle of elevation to the bottom of the tower is 8.3^o. Find the height of the hill.

8 From the foot of a building I have to look upwards at an angle of 22^o to sight the top of a tree. From the top of the building, 150 metres above ground level, I have to look down at an angle of 50^o below the horizontal to sight the tree top.

- **a** How high is the tree?
- **b** How far from the building is this tree?

9

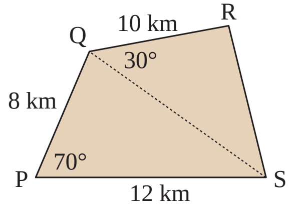


Find the measure of angle PQR in the rectangular box shown.

10 Two observation posts are 12 km apart at A and B. A third observation post C is located such that angle CAB is 42^o and angle CBA is 67^o. Find the distance of C from both A and B.

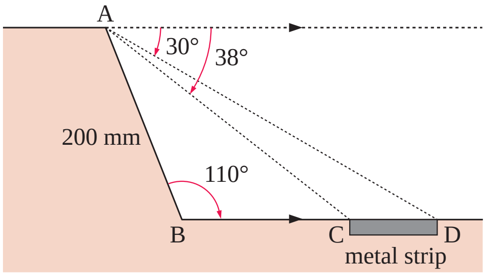
11 Stan and Olga are considering buying a sheep farm. A surveyor has supplied them with the given accurate sketch. Find the area of the property, giving your answer in:

a km^2 **b** hectares.



12 Thabo and Palesa start at point A. They each walk in a straight line at an angle of 120^o to each other. Thabo walks at 6 km h^{-1} and Palesa walks at 8 km h^{-1}. How far apart are they after 45 minutes?

13 The cross-section design of the kerbing for a driverless-bus roadway is shown opposite. The metal strip is inlaid into the concrete and is used to control the direction and speed of the bus. Find the width of the metal strip.



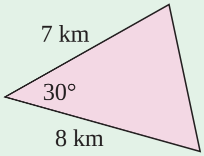
14 An orienteer runs for $4\frac{1}{2}$ km, then turns through an angle of 32^o and runs for another 6 km. How far is she from her starting point?

15 Sam and Markus are standing on level ground 100 metres apart. A large tree is due north of Markus and on a bearing of 065^o from Sam. The top of the tree appears at an angle of elevation of 25^o to Sam and 15^o to Markus. Find the height of the tree.

16 A helicopter A observes two ships B and C. B is 23.8 km from the helicopter and C is 31.9 km from it. The angle of view from the helicopter to B and C (angle BAC) is 83.6^o. How far are the ships apart?

REVIEW SET 9A NON-CALCULATOR

1 Determine the area of:

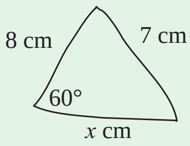


2 You are given details of a triangle such that you could use either the cosine rule or the sine rule to find an unknown. Which rule should you use? Explain your answer.

3 Kady was asked to draw the illustrated triangle exactly.

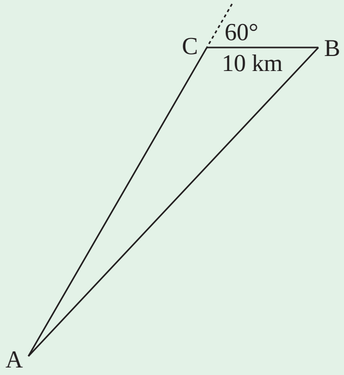
a Use the cosine rule to find x.

b What should Kady's response be?



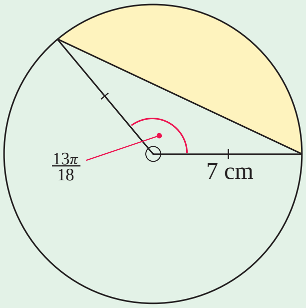
4 A triangle has sides of length 7 cm and 13 cm and its area is 42 cm^2. Find the sine of the included angle.

5



A boat is meant to be sailing directly from A to B. However, it travels in a straight line to C before the captain realises he is off course. The boat is turned through an angle of 60^o, then travels another 10 km to B. The trip would have been 4 km shorter if the boat had gone straight from A to B. How far did the boat travel?

6 Show that the yellow shaded area is given by $A = \frac{49}{2}\left(\frac{13\pi}{18} - \sin(\frac{13\pi}{18})\right)$:

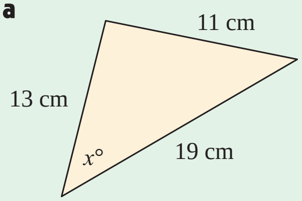


REVIEW SET 9B

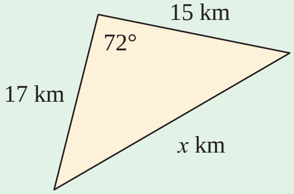
CALCULATOR

1 Determine the value of x:

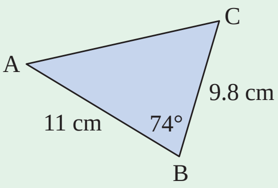
a



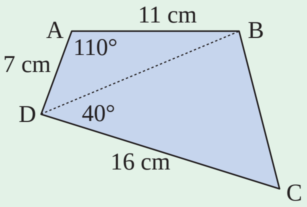
b



2 Find the unknown side and angles:



3 Find the area of quadrilateral ABCD:



4 A vertical tree is growing on the side of a hill with gradient of 10^o to the horizontal. From a point 50 m downhill from the tree, the angle of elevation to the top of the tree is 18^o. Find the height of the tree.

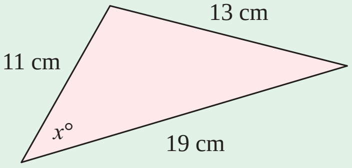
5 From point A, the angle of elevation to the top of a tall building is 20^o. On walking 80 m towards the building the angle of elevation is now 23^o. How tall is the building?

6 Peter, Sue and Alix are sea-kayaking. Peter is 430 m from Sue on a bearing of 113^o while Alix is on a bearing of 210^o and a distance 310 m from Sue. Find the distance and bearing of Peter from Alix.

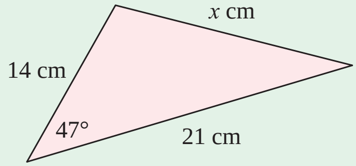
REVIEW SET 9C

1 Find the value of x:

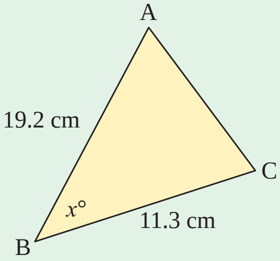
a



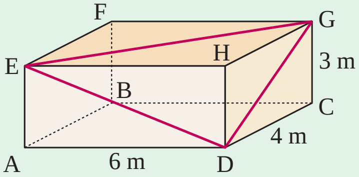
b



2 Find the value of x if the area is 80 cm^2.

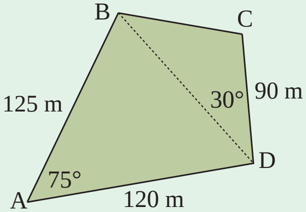


3 Find the measure of angle EDG:



4 Anke and Lucas are considering buying a block of land. The land agent supplies them with the given accurate sketch. Find the area of the property, giving your answer in:

a m^2 **b** hectares.



5 A family in Germany drives at 140 $\text{km}\,\text{h}^{-1}$ for 45 minutes on a bearing of 032^o and then 180 $\text{km}\,\text{h}^{-1}$ for 40 minutes on a bearing 317^o. Find the distance and bearing of the car from its starting point.

6 Soil contractor Frank was given the following dimensions over the telephone:
The triangular garden plot ABC has angle CAB measuring 44^o, [AC] is 8 m long and [BC] is 6 m long. Soil to a depth of 10 cm is required.

a Explain why Frank needs extra information from his client.

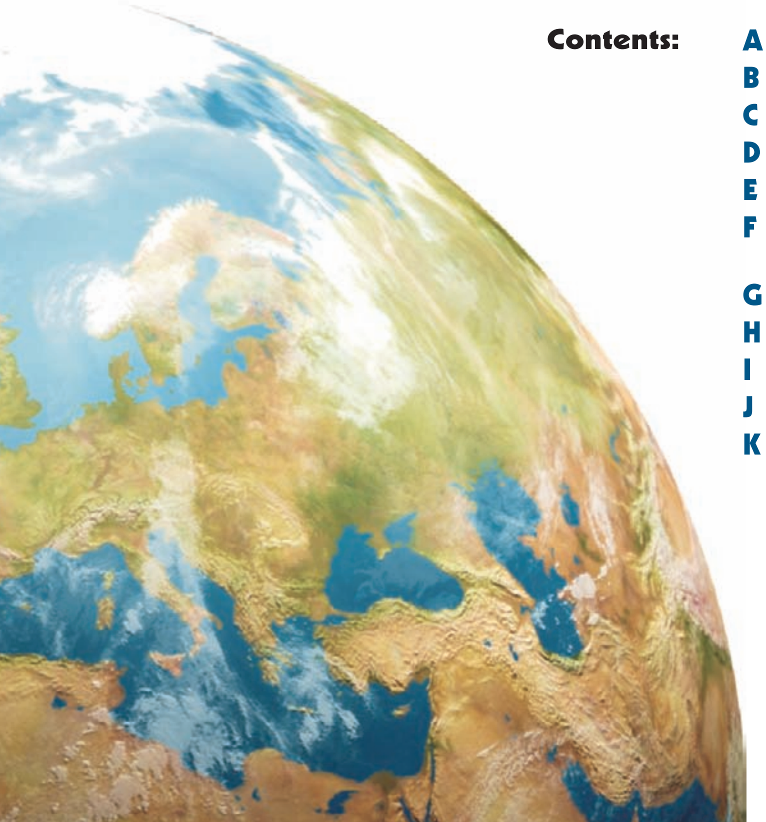
b What is the maximum volume of soil needed if his client is unable to supply the necessary information?

Chapter 10

Advanced trigonometry

Syllabus reference: 3.3, 3.4, 3.5

Contents:



INTRODUCTION

Periodic phenomena occur all the time in the physical world. For example, in:

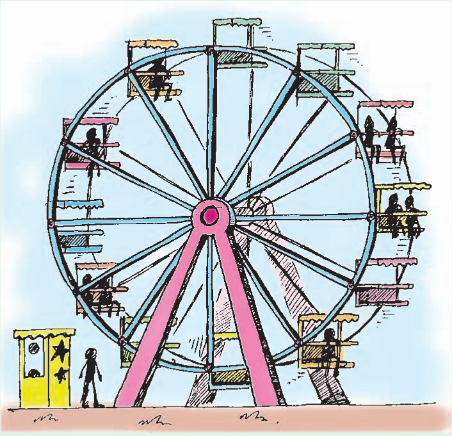
- seasonal variations in our climate
- variations in average maximum and minimum monthly temperatures
- the number of daylight hours at a particular location
- tidal variations in the depth of water in a harbour
- the phases of the moon
- animal populations.

These phenomena illustrate variable behaviour which is repeated over time. The repetition may be called **periodic**, **oscillatory**, or **cyclic** in different situations.

In this chapter we will consider how trigonometric functions can be used to model periodic phenomena. We will then extend our knowledge of the trigonometric functions by considering formulae that connect them.

OPENING PROBLEM

A Ferris wheel rotates at a constant speed. The wheel's radius is 10 m and the bottom of the wheel is 2 m above ground level. From a point in front of the wheel, Andrew is watching a green light on the perimeter of the wheel. Andrew notices that the green light moves in a circle. He estimates how high the light is above ground level at two second intervals and draws a scatterplot of his results.



Things to think about:

- What does his scatterplot look like?
- Could a known function be used to model the data?
- How could this function be used to find the light's position at any point in time?
- How could this function be used to find the times when the light is at its maximum and minimum heights?
- What part of the function would indicate the time interval over which one complete cycle occurs?

Click on the icon to visit a simulation of the Ferris wheel. You will be able to view the light from:

DEMO

- in front of the wheel
- a side-on position
- above the wheel.

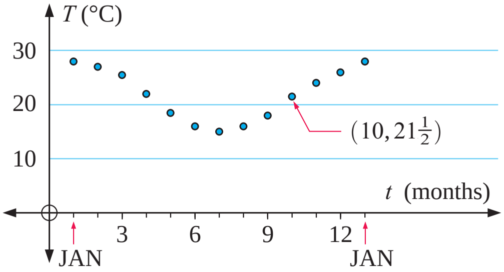
You can then observe the graph of height above or below the wheel's axis as the wheel rotates at a constant rate.

A OBSERVING PERIODIC BEHAVIOUR

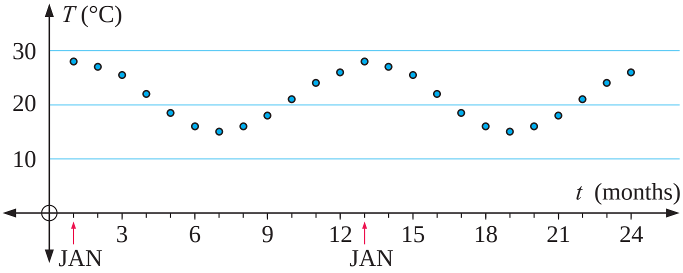
Consider the table below which shows the mean monthly maximum temperature for Cape Town, South Africa.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temp (°C)	28	27	$25\frac{1}{2}$	22	$18\frac{1}{2}$	16	15	16	18	$21\frac{1}{2}$	24	26

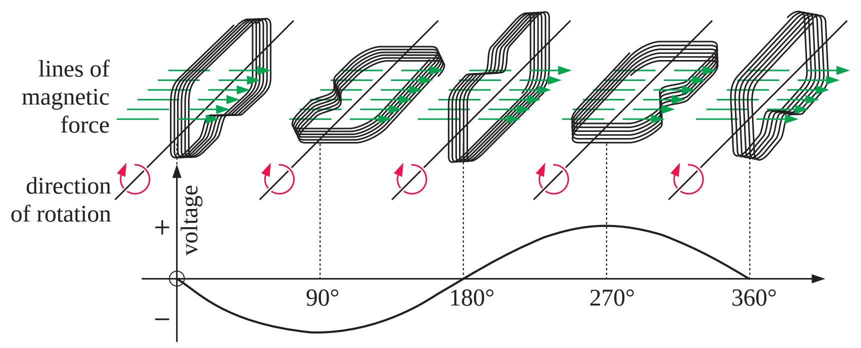
On the scatterplot alongside we plot the temperature T on the vertical axis. We assign January as $t = 1$ month, February as $t = 2$ months, and so on for the 12 months of the year.



The temperature shows a variation from an average of 28°C in January through a range of values across the months. The cycle will repeat itself for the next 12 month period. By the end of the chapter we will be able to establish a function which approximately fits this set of points.



HISTORICAL NOTE



In 1831 **Michael Faraday** discovered that an electric current was generated by rotating a coil of wire in a magnetic field. The electric current produced showed a voltage which varied between positive and negative values as the coil rotated through 360°.

Graphs with this basic shape where the cycle is repeated over and over are called **sine waves**.

GATHERING PERIODIC DATA

Data on a number of periodic phenomena can be found online or in other publications. For example:

- Maximum and minimum monthly temperatures can be found at http://www.bom.gov.au/silo/
- Tidal details can be obtained from daily newspapers or internet sites such as http://tidesandcurrents.noaa.gov or http://www.bom.gov.au/oceanography

TERMINOLOGY USED TO DESCRIBE PERIODICITY

A **periodic function** is one which repeats itself over and over in a horizontal direction.

The **period** of a periodic function is the length of one repetition or cycle.

$f(x)$ is a periodic function with period $p \Leftrightarrow \boldsymbol{f(x+p)=f(x)}$ for all x, and p is the smallest positive value for this to be true.

GRAPHING PACKAGE

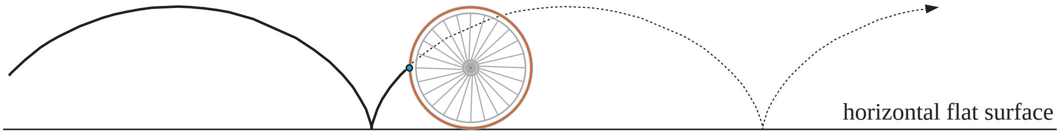
Use a **graphing package** to examine the function: $f : x \mapsto x - [x]$

where $[x]$ is "the largest integer less than or equal to x".

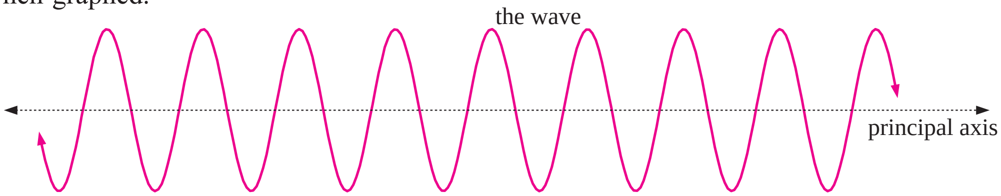
Is $f(x)$ periodic? What is its period?

A **cycloid** is another example of a periodic function. It is the curve traced out by a point on a circle as the circle rolls across a flat surface in a straight line. However, the cycloid function cannot be written as a Cartesian equation in the form $y = ...$ or $f(x) = ...$

DEMO



In this course we are mainly concerned with periodic phenomena which show a wave pattern when graphed.

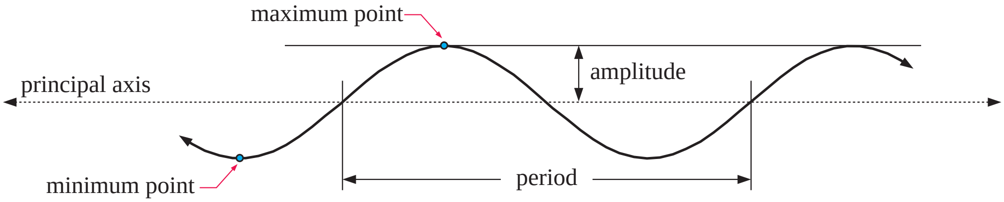


The wave oscillates about a horizontal line called the **principal axis** or **mean line**.

A **maximum point** occurs at the top of a crest and a **minimum point** at the bottom of a trough.

The **amplitude** is the distance between a maximum (or minimum) point and the principal axis.

$$\textbf{amplitude} = \frac{\textbf{max} - \textbf{min}}{\textbf{2}} \qquad \textbf{principal axis } y = \frac{\textbf{max} + \textbf{min}}{\textbf{2}}$$



EXERCISE 10A

1 For each set of data below, draw a scatterplot and decide whether or not the data exhibits approximately periodic behaviour.

a

x	0	1	2	3	4	5	6	7	8	9	10	11	12
y	0	1	1.4	1	0	-1	-1.4	-1	0	1	1.4	1	0

b

x	0	1	2	3	4
y	4	1	0	1	4

TI-*n*spire
TI-84
Casio

c

x	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5
y	0	1.9	3.5	4.5	4.7	4.3	3.4	2.4

d

x	0	2	3	4	5	6	7	8	9	10	12
y	0	4.7	3.4	1.7	2.1	5.2	8.9	10.9	10.2	8.4	10.4

2 The following tabled values show the height above the ground of a point on a bicycle wheel as the bicycle is wheeled along a flat surface.

Distance travelled (cm)	0	20	40	60	80	100	120	140	160	180	200
Height above ground (cm)	0	6	23	42	57	64	59	43	23	7	1

Distance travelled (cm)	220	240	260	280	300	320	340	360	380	400
Height above ground (cm)	5	27	40	55	63	60	44	24	9	3

a Plot the graph of height against distance.

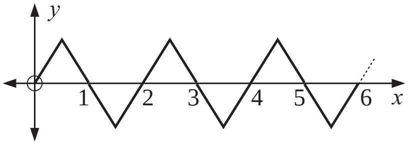
b Is the data periodic? If so, estimate:

- **i** the equation of the principal axis
- **ii** the maximum value
- **iii** the period
- **iv** the amplitude.

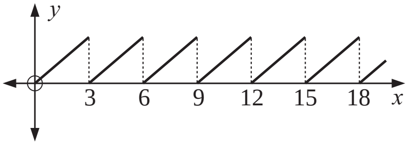
c Is it reasonable to fit a curve to this data, or should we leave it as discrete points?

3 Which of these graphs show periodic behaviour?

a



b



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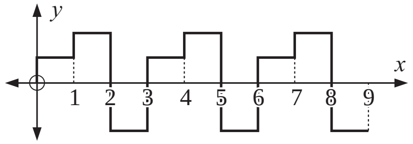
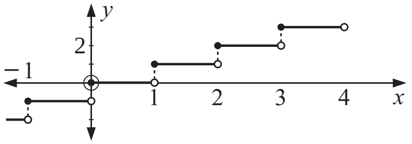
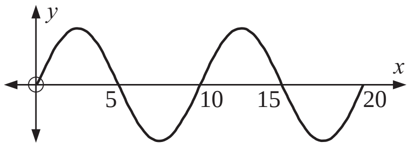


d



e



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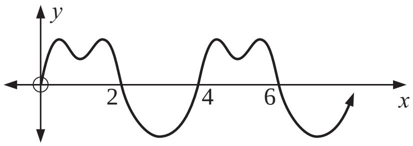
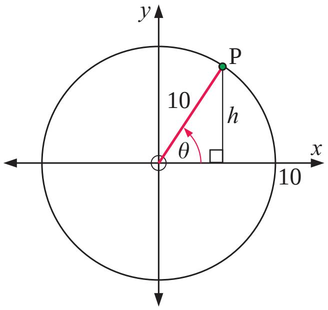


B THE SINE FUNCTION

In previous studies of trigonometry we have only considered right angled triangles, or static situations where an angle θ is fixed. However, when an object moves around a circle, the situation is dynamic. The angle between the radius [OP] and the horizontal axis continually changes with time.

Consider again the **Opening Problem** in which a Ferris wheel of radius 10 m revolves at constant speed.

The height of P, the point representing the green light on the wheel relative to the principal axis O, can be determined using right angled triangle trigonometry.

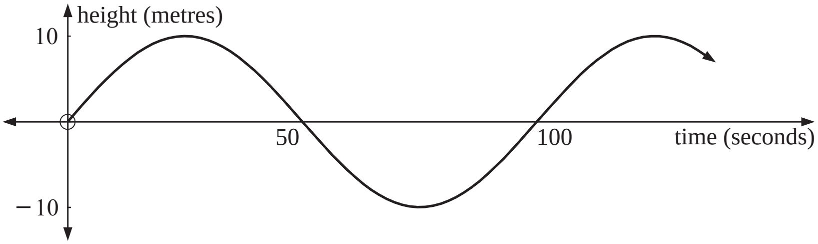


$$\sin\theta = \frac{h}{10}, \quad \text{so} \quad h = 10\sin\theta.$$

As time goes by, θ changes and so does h.

So, h is a function of θ, but more importantly h is a function of time t.

Suppose the Ferris wheel observed by Andrew takes 100 seconds for a full revolution. The graph below shows the height of the light above or below the principal axis against the time in seconds.



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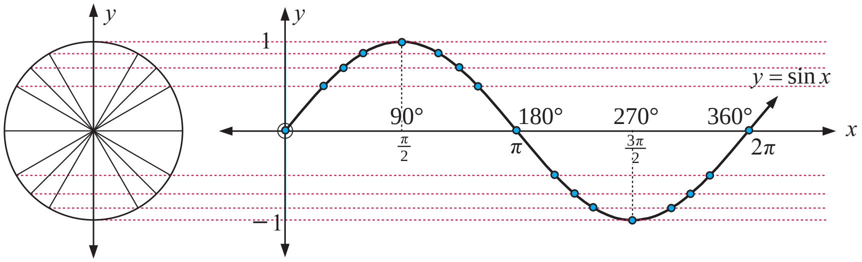
We observe that the amplitude is 10 metres and the period is 100 seconds.

THE BASIC SINE CURVE

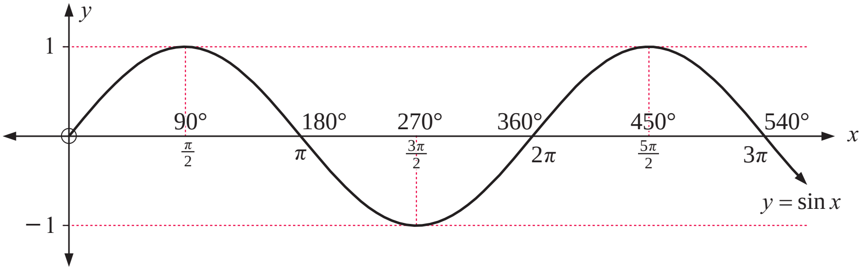
Suppose point P moves around the unit circle so the angle [OP] makes with the horizontal axis is x. In this case P has coordinates $(\cos x, \sin x)$.

If we project the values of $\sin x$ from the unit circle to the set of axes on the right we obtain the graph of $y = \sin x$.

Unless indicated otherwise, you should assume that x is measured in radians. Degrees are only included on this graph for the sake of completeness.



The wave of course can be continued beyond $0 \leqslant x \leqslant 2\pi$.



We expect the *period* to be 2π, since the unit circle repeats its positioning after one full revolution.

The *maximum* value is 1 and the *minimum* is -1, as $-1 \leqslant y \leqslant 1$ on the unit circle.

The *amplitude* is 1.

Use your **graphics calculator** or **graphing package** to obtain the graph of $y = \sin x$ to check these features.

GRAPHING PACKAGE

TRANSFORMATIONS OF THE SINE CURVE

In the investigations that follow, we will consider applying transformations to the sine curve $y = \sin x$.

Using the transformations we learnt in **Chapter 5**, we can generate the curve for the general sine function $y = a \sin b(x - c) + d$.

INVESTIGATION 1 THE FAMILY $y = a \sin x$

What to do:

1 Use technology to graph on the same set of axes:

a $y = \sin x$ and $y = 2 \sin x$

b $y = \sin x$ and $y = 0.5 \sin x$

c $y = \sin x$ and $y = -\sin x$

GRAPHING PACKAGE

If using a graphics calculator, make sure that the mode is set in **radians** and that your viewing window is appropriate.

2 For each of $y = \sin x$, $y = 2\sin x$, $y = 0.5\sin x$, and $y = -\sin x$:

- **a** Record the maximum and minimum values.
- **b** State the period and amplitude.

3 How does a affect the function $y = a\sin x$?

4 State the amplitude of:

a $y = 3\sin x$ **b** $y = \sqrt{7}\sin x$ **c** $y = -2\sin x$

INVESTIGATION 2 THE FAMILY $y = \sin bx,\ b > 0$

What to do:

1 Use technology to graph on the same set of axes:

a $y = \sin x$ and $y = \sin 2x$ **b** $y = \sin x$ and $y = \sin(\frac{1}{2}x)$

2 For each of $y = \sin x$, $y = \sin 2x$, and $y = \sin(\frac{1}{2}x)$:

- **a** Record the maximum and minimum values.
- **b** State the period and amplitude.

GRAPHING PACKAGE

3 How does b affect the function $y = \sin bx$?

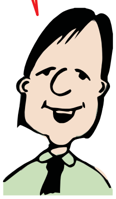
4 State the period of:

a $y = \sin 3x$ **b** $y = \sin(\frac{1}{3}x)$ **c** $y = \sin(1.2x)$ **d** $y = \sin bx$

From the previous investigations you should have observed that:

- in $y = a\sin x$, $|a|$ determines the amplitude
- in $y = \sin bx$, $b > 0$, b affects the period and the period is $\frac{2\pi}{b}$.

$|a|$ is the modulus of a. It is the *size* of a, and cannot be negative.



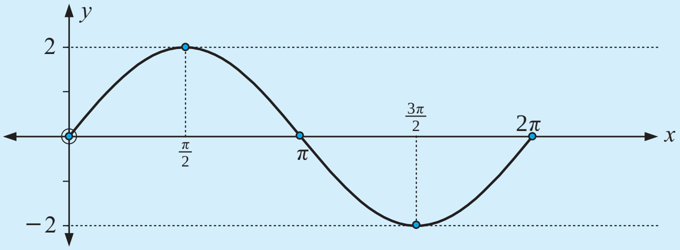
Example 1

Self Tutor

Without using technology, sketch the graphs of:

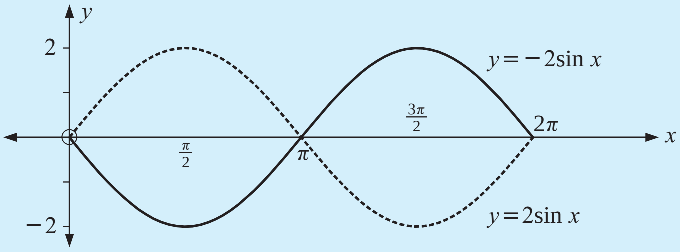
a $y = 2\sin x$ **b** $y = -2\sin x$ for $0 \leqslant x \leqslant 2\pi$.

a The amplitude is 2 and the period is 2π.



We place the 5 points as shown and fit the sine wave to them.

b The amplitude is 2, the period is 2π, and it is the reflection of $y = 2\sin x$ in the x-axis.

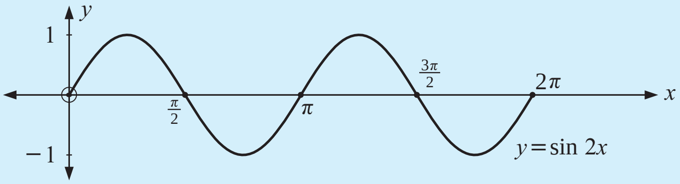


Example 2 **Self Tutor**

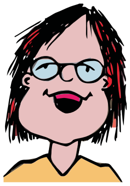
Without using technology, sketch the graph of $y = \sin 2x$ for $0 \leqslant x \leqslant 2\pi$.

The period is $\frac{2\pi}{2} = \pi$.

The maximum values are therefore π units apart.



As $\sin 2x$ has half the period of $\sin x$, the first maximum is at $\frac{\pi}{4}$ not $\frac{\pi}{2}$.



EXERCISE 10B.1

1 Without using technology, sketch the graphs of the following for $0 \leqslant x \leqslant 2\pi$:

a $y = 3\sin x$ **b** $y = -3\sin x$ **c** $y = \frac{3}{2}\sin x$ **d** $y = -\frac{3}{2}\sin x$

2 Without using technology, sketch the graphs of the following for $0 \leqslant x \leqslant 3\pi$:

a $y = \sin 3x$ **b** $y = \sin\left(\frac{x}{2}\right)$ **c** $y = \sin(-2x)$

3 State the period of:

a $y = \sin 4x$ **b** $y = \sin(-4x)$ **c** $y = \sin\left(\frac{x}{3}\right)$ **d** $y = \sin(0.6x)$

4 Find b given that the function $y = \sin bx$, $b > 0$ has period:

a 5π **b** $\frac{2\pi}{3}$ **c** 12π **d** 4 **e** 100

INVESTIGATION 3 THE FAMILIES $y = \sin(x - c)$ AND $y = \sin x + d$

What to do:

GRAPHING PACKAGE

1 Use technology to graph on the same set of axes:

- **a** $y = \sin x$ and $y = \sin(x - 2)$
- **b** $y = \sin x$ and $y = \sin(x + 2)$
- **c** $y = \sin x$ and $y = \sin(x - \frac{\pi}{3})$

2 For each of $y = \sin x$, $y = \sin(x - 2)$, $y = \sin(x + 2)$, $y = \sin(x - \frac{\pi}{3})$:

- **a** Record the maximum and minimum values.
- **b** State the period and amplitude.

3 What transformation moves $y = \sin x$ to $y = \sin(x - c)$?

4 Use technology to graph on the same set of axes:

- **a** $y = \sin x$ and $y = \sin x + 3$
- **b** $y = \sin x$ and $y = \sin x - 2$

5 For each of $y = \sin x$, $y = \sin x + 3$ and $y = \sin x - 2$:

- **a** Record the maximum and minimum values.
- **b** State the period and amplitude.

6 What transformation moves $y = \sin x$ to $y = \sin x + d$?

7 What transformation moves $y = \sin x$ to $y = \sin(x - c) + d$?

From **Investigation 3** we observe that:

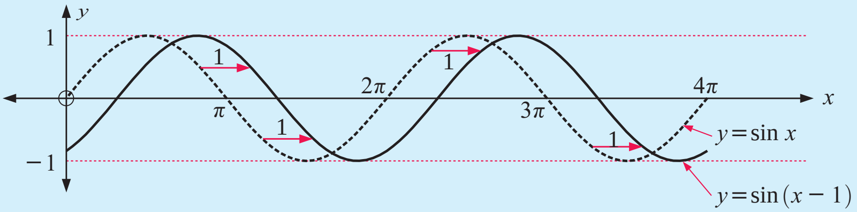
- $y = \sin(x - c)$ is a **horizontal translation** of $y = \sin x$ through c units.
- $y = \sin x + d$ is a **vertical translation** of $y = \sin x$ through d units.
- $y = \sin(x - c) + d$ is a **translation** of $y = \sin x$ through vector $\binom{c}{d}$.

Example 3 **Self Tutor**

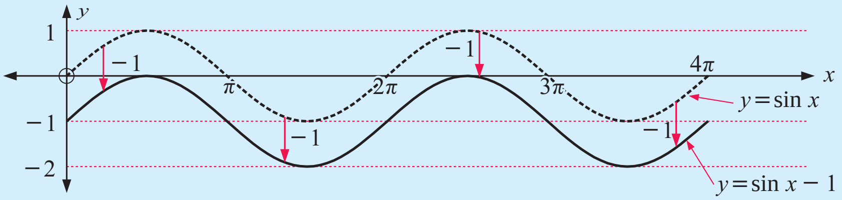
On the same set of axes graph for $0 \leqslant x \leqslant 4\pi$:

a $y = \sin x$ and $y = \sin(x - 1)$ **b** $y = \sin x$ and $y = \sin x - 1$

a



b



THE GENERAL FUNCTIONS

$\boldsymbol{y = a\sin b(x-c)+d}$ is called the **general sine function**.

affects **amplitude** (a)

affects **period** (b)

affects **horizontal translation** (c)

affects **vertical translation** (d)

The **principal axis** of the general sine function is $\boldsymbol{y = d}$.

The **period** of the general sine function is $\dfrac{\boldsymbol{2\pi}}{\boldsymbol{b}}$.

Consider $y = 2\sin 3\left(x-\frac{\pi}{4}\right)+1$. It is a translation of $y = 2\sin 3x$ under $\begin{pmatrix}\frac{\pi}{4}\\ 1\end{pmatrix}$.

So, starting with $y = \sin x$ we would:

- double the amplitude to produce $y = 2\sin x$, then
- divide the period by 3 to produce $y = 2\sin 3x$, then
- translate by $\begin{pmatrix}\frac{\pi}{4}\\ 1\end{pmatrix}$ to produce $y = 2\sin 3\left(x-\frac{\pi}{4}\right)+1$.

EXERCISE 10B.2

1 Draw sketch graphs of:

a $y = \sin x - 2$
b $y = \sin(x-2)$
c $y = \sin(x+2)$
d $y = \sin x + 2$
e $y = \sin(x+\frac{\pi}{4})$
f $y = \sin(x-\frac{\pi}{6})+1$

GRAPHING PACKAGE

Check your answers using technology.

2 State the period of:

a $y = \sin 5t$
b $y = \sin\left(\frac{t}{4}\right)$
c $y = \sin(-2t)$

3 Find b in $y = \sin bx$ if $b > 0$ and the period is:

a 3π
b $\frac{\pi}{10}$
c 100π
d 50

4 State the transformation(s) which map:

a $y = \sin x$ onto $y = \sin x - 1$
b $y = \sin x$ onto $y = \sin(x-\frac{\pi}{4})$
c $y = \sin x$ onto $y = 2\sin x$
d $y = \sin x$ onto $y = \sin 4x$
e $y = \sin x$ onto $y = \frac{1}{2}\sin x$
f $y = \sin x$ onto $y = \sin\left(\frac{x}{4}\right)$
g $y = \sin x$ onto $y = -\sin x$
h $y = \sin x$ onto $y = -3+\sin(x+2)$
i $y = \sin x$ onto $y = 2\sin 3x$
j $y = \sin x$ onto $y = \sin(x-\frac{\pi}{3})+2$

C MODELLING USING SINE FUNCTIONS

When patterns of variation can be identified and quantified in terms of a formula or equation, predictions may be made about behaviour in the future. Examples of this include tidal movement which can be predicted many months ahead, and the date of a future full moon.

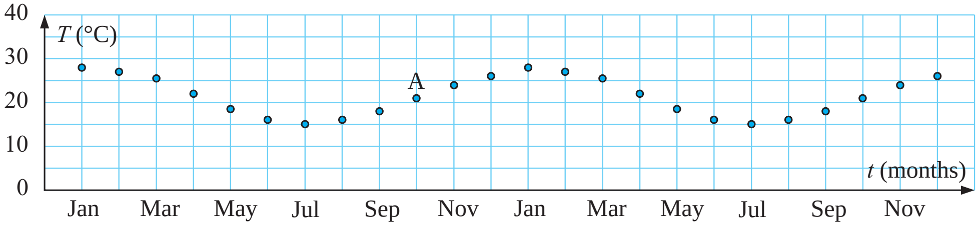
In this section we use sine functions to model certain biological and physical phenomena that are periodic in nature.

MEAN MONTHLY TEMPERATURE

Consider again the mean monthly maximum temperature for Cape Town:

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temp (°C)	28	27	$25\frac{1}{2}$	22	$18\frac{1}{2}$	16	15	16	18	$21\frac{1}{2}$	24	26

The graph over a two year period is shown below:



We attempt to model this data using the general sine function $y = a\sin b(x-c)+d$,
or in this case $T = a\sin b(t-c)+d$.

The period is 12 months, so $\dfrac{2\pi}{b} = 12$ and $\therefore$ $b = \frac{\pi}{6}$.

The amplitude $= \dfrac{\text{max} - \text{min}}{2} \approx \dfrac{28-15}{2} \approx 6.5$, so $a \approx 6.5$.

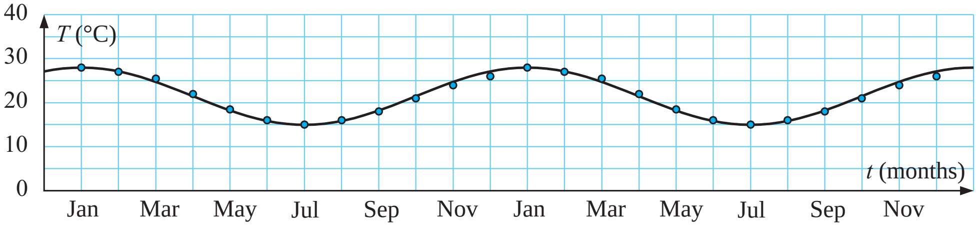
The principal axis is midway between the maximum and minimum,

so $d \approx \dfrac{28+15}{2} \approx 21.5$.

So, the model is $T \approx 6.5\sin\frac{\pi}{6}(t-c)+21.5$ for some constant c.

We notice that point A on the original graph lies on the principal axis and is a point at which we are starting a new period. Since A is at $(10, 21.5)$, $c = 10$.

The model is therefore $T \approx 6.5\sin\frac{\pi}{6}(t-10)+21.5$ and we can superimpose it on the original data as follows.



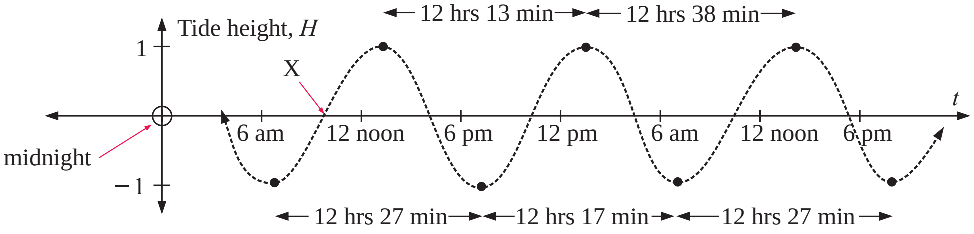
TIDAL MODELS

The tides at Juneau, Alaska were recorded over a two day period. The results are shown in the table opposite:

Day 1	high tide	1.18 pm
	low tide	6.46 am, 7.13 pm
Day 2	high tide	1.31 am, 2.09 pm
	low tide	7.30 am, 7.57 pm

Suppose high tide corresponds to height 1 and low tide to height -1.

Plotting these times with t being the time after midnight before the first low tide, we get:



We attempt to model this periodic data using $H = a \sin b(t - c) + d$.

The principal axis is $H = 0$, so $d = 0$. The amplitude is 1, so $a = 1$.

The graph shows that the 'average' period is about 12 hours 24 min $\approx$ 12.4 hours.

But the period is $\dfrac{2\pi}{b}$. $\therefore$ $\dfrac{2\pi}{b} \approx 12.4$ and so $b \approx \dfrac{2\pi}{12.4} \approx 0.507$.

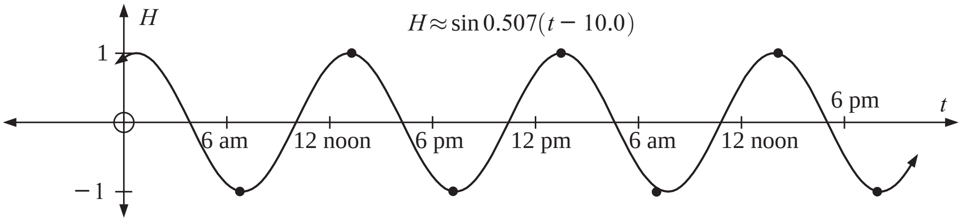
The model is now $H \approx \sin 0.507(t - c)$ for some constant c.

We find point X which is midway between the *first minimum* and the *following maximum*,

$$c \approx \frac{6.77 + 13.3}{2} \approx 10.0\,.$$

So, the model is $H \approx \sin 0.507(t - 10.0)$.

Below is our original graph of seven plotted points and our model which attempts to fit them.



Use your **graphics calculator** to check this result.
The times must be given in hours after midnight, so
the low tide at 6.46 am is $(6.77, -1)$,
the high tide at 1.18 pm is $(13.3, 1)$, and so on.

MODELLING

TI-$n$spire
TI-84
Casio

EXERCISE 10C

1 Below is a table which shows the mean monthly maximum temperatures for a city in Greece.

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temperature (°C)	15	14	15	18	21	25	27	26	24	20	18	16

- **a** Use a sine function of the form $T \approx a \sin b(t-c) + d$ to model the data. Find good estimates of the constants a, b, c and d without using technology. Use Jan $\equiv 1$, Feb $\equiv 2$, and so on.
- **b** Use technology to check your answer to **a**. How well does your model fit?

2 The data in the table shows the mean monthly temperatures for Christchurch.

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temperature (°C)	15	16	$14\frac{1}{2}$	12	10	$7\frac{1}{2}$	7	$7\frac{1}{2}$	$8\frac{1}{2}$	$10\frac{1}{2}$	$12\frac{1}{2}$	14

- **a** Find a sine model for this data in the form $T \approx a \sin b(t-c) + d$. Assume Jan $\equiv 1$, Feb $\equiv 2$, and so on. Do not use technology.
- **b** Use technology to check your answer to **a**.

3 At the Mawson base in Antarctica, the mean monthly temperatures for the last 30 years are as follows:

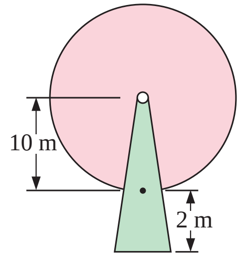
Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temperature (°C)	0	-4	-10	-15	-16	-17	-18	-19	-17	-13	-6	-1

Find a sine model for this data without using technology. Use Jan $\equiv 1$, Feb $\equiv 2$, and so on. How appropriate is the model?

4 Some of the largest tides in the world are observed in Canada's Bay of Fundy. The difference between high and low tides is 14 metres and the average time difference between high tides is about 12.4 hours.

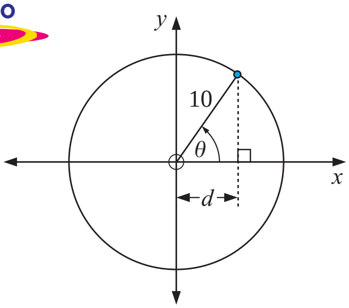
- **a** Find a sine model for the height of the tide H in terms of the time t.
- **b** Sketch the graph of the model over one period.

5 Revisit the **Opening Problem** on page **236**.
The wheel takes 100 seconds to complete one revolution. Find the sine model which gives the height of the light above the ground at any point in time. Assume that at time $t = 0$, the light is at its lowest point.



D THE COSINE FUNCTION

DEMO



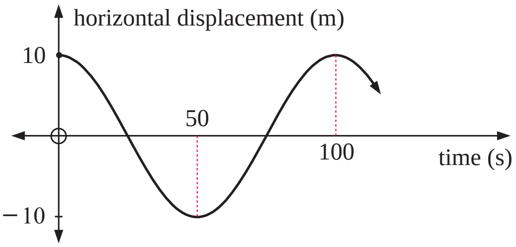
We return to the Ferris wheel to see the cosine function being generated.

Click on the icon to inspect a simulation of the view from above the wheel.

The graph being generated over time is a **cosine function**.

This is no surprise as $\cos\theta = \dfrac{d}{10}$ and so $d = 10\cos\theta$.

The graph below shows the horizontal displacement of the light on the Ferris wheel over time.



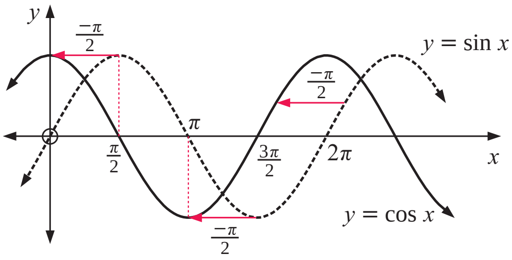
The cosine curve $y = \cos x$, like the sine curve $y = \sin x$, has a **period** of 2π, an **amplitude** of 1, and its **range** is $-1 \leqslant y \leqslant 1$.

Use your graphics calculator or graphing package to check these features.

TI-*n*spire
TI-84
Casio

Now view the relationship between the sine and cosine functions.

GRAPHING PACKAGE



The functions are identical in shape, but the cosine function is $\frac{\pi}{2}$ units left of the sine function under a horizontal translation.

This suggests that $\boldsymbol{\cos x = \sin\left(x + \frac{\pi}{2}\right)}$.

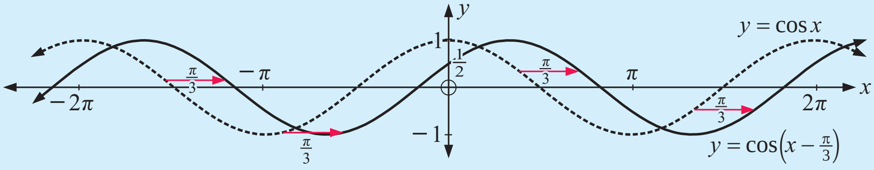
Use your graphing package or graphics calculator to check this by graphing $y = \cos x$ and $y = \sin\left(x + \frac{\pi}{2}\right)$.

GRAPHING PACKAGE

Example 4 **Self Tutor**

On the same set of axes graph $y = \cos x$ and $y = \cos\left(x - \frac{\pi}{3}\right)$.

$y = \cos\left(x - \frac{\pi}{3}\right)$ comes from $y = \cos x$ under a horizontal translation through $\frac{\pi}{3}$.

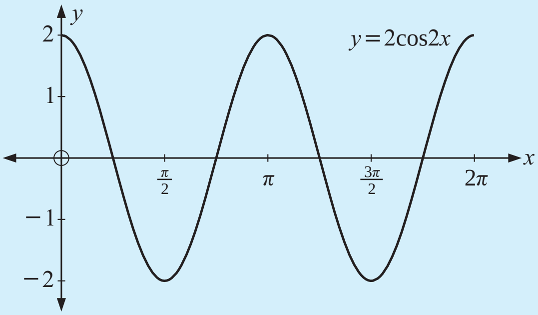


Example 5 **Self Tutor**

Without using technology, sketch the graph of $y = 2\cos 2x$ for $0 \leqslant x \leqslant 2\pi$.

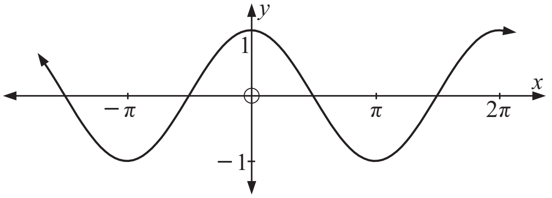
$a = 2$, so the amplitude is $|2| = 2$.

$b = 2$, so the period is $\dfrac{2\pi}{b} = \dfrac{2\pi}{2} = \pi$.



EXERCISE 10D

1 Given the graph of $y = \cos x$, sketch the graphs of:



a $y = \cos x + 2$
b $y = \cos x - 1$
c $y = \cos(x - \frac{\pi}{4})$
d $y = \cos(x + \frac{\pi}{6})$
e $y = \frac{2}{3}\cos x$
f $y = \frac{3}{2}\cos x$
g $y = -\cos x$
h $y = \cos(x - \frac{\pi}{6}) + 1$
i $y = \cos(x + \frac{\pi}{4}) - 1$
j $y = \cos 2x$
k $y = \cos\left(\frac{x}{2}\right)$
l $y = 3\cos 2x$

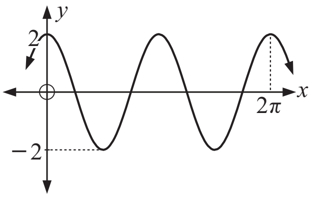
2 Without graphing them, state the periods of:

a $y = \cos 3x$
b $y = \cos\left(\frac{x}{3}\right)$
c $y = \cos\left(\frac{\pi}{50}x\right)$

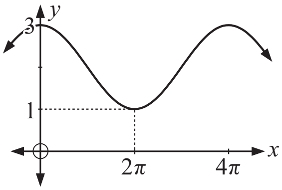
3 The general cosine function is $y = a\cos b(x - c) + d$.
State the geometrical significance of a, b, c and d.

4 For the following graphs, find the cosine function representing them:

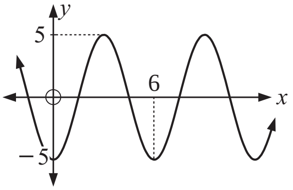
a



b



c



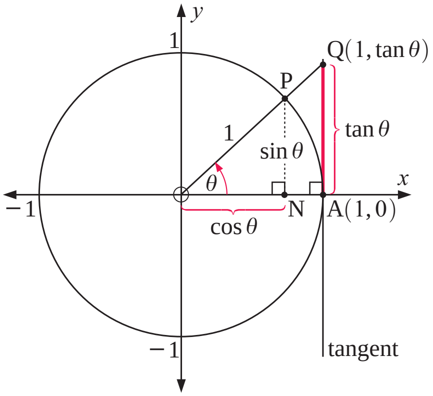
E THE TANGENT FUNCTION

Consider the unit circle diagram given.

P($\cos\theta$, $\sin\theta$) is a point which is free to move around the circle.

In the first quadrant we extend [OP] to meet the tangent at A(1, 0). The intersection between these lines occurs at Q, and as P moves so does Q.

The position of Q relative to A is defined as the **tangent function**.

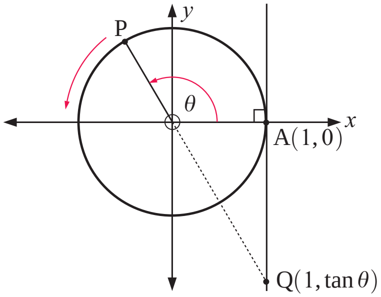


Notice that Δs ONP and OAQ are equiangular and therefore similar.

Consequently $\dfrac{\text{AQ}}{\text{OA}} = \dfrac{\text{NP}}{\text{ON}}$ and hence $\dfrac{\text{AQ}}{1} = \dfrac{\sin\theta}{\cos\theta}$.

Under the definition that AQ $= \tan\theta$,

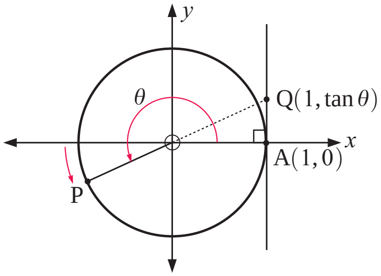
$$\boldsymbol{\tan\theta = \frac{\sin\theta}{\cos\theta}}.$$



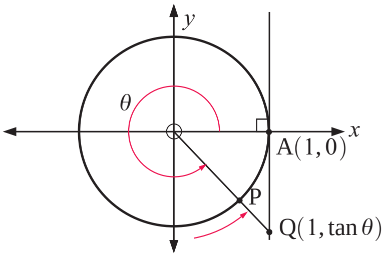
The question arises: "If P does not lie in the first quadrant, how is $\tan\theta$ defined?"

For θ obtuse, since $\sin\theta$ is positive and $\cos\theta$ is negative, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ is negative.

As before, [OP] is extended to meet the tangent at A at Q(1, $\tan\theta$).



For θ in quadrant 3, $\sin\theta$ and $\cos\theta$ are both negative and so $\tan\theta$ is positive. This is clearly demonstrated as Q is back above the x-axis.



For θ in quadrant 4, $\sin\theta$ is negative and $\cos\theta$ is positive. $\tan\theta$ is again negative.

DISCUSSION

- What is $\tan\theta$ when P is at $(0,\ 1)$?
- What is $\tan\theta$ when P is at $(0,\ -1)$?

EXERCISE 10E.1

1 Use your calculator to find the value of:

a $\tan 0^o$	**b** $\tan 15^o$	**c** $\tan 20^o$	**d** $\tan 25^o$
e $\tan 35^o$	**f** $\tan 45^o$	**g** $\tan 50^o$	**h** $\tan 55^o$

2 Explain why $\tan 45^o = 1$ exactly.

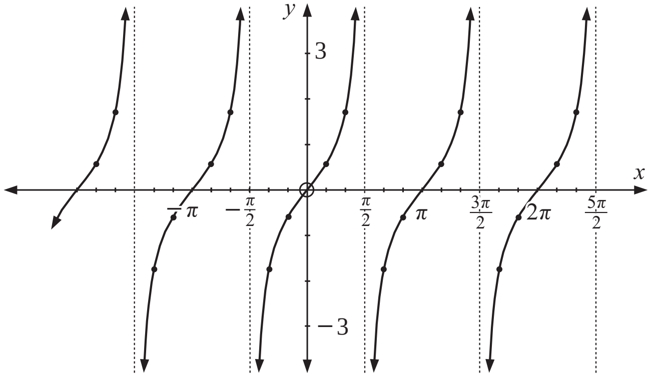
THE GRAPH OF $y = \tan x$

Since $\tan x = \dfrac{\sin x}{\cos x}$, $\tan x$ will be undefined whenever $\cos x = 0$.

The zeros of the function $y = \cos x$ correspond to vertical asymptotes of the function $y = \tan x$.

The graph of $y = \tan x$ is

DEMO



We observe that $y = \tan x$ has a **period** of π, **range** $y \in \mathbb{R}$, and **vertical asymptotes** $x = \frac{\pi}{2} + k\pi$ for all $k \in \mathbb{Z}$.

DISCUSSION

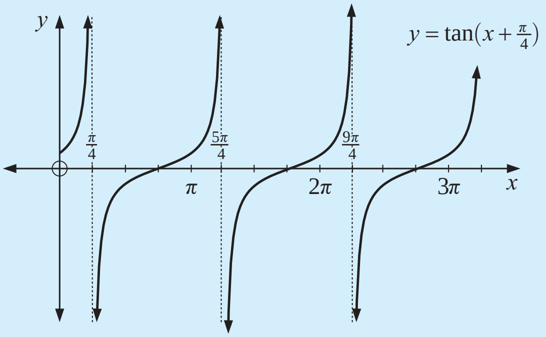
- Discuss how to find the x-intercepts of $y = \tan x$.
- What must $\tan(x - \pi)$ simplify to?
- How many solutions does the equation $\tan x = 2$ have?

Example 6 **Self Tutor**

Without using technology, sketch the graph of $y = \tan(x + \frac{\pi}{4})$ for $0 \leqslant x \leqslant 3\pi$.

$y = \tan(x + \frac{\pi}{4})$
is a horizontal translation of
$y = \tan x$ through $-\frac{\pi}{4}$

$\therefore$ $y = \tan(x + \frac{\pi}{4})$
has vertical asymptotes
$x = \frac{\pi}{4}$, $x = \frac{5\pi}{4}$, $x = \frac{9\pi}{4}$.

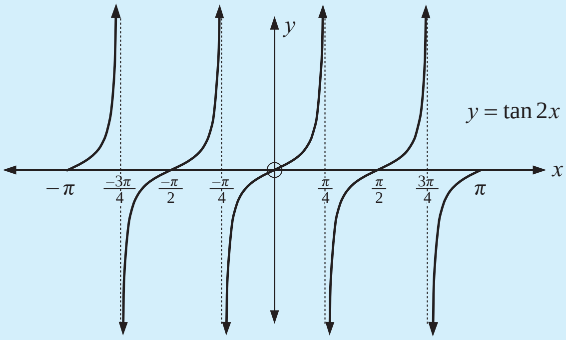


Example 7 **Self Tutor**

Without using technology, sketch the graph of $y = \tan 2x$ for $-\pi \leqslant 0 \leqslant \pi$.

Since $b = 2$, the period is $\frac{\pi}{2}$.

The vertical asymptotes are
$x = \pm\frac{\pi}{4}$, $\pm\frac{3\pi}{4}$.



EXERCISE 10E.2

1 **a** Use a transformation approach to *sketch* graphs of the following functions for $0 \leqslant x \leqslant 3\pi$:

  i $y = \tan(x - \frac{\pi}{2})$ **ii** $y = -\tan x$ **iii** $y = \tan 3x$

b Use technology to check your answers to **a**.
Look in particular for asymptotes and the x-intercepts.

GRAPHING PACKAGE

2 Use the graphing package to graph, on the same set of axes:

a $y = \tan x$ and $y = \tan(x - 1)$ **b** $y = \tan x$ and $y = -\tan x$

c $y = \tan x$ and $y = \tan\left(\frac{x}{2}\right)$

Describe the transformation which moves the first curve to the second in each case.

GRAPHING PACKAGE

3 State the period of:

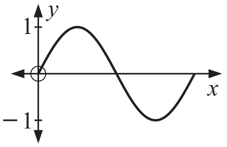
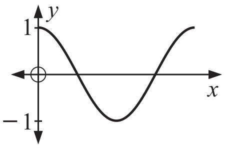
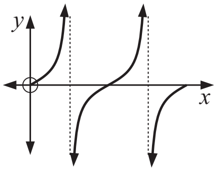
a $y = \tan x$ **b** $y = \tan 2x$ **c** $y = \tan nx$.

F GENERAL TRIGONOMETRIC FUNCTIONS

In the previous sections we have explored properties of the sine, cosine, and tangent functions. We have also observed how they can be transformed into more general trigonometric functions.

The following tables summarise our observations:

FEATURES OF CIRCULAR FUNCTIONS

Function	*Sketch for* $0 \leqslant x \leqslant 2\pi$	*Period*	*Amplitude*	*Domain*	*Range*
$y = \sin x$		2π	1	$x \in \mathbb{R}$	$-1 \leqslant y \leqslant 1$
$y = \cos x$		2π	1	$x \in \mathbb{R}$	$-1 \leqslant y \leqslant 1$
$y = \tan x$		π	undefined	$x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, ...$	$y \in \mathbb{R}$

GENERAL TRIGONOMETRIC FUNCTIONS

General function	a affects vertical stretch	$b > 0$ affects horizontal stretch	c affects horizontal translation	d affects vertical translation
$y = a\sin b(x-c) + d$ $y = a\cos b(x-c) + d$	amplitude $= \lvert a \rvert$	period $= \frac{2\pi}{b}$	• $c > 0$ moves the graph right • $c < 0$ moves the graph left	• $d > 0$ moves the graph up • $d < 0$ moves the graph down • principal axis is $y = d$
$y = a\tan b(x-c) + d$	amplitude undefined	period $= \frac{\pi}{b}$		

EXERCISE 10F

1 State the amplitude, where appropriate, of:

a $y = \sin 4x$ **b** $y = 2\tan(\frac{x}{2})$ **c** $y = -\cos 3(x - \frac{\pi}{4})$

2 State the period of:

a $y = -\tan x$ **b** $y = \cos(\frac{x}{3}) - 1$ **c** $y = \sin 2(x - \frac{\pi}{4})$

3 Find b given:

- **a** $y = \sin bx$ has period 2π
- **b** $y = \cos bx$ has period $\frac{2\pi}{3}$
- **c** $y = \tan bx$ has period $\frac{\pi}{2}$
- **d** $y = \sin bx$ has period 4

4 Without using technology, sketch the graphs of these functions for $0 \leqslant x \leqslant 2\pi$:

- **a** $y = \frac{2}{3}\cos x$
- **b** $y = \sin x + 1$
- **c** $y = \tan(x + \frac{\pi}{2})$
- **d** $y = 3\cos 2x$
- **e** $y = \sin(x + \frac{\pi}{4}) - 1$
- **f** $y = \tan x - 2$

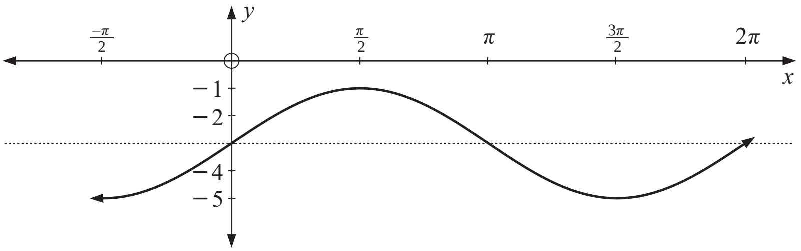
5 State the maximum and minimum values, where appropriate, of:

- **a** $y = -\sin 5x$
- **b** $y = 3\cos x$
- **c** $y = 2\tan x$
- **d** $y = -\cos 2x + 3$
- **e** $y = 1 + 2\sin x$
- **f** $y = \sin(x - \frac{\pi}{2}) - 3$

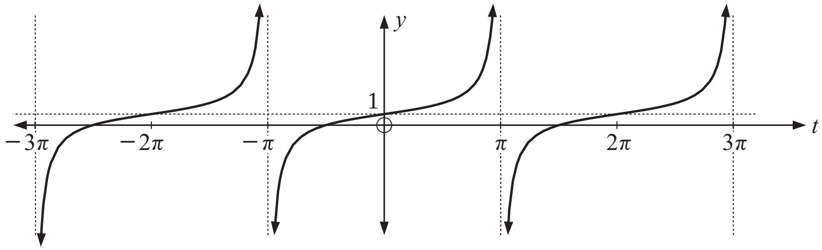
6 State the transformation(s) which map(s):

- **a** $y = \sin x$ onto $y = \frac{1}{2}\sin x$
- **b** $y = \cos x$ onto $y = \cos(\frac{x}{4})$
- **c** $y = \sin x$ onto $y = -\sin x$
- **d** $y = \cos x$ onto $y = \cos x - 2$
- **e** $y = \tan x$ onto $y = \tan(x + \frac{\pi}{4})$
- **f** $y = \sin x$ onto $y = \sin(-x)$

7 Find m and n given the following graph is of the function $y = m\sin x + n$.



8 Find p and q given the following graph is of the function $y = \tan pt + q$.



G TRIGONOMETRIC EQUATIONS

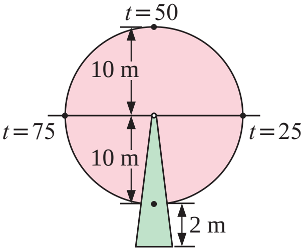
Linear equations such as $2x + 3 = 11$ have exactly one solution. Quadratic equations of the form $ax^2 + bx + c = 0$, $a \neq 0$ have at most two real solutions.

Trigonometric equations generally have infinitely many solutions unless a restrictive domain such as $0 \leqslant x \leqslant 3\pi$ is given.

We will examine solving trigonometric equations using:

- pre-prepared graphs
- technology
- algebraic methods.

For the Ferris wheel **Opening Problem** the model is $H = 10\sin\frac{\pi}{50}(t-25)+12$.



We can easily check this by substituting $t = 0$, 25, 50, 75

$$H(0) = 10\sin\left(-\tfrac{\pi}{2}\right)+12 = -10+12 = 2 \quad ✓$$
$$H(25) = 10\sin 0 + 12 = 12 \quad ✓$$
$$H(50) = 10\sin\left(\tfrac{\pi}{2}\right)+12 = 22 \quad ✓$$
$$H(75) = 10\sin\pi + 12 = 12 \quad ✓$$

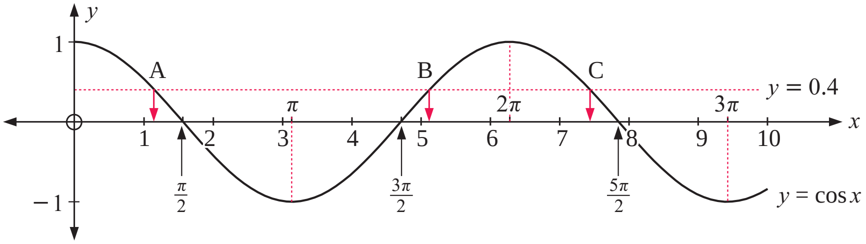
However, we may be interested in the times when the light is some other height above the ground, for example 16 m. We would then need to solve the equation

$10\sin\frac{\pi}{50}(t-25)+12 = 16$ which is called a **sine equation**.

GRAPHICAL SOLUTION OF TRIGONOMETRIC EQUATIONS

Sometimes simple trigonometric graphs are available on grid paper. In such cases we can estimate solutions straight from the graph.

For example, we could use a graph to find approximate solutions for trigonometric equations such as $\cos x = 0.4$ for $0 \leqslant x \leqslant 10$ radians.

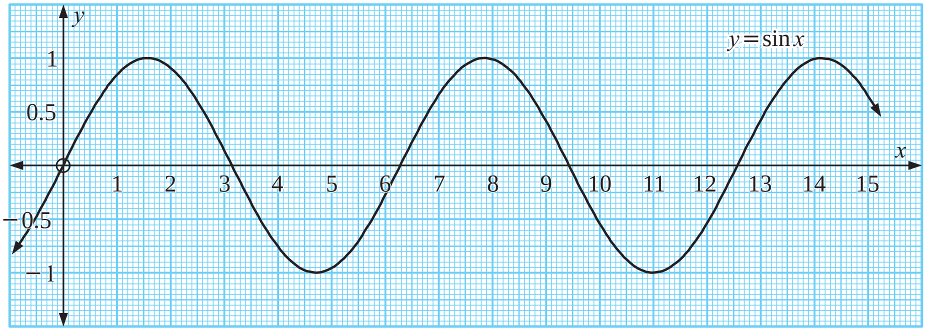


$y = 0.4$ meets $y = \cos x$ at A, B and C. Hence $x \approx 1.2$, 5.1 or 7.4.

So, the solutions of $\cos x = 0.4$ for $0 \leqslant x \leqslant 10$ radians are 1.2, 5.1 and 7.4.

EXERCISE 10G.1

1

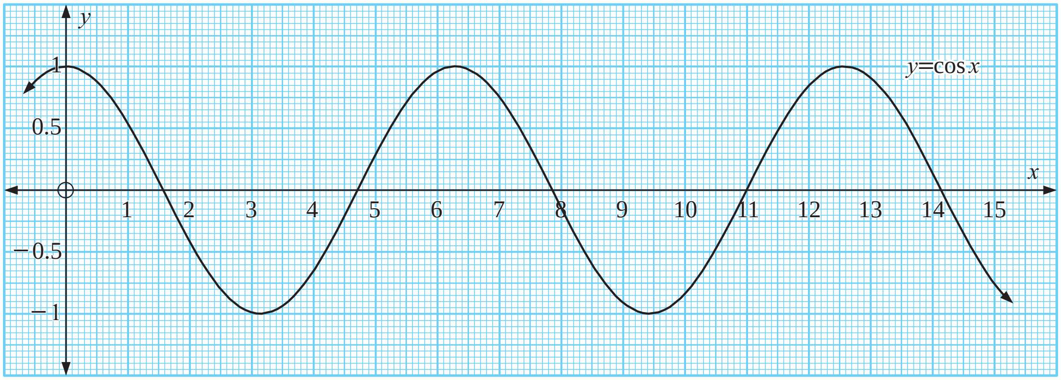


Use the graph of $y = \sin x$ to find, correct to 1 decimal place, the solutions of:

a $\sin x = 0.3$ for $0 \leqslant x \leqslant 15$

b $\sin x = -0.4$ for $5 \leqslant x \leqslant 15$.

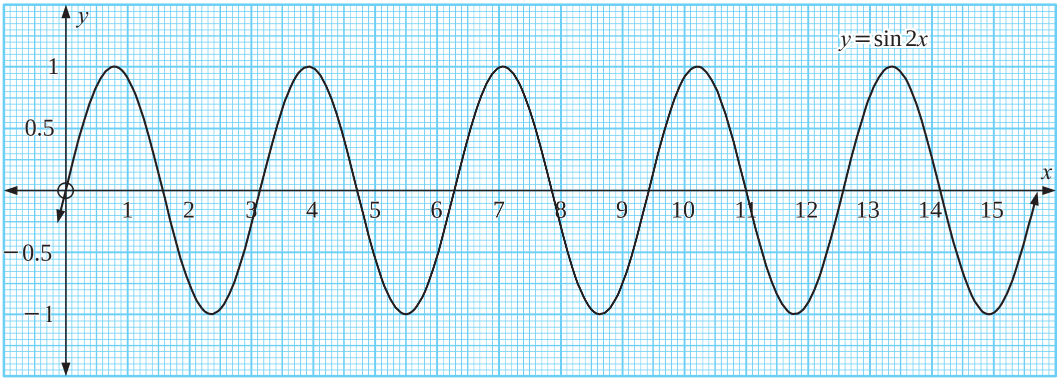
2



Use the graph of $y = \cos x$ to find, to 1 decimal place, approximate solutions of:

a $\cos x = 0.4,\ 0 \leqslant x \leqslant 10$ **b** $\cos x = -0.3,\ 4 \leqslant x \leqslant 12.$

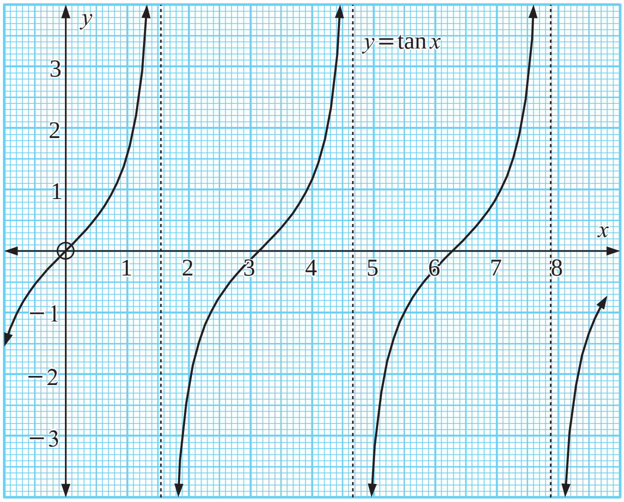
3



Use the graph of $y = \sin 2x$ to find, correct to 1 decimal place, the solutions of:

a $\sin 2x = 0.7,\ 0 \leqslant x \leqslant 16$ **b** $\sin 2x = -0.3,\ 0 \leqslant x \leqslant 16.$

4 The graph of $y = \tan x$ is illustrated.



a Use the graph to estimate:

  i $\tan 1$

  ii $\tan 2.3$

  Check your answers with a calculator.

b Find, correct to 1 decimal place, the solutions of:

  i $\tan x = 2$ for $0 \leqslant x \leqslant 8$

  ii $\tan x = -1.4$ for $2 \leqslant x \leqslant 7.$

SOLVING TRIGONOMETRIC EQUATIONS USING TECHNOLOGY

Trigonometric equations may be solved using either a **graphing package** or a **graphics calculator**.

GRAPHING PACKAGE

When using a graphics calculator make sure that the **mode** is set to **radians**.

Example 8

Self Tutor

Solve $2\sin x - \cos x = 4 - x$ for $0 \leqslant x \leqslant 2\pi$.

We graph the functions $\text{Y}_1 = 2\sin \text{X} - \cos \text{X}$ and $\text{Y}_2 = 4 - \text{X}$ on the same set of axes.

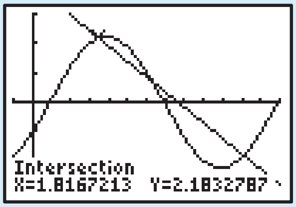
We need to use **window** settings just larger than the domain.

In this case, $X\text{min} = -\frac{\pi}{6}$ $X\text{max} = \frac{13\pi}{6}$ $X\text{scale} = \frac{\pi}{6}$

The grid facility on the graphics calculator can also be helpful, particularly when a sketch is required.

Using the appropriate function on the calculator gives the following solutions:

$x \approx 1.82, 3.28, 5.81$



```
Intersection
X=1.8167213  Y=2.1832787
```

EXERCISE 10G.2

1 Solve each of the following for $0 \leqslant x \leqslant 2\pi$:

- **a** $\sin(x + 2) = 0.0652$
- **b** $\sin^2 x + \sin x - 1 = 0$
- **c** $x \tan\left(\frac{x^2}{10}\right) = x^2 - 6x + 1$
- **d** $2\sin(2x)\cos x = \ln x$

2 Solve for x, $-2 \leqslant x \leqslant 6$: $\cos(x - 1) + \sin(x + 1) = 6x + 5x^2 - x^3$

SOLVING TRIGONOMETRIC EQUATIONS ALGEBRAICALLY

Using a calculator we get approximate decimal or **numerical** solutions to trigonometric equations.

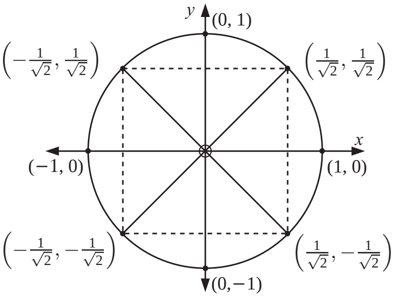
Sometimes exact solutions are needed in terms of π, and these arise when the solutions are multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$. Exact solutions obtained using algebra are called **analytical** solutions.

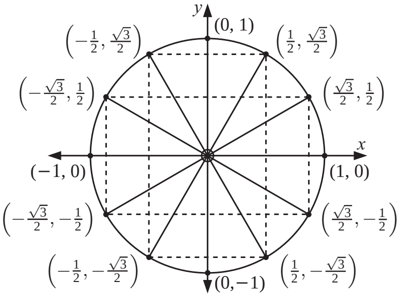
We use the periodicity of the trigonometric functions to give us all solutions in the required domain.

For example, one solution to the equation $\sin x = 1$ is $x = \frac{\pi}{2}$. Since the period of $y = \sin x$ is 2π, the general solution of $\sin x = 1$ is $x = \frac{\pi}{2} + k2\pi$ for all $k \in \mathbb{Z}$.

We then find the values of k that give us solutions which lie in the required domain.

Reminder:





Example 9

Self Tutor

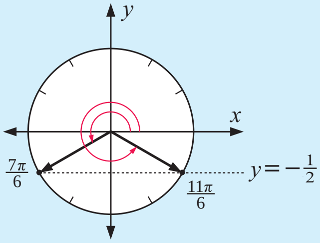
Use the unit circle to find the exact solutions of x, $0 \leqslant x \leqslant 3\pi$ for:

a $\sin x = -\frac{1}{2}$ **b** $\sin 2x = -\frac{1}{2}$ **c** $\sin\left(x - \frac{\pi}{6}\right) = -\frac{1}{2}$

a $\sin x = -\frac{1}{2}$, so from the unit circle

$$x = \left.\begin{array}{c}\frac{7\pi}{6} \\ \frac{11\pi}{6}\end{array}\right\} + k2\pi, \quad k \text{ an integer}$$

$$\therefore \quad x = \underset{\overset{\uparrow}{k=0}}{\frac{7\pi}{6}}, \ \underset{\overset{\uparrow}{k=0}}{\frac{11\pi}{6}}, \ \underset{\overset{\uparrow}{k=1}}{\cancel{\frac{19\pi}{6}}} \quad \text{is too big}$$



Substituting $k = 1, 2, 3, \ldots..$ gives answers outside the required domain.
Likewise $k = -1, -2, \ldots..$ gives answers outside the required domain.

$\therefore$ there are *two solutions*: $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$.

b $\sin 2x = -\frac{1}{2}$ is solved in exactly the same way:

$$2x = \left.\begin{array}{c}\frac{7\pi}{6} \\ \frac{11\pi}{6}\end{array}\right\} + k2\pi, \quad k \text{ an integer}$$

$$\therefore \quad x = \left.\begin{array}{c}\frac{7\pi}{12} \\ \frac{11\pi}{12}\end{array}\right\} + k\pi \qquad \{\text{dividing each term by } 2\}$$

$$\therefore \quad x = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}, \frac{31\pi}{12}, \frac{35\pi}{12} \qquad \{\text{obtained by letting } k = 0, 1, 2\}$$

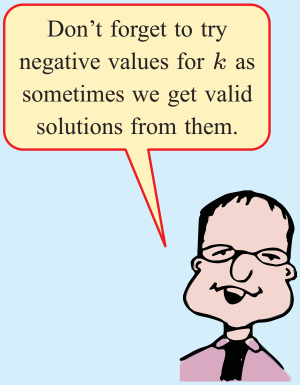
c $\sin(x - \frac{\pi}{6}) = -\frac{1}{2}$ is also solved in the same way:

$$x - \frac{\pi}{6} = \left.\begin{array}{c}\frac{7\pi}{6} \\ \frac{11\pi}{6}\end{array}\right\} + k2\pi$$

$$\therefore \quad x = \left.\begin{array}{c}\frac{8\pi}{6} \\ 2\pi\end{array}\right\} + k2\pi \qquad \{\text{adding } \tfrac{\pi}{6} \text{ to both sides}\}$$

$$\therefore \quad x = \underset{\overset{\uparrow}{k=0}}{\frac{4\pi}{3}}, \ \underset{\overset{\uparrow}{k=0}}{2\pi}, \ \underset{\overset{\uparrow}{k=1}}{\cancel{\frac{10\pi}{3}}} \text{ is too big, } \ \underset{\overset{\uparrow}{k=-1}}{0}$$

So, $x = 0$, $\frac{4\pi}{3}$ or 2π which is *three solutions*.



Example 10

Self Tutor

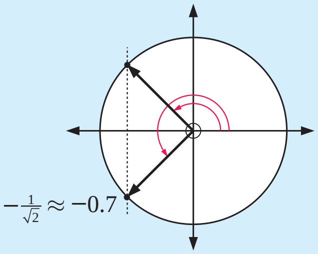
Find exact solutions of $\sqrt{2}\cos\left(x-\frac{3\pi}{4}\right)+1=0$ for $0 \leqslant x \leqslant 6\pi$.

Rearranging $\sqrt{2}\cos\left(x-\frac{3\pi}{4}\right)+1=0$, we find $\cos\left(x-\frac{3\pi}{4}\right)=-\frac{1}{\sqrt{2}}$.

We recognise $\frac{1}{\sqrt{2}}$ as a special fraction for multiples of $\frac{\pi}{4}$.

$$\therefore \quad x-\frac{3\pi}{4}=\left.\begin{array}{c}\frac{3\pi}{4}\\ \frac{5\pi}{4}\end{array}\right\}+k2\pi, \quad k \text{ an integer}$$

$$\therefore \quad x=\left.\begin{array}{c}\frac{3\pi}{2}\\ 2\pi\end{array}\right\}+k2\pi$$



If $k=-1$, $x=-\frac{\pi}{2}$ or 0. If $k=0$, $x=\frac{3\pi}{2}$ or 2π.

If $k=1$, $x=\frac{7\pi}{2}$ or 4π. If $k=2$, $x=\frac{11\pi}{2}$ or 6π.

If $k=3$, the answers are greater than 6π.

So, the solutions are: $x=0, \frac{3\pi}{2}, 2\pi, \frac{7\pi}{2}, 4\pi, \frac{11\pi}{2}$ or 6π.

Since the tangent function is periodic with period π we see that $\tan(x+\pi)=\tan x$ for all values of x. This means that equal $\tan$ values are π units apart.

Example 11

Self Tutor

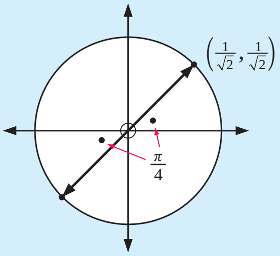
Find exact solutions of $\tan\left(2x-\frac{\pi}{3}\right)=1$ for $-\pi \leqslant x \leqslant \pi$.

$2x-\frac{\pi}{3}=\frac{\pi}{4}+k\pi$ {since $\tan\frac{\pi}{4}=1$}

$\therefore \quad 2x=\frac{7\pi}{12}+k\pi$

$\therefore \quad x=\frac{7\pi}{24}+k\,\frac{\pi}{2}$

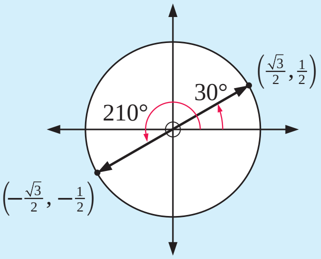
$$\therefore \quad x=\underset{\substack{\uparrow\\ k=-2}}{-\frac{17\pi}{24}}, \quad \underset{\substack{\uparrow\\ k=-1}}{-\frac{5\pi}{24}}, \quad \underset{\substack{\uparrow\\ k=0}}{\frac{7\pi}{24}}, \quad \underset{\substack{\uparrow\\ k=1}}{\frac{19\pi}{24}}$$



Example 12

Self Tutor

Find the exact solutions of $\sqrt{3}\sin x=\cos x$ for $0^o \leqslant x \leqslant 360^o$.



$\sqrt{3}\sin x=\cos x$

$\therefore \quad \frac{\sin x}{\cos x}=\frac{1}{\sqrt{3}}$ {dividing both sides by $\sqrt{3}\cos x$}

$\therefore \quad \tan x=\frac{1}{\sqrt{3}}$

$\therefore \quad x=30^o$ or 210^o.

EXERCISE 10G.3

1 List the possible solutions for x if k is an integer and:

- **a** $x = \frac{\pi}{6} + k2\pi, \ 0 \leqslant x \leqslant 6\pi$
- **b** $x = -\frac{\pi}{3} + k2\pi, \ -2\pi \leqslant x \leqslant 2\pi$
- **c** $x = -\frac{\pi}{2} + k\pi, \ -4\pi \leqslant x \leqslant 4\pi$
- **d** $x = \frac{5\pi}{6} + k\left(\frac{\pi}{2}\right), \ 0 \leqslant x \leqslant 4\pi$

2 Find the exact solutions of:

- **a** $\cos x = -\frac{1}{2}, \ 0 \leqslant x \leqslant 5\pi$
- **b** $2\sin x - 1 = 0, \ -360^o \leqslant x \leqslant 360^o$
- **c** $2\cos x + \sqrt{3} = 0, \ 0 \leqslant x \leqslant 3\pi$
- **d** $\cos\left(x - \frac{2\pi}{3}\right) = \frac{1}{2}, \ -2\pi \leqslant x \leqslant 2\pi$
- **e** $2\sin\left(x + \frac{\pi}{3}\right) = 1, \ -3\pi \leqslant x \leqslant 3\pi$
- **f** $\sqrt{2}\sin\left(x - \frac{\pi}{4}\right) + 1 = 0, \ 0 \leqslant x \leqslant 3\pi$
- **g** $3\cos 2x + 3 = 0, \ 0 \leqslant x \leqslant 3\pi$
- **h** $\sin\left(4(x - \frac{\pi}{4})\right) = 0, \ 0 \leqslant x \leqslant \pi$
- **i** $4\cos 3x + 2 = 0, \ -\pi \leqslant x \leqslant \pi$
- **j** $2\sin\left(2(x - \frac{\pi}{3})\right) = -\sqrt{3}, \ 0 \leqslant x \leqslant 2\pi$

3 Find the exact solutions of $\tan X = \sqrt{3}$ in terms of π. Hence solve the following equations for $0 \leqslant x \leqslant 2\pi$:

- **a** $\tan\left(x - \frac{\pi}{6}\right) = \sqrt{3}$
- **b** $\tan 4x = \sqrt{3}$
- **c** $\tan^2 x = 3$

4 Find the zeros of:

- **a** $y = \sin 2x$ between 0^o and 180^o (inclusive)
- **b** $y = \sin(x - \frac{\pi}{4})$ between 0 and 3π (inclusive)

5
- **a** Use your graphics calculator to sketch the graphs of $y = \sin x$ and $y = \cos x$ on the same set of axes on the domain $0 \leqslant x \leqslant 2\pi$.
- **b** Find the x values of the points of intersection of the two graphs.
- **c** Confirm that these values are the solutions of $\sin x = \cos x$ on $0 \leqslant x \leqslant 2\pi$.

6 Find the exact solutions to these equations for $0 \leqslant x \leqslant 2\pi$:

- **a** $\sin x = -\cos x$
- **b** $\sin(3x) = \cos(3x)$
- **c** $\sin(2x) = \sqrt{3}\cos(2x)$

7 Check your answers to question **6** using a graphics calculator.
Find the points of intersection of the appropriate graphs.

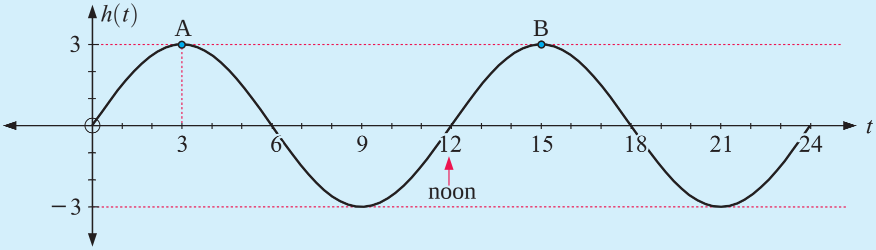
H USING TRIGONOMETRIC MODELS

Example 13 **Self Tutor**

The height $h(t)$ metres of the tide above mean sea level on January 24th at Cape Town is modelled approximately by $h(t) = 3\sin\left(\frac{\pi t}{6}\right)$ where t is the number of hours after midnight.

- **a** Graph $y = h(t)$ for $0 \leqslant t \leqslant 24$.
- **b** When is high tide and what is the maximum height?
- **c** What is the height at 2 pm?
- **d** If a ship can cross the harbour provided the tide is at least 2 m above mean sea level, when is crossing possible on January 24?

a $h(t) = 3\sin\left(\frac{\pi t}{6}\right)$ has period $= \dfrac{2\pi}{\frac{\pi}{6}} = 2\pi \times \frac{6}{\pi} = 12$ hours and $h(0) = 0$

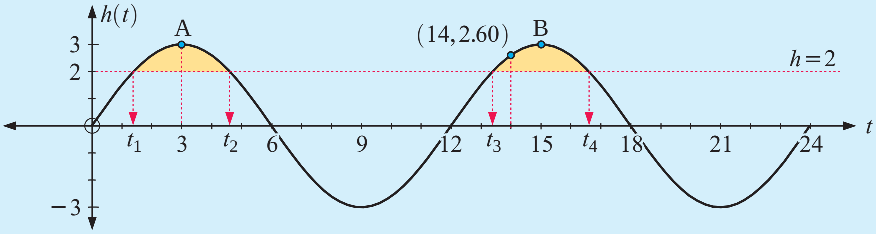


b High tide is at 3 am and 3 pm. The maximum height is 3 m above the mean as seen at points A and B.

c At 2 pm, $t = 14$ and $h(14) = 3\sin\left(\frac{14\pi}{6}\right) \approx 2.60$ m

So, the tide is 2.6 m above the mean.

d



We need to solve $h(t) = 2$, so $3\sin\left(\frac{\pi t}{6}\right) = 2$.

Using a graphics calculator with $Y_1 = 3\sin\left(\frac{\pi X}{6}\right)$ and $Y_2 = 2$

we obtain $t_1 = 1.39$, $t_2 = 4.61$, $t_3 = 13.39$, $t_4 = 16.61$

or you could **trace** across the graph to find these values.

Now 1.39 hours = 1 hour 23 minutes, and so on.

So, the ship can cross between 1:23 am and 4:37 am or 1:23 pm and 4:37 pm.

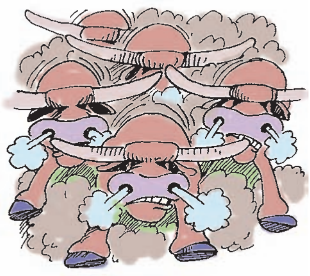
EXERCISE 10H

1 The population of grasshoppers after t weeks where $0 \leqslant t \leqslant 12$ is estimated by $P(t) = 7500 + 3000\sin\left(\frac{\pi t}{8}\right)$.

- **a** Find: **i** the initial estimate **ii** the estimate after 5 weeks.
- **b** What is the greatest population size over this interval and when does it occur?
- **c** When is the population: **i** 9000 **ii** 6000?
- **d** During what time interval(s) does the population size exceed 10 000?

2 The model for the height of a light on a Ferris wheel is $H(t) = 20 - 19\sin\left(\frac{2\pi t}{3}\right)$, where H is the height in metres above the ground, and t is in minutes.

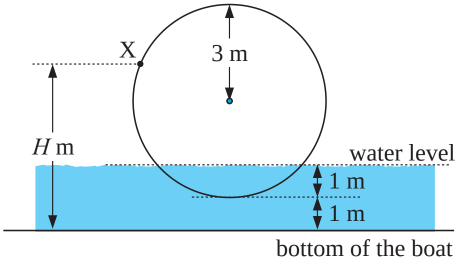
- **a** Where is the light at time $t = 0$?
- **b** At what time is the light at its lowest in the first revolution of the wheel?
- **c** How long does the wheel take to complete one revolution?
- **d** Sketch the graph of the function $H(t)$ over one revolution.



3 The population of water buffalo is given by $P(t) = 400 + 250\sin\left(\frac{\pi t}{2}\right)$ where t is the number of years since the first estimate was made.

- **a** What was the initial estimate?
- **b** What was the population size after:
  - **i** 6 months
  - **ii** two years?
- **c** Find $P(1)$. What is the significance of this value?
- **d** Find the smallest population size and when it first occurred.
- **e** Find the first time when the herd exceeded 500.

4 A paint spot X lies on the outer rim of the wheel of a paddle-steamer. The wheel has radius 3 m and as it rotates at a constant rate, X is seen entering the water every 4 seconds. H is the distance of X above the bottom of the boat. At time $t = 0$, X is at its highest point.



- **a** Find a cosine model for H in the form $H(t) = a\cos b(t - c) + d$.
- **b** At what time does X first enter the water?

5 Over a 28 day period, the cost per litre of petrol was modelled by $C(t) = 9.2\sin\frac{\pi}{7}(t - 4) + 107.8$ cents L^{-1}.

- **a** True or false?
  - **i** "The cost per litre oscillates about 107.8 cents with maximum price \$1.17."
  - **ii** "Every 14 days, the cycle repeats itself."
- **b** What was the cost of petrol on day 7, to the nearest tenth of a cent per litre?
- **c** On what days was the petrol priced at \$1.10 per litre?
- **d** What was the minimum cost per litre and when did it occur?

I TRIGONOMETRIC RELATIONSHIPS

There are a vast number of trigonometric relationships. However, we only need to remember a few because we can obtain the rest by rearrangement or substitution.

SIMPLIFYING TRIGONOMETRIC EXPRESSIONS

For any given angle θ, $\sin\theta$ and $\cos\theta$ are real numbers, so the algebra of trigonometry is identical to the algebra of real numbers.

An expression like $2\sin\theta + 3\sin\theta$ compares with $2x + 3x$ when we wish to do simplification, and so $2\sin\theta + 3\sin\theta = 5\sin\theta$.

To simplify complicated trigonometric expressions, we often use the identities:

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Notice that we can also use rearrangements of these formulae, such as:

$$\sin^2\theta = 1 - \cos^2\theta \qquad 1 - \cos^2\theta = \sin^2\theta$$
$$\cos^2\theta = 1 - \sin^2\theta \qquad 1 - \sin^2\theta = \cos^2\theta$$

Example 14 **Self Tutor**

Simplify: **a** $3\cos\theta + 4\cos\theta$ **b** $\sin\alpha - 3\sin\alpha$

a Since $3x + 4x = 7x$,
$3\cos\theta + 4\cos\theta = 7\cos\theta$

b Since $x - 3x = -2x$,
$\sin\alpha - 3\sin\alpha = -2\sin\alpha$

Example 15 **Self Tutor**

Simplify: **a** $2 - 2\sin^2\theta$ **b** $\cos^2\theta\sin\theta + \sin^3\theta$

a
$$\begin{aligned} &2 - 2\sin^2\theta \\ &= 2(1 - \sin^2\theta) \\ &= 2\cos^2\theta \\ &\{\text{as } \cos^2\theta + \sin^2\theta = 1\} \end{aligned}$$

b
$$\begin{aligned} &\cos^2\theta\sin\theta + \sin^3\theta \\ &= \sin\theta(\cos^2\theta + \sin^2\theta) \\ &= \sin\theta \times 1 \\ &= \sin\theta \end{aligned}$$

Example 16 **Self Tutor**

Expand and simplify: $(\cos\theta - \sin\theta)^2$

$$\begin{aligned} &(\cos\theta - \sin\theta)^2 \\ &= \cos^2\theta - 2\cos\theta\sin\theta + \sin^2\theta \qquad \{\text{using } (a-b)^2 = a^2 - 2ab + b^2\} \\ &= \cos^2\theta + \sin^2\theta - 2\cos\theta\sin\theta \\ &= 1 - 2\cos\theta\sin\theta \end{aligned}$$

EXERCISE 10I.1

1 Simplify:

a $\sin\theta + \sin\theta$ **b** $2\cos\theta + \cos\theta$ **c** $3\sin\theta - \sin\theta$

d $3\sin\theta - 2\sin\theta$ **e** $\cos\theta - 3\cos\theta$ **f** $2\cos\theta - 5\cos\theta$

2 Simplify:

a $3\sin^2\theta + 3\cos^2\theta$ **b** $-2\sin^2\theta - 2\cos^2\theta$ **c** $-\cos^2\theta - \sin^2\theta$

d $3 - 3\sin^2\theta$ **e** $4 - 4\cos^2\theta$ **f** $\cos^3\theta + \cos\theta\sin^2\theta$

g $\cos^2\theta - 1$ **h** $\sin^2\theta - 1$ **i** $2\cos^2\theta - 2$

j $\dfrac{1-\sin^2\theta}{\cos^2\theta}$ **k** $\dfrac{1-\cos^2\theta}{\sin\theta}$ **l** $\dfrac{\cos^2\theta - 1}{-\sin\theta}$

3 Simplify:

a $3\tan x - \tan x$ **b** $\tan x - 4\tan x$ **c** $\tan x \cos x$

d $\dfrac{\sin x}{\tan x}$ **e** $3\sin x + 2\cos x \tan x$ **f** $\dfrac{2\tan x}{\sin x}$

4 Expand and simplify if possible:

a $(1+\sin\theta)^2$ **b** $(\sin\alpha - 2)^2$ **c** $(\tan\alpha - 1)^2$

d $(\sin\alpha + \cos\alpha)^2$ **e** $(\sin\beta - \cos\beta)^2$ **f** $-(2-\cos\alpha)^2$

FACTORISING TRIGONOMETRIC EXPRESSIONS

Example 17 **Self Tutor**

Factorise: **a** $\cos^2\alpha - \sin^2\alpha$ **b** $\tan^2\theta - 3\tan\theta + 2$

a $\cos^2\alpha - \sin^2\alpha$

$= (\cos\alpha + \sin\alpha)(\cos\alpha - \sin\alpha)$ $\{a^2 - b^2 = (a+b)(a-b)\}$

b $\tan^2\theta - 3\tan\theta + 2$

$= (\tan\theta - 2)(\tan\theta - 1)$ $\{x^2 - 3x + 2 = (x-2)(x-1)\}$

Example 18 **Self Tutor**

Simplify: **a** $\dfrac{2-2\cos^2\theta}{1+\cos\theta}$ **b** $\dfrac{\cos\theta - \sin\theta}{\cos^2\theta - \sin^2\theta}$

a

$$\frac{2-2\cos^2\theta}{1+\cos\theta}$$

$$= \frac{2(1-\cos^2\theta)}{1+\cos\theta}$$

$$= \frac{2\cancel{(1+\cos\theta)}(1-\cos\theta)}{\cancel{(1+\cos\theta)}}$$

$$= 2(1-\cos\theta)$$

b

$$\frac{\cos\theta - \sin\theta}{\cos^2\theta - \sin^2\theta}$$

$$= \frac{\cancel{(\cos\theta - \sin\theta)}}{(\cos\theta + \sin\theta)\cancel{(\cos\theta - \sin\theta)}}$$

$$= \frac{1}{\cos\theta + \sin\theta}$$

EXERCISE 10I.2

1 Factorise:

a $1 - \sin^2\theta$ **b** $\sin^2\alpha - \cos^2\alpha$ **c** $\tan^2\alpha - 1$

d $2\sin^2\beta - \sin\beta$ **e** $2\cos\phi + 3\cos^2\phi$ **f** $3\sin^2\theta - 6\sin\theta$

g $\tan^2\theta + 5\tan\theta + 6$ **h** $2\cos^2\theta + 7\cos\theta + 3$ **i** $6\cos^2\alpha - \cos\alpha - 1$

2 Simplify:

a $\dfrac{1-\sin^2\alpha}{1-\sin\alpha}$ **b** $\dfrac{\tan^2\beta-1}{\tan\beta+1}$ **c** $\dfrac{\cos^2\phi-\sin^2\phi}{\cos\phi+\sin\phi}$

d $\dfrac{\cos^2\phi-\sin^2\phi}{\cos\phi-\sin\phi}$ **e** $\dfrac{\sin\alpha+\cos\alpha}{\sin^2\alpha-\cos^2\alpha}$ **f** $\dfrac{3-3\sin^2\theta}{6\cos\theta}$

3 Show that:

a $(\cos\theta+\sin\theta)^2+(\cos\theta-\sin\theta)^2$ simplifies to 2

b $(2\sin\theta+3\cos\theta)^2+(3\sin\theta-2\cos\theta)^2$ simplifies to 13

c $(1-\cos\theta)\left(1+\dfrac{1}{\cos\theta}\right)$ simplifies to $\tan\theta\sin\theta$

d $\left(1+\dfrac{1}{\sin\theta}\right)(\sin\theta-\sin^2\theta)$ simplifies to $\cos^2\theta$

e $\dfrac{\sin\theta}{1+\cos\theta}+\dfrac{1+\cos\theta}{\sin\theta}$ simplifies to $\dfrac{2}{\sin\theta}$

f $\dfrac{\sin\theta}{1-\cos\theta}-\dfrac{\sin\theta}{1+\cos\theta}$ simplifies to $\dfrac{2}{\tan\theta}$

g $\dfrac{1}{1-\sin\theta}+\dfrac{1}{1+\sin\theta}$ simplifies to $\dfrac{2}{\cos^2\theta}$

Use a graphing package to check these simplifications by graphing each function on the same set of axes.

GRAPHING PACKAGE

J DOUBLE ANGLE FORMULAE

INVESTIGATION 4 DOUBLE ANGLE FORMULAE

What to do:

1 Copy and complete, using angles of your choice as well:

θ	$\sin 2\theta$	$2\sin\theta$	$2\sin\theta\cos\theta$	$\cos 2\theta$	$2\cos\theta$	$\cos^2\theta-\sin^2\theta$
0.631						
57.81^o						
-3.697						

2 Write down any discoveries from your table of values in **1**.

The **double angle formulae** are:

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ \cos 2\theta &= \cos^2\theta - \sin^2\theta\\ &= 1 - 2\sin^2\theta\\ &= 2\cos^2\theta - 1\end{aligned}$$

GRAPHING PACKAGE

To show the equivalence of these formulae you can plot them on the same set of axes.

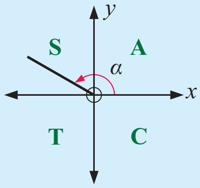
Example 19 **Self Tutor**

If $\sin\alpha = \frac{5}{13}$ where $\frac{\pi}{2} < \alpha < \pi$, find the value of $\sin 2\alpha$ without using a calculator.

α is in quadrant 2, so $\cos\alpha$ is negative.

$$\begin{aligned}\text{Now}\quad \cos^2\alpha + \sin^2\alpha &= 1\\ \therefore\quad \cos^2\alpha + \tfrac{25}{169} &= 1\\ \therefore\quad \cos^2\alpha &= \tfrac{144}{169}\\ \therefore\quad \cos\alpha &= \pm\tfrac{12}{13}\\ \therefore\quad \cos\alpha &= -\tfrac{12}{13}\end{aligned}$$

$$\begin{aligned}\text{But}\quad \sin 2\alpha &= 2\sin\alpha\cos\alpha\\ &= 2(\tfrac{5}{13})(-\tfrac{12}{13})\\ &= -\tfrac{120}{169}\end{aligned}$$



Example 20 **Self Tutor**

Given that $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = -\frac{4}{5}$ find:

a $\sin 2\alpha$ **b** $\cos 2\alpha$

a
$$\begin{aligned}&\sin 2\alpha\\ &= 2\sin\alpha\cos\alpha\\ &= 2(\tfrac{3}{5})(-\tfrac{4}{5})\\ &= -\tfrac{24}{25}\end{aligned}$$

b
$$\begin{aligned}&\cos 2\alpha\\ &= \cos^2\alpha - \sin^2\alpha\\ &= (-\tfrac{4}{5})^2 - (\tfrac{3}{5})^2\\ &= \tfrac{7}{25}\end{aligned}$$

Example 21 **Self Tutor**

If α is acute and $\cos 2\alpha = \frac{3}{4}$ find the values of: **a** $\cos\alpha$ **b** $\sin\alpha$.

a
$$\begin{aligned}\cos 2\alpha &= 2\cos^2\alpha - 1\\ \therefore\quad \tfrac{3}{4} &= 2\cos^2\alpha - 1\\ \therefore\quad \cos^2\alpha &= \tfrac{7}{8}\\ \therefore\quad \cos\alpha &= \pm\tfrac{\sqrt{7}}{2\sqrt{2}}\\ \therefore\quad \cos\alpha &= \tfrac{\sqrt{7}}{2\sqrt{2}}\end{aligned}$$
{as α is acute, $\cos\alpha > 0$}

b
$$\sin\alpha = \sqrt{1-\cos^2\alpha}$$
{as α is acute, $\sin\alpha > 0$}
$$\begin{aligned}\therefore\quad \sin\alpha &= \sqrt{1-\tfrac{7}{8}}\\ \therefore\quad \sin\alpha &= \sqrt{\tfrac{1}{8}}\\ \therefore\quad \sin\alpha &= \tfrac{1}{2\sqrt{2}}\end{aligned}$$

Example 22 **Self Tutor**

Use an appropriate 'double angle formula' to simplify:

a $3\sin\theta\cos\theta$ **b** $4\cos^2 2B - 2$

a
$$\begin{aligned} &3\sin\theta\cos\theta \\ &= \tfrac{3}{2}(2\sin\theta\cos\theta) \\ &= \tfrac{3}{2}\sin 2\theta \end{aligned}$$

b
$$\begin{aligned} &4\cos^2 2B - 2 \\ &= 2(2\cos^2 2B - 1) \\ &= 2\cos 2(2B) \\ &= 2\cos 4B \end{aligned}$$

EXERCISE 10J

1 If $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$ find the values of: **a** $\sin 2\theta$ **b** $\cos 2\theta$.

2 **a** If $\cos A = \frac{1}{3}$, find $\cos 2A$. **b** If $\sin\phi = -\frac{2}{3}$, find $\cos 2\phi$.

3 **a** If $\sin\alpha = -\frac{2}{3}$ where $\pi < \alpha < \frac{3\pi}{2}$, find the value of $\cos\alpha$ and hence the value of $\sin 2\alpha$.

b If $\cos\beta = \frac{2}{5}$ where $270^o < \beta < 360^o$, find the value of $\sin\beta$ and hence the value of $\sin 2\beta$.

4 If α is acute and $\cos 2\alpha = -\frac{7}{9}$, find without a calculator: **a** $\cos\alpha$ **b** $\sin\alpha$.

5 Find the exact value of $\left[\cos\left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{12}\right)\right]^2$.

6 Use an appropriate 'double angle' formula to simplify:

a $2\sin\alpha\cos\alpha$ **b** $4\cos\alpha\sin\alpha$ **c** $\sin\alpha\cos\alpha$

d $2\cos^2\beta - 1$ **e** $1 - 2\cos^2\phi$ **f** $1 - 2\sin^2 N$

g $2\sin^2 M - 1$ **h** $\cos^2\alpha - \sin^2\alpha$ **i** $\sin^2\alpha - \cos^2\alpha$

j $2\sin 2A\cos 2A$ **k** $2\cos 3\alpha\sin 3\alpha$ **l** $2\cos^2 4\theta - 1$

m $1 - 2\cos^2 3\beta$ **n** $1 - 2\sin^2 5\alpha$ **o** $2\sin^2 3D - 1$

p $\cos^2 2A - \sin^2 2A$ **q** $\cos^2(\frac{\alpha}{2}) - \sin^2(\frac{\alpha}{2})$ **r** $2\sin^2 3P - 2\cos^2 3P$

7 Show that:

a $(\sin\theta + \cos\theta)^2 = 1 + \sin 2\theta$ **b** $\cos^4\theta - \sin^4\theta = \cos 2\theta$

GRAPHING PACKAGE

8 Solve for x where $0 \leqslant x \leqslant 2\pi$:

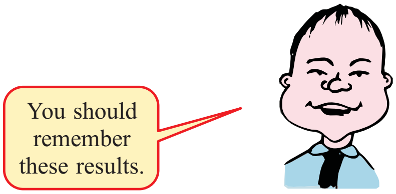
a $\sin 2x + \sin x = 0$ **b** $\sin 2x - 2\cos x = 0$

c $\sin 2x + 3\sin x = 0$

9 Prove that:

a $\boldsymbol{\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta}$

b $\boldsymbol{\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta}$



K TRIGONOMETRIC EQUATIONS IN QUADRATIC FORM

Sometimes we may be given trigonometric equations in quadratic form.

For example, $2\sin^2 x + \sin x = 0$ and $2\cos^2 x + \cos x - 1 = 0$ are clearly quadratic equations where the variables are $\sin x$ and $\cos x$ respectively.

These equations can be factorised and then solved:

$$\begin{aligned} 2\sin^2 x + \sin x &= 0 \\ \therefore\ \sin x(2\sin x + 1) &= 0 \\ \therefore\ \sin x &= 0 \text{ or } -\tfrac{1}{2} \end{aligned} \qquad \text{and} \qquad \begin{aligned} 2\cos^2 x + \cos x + 1 &= 0 \\ \therefore\ (2\cos x - 1)(\cos x + 1) &= 0 \\ \therefore\ \cos x &= \tfrac{1}{2} \text{ or } -1 \end{aligned}$$

EXERCISE 10K

1 Solve for $0 \leqslant x \leqslant 2\pi$ giving your answers as **exact** values:

- **a** $2\sin^2 x + \sin x = 0$
- **b** $2\cos^2 x = \cos x$
- **c** $2\cos^2 x + \cos x - 1 = 0$
- **d** $2\sin^2 x + 3\sin x + 1 = 0$
- **e** $\sin^2 x = 2 - \cos x$

2 Solve for $0 \leqslant x \leqslant 2\pi$ giving your answers as **exact** values:

- **a** $\cos 2x - \cos x = 0$
- **b** $\cos 2x + 3\cos x = 1$
- **c** $\cos 2x + \sin x = 0$
- **d** $\sin 4x = \sin 2x$
- **e** $\sin x + \cos x = \sqrt{2}$
- **f** $2\cos^2 x = 3\sin x$

REVIEW SET 10A NON-CALCULATOR

1 Draw the graph of $y = 4\sin x$ for $0 \leqslant x \leqslant 2\pi$.

2 State the minimum and maximum values of:

- **a** $1 + \sin x$
- **b** $-2\cos 3x$

3 Solve algebraically in terms of π:

- **a** $2\sin x = -1$ for $0 \leqslant x \leqslant 4\pi$
- **b** $\sqrt{2}\sin x - 1 = 0$ for $-2\pi \leqslant x \leqslant 2\pi$

4 Find the exact x-intercepts of:

- **a** $y = 2\sin 3x + \sqrt{3}$ for $0 \leqslant x \leqslant 2\pi$
- **b** $y = \sqrt{2}\sin(x + \frac{\pi}{4})$ for $0 \leqslant x \leqslant 3\pi$

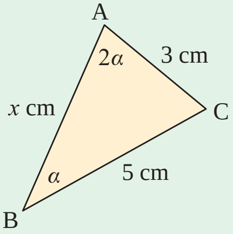
5 Find the exact solutions of $\sqrt{2}\cos\left(x + \frac{\pi}{4}\right) - 1 = 0,\ 0 \leqslant x \leqslant 4\pi$.

6 Simplify:

- **a** $\dfrac{1 - \cos^2\theta}{1 + \cos\theta}$
- **b** $\dfrac{\sin\alpha - \cos\alpha}{\sin^2\alpha - \cos^2\alpha}$
- **c** $\dfrac{4\sin^2\alpha - 4}{8\cos\alpha}$

7 If $\sin\alpha = -\frac{3}{4}$, $\pi \leqslant \alpha \leqslant \frac{3\pi}{2}$, find the value of $\cos\alpha$ and hence the value of $\sin 2\alpha$.

8 Show that $\dfrac{\sin 2\alpha - \sin\alpha}{\cos 2\alpha - \cos\alpha + 1}$ simplifies to $\tan\alpha$.

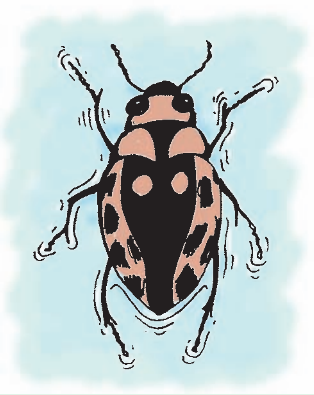
9 

a Show that $\cos\alpha = \frac{5}{6}$.

b Show that x is a solution of $3x^2 - 25x + 48 = 0$.

c Find x by solving the equation in **b**.

10 An ecologist studying a species of water beetle estimates the population of a colony over an eight week period. If t is the number of weeks after the initial estimate is made, then the population in thousands can be modelled by $P(t) = 5 + 2\sin\left(\frac{\pi t}{3}\right)$ where $0 \leqslant t \leqslant 8$.



a What was the initial population?

b What were the smallest and largest populations?

c During what time interval(s) did the population exceed 6000?

REVIEW SET 10B CALCULATOR

1 Draw the graph of $y = \sin 3x$ for $0 \leqslant x \leqslant 2\pi$.

2 State the period of: **a** $y = 4\sin\left(\frac{x}{3}\right)$ **b** $y = -2\tan 4x$

3 Use technology to solve for $0 \leqslant x \leqslant 8$:

a $\sin x = 0.382$ **b** $\tan\left(\frac{x}{2}\right) = -0.458$

4 On the same set of axes, for the domain $0 \leqslant x \leqslant 2\pi$, sketch:

a $y = \cos x$ and $y = \cos x - 3$

b $y = \cos x$ and $y = \cos\left(x - \frac{\pi}{4}\right)$

c $y = \cos x$ and $y = 3\cos 2x$

d $y = \cos x$ and $y = 2\cos\left(x - \frac{\pi}{3}\right) + 3$.

5 Use technology to solve:

a $\cos x = 0.4379$ for $0 \leqslant x \leqslant 10$

b $\cos(x - 2.4) = -0.6014$ for $0 \leqslant x \leqslant 6$

6 If $\sin A = \frac{5}{13}$ and $\cos A = \frac{12}{13}$ find the values of: **a** $\sin 2A$ **b** $\cos 2A$.

7 **a** Use technology to solve for $0 \leqslant x \leqslant 10$:

i $\tan x = 4$ **ii** $\tan\left(\frac{x}{4}\right) = 4$ **iii** $\tan(x - 1.5) = 4$

b Find exact solutions for x given $-\pi \leqslant x \leqslant \pi$:

i $\tan\left(x + \frac{\pi}{6}\right) = -\sqrt{3}$ **ii** $\tan 2x = -\sqrt{3}$ **iii** $\tan^2 x - 3 = 0$

c Use technology to solve $3\tan(x - 1.2) = -2$ for $0 \leqslant x \leqslant 10$.

8 The table below gives the mean monthly maximum temperature for Perth Airport in Western Australia.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temp (°C)	31.5	31.8	29.5	25.4	21.5	18.8	17.7	18.3	20.1	22.4	25.5	28.8

a A sine function of the form $T \approx a\sin b(t-c)+d$ is used to model the data. Find good estimates of the constants a, b, c and d without using technology. Use Jan $\equiv 1$, Feb $\equiv 2$, and so on.

b Check your answer to **a** using technology. How well does your model fit?

REVIEW SET 10C

1 Solve algebraically for $0 \leqslant x \leqslant 2\pi$, giving answers in terms of π:

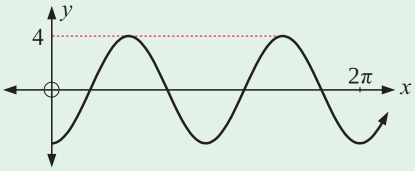
a $\sin^2 x - \sin x - 2 = 0$ **b** $4\sin^2 x = 1$

2 **a** Without using technology draw the graph of $f(x) = \sin\left(x - \frac{\pi}{3}\right) + 2$.

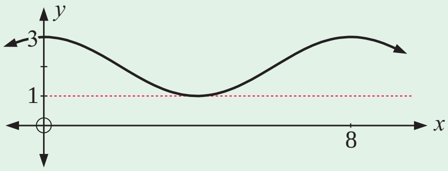
b For what values of k will $f(x) = k$ have solutions?

3 Find the cosine function represented in the following graphs:

a



b



4 Find the exact solutions of:

a $\tan\left(x - \frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}$, $0 \leqslant x \leqslant 4\pi$ **b** $\cos\left(x + \frac{2\pi}{3}\right) = \frac{1}{2}$, $-2\pi \leqslant x \leqslant 2\pi$

5 Simplify:

a $\cos^3\theta + \sin^2\theta\cos\theta$ **b** $\dfrac{\cos^2\theta - 1}{\sin\theta}$

c $5 - 5\sin^2\theta$ **d** $\dfrac{\sin^2\theta - 1}{\cos\theta}$

6 Expand and simplify if possible:

a $(2\sin\alpha - 1)^2$ **b** $(\cos\alpha - \sin\alpha)^2$

7 Show that:

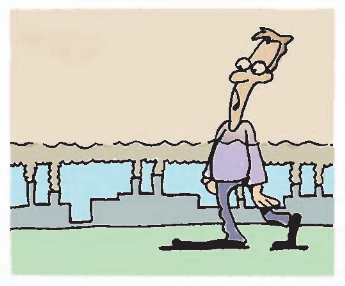
a $\dfrac{\cos\theta}{1+\sin\theta} + \dfrac{1+\sin\theta}{\cos\theta} = \dfrac{2}{\cos\theta}$ **b** $\left(1 + \dfrac{1}{\cos\theta}\right)(\cos\theta - \cos^2\theta) = \sin^2\theta$.

8 If $\tan\theta = -\frac{2}{3}$, $\frac{\pi}{2} < \theta < \pi$, find $\sin\theta$ and $\cos\theta$ without using a calculator.

9 In an industrial city, the amount of pollution in the air becomes greater during the working week when factories are operating, and lessens over the weekend. The number of milligrams of pollutants in a cubic metre of air is given by

$$P(t) = 40 + 12\sin\frac{2\pi}{7}\left(t - \frac{37}{12}\right)$$

where t is the number of days after midnight on Saturday night.

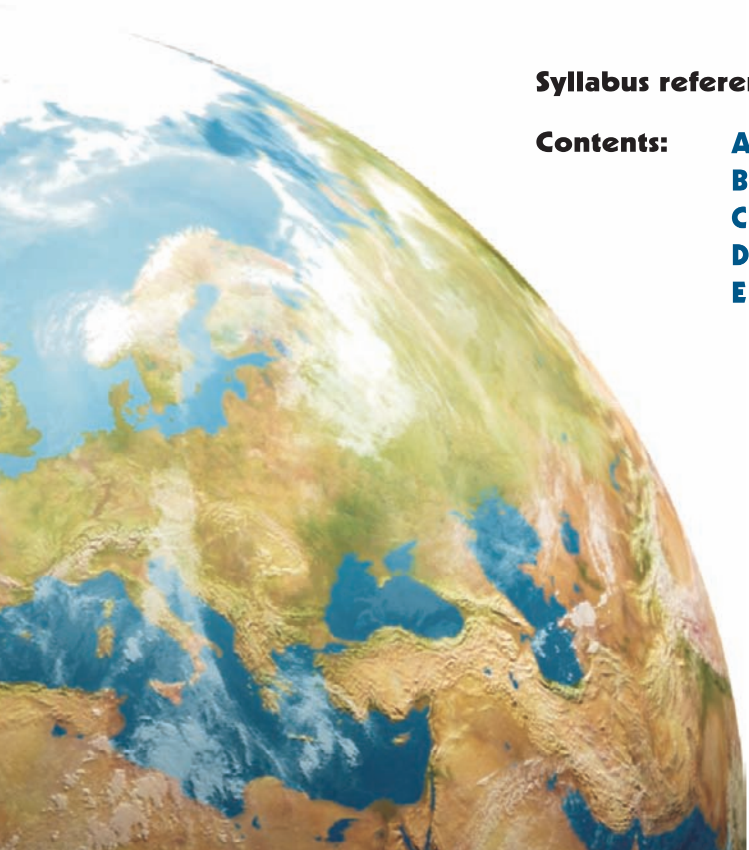


a What is the minimum level of pollution?

b At what time during the week does this minimum level occur?

Chapter 11

Matrices



Syllabus reference: 4.1, 4.2, 4.3, 4.4

Contents:

Matrices are rectangular arrays of numbers which are used to organise information of a numeric nature. They are used in a wide array of fields extending far beyond mathematics, including:

- **Solving systems of equations** in business, physics, and engineering.
- **Linear programming** where we may wish to optimise a linear expression subject to linear constraints. For example, optimising profits of a business.
- **Business inventories** involving stock control, cost, revenue, and profit calculations. Matrices form the basis of business computer software.
- **Markov chains** for predicting long term probabilities such as in weather.
- **Strategies in games** where we wish to maximise our chance of winning.
- **Economic modelling** where the input from various suppliers is needed to help a business be successful.
- **Network theory** used to determine routes for trucks and airlines to minimise distance travelled and therefore costs.
- **Assignment problems** to direct resources in the most cost effective way.
- **Forestry and fisheries management** where we need to select an appropriate sustainable harvesting policy.
- **Cubic spline interpolation** used to construct curves and fonts. Each font is stored in matrix form in the memory of a computer.
- **Computer graphics**, **flight simulation**, **Computer Aided Tomography** (CAT scanning) and **Magnetic Resonance Imaging** (MRI), **Fractals**, **Chaos**, **Genetics**, **Cryptography** (coding, code breaking, computer confidentiality), and much more.

A MATRIX STRUCTURE

A **matrix** is a rectangular array of numbers arranged in **rows** and **columns**.
Each number is an **element** of the matrix.

In general the numbers within a matrix represent specific quantities. You have been using matrices for many years without realising it.

For example:

	Won	*Lost*	*Drew*	*Points*
Arsenal	24	2	4	76
Liverpool	23	3	4	73
Chelsea	21	4	5	68
Leeds	20	5	5	65
⋮				

Ingredients	*Amount*
sugar	1 tspn
flour	1 cup
milk	200 mL
salt	1 pinch

Consider these two items of information:

Shopping list

Bread	2 loaves
Juice	1 carton
Eggs	6
Cheese	1

Furniture inventory

	chairs	tables	beds
Flat	6	1	2
Unit	9	2	3
House	10	3	4

We can write these tables as matrices by extracting the numbers and placing them in brackets:

$$\begin{array}{cc} & \text{number} \\ \text{B} & \\ \text{J} & \\ \text{E} & \\ \text{C} & \end{array}\!\!\begin{pmatrix} 2 \\ 1 \\ 6 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{array}{c|ccc} & \text{C} & \text{T} & \text{B} \\ \text{F} & 6 & 1 & 2 \\ \text{U} & 9 & 2 & 3 \\ \text{H} & 10 & 3 & 4 \end{array} \quad \text{or simply} \quad \begin{pmatrix} 2 \\ 1 \\ 6 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 6 & 1 & 2 \\ 9 & 2 & 3 \\ 10 & 3 & 4 \end{pmatrix}$$

Notice how the organisation of the data is maintained in matrix form.

If a matrix has m rows and n columns, the **order** of the matrix is $m \times n$.

$$\begin{pmatrix} 2 \\ 1 \\ 6 \\ 1 \end{pmatrix}$$

has 4 rows and 1 column. We say this is a 4×1 **column matrix** or **column vector**.

column 2

$$\text{row 3} \longrightarrow \begin{pmatrix} 6 & 1 & 2 \\ 9 & 2 & 3 \\ 10 & 3 & 4 \end{pmatrix}$$

this element, 3, is in row 3, column 2

has 3 rows and 3 columns and is called a 3×3 **square matrix**.

$$\begin{pmatrix} 3 & 0 & -1 & 2 \end{pmatrix}$$

has 1 row and 4 columns and is called a 1×4 **row matrix** or **row vector**.

In business, a matrix can be used to represent numbers of items to be purchased, prices of items to be purchased, and so on.

Example 1 **Self Tutor**

Lisa goes shopping at store A to buy 2 loaves of bread at \$2.65 each, 3 litres of milk at \$1.55 per litre, and one 500 g tub of butter at \$2.35.

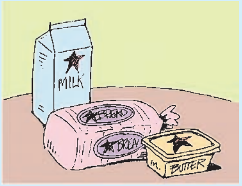
a Represent the quantities purchased in a row matrix **Q** and the costs in a column matrix **A**.

b Lisa goes to a different supermarket (store B) and finds that the prices for the same items are \$2.25 for bread, \$1.50 for milk, and \$2.20 for butter. Write the costs for both stores in a single costs matrix **C**.

a The quantities matrix is $\mathbf{Q} = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$

$$\begin{array}{ccc} \nearrow & \uparrow & \nwarrow \\ \text{bread} & \text{milk} & \text{butter} \end{array}$$

The costs matrix is $\mathbf{A} = \begin{pmatrix} 2.65 \\ 1.55 \\ 2.35 \end{pmatrix} \begin{array}{l} \longleftarrow \text{bread} \\ \longleftarrow \text{milk} \\ \longleftarrow \text{butter} \end{array}$



b We write the costs for each store in separate columns.

The new costs matrix is $\mathbf{C} = \begin{pmatrix} 2.65 & 2.25 \\ 1.55 & 1.50 \\ 2.35 & 2.20 \end{pmatrix} \begin{array}{l} \longleftarrow \text{bread} \\ \longleftarrow \text{milk} \\ \longleftarrow \text{butter} \end{array}$

$$\begin{array}{cc} \uparrow & \uparrow \\ \text{store A} & \text{store B} \end{array}$$

EXERCISE 11A

1 Write down the order of:

a $\begin{pmatrix} 5 & 1 & 0 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ 7 \end{pmatrix}$ **c** $\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$ **d** $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 5 & 1 & 0 \end{pmatrix}$

2 A grocery list consists of 2 loaves of bread, 1 kg of butter, 6 eggs, and 1 carton of cream. The cost of each grocery item is $1.95, $2.35, $0.15 and $0.95 respectively.

a Construct a row matrix showing quantities.

b Construct a column matrix showing prices.

c What is the significance of $(2 \times 1.95) + (1 \times 2.35) + (6 \times 0.15) + (1 \times 0.95)$?

3 Big Bart's Baked Beans factory produces cans of baked beans in 3 sizes: 200 g, 300 g, and 500 g. In February they produced respectively:
1000, 1500, and 1250 cans of each in week 1; 1500, 1000, and 1000 of each in week 2; 800, 2300, and 1300 cans of each in week 3; 1200 cans of each in week 4.
Construct a matrix to show February's production levels.

4 Over a long weekend holiday, a baker produced the following food items: On Friday he baked 40 dozen pies, 50 dozen pasties, 55 dozen rolls, and 40 dozen buns. On Saturday, 25 dozen pies, 65 dozen pasties, 30 dozen buns, and 44 dozen rolls were made. On Sunday, 40 dozen pasties, 40 dozen rolls, and 35 dozen of each of pies and buns were made. On Monday the totals were 40 dozen pasties, 50 dozen buns, and 35 dozen of each of pies and rolls. Represent this information as a matrix.

B MATRIX OPERATIONS AND DEFINITIONS

EQUALITY

Two matrices are **equal** if they have exactly the same order *and* the elements in corresponding positions are equal.

For example, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ then $a = w$, $b = x$, $c = y$ and $d = z$.

MATRIX ADDITION

Thao has three stores (A, B and C). Her stock levels for dresses, skirts and blouses are given by the matrix:

$$\begin{array}{c} \text{Store} \\ \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ \begin{pmatrix} 23 & 41 & 68 \\ 28 & 39 & 79 \\ 46 & 17 & 62 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \text{dresses} \\ \text{skirts} \\ \text{blouses} \end{array}$$

Some newly ordered stock has just arrived. For each store 20 dresses, 30 skirts, and 50 blouses must be added to stock levels. Her stock order is given by the matrix: $\begin{pmatrix} 20 & 20 & 20 \\ 30 & 30 & 30 \\ 50 & 50 & 50 \end{pmatrix}$

Clearly the new levels are shown as: $\begin{pmatrix} 23+20 & 41+20 & 68+20 \\ 28+30 & 39+30 & 79+30 \\ 46+50 & 17+50 & 62+50 \end{pmatrix}$

$$\text{or} \quad \begin{pmatrix} 23 & 41 & 68 \\ 28 & 39 & 79 \\ 46 & 17 & 62 \end{pmatrix} + \begin{pmatrix} 20 & 20 & 20 \\ 30 & 30 & 30 \\ 50 & 50 & 50 \end{pmatrix} = \begin{pmatrix} 43 & 61 & 88 \\ 58 & 69 & 109 \\ 96 & 67 & 112 \end{pmatrix}$$

To **add** two matrices they must be of the **same order**.

We then **add corresponding elements**.

Example 2 **Self Tutor**

If $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 3 & 5 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$ find:

a $\mathbf{A} + \mathbf{B}$

b $\mathbf{A} + \mathbf{C}$

a
$$\begin{aligned} \mathbf{A} + \mathbf{B} &= \begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 6 \\ 0 & 3 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 1+2 & 2+1 & 3+6 \\ 6+0 & 5+3 & 4+5 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 3 & 9 \\ 6 & 8 & 9 \end{pmatrix} \end{aligned}$$

b $\mathbf{A}$ and $\mathbf{C}$ are not the same sized matrices.
Since they have different orders,
$\mathbf{A} + \mathbf{C}$ cannot be found.

MATRIX SUBTRACTION

If Thao's stock levels were $\begin{pmatrix} 29 & 51 & 19 \\ 31 & 28 & 32 \\ 40 & 17 & 29 \end{pmatrix}$ and her sales matrix for the week was $\begin{pmatrix} 15 & 12 & 6 \\ 20 & 16 & 19 \\ 19 & 8 & 14 \end{pmatrix}$, what are the current stock levels?

What Thao has left is her original stock levels less what she has sold. Clearly, we need to subtract corresponding elements:

$$\begin{pmatrix} 29 & 51 & 19 \\ 31 & 28 & 32 \\ 40 & 17 & 29 \end{pmatrix} - \begin{pmatrix} 15 & 12 & 6 \\ 20 & 16 & 19 \\ 19 & 8 & 14 \end{pmatrix} = \begin{pmatrix} 14 & 39 & 13 \\ 11 & 12 & 13 \\ 21 & 9 & 15 \end{pmatrix}$$

To **subtract** matrices they must be of the **same order**.

We then **subtract corresponding elements**.

Summary:

- We can only add or subtract matrices of the same order.
- We add or subtract corresponding elements.
- The result of addition or subtraction is another matrix of the same order.

Example 3 **Self Tutor**

If $\mathbf{A} = \begin{pmatrix} 3 & 4 & 8 \\ 2 & 1 & 0 \\ 1 & 4 & 7 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 0 & 6 \\ 3 & 0 & 4 \\ 5 & 2 & 3 \end{pmatrix}$, find $\mathbf{A} - \mathbf{B}$.

$$\begin{aligned}\mathbf{A} - \mathbf{B} &= \begin{pmatrix} 3 & 4 & 8 \\ 2 & 1 & 0 \\ 1 & 4 & 7 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 6 \\ 3 & 0 & 4 \\ 5 & 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 3-2 & 4-0 & 8-6 \\ 2-3 & 1-0 & 0-4 \\ 1-5 & 4-2 & 7-3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 4 & 2 \\ -1 & 1 & -4 \\ -4 & 2 & 4 \end{pmatrix}\end{aligned}$$

EXERCISE 11B.1

1 If $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -3 & 7 \\ -4 & -2 \end{pmatrix}$, find:

a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{A} + \mathbf{B} + \mathbf{C}$ **c** $\mathbf{B} + \mathbf{C}$ **d** $\mathbf{C} + \mathbf{B} - \mathbf{A}$

2 If $\mathbf{P} = \begin{pmatrix} 3 & 5 & -11 \\ 10 & 2 & 6 \\ -2 & -1 & 7 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 17 & -4 & 3 \\ -2 & 8 & -8 \\ 3 & -4 & 11 \end{pmatrix}$, find:

a $\mathbf{P} + \mathbf{Q}$ **b** $\mathbf{P} - \mathbf{Q}$ **c** $\mathbf{Q} - \mathbf{P}$

3 A restaurant served 85 men, 92 women, and 52 children on Friday night. On Saturday night they served 102 men, 137 women, and 49 children.

a Express this information in *two* column matrices.

b Use the matrices to find the totals of men, women and children served over the Friday-Saturday period.

4 On Monday David bought shares in five companies and on Friday he sold them. The details are:

	Cost price per share	*Selling price per share*
A	\$1.72	\$1.79
B	\$27.85	\$28.75
C	\$0.92	\$1.33
D	\$2.53	\$2.25
E	\$3.56	\$3.51

a Write down David's column matrix for:

i cost price **ii** selling price

b What matrix operation is needed to find David's profit or loss matrix?

c Find David's profit or loss matrix.

5 In November, Lou E Gee sold 23 fridges, 17 stoves, and 31 microwave ovens. His partner Rose A Lee sold 19 fridges, 29 stoves, and 24 microwave ovens.

In December, Lou's sales were 18 fridges, 7 stoves, and 36 microwaves, while Rose's sales were 25 fridges, 13 stoves, and 19 microwaves.

- **a** Write their sales for November as a 3×2 matrix.
- **b** Write their sales for December as a 3×2 matrix.
- **c** Write their total sales for November and December as a 3×2 matrix.

6 Find x and y if: **a** $\begin{pmatrix} x & x^2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} y & 4 \\ 3 & y+1 \end{pmatrix}$ **b** $\begin{pmatrix} x & y \\ y & x \end{pmatrix} = \begin{pmatrix} -y & x \\ x & -y \end{pmatrix}$

7 **a** If $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 2 \\ 2 & 3 \end{pmatrix}$ find $\mathbf{A} + \mathbf{B}$ and $\mathbf{B} + \mathbf{A}$.

b Explain why $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ for all 2×2 matrices **A** and **B**.

8 **a** For $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 1 & 5 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 4 & -1 \\ -1 & 3 \end{pmatrix}$ find $(\mathbf{A} + \mathbf{B}) + \mathbf{C}$ and $\mathbf{A} + (\mathbf{B} + \mathbf{C})$.

b Prove that, if **A**, **B** and **C** are any 2×2 matrices then $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.

Hint: Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$.

MULTIPLES OF MATRICES

In the pantry there are 6 cans of peaches, 4 cans of apricots, and 8 cans of pears.

This information could be represented by the column vector $\mathbf{C} = \begin{pmatrix} 6 \\ 4 \\ 8 \end{pmatrix}$.

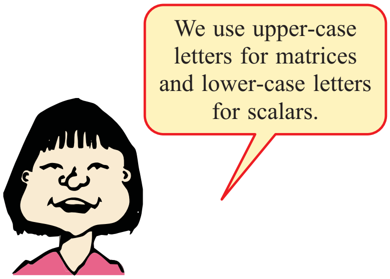
Doubling these cans in the pantry gives $\begin{pmatrix} 12 \\ 8 \\ 16 \end{pmatrix}$ which is $\mathbf{C} + \mathbf{C}$ or $2\mathbf{C}$.

Notice that to get $2\mathbf{C}$ from **C** we simply multiply all matrix elements by 2.

Likewise, trebling the fruit cans in the pantry gives: $3\mathbf{C} = \begin{pmatrix} 3 \times 6 \\ 3 \times 4 \\ 3 \times 8 \end{pmatrix} = \begin{pmatrix} 18 \\ 12 \\ 24 \end{pmatrix}$ and halving them gives: $\frac{1}{2}\mathbf{C} = \begin{pmatrix} \frac{1}{2} \times 6 \\ \frac{1}{2} \times 4 \\ \frac{1}{2} \times 8 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$

If a scalar k is multiplied by matrix **A**, the resulting matrix $k\mathbf{A}$ is found by multiplying every element of **A** by k.

The result is another matrix of the same order as **A**.



Example 4 **Self Tutor**

If $\mathbf{A}$ is $\begin{pmatrix} 1 & 2 & 5 \\ 2 & 0 & 1 \end{pmatrix}$ find:

a $3\mathbf{A}$ **b** $\frac{1}{2}\mathbf{A}$

a $$3\mathbf{A} = 3\begin{pmatrix} 1 & 2 & 5 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 15 \\ 6 & 0 & 3 \end{pmatrix}$$

b $$\tfrac{1}{2}\mathbf{A} = \tfrac{1}{2}\begin{pmatrix} 1 & 2 & 5 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 & 2\frac{1}{2} \\ 1 & 0 & \frac{1}{2} \end{pmatrix}$$

EXERCISE 11B.2

1 If $\mathbf{B} = \begin{pmatrix} 6 & 12 \\ 24 & 6 \end{pmatrix}$ find: **a** $2\mathbf{B}$ **b** $\frac{1}{3}\mathbf{B}$ **c** $\frac{1}{12}\mathbf{B}$ **d** $-\frac{1}{2}\mathbf{B}$

2 If $\mathbf{A} = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$ find:

a $\mathbf{A}+\mathbf{B}$ **b** $\mathbf{A}-\mathbf{B}$ **c** $2\mathbf{A}+\mathbf{B}$ **d** $3\mathbf{A}-\mathbf{B}$

3 Frank's order for hardware items is given by the matrix $\mathbf{H} = \begin{pmatrix} 6 \\ 12 \\ 60 \\ 30 \end{pmatrix} \begin{array}{l} \text{hammers} \\ \text{screwdriver sets} \\ \text{packets of nails} \\ \text{packets of screws} \end{array}$

Find the order matrix if Frank:

a doubles his order **b** halves his order **c** increases his order by 50%.

4 Isabelle sells clothing made by four different companies which we will call A, B, C and D. Her usual monthly order is:

$$\begin{array}{l} \\ \text{skirt} \\ \text{dress} \\ \text{evening} \\ \text{suit} \end{array} \begin{array}{c} \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\ \begin{pmatrix} 30 & 40 & 40 & 60 \\ 50 & 40 & 30 & 75 \\ 40 & 40 & 50 & 50 \\ 10 & 20 & 20 & 15 \end{pmatrix} \end{array}$$

Find her order, to the nearest whole number, if:

a she increases her total order by 15%

b she decreases her total order by 15%.

5 During weekdays a video store finds that its average hirings are: 75 DVD movies, 27 VHS movies, and 102 games. On the weekends the average figures are: 43 VHS movies, 136 DVD movies, and 129 games.

a Represent the data using *two* column matrices.

b Find the sum of the matrices in **a**.

c What does the sum matrix in **b** represent?

$$\begin{pmatrix} \\ \\ \end{pmatrix} \begin{array}{l} \leftarrow \text{DVD} \\ \leftarrow \text{VHS} \\ \leftarrow \text{games} \end{array}$$

6 A builder builds a block of 12 identical flats. Each flat is to contain 1 table, 4 chairs, 2 beds, and 1 wardrobe.

Suppose $\mathbf{F} = \begin{pmatrix} 1 \\ 4 \\ 2 \\ 1 \end{pmatrix}$ is the matrix representing the furniture in one flat. State, in terms of $\mathbf{F}$, the matrix representing the furniture in **all** flats.

ZERO MATRIX

For real numbers, it is true that $a + 0 = 0 + a = a$ for all values of a.

So, is there a matrix $\mathbf{O}$ such that $\mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$ for any matrix $\mathbf{A}$?

Simple examples like: $\begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$ suggest that $\mathbf{O}$ consists of all zeros.

A **zero matrix** is a matrix in which all elements are zero.

For example, the 2×2 zero matrix is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; the 2×3 zero matrix is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Zero matrices have the property that:

If $\mathbf{A}$ is a matrix of any order and $\mathbf{O}$ is the corresponding **zero matrix**, then $\mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$.

NEGATIVE MATRICES

The **negative** matrix $\mathbf{A}$, denoted $-\mathbf{A}$, is actually $-1\mathbf{A}$.

So, if $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}$, then $-\mathbf{A} = \begin{pmatrix} -1 \times 3 & -1 \times -1 \\ -1 \times 2 & -1 \times 4 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -2 & -4 \end{pmatrix}$.

Thus $-\mathbf{A}$ is obtained from $\mathbf{A}$ by reversing the sign of each element of $\mathbf{A}$.

The addition of a matrix and its negative always produces a zero matrix. For example:

$$\begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} -3 & 1 \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Thus, in general, $\mathbf{A} + (-\mathbf{A}) = (-\mathbf{A}) + \mathbf{A} = \mathbf{O}$.

MATRIX ALGEBRA

Compare our discoveries about matrices so far with ordinary algebra.

We will assume that $\mathbf{A}$ and $\mathbf{B}$ are matrices of the same order.

Ordinary algebra	Matrix algebra
• If a and b are real numbers then $a + b$ is also a real number.	• If $\mathbf{A}$ and $\mathbf{B}$ are matrices then $\mathbf{A} + \mathbf{B}$ is a matrix of the same order.
• $a + b = b + a$	• $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
• $(a + b) + c = a + (b + c)$	• $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
• $a + 0 = 0 + a = a$	• $\mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$
• $a + (-a) = (-a) + a = 0$	• $\mathbf{A} + (-\mathbf{A}) = (-\mathbf{A}) + \mathbf{A} = \mathbf{O}$
• a half of a is $\dfrac{a}{2}$	• a half of $\mathbf{A}$ is $\frac{1}{2}\mathbf{A}$ $\left(\text{not } \dfrac{\mathbf{A}}{2}\right)$

Example 5 **Self Tutor**

Explain why it is true that:

a if $\mathbf{X}+\mathbf{A}=\mathbf{B}$ then $\mathbf{X}=\mathbf{B}-\mathbf{A}$ **b** if $3\mathbf{X}=\mathbf{A}$ then $\mathbf{X}=\frac{1}{3}\mathbf{A}$.

a

$$\begin{aligned} \text{If} \quad & \mathbf{X}+\mathbf{A}=\mathbf{B} \\ \text{then} \quad & \mathbf{X}+\mathbf{A}+(-\mathbf{A})=\mathbf{B}+(-\mathbf{A}) \\ \therefore \quad & \mathbf{X}+\mathbf{O}=\mathbf{B}-\mathbf{A} \\ \therefore \quad & \mathbf{X}=\mathbf{B}-\mathbf{A} \end{aligned}$$

b

$$\begin{aligned} \text{If} \quad & 3\mathbf{X}=\mathbf{A} \\ \text{then} \quad & \tfrac{1}{3}(3\mathbf{X})=\tfrac{1}{3}\mathbf{A} \\ \therefore \quad & 1\mathbf{X}=\tfrac{1}{3}\mathbf{A} \\ \therefore \quad & \mathbf{X}=\tfrac{1}{3}\mathbf{A} \end{aligned}$$

EXERCISE 11B.3

1 Simplify:

a $\mathbf{A}+2\mathbf{A}$ **b** $3\mathbf{B}-3\mathbf{B}$ **c** $\mathbf{C}-2\mathbf{C}$

d $-\mathbf{B}+\mathbf{B}$ **e** $2(\mathbf{A}+\mathbf{B})$ **f** $-(\mathbf{A}+\mathbf{B})$

g $-(2\mathbf{A}-\mathbf{C})$ **h** $3\mathbf{A}-(\mathbf{B}-\mathbf{A})$ **i** $\mathbf{A}+2\mathbf{B}-(\mathbf{A}-\mathbf{B})$

2 Find **X** in terms of **A**, **B** and **C** if:

a $\mathbf{X}+\mathbf{B}=\mathbf{A}$ **b** $\mathbf{B}+\mathbf{X}=\mathbf{C}$ **c** $4\mathbf{B}+\mathbf{X}=2\mathbf{C}$

d $2\mathbf{X}=\mathbf{A}$ **e** $3\mathbf{X}=\mathbf{B}$ **f** $\mathbf{A}-\mathbf{X}=\mathbf{B}$

g $\frac{1}{2}\mathbf{X}=\mathbf{C}$ **h** $2(\mathbf{X}+\mathbf{A})=\mathbf{B}$ **i** $\mathbf{A}-4\mathbf{X}=\mathbf{C}$

3 **a** Find $\mathbf{X}$ if $\frac{1}{3}\mathbf{X}=\mathbf{M}$ and $\mathbf{M}=\begin{pmatrix}1 & 2\\ 3 & 6\end{pmatrix}$.

b Find $\mathbf{X}$ if $4\mathbf{X}=\mathbf{N}$ and $\mathbf{N}=\begin{pmatrix}2 & -1\\ 3 & 5\end{pmatrix}$.

c Find $\mathbf{X}$ if $\mathbf{A}-2\mathbf{X}=3\mathbf{B}$, $\mathbf{A}=\begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}1 & 4\\ -1 & 1\end{pmatrix}$.

MATRIX MULTIPLICATION

Suppose you go to a shop and purchase 3 soft drink cans, 4 chocolate bars, and 2 icecreams. We can represent this by the quantities matrix $\mathbf{A}=\begin{pmatrix}3 & 4 & 2\end{pmatrix}$.

If the prices are:

soft drink cans	chocolate bars	ice creams
\$1.30	\$0.90	\$1.20

then we can represent these using the costs matrix $\mathbf{B}=\begin{pmatrix}1.30\\ 0.90\\ 1.20\end{pmatrix}$.

We can find the total cost of the items by multiplying the number of each item by its respective cost, and then adding the results. The total cost is thus

$$3\times\$1.30+4\times\$0.90+2\times\$1.20=\$9.90$$

We can also determine the total cost by the **matrix multiplication**:

$$\begin{aligned}\mathbf{AB} &= \begin{pmatrix} 3 & 4 & 2 \end{pmatrix}\begin{pmatrix} 1.30 \\ 0.90 \\ 1.20 \end{pmatrix} \\ &= (3 \times 1.30) + (4 \times 0.90) + (2 \times 1.20) \\ &= 9.90\end{aligned}$$

Notice that we write the **row matrix** first and the **column matrix** second.

In general, $\begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = ap + bq + cr.$

EXERCISE 11B.4

1 Determine:

- **a** $\begin{pmatrix} 3 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix}$
- **b** $\begin{pmatrix} 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \\ 7 \end{pmatrix}$
- **c** $\begin{pmatrix} 6 & -1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 4 \end{pmatrix}$

2 Show that the *sum* of w, x, y and z is given by $\begin{pmatrix} w & x & y & z \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

Represent the *average* of w, x, y and z in the same way.

3 Lucy buys 4 shirts, 3 skirts, and 2 blouses costing \$27, \$35, and \$39 respectively.

- **a** Write down a quantities matrix **Q** and a price matrix **P**.
- **b** Show how to use **P** and **Q** to determine the total cost.

4 In the interschool public speaking competition a first place is awarded 10 points, second place 6 points, third place 3 points, and fourth place 1 point. One school won 3 first places, 2 seconds, 4 thirds, and 2 fourths.

- **a** Write down this information in terms of points matrix **P** and a numbers matrix **N**.
- **b** Use **P** and **N** to find the total number of points awarded to the school.

MORE COMPLICATED MULTIPLICATIONS

Consider again **Example 1** on page **275** where Lisa needed 2 loaves of bread, 3 litres of milk, and 1 tub of butter.

We represented this by the quantities matrix $\mathbf{Q} = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$.

The prices for each store were summarised in the costs matrix $\mathbf{C} = \begin{pmatrix} 2.65 & 2.25 \\ 1.55 & 1.50 \\ 2.35 & 2.20 \end{pmatrix}$.

To find the *total cost* of the items in each store, Lisa needs to multiply the number of items by their respective cost.

In Store A a loaf of bread is \$2.65, a litre of milk is \$1.55 and a tub of butter is \$2.35, so the total cost is $2 \times \$2.65 + 3 \times \$1.55 + 1 \times \$2.35 = \$12.30$.

In Store B a loaf of bread is \$2.25, a litre of milk is \$1.50 and a tub of butter is \$2.20, so the total cost is $2 \times \$2.25 + 3 \times \$1.50 + 1 \times \$2.20 = \$11.20$.

To do this using matrices, notice that:

$$\mathbf{QC} = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix} \times \begin{pmatrix} 2.65 & 2.25 \\ 1.55 & 1.50 \\ 2.35 & 2.20 \end{pmatrix} = \begin{pmatrix} 12.30 & 11.20 \end{pmatrix}$$

orders: 1×3 ← the same → 3×2 $\qquad 1 \times 2$

resultant matrix

Now suppose Lisa's friend Olu needs 1 loaf of bread, 2 litres of milk, and 2 tubs of butter.

The quantities matrix for both Lisa and Olu would be

$$\begin{array}{cc} \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} & \begin{matrix} \leftarrow \text{Lisa} \\ \leftarrow \text{Olu} \end{matrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ \text{bread} & \text{milk} & \text{butter} \end{matrix} & \end{array}$$

Lisa's *total cost* at Store A is \$12.30 and at store B is \$11.20.

Olu's *total cost* at Store A is $1 \times \$2.65 + 2 \times \$1.55 + 2 \times \$2.35 = \$10.45$

and at Store B is $1 \times \$2.25 + 2 \times \$1.50 + 2 \times \$2.20 = \$9.65$.

So, using matrices we require that

$$\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} \times \begin{pmatrix} 2.65 & 2.25 \\ 1.55 & 1.50 \\ 2.35 & 2.20 \end{pmatrix} = \begin{pmatrix} \overset{\text{row 1} \times \text{column 1}}{\overset{\downarrow}{12.30}} & \overset{\text{row 1} \times \text{column 2}}{\overset{\swarrow}{11.20}} \\ \underset{\text{row 2} \times \text{column 1}}{\underset{\uparrow}{10.45}} & \underset{\text{row 2} \times \text{column 2}}{\underset{\nwarrow}{9.65}} \end{pmatrix}$$

2×3 ← the same → 3×2 $\qquad 2 \times 2$

resultant matrix

Having observed the usefulness of multiplying matrices in the examples above, we are now in a position to define matrix multiplication more formally.

The **product** of an $m \times n$ matrix $\mathbf{A}$ with an $n \times p$ matrix $\mathbf{B}$, is the $m \times p$ matrix $\mathbf{AB}$ in which the element in the rth row and cth column is the sum of the products of the elements in the rth row of $\mathbf{A}$ with the corresponding elements in the cth column of $\mathbf{B}$.

The product $\mathbf{AB}$ exists *only* if the number of columns of $\mathbf{A}$ equals the number of rows of $\mathbf{B}$.

For example:

If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, then $\mathbf{AB} = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}$.

If $\mathbf{C} = \underset{2\times 3}{\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}}$ and $\mathbf{D} = \underset{3\times 1}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}$, then $\mathbf{CD} = \underset{2\times 1}{\begin{pmatrix} ax+by+cz \\ dx+ey+fz \end{pmatrix}}$.

To get the matrix **AB** you multiply **rows by columns**. To get the element in the 5th row and 3rd column of **AB** (if it exists), multiply the 5th row of **A** by the 3rd column of **B**.

Example 6 **Self Tutor**

If $\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 1 & 3 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 4 \end{pmatrix}$ find:

a **AC**

b **BC**

a **A** is 1×3 and **C** is 3×2 $\quad\therefore$ **AC** is 1×2

✓

$$\mathbf{AC} = \begin{pmatrix} 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times 1 + 3\times 2 + 5\times 1 & 1\times 0 + 3\times 3 + 5\times 4 \end{pmatrix}$$

$$= \begin{pmatrix} 12 & 29 \end{pmatrix}$$

b **B** is 2×3 and **C** is 3×2 $\quad\therefore$ **BC** is 2×2

✓

$$\mathbf{BC} = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times 1 + 3\times 2 + 5\times 1 & 1\times 0 + 3\times 3 + 5\times 4 \\ 2\times 1 + 1\times 2 + 3\times 1 & 2\times 0 + 1\times 3 + 3\times 4 \end{pmatrix}$$

$$= \begin{pmatrix} 12 & 29 \\ 7 & 15 \end{pmatrix}$$

EXERCISE 11B.5

1 Explain why **AB** cannot be found for $\mathbf{A} = \begin{pmatrix} 4 & 2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

2 Suppose **A** is $2 \times n$ and **B** is $m \times 3$.

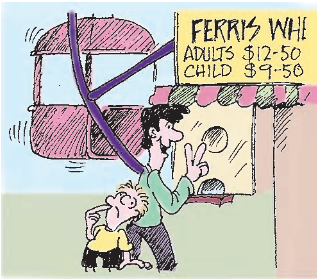
- **a** When can we find **AB**?
- **b** If **AB** can be found, what is its order?
- **c** Why can **BA** never be found?

3 **a** For $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 6 \end{pmatrix}$ find **i** **AB** **ii** **BA**.

b For $\mathbf{A} = \begin{pmatrix} 2 & 0 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$ find **i** **AB** **ii** **BA**.

4 Find: **a** $\begin{pmatrix}1 & 2 & 1\end{pmatrix}\begin{pmatrix}2 & 3 & 1\\0 & 1 & 0\\1 & 0 & 2\end{pmatrix}$ **b** $\begin{pmatrix}1 & 0 & -1\\-1 & 1 & 0\\0 & -1 & 1\end{pmatrix}\begin{pmatrix}2\\3\\4\end{pmatrix}$

5



At a fair, tickets for the Ferris wheel are $12.50 per adult and $9.50 per child. On the first day of the fair, 2375 adults and 5156 children ride this wheel. On the second day, the figures are 2502 adults and 3612 children.

a Write the costs matrix **C** as a 2 × 1 matrix and the numbers matrix **N** as a 2 × 2 matrix.

b Find **NC** and interpret the resulting matrix.

c Find the total income for the two days.

6 You and your friend each go to your local hardware stores A and B to price items you wish to purchase. You want to buy 1 hammer, 1 screwdriver, and 2 cans of white paint. Your friend wants 1 hammer, 2 screwdrivers, and 3 cans of white paint. The prices of these goods are:

	Hammer	*Screwdriver*	*Can of paint*
Store A	€7	€3	€19
Store B	€6	€2	€22

a Write the requirements matrix **R** as a 3 × 2 matrix.

b Write the prices matrix **P** as a 2 × 3 matrix.

c Find **PR**.

d What are your costs at store A and your friend's costs at store B?

e Should you buy from store A or store B?

USING TECHNOLOGY FOR MATRIX OPERATIONS

Instructions for performing matrix operations can be found in the graphics calculator chapter at the start of the book.

MATRIX OPERATIONS

Alternatively, click on the **Matrix Operations** icon to obtain computer software for these tasks.

EXERCISE 11B.6

1 Use technology to find:

a $\begin{pmatrix}13 & 12 & 4\\11 & 12 & 8\\7 & 9 & 7\end{pmatrix}+\begin{pmatrix}3 & 6 & 11\\2 & 9 & 8\\3 & 13 & 17\end{pmatrix}$

b $\begin{pmatrix}13 & 12 & 4\\11 & 12 & 8\\7 & 9 & 7\end{pmatrix}-\begin{pmatrix}3 & 6 & 11\\2 & 9 & 8\\3 & 13 & 17\end{pmatrix}$

c $22\begin{pmatrix}1 & 0 & 6 & 8 & 9\\2 & 7 & 4 & 5 & 0\\8 & 2 & 4 & 4 & 6\end{pmatrix}$

d $\begin{pmatrix}2 & 6 & 0 & 7\\3 & 2 & 8 & 6\\1 & 4 & 0 & 2\\3 & 0 & 1 & 8\end{pmatrix}\begin{pmatrix}4\\5\\6\\11\end{pmatrix}$

Use technology to assist in solving the following problems:

2 For their holiday, Lars and Simke are planning to spend time at a popular tourist resort. They will need accommodation at one of the local motels and they are not certain how long they will stay. Their initial planning is for three nights and includes three breakfasts and two dinners. They have gathered prices from three different motels.

The Bay View has rooms at \$125 per night. A full breakfast costs \$22 per person, and therefore \$44 for them both. An evening meal for two usually costs \$75 including drinks.

By contrast, 'The Terrace' has rooms at \$150 per night, breakfast at \$40 per double, and dinner costs on average \$80.

Things seem to be a little better at the Staunton Star Motel. Accommodation is \$140 per night, full breakfast for two is \$40, while an evening meal for two usually costs \$65.

- **a** Write down a 'numbers' matrix as a 1×3 row matrix.
- **b** Write down a 'prices' matrix in 3×3 form.
- **c** Use matrix multiplication to establish total prices for each venue.
- **d** Instead of the couple staying three nights, the alternative is to spend just two nights. In that event Lars and Simke decide on having breakfast just once and one evening meal before moving on. Recalculate prices for each venue.
- **e** Now remake the 'numbers' matrix (2×3) so that it includes both scenarios. Calculate the product with the 'prices' matrix to check your answers to **c** and **d**.

3 A bus company runs four tours. Tour A costs \$125, Tour B costs \$315, Tour C costs \$405, and Tour D costs \$375. The numbers of clients they had over the summer period are shown in the table below.

$$\begin{array}{rc} & \begin{array}{cccc}\text{Tour A} & \text{Tour B} & \text{Tour C} & \text{Tour D}\end{array} \\ \begin{array}{r}\text{November}\\ \text{December}\\ \text{January}\\ \text{February}\end{array} & \begin{pmatrix} 50 & 42 & 18 & 65 \\ 65 & 37 & 25 & 82 \\ 120 & 29 & 23 & 75 \\ 42 & 36 & 19 & 72 \end{pmatrix}\end{array}$$

Use the information and matrix methods to find the total income for the tour company.

4 The Oregon Motel has three types of suites for guests.

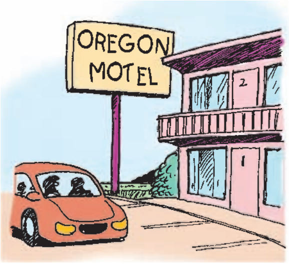
Standard suites cost \$125 per night. They have 20 suites.
Deluxe suites cost \$195 per night. They have 15 suites.
Executive suites cost \$225 per night. They have 5 suites.

The rooms which are occupied also have a maintenance cost:

Standard suites cost \$85 per day to maintain.
Deluxe suites cost \$120 per day to maintain.
Executive suites cost \$130 per day to maintain.

The hotel has confirmed room bookings for the next week:

$$\begin{array}{rc} & \begin{array}{ccccccc}\text{M} & \text{T} & \text{W} & \text{Th} & \text{F} & \text{S} & \text{Su}\end{array} \\ \begin{array}{r}\text{Standard}\\ \text{Deluxe}\\ \text{Executive}\end{array} & \begin{pmatrix} 15 & 12 & 13 & 11 & 14 & 16 & 8 \\ 4 & 3 & 6 & 2 & 0 & 4 & 7 \\ 3 & 1 & 4 & 4 & 3 & 2 & 0 \end{pmatrix}\end{array}$$



a The profit per day is given by
(income from room) × (bookings per day)
− (maintenance cost per room) × (bookings per day)
Create the matrices required to show how the profit per week can be found.

b How would the results alter if the hotel maintained (cleaned) all rooms every day? Show your calculations.

c Produce a profit per room matrix and show how **a** could be done with a single matrix product.

PROPERTIES OF MATRIX MULTIPLICATION

In the following exercise we should discover the properties of 2×2 matrix multiplication which *are* like those of ordinary number multiplication, and those which *are not*.

EXERCISE 11B.7

1 For ordinary arithmetic $2 \times 3 = 3 \times 2$ and in algebra $ab = ba$.
For matrices, does $\mathbf{AB}$ always equal $\mathbf{BA}$?

Hint: Try $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 1 \\ 0 & 3 \end{pmatrix}$.

2 If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ find $\mathbf{AO}$ and $\mathbf{OA}$.

3 For all real numbers a, b and c it is true that $a(b+c) = ab + ac$. This is known as the **distributive law**.

a Use any three 2×2 matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ to verify that $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC}$.

b Now let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$.
Prove that in general, $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC}$.

c Use the matrices you 'made up' in **a** to verify that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

d Prove that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

4 **a** Suppose that $\mathbf{AX} = \mathbf{A}$, where $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$.
Show that $w = z = 1$ and $x = y = 0$ is a solution.

b For any real number a, is it true that $a \times 1 = 1 \times a = a$?
Is there a matrix $\mathbf{I}$ such that $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for all 2×2 matrices $\mathbf{A}$?

5 Suppose $\mathbf{A}^2 = \mathbf{AA}$ or $\mathbf{A} \times \mathbf{A}$ and that $\mathbf{A}^3 = \mathbf{AAA}$.

a Find $\mathbf{A}^2$ if $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix}$

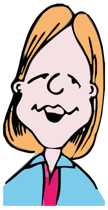
b Find $\mathbf{A}^3$ if $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 2 & 4 \end{pmatrix}$.

6 **a** If $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ try to find $\mathbf{A}^2$.

b Under what conditions can we square a matrix?

7 Show that if $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $\mathbf{I}^2 = \mathbf{I}$ and $\mathbf{I}^3 = \mathbf{I}$.

$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the **identity matrix**.



You should have discovered from the above exercise that:

Ordinary algebra	Matrix algebra
• If a and b are real numbers then so is ab.	• If $\mathbf{A}$ and $\mathbf{B}$ are matrices that can be multiplied then $\mathbf{AB}$ is also a matrix. {closure}
• $ab = ba$ for all a, b	• In general $\mathbf{AB} \neq \mathbf{BA}$. {non-commutative}
• $a0 = 0a = 0$ for all a	• If $\mathbf{O}$ is a zero matrix then $\mathbf{AO} = \mathbf{OA} = \mathbf{O}$ for all $\mathbf{A}$.
• $a(b+c) = ab + ac$	• $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ {distributive law}
• $a \times 1 = 1 \times a = a$	• If $\mathbf{I}$ is the **identity matrix** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for all 2×2 matrices $\mathbf{A}$. {identity law}
• a^n exists for all $a \geqslant 0$ and $n \in \mathbb{R}$.	• $\mathbf{A}^n$ exists provided $\mathbf{A}$ is square and $n \in \mathbb{Z}^+$.

Notice that in general, $\mathbf{A}(k\mathbf{B}) = k(\mathbf{AB}) \neq k\mathbf{BA}$. We can change the order in which we multiply by a scalar, but in general we cannot reverse the order in which we multiply matrices.

Example 7 **Self Tutor**

Expand and simplify where possible: **a** $(\mathbf{A} + 2\mathbf{I})^2$ **b** $(\mathbf{A} - \mathbf{B})^2$

a $(\mathbf{A} + 2\mathbf{I})^2$

$= (\mathbf{A} + 2\mathbf{I})(\mathbf{A} + 2\mathbf{I})$ {$\mathbf{X}^2 = \mathbf{XX}$ by definition}

$= (\mathbf{A} + 2\mathbf{I})\mathbf{A} + (\mathbf{A} + 2\mathbf{I})2\mathbf{I}$ {$\mathbf{B}(\mathbf{C} + \mathbf{D}) = \mathbf{BC} + \mathbf{BD}$}

$= \mathbf{A}^2 + 2\mathbf{IA} + 2\mathbf{AI} + 4\mathbf{I}^2$ {$\mathbf{B}(\mathbf{C} + \mathbf{D}) = \mathbf{BC} + \mathbf{BD}$ again, twice}

$= \mathbf{A}^2 + 2\mathbf{A} + 2\mathbf{A} + 4\mathbf{I}$ {$\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ and $\mathbf{I}^2 = \mathbf{I}$}

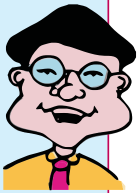
$= \mathbf{A}^2 + 4\mathbf{A} + 4\mathbf{I}$

b $(\mathbf{A} - \mathbf{B})^2$

$= (\mathbf{A} - \mathbf{B})(\mathbf{A} - \mathbf{B})$ {$\mathbf{X}^2 = \mathbf{XX}$ by definition}

$= (\mathbf{A} - \mathbf{B})\mathbf{A} - (\mathbf{A} - \mathbf{B})\mathbf{B}$ {$\mathbf{C}(\mathbf{D} - \mathbf{E}) = \mathbf{CD} - \mathbf{CE}$ twice}

$= \mathbf{A}^2 - \mathbf{BA} - \mathbf{AB} + \mathbf{B}^2$



b cannot be simplified further since, in general, $\mathbf{AB} \neq \mathbf{BA}$.

Example 8 Self Tutor

If $\mathbf{A}^2 = 2\mathbf{A} + 3\mathbf{I}$, find $\mathbf{A}^3$ and $\mathbf{A}^4$ in the form $k\mathbf{A} + l\mathbf{I}$ where k and l are scalars.

$$\begin{aligned}\mathbf{A}^3 &= \mathbf{A} \times \mathbf{A}^2 \\ &= \mathbf{A}(2\mathbf{A} + 3\mathbf{I}) \\ &= 2\mathbf{A}^2 + 3\mathbf{AI} \\ &= 2(2\mathbf{A} + 3\mathbf{I}) + 3\mathbf{AI} \\ &= 7\mathbf{A} + 6\mathbf{I}\end{aligned}$$

$$\begin{aligned}\mathbf{A}^4 &= \mathbf{A} \times \mathbf{A}^3 \\ &= \mathbf{A}(7\mathbf{A} + 6\mathbf{I}) \\ &= 7\mathbf{A}^2 + 6\mathbf{AI} \\ &= 7(2\mathbf{A} + 3\mathbf{I}) + 6\mathbf{A} \\ &= 20\mathbf{A} + 21\mathbf{I}\end{aligned}$$

Example 9 Self Tutor

Find constants a and b such that $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$, for $\mathbf{A}$ equal to $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Since $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = a\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1+6 & 2+8 \\ 3+12 & 6+16 \end{pmatrix} = \begin{pmatrix} a & 2a \\ 3a & 4a \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$$

$$\therefore \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} = \begin{pmatrix} a+b & 2a \\ 3a & 4a+b \end{pmatrix}$$

Thus $a + b = 7$ and $2a = 10$

$\therefore$ $a = 5$ and $b = 2$

Checking for consistency: $3a = 3(5) = 15$ ✓ $4a + b = 4(5) + (2) = 22$ ✓

EXERCISE 11B.8

1 Given that all matrices are 2×2 and $\mathbf{I}$ is the identity matrix, expand and simplify:

a $\mathbf{A}(\mathbf{A} + \mathbf{I})$ **b** $(\mathbf{B} + 2\mathbf{I})\mathbf{B}$ **c** $\mathbf{A}(\mathbf{A}^2 - 2\mathbf{A} + \mathbf{I})$

d $\mathbf{A}(\mathbf{A}^2 + \mathbf{A} - 2\mathbf{I})$ **e** $(\mathbf{A} + \mathbf{B})(\mathbf{C} + \mathbf{D})$ **f** $(\mathbf{A} + \mathbf{B})^2$

g $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B})$ **h** $(\mathbf{A} + \mathbf{I})^2$ **i** $(3\mathbf{I} - \mathbf{B})^2$

2 **a** If $\mathbf{A}^2 = 2\mathbf{A} - \mathbf{I}$, find $\mathbf{A}^3$ and $\mathbf{A}^4$ in the linear form $k\mathbf{A} + l\mathbf{I}$ where k and l are scalars.

b If $\mathbf{B}^2 = 2\mathbf{I} - \mathbf{B}$, find $\mathbf{B}^3$, $\mathbf{B}^4$ and $\mathbf{B}^5$ in linear form.

c If $\mathbf{C}^2 = 4\mathbf{C} - 3\mathbf{I}$, find $\mathbf{C}^3$ and $\mathbf{C}^5$ in linear form.

3 **a** If $\mathbf{A}^2 = \mathbf{I}$, simplify:

i $\mathbf{A}(\mathbf{A} + 2\mathbf{I})$ **ii** $(\mathbf{A} - \mathbf{I})^2$ **iii** $\mathbf{A}(\mathbf{A} + 3\mathbf{I})^2$

b If $\mathbf{A}^3 = \mathbf{I}$, simplify $\mathbf{A}^2(\mathbf{A} + \mathbf{I})^2$.

c If $\mathbf{A}^2 = \mathbf{O}$, simplify:

i $\mathbf{A}(2\mathbf{A} - 3\mathbf{I})$ **ii** $\mathbf{A}(\mathbf{A} + 2\mathbf{I})(\mathbf{A} - \mathbf{I})$ **iii** $\mathbf{A}(\mathbf{A} + \mathbf{I})^3$

4 The Null Factor law "if $ab = 0$ then $a = 0$ or $b = 0$" for real numbers does not have an equivalent result for matrices.

a If $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ find $\mathbf{AB}$.

This example provides us with evidence that
"If $\mathbf{AB} = \mathbf{O}$ then $\mathbf{A} = \mathbf{O}$ or $\mathbf{B} = \mathbf{O}$" is a false statement.

b If $\mathbf{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ determine $\mathbf{A}^2$.

c Comment on the following argument for a 2 × 2 matrix $\mathbf{A}$:

It is known that $\mathbf{A}^2 = \mathbf{A}$, so $\mathbf{A}^2 - \mathbf{A} = \mathbf{O}$

$$\therefore \quad \mathbf{A}(\mathbf{A} - \mathbf{I}) = \mathbf{O}$$
$$\therefore \quad \mathbf{A} = \mathbf{O} \text{ or } \mathbf{A} - \mathbf{I} = \mathbf{O}$$
$$\therefore \quad \mathbf{A} = \mathbf{O} \text{ or } \mathbf{I}$$

d Find **all** 2 × 2 matrices $\mathbf{A}$ for which $\mathbf{A}^2 = \mathbf{A}$. **Hint:** Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

5 Give **one** example which shows that the statement "if $\mathbf{A}^2 = \mathbf{O}$ then $\mathbf{A} = \mathbf{O}$" is *false*.

6 Find constants a and b such that $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$ for $\mathbf{A}$ equal to:

a $\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & 1 \\ 2 & -2 \end{pmatrix}$

7 If $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & -3 \end{pmatrix}$, find constants p and q such that $\mathbf{A}^2 = p\mathbf{A} + q\mathbf{I}$.

a Hence, write $\mathbf{A}^3$ in the linear form $r\mathbf{A} + s\mathbf{I}$ where r and s are scalars.

b Write $\mathbf{A}^4$ in linear form.

C THE INVERSE OF A 2 × 2 MATRIX

We can solve $\begin{cases} 2x + 3y = 4 \\ 5x + 4y = 17 \end{cases}$ algebraically to get $x = 5$, $y = -2$.

This system can be written as the matrix equation $\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 17 \end{pmatrix}$.

The solution $x = 5$, $y = -2$ is easily checked as

$$\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 2(5) + 3(-2) \\ 5(5) + 4(-2) \end{pmatrix} = \begin{pmatrix} 4 \\ 17 \end{pmatrix} \quad \checkmark$$

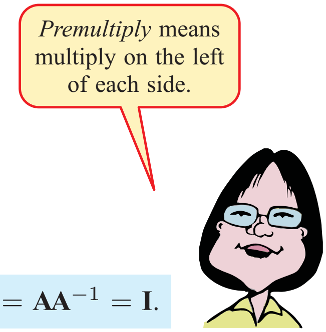
In general, a system of linear equations can be written in the form $\mathbf{AX} = \mathbf{B}$ where $\mathbf{A}$ is the matrix of coefficients, $\mathbf{X}$ is the unknown column matrix, and $\mathbf{B}$ is the column matrix of constants.

The question arises: If $\mathbf{AX} = \mathbf{B}$, how can we find $\mathbf{X}$ using matrices?

To answer this question, suppose there exists a matrix $\mathbf{C}$ such that $\mathbf{CA} = \mathbf{I}$.

If we *premultiply* each side of $\mathbf{AX} = \mathbf{B}$ by $\mathbf{C}$ we get

$$\begin{aligned} \mathbf{C}(\mathbf{AX}) &= \mathbf{CB} \\ \therefore \quad (\mathbf{CA})\mathbf{X} &= \mathbf{CB} \\ \therefore \quad \mathbf{IX} &= \mathbf{CB} \\ \text{and so} \quad \mathbf{X} &= \mathbf{CB} \end{aligned}$$



If $\mathbf{C}$ exists such that $\mathbf{CA} = \mathbf{I}$ then $\mathbf{C}$ is said to be the **multiplicative inverse** of $\mathbf{A}$, and we denote $\mathbf{C}$ by $\mathbf{A}^{-1}$.

The **multiplicative inverse** of $\mathbf{A}$, denoted $\mathbf{A}^{-1}$, satisfies $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

Suppose $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{A}^{-1} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$

so $\mathbf{AA}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \mathbf{I}$

$$\therefore \quad \begin{pmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \quad \begin{cases} aw+by=1 & \ldots\ldots (1) \\ cw+dy=0 & \ldots\ldots (2) \end{cases} \quad \text{and} \quad \begin{cases} ax+bz=0 & \ldots\ldots (3) \\ cx+dz=1 & \ldots\ldots (4) \end{cases}$$

Solving (1) and (2) simultaneously for w and y gives: $w = \dfrac{d}{ad-bc}$ and $y = \dfrac{-c}{ad-bc}$.

Solving (3) and (4) simultaneously for x and z gives: $x = \dfrac{-b}{ad-bc}$ and $z = \dfrac{a}{ad-bc}$.

So, if $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\mathbf{A}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

If $ad - bc \neq 0$ then $\mathbf{A}^{-1}$ exists and we say that $\mathbf{A}$ is **invertible** or **non-singular**.

If $ad - bc = 0$ then $\mathbf{A}^{-1}$ does not exist and we say that $\mathbf{A}$ is **singular**.

If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the value $ad - bc$ is called the **determinant** of $\mathbf{A}$, denoted $|\mathbf{A}|$ or $\det\mathbf{A}$.

If $|\mathbf{A}| = 0$ then $\mathbf{A}^{-1}$ does not exist and $\mathbf{A}$ is **singular**.

If $|\mathbf{A}| \neq 0$ then $\mathbf{A}$ is **invertible** and $\mathbf{A}^{-1} = \dfrac{1}{|\mathbf{A}|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

EXERCISE 11C.1

1 **a** Find $\begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & -6 \\ -2 & 5 \end{pmatrix}$ and hence find the inverse of $\begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix}$.

b Find $\begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}$ and hence find the inverse of $\begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix}$.

2 Find $|\mathbf{A}|$ for $\mathbf{A}$ equal to:

a $\begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix}$ **b** $\begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$ **c** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ **d** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

3 Find $\det\mathbf{B}$ for $\mathbf{B}$ equal to:

a $\begin{pmatrix} 3 & -2 \\ 7 & 4 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ **c** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **d** $\begin{pmatrix} a & -a \\ 1 & a \end{pmatrix}$

4 For $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & -1 \end{pmatrix}$ find: **a** $|\mathbf{A}|$ **b** $|\mathbf{A}|^2$ **c** $|2\mathbf{A}|$

5 Prove that if $\mathbf{A}$ is any 2×2 matrix and k is a constant, then $|k\mathbf{A}| = k^2\,|\mathbf{A}|$.

6 Suppose $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$.

a Find $|\mathbf{A}|$ and $|\mathbf{B}|$. **b** Find $\mathbf{AB}$ and $|\mathbf{AB}|$.

c Hence show that $|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|$ for all 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$.

7 $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$.

a Using the results of **5** and **6** above and the calculated values of $|\mathbf{A}|$ and $|\mathbf{B}|$, find:

i $|\mathbf{A}|$ **ii** $|2\mathbf{A}|$ **iii** $|-\mathbf{A}|$ **iv** $|-3\mathbf{B}|$ **v** $|\mathbf{AB}|$

b Check your answers without using the results of **5** and **6** above.

8 Find, if it exists, the inverse matrix of:

a $\begin{pmatrix} 2 & 4 \\ -1 & 5 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ **c** $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$ **d** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

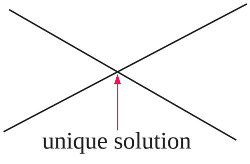
e $\begin{pmatrix} 3 & 5 \\ -6 & -10 \end{pmatrix}$ **f** $\begin{pmatrix} -1 & 2 \\ 4 & 7 \end{pmatrix}$ **g** $\begin{pmatrix} 3 & 4 \\ -1 & 2 \end{pmatrix}$ **h** $\begin{pmatrix} -1 & -1 \\ 2 & 3 \end{pmatrix}$

SOLVING A PAIR OF LINEAR EQUATIONS

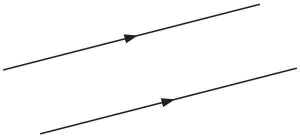
We have already seen how a system of linear equations can be written in matrix form. We can solve the system using an inverse matrix if such an inverse exists.

If two lines are **parallel** then the resulting matrix will be singular and no inverse exists. This indicates that either the lines are **coincident** and there are infinitely many solutions, or the lines never meet and there are no solutions.

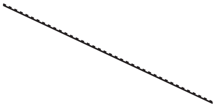
If the lines are not parallel then the resulting matrix will be invertible. We premultiply by the inverse to find the unique solution which is the point of intersection.



unique solution



never meet
(no solution)



coincident
(infinitely many solutions)

Example 10 **Self Tutor**

a If $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ find $|\mathbf{A}|$. **b** Does $\begin{cases} 2x + y = 4 \\ 3x + 4y = -1 \end{cases}$ have a unique solution?

a $|\mathbf{A}| = 2(4) - 1(3)$
$= 8 - 3$
$= 5$

b The system in matrix form is: $\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

Since $|\mathbf{A}| = 5 \neq 0$, $\mathbf{A}^{-1}$ exists and there is a unique solution.

Example 11 **Self Tutor**

If $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$, find $\mathbf{A}^{-1}$ and hence solve $\begin{cases} 2x + 3y = 4 \\ 5x + 4y = 17. \end{cases}$

In the matrix form $\mathbf{AX} = \mathbf{B}$, the system is:

$$\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 17 \end{pmatrix} \quad \text{where} \quad |\mathbf{A}| = 8 - 15 = -7$$

Now $\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}$

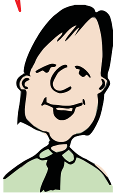
$$\therefore \quad \mathbf{IX} = \tfrac{1}{-7}\begin{pmatrix} 4 & -3 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 17 \end{pmatrix}$$

$$\therefore \quad \mathbf{X} = \tfrac{1}{-7}\begin{pmatrix} -35 \\ 14 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

and so $x = 5$, $y = -2$

Check: $2(5) + 3(-2) = 10 - 6 = 4$
$5(5) + 4(-2) = 25 - 8 = 17$

We **premultiply** with the inverse matrix on both sides.



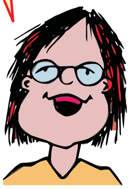
Example 12 **Self Tutor**

If $\mathbf{A} = \begin{pmatrix} 4 & k \\ 2 & -1 \end{pmatrix}$, find $\mathbf{A}^{-1}$ and state the values of k for which this inverse exists.

$$\mathbf{A}^{-1} = \frac{1}{-4 - 2k}\begin{pmatrix} -1 & -k \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{2k+4} & \dfrac{k}{2k+4} \\ \dfrac{2}{2k+4} & \dfrac{-4}{2k+4} \end{pmatrix}$$

So, $\mathbf{A}^{-1}$ exists provided that $2k + 4 \neq 0$
which is when $k \neq -2$.

If $|\mathbf{A}| = 0$, the matrix $\mathbf{A}$ is singular and is not invertible.



EXERCISE 11C.2

1 Find the following products:

a $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ **b** $\begin{pmatrix} 2 & 3 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$

2 Convert into matrix equations:

a $3x - y = 8$, $2x + 3y = 6$ **b** $4x - 3y = 11$, $3x + 2y = -5$ **c** $3a - b = 6$, $2a + 7b = -4$

3 Use matrix algebra to solve the system:

a $2x - y = 6$, $x + 3y = 14$ **b** $5x - 4y = 5$, $2x + 3y = -13$ **c** $x - 2y = 7$, $5x + 3y = -2$

d $3x + 5y = 4$, $2x - y = 11$ **e** $4x - 7y = 8$, $3x - 5y = 0$ **f** $7x + 11y = 18$, $11x - 7y = -11$

4 **a** Show that if $\mathbf{AX} = \mathbf{B}$ then $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$, whereas if $\mathbf{XA} = \mathbf{B}$ then $\mathbf{X} = \mathbf{BA}^{-1}$.

b Find $\mathbf{X}$ if:

i $\mathbf{X}\begin{pmatrix} 1 & 2 \\ 5 & -1 \end{pmatrix} = \begin{pmatrix} 14 & -5 \\ 22 & 0 \end{pmatrix}$ **ii** $\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 1 & -3 \\ 4 & 2 \end{pmatrix}$

5 For **a** $\mathbf{A} = \begin{pmatrix} k & 1 \\ -6 & 2 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 0 & k \end{pmatrix}$ **c** $\mathbf{A} = \begin{pmatrix} k+1 & 2 \\ 1 & k \end{pmatrix}$

i find the values of k for which the matrix $\mathbf{A}$ is singular

ii find $\mathbf{A}^{-1}$ when $\mathbf{A}$ is non-singular.

6 **a** Consider the system $\begin{cases} 2x - 3y = 8 \\ 4x - y = 11. \end{cases}$

i Write the equations in the form $\mathbf{AX} = \mathbf{B}$ and find $|\mathbf{A}|$.

ii Does the system have a unique solution? If so, find it.

b Consider the system $\begin{cases} 2x + ky = 8 \\ 4x - y = 11. \end{cases}$

i Write the system in the form $\mathbf{AX} = \mathbf{B}$ and find $|\mathbf{A}|$.

ii For what value(s) of k does the system have a unique solution? Find the unique solution.

iii Find k when the system does not have a unique solution. How many solutions does it have in this case?

FURTHER MATRIX ALGEBRA

The following exercise requires matrix algebra with inverse matrices. Be careful that you use multiplication correctly. In particular, remember that:

- We can only perform matrix multiplication if the orders of the matrices allow it.
- If we *pre*multiply on one side then we must *pre*multiply on the other. This is important because, in general, $\mathbf{AB} \neq \mathbf{BA}$. The same applies if we *post*multiply.

Example 13 **Self Tutor**

If $\mathbf{A}^2 = 2\mathbf{A} + 3\mathbf{I}$, find $\mathbf{A}^{-1}$ in the linear form $r\mathbf{A} + s\mathbf{I}$ where r and s are scalars.

$$\begin{aligned} \mathbf{A}^2 &= 2\mathbf{A} + 3\mathbf{I} \\ \therefore\quad \mathbf{A}^{-1}\mathbf{A}^2 &= \mathbf{A}^{-1}(2\mathbf{A} + 3\mathbf{I}) && \{\text{premultiply both sides by } \mathbf{A}^{-1}\} \\ \therefore\quad \mathbf{A}^{-1}\mathbf{A}\mathbf{A} &= 2\mathbf{A}^{-1}\mathbf{A} + 3\mathbf{A}^{-1}\mathbf{I} \\ \therefore\quad \mathbf{I}\mathbf{A} &= 2\mathbf{I} + 3\mathbf{A}^{-1} \\ \therefore\quad \mathbf{A} - 2\mathbf{I} &= 3\mathbf{A}^{-1} \\ \therefore\quad \mathbf{A}^{-1} &= \tfrac{1}{3}(\mathbf{A} - 2\mathbf{I}) && \text{So,}\quad \mathbf{A}^{-1} = \tfrac{1}{3}\mathbf{A} - \tfrac{2}{3}\mathbf{I}. \end{aligned}$$

EXERCISE 11C.3

1 Given $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$, find $\mathbf{X}$ if $\mathbf{AXB} = \mathbf{C}$.

2 If a matrix $\mathbf{A}$ is its own inverse, then $\mathbf{A} = \mathbf{A}^{-1}$.

For example, if $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ then $\mathbf{A}^{-1} = \dfrac{1}{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \mathbf{A}$.

a Show that if $\mathbf{A} = \mathbf{A}^{-1}$, then $\mathbf{A}^2 = \mathbf{I}$.

b If $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ is its own inverse, show that there are exactly 4 matrices of this form.

3 **a** If $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$ find $\mathbf{A}^{-1}$ and $(\mathbf{A}^{-1})^{-1}$.

b If $\mathbf{A}$ is any square matrix which has inverse $\mathbf{A}^{-1}$, simplify $(\mathbf{A}^{-1})^{-1}\,(\mathbf{A}^{-1})$ and $(\mathbf{A}^{-1})(\mathbf{A}^{-1})^{-1}$ by replacing $\mathbf{A}^{-1}$ by $\mathbf{B}$.

c What can be deduced from **b**?

4 **a** If $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 2 & -3 \end{pmatrix}$ find in simplest form:

i $\mathbf{A}^{-1}$ **ii** $\mathbf{B}^{-1}$ **iii** $(\mathbf{AB})^{-1}$

iv $(\mathbf{BA})^{-1}$ **v** $\mathbf{A}^{-1}\mathbf{B}^{-1}$ **vi** $\mathbf{B}^{-1}\mathbf{A}^{-1}$

b Choose any two invertible matrices and repeat question **a**.

c What do the results of **a** and **b** suggest?

d Simplify $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1})$ and $(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB})$ given that $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ exist. What conclusion follows from your results?

5 If k is a non-zero number and $\mathbf{A}^{-1}$ exists, simplify $(k\mathbf{A})(\dfrac{1}{k}\mathbf{A}^{-1})$ and $(\dfrac{1}{k}\mathbf{A}^{-1})(k\mathbf{A})$.

What conclusion follows from your results?

6 Suppose $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ are 2×1 matrices and $\mathbf{A}$, $\mathbf{B}$ are 2×2 matrices.
If $\mathbf{X} = \mathbf{AY}$ and $\mathbf{Y} = \mathbf{BZ}$ where $\mathbf{A}$ and $\mathbf{B}$ are invertible, find:

a $\mathbf{X}$ in terms of $\mathbf{Z}$ **b** $\mathbf{Z}$ in terms of $\mathbf{X}$.

7 If $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$, write $\mathbf{A}^2$ in the form $p\mathbf{A} + q\mathbf{I}$ where p and q are scalars.

Hence write $\mathbf{A}^{-1}$ in the form $r\mathbf{A} + s\mathbf{I}$ where r and s are scalars.

8 Find $\mathbf{A}^{-1}$ in linear form given that

- **a** $\mathbf{A}^2 = 4\mathbf{A} - \mathbf{I}$
- **b** $5\mathbf{A} = \mathbf{I} - \mathbf{A}^2$
- **c** $2\mathbf{I} = 3\mathbf{A}^2 - 4\mathbf{A}$

9 It is known that $\mathbf{AB} = \mathbf{A}$ and $\mathbf{BA} = \mathbf{B}$ where the matrices $\mathbf{A}$ and $\mathbf{B}$ are not necessarily invertible.
Prove that $\mathbf{A}^2 = \mathbf{A}$. (**Note:** From $\mathbf{AB} = \mathbf{A}$, you cannot deduce that $\mathbf{B} = \mathbf{I}$. Why?)

10 Under what condition is it true that "if $\mathbf{AB} = \mathbf{AC}$ then $\mathbf{B} = \mathbf{C}$"?

11 If $\mathbf{X} = \mathbf{P}^{-1}\mathbf{AP}$ and $\mathbf{A}^3 = \mathbf{I}$, prove that $\mathbf{X}^3 = \mathbf{I}$.

12 If $a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I} = \mathbf{O}$ and $\mathbf{X} = \mathbf{P}^{-1}\mathbf{AP}$, prove that $a\mathbf{X}^2 + b\mathbf{X} + c\mathbf{I} = \mathbf{O}$.

Summary: During this exercise you should have discovered that:

- $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

D 3 × 3 MATRICES

The principles of determinants and inverses are equally applicable to 3×3 and other larger square matrices.

THE DETERMINANT OF A 3 × 3 MATRIX

The **determinant** of $\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$ is defined as

$$|\mathbf{A}| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Example 14 **Self Tutor**

Find $|\mathbf{A}|$ for

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \\ 3 & -1 & 2 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \\ 3 & -1 & 2 \end{pmatrix}$$

$$\begin{aligned} |\mathbf{A}| &= 1\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 4\begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix} \\ &= 1(0 - -1) - 2(4 - 3) + 4(-2 - 0) \\ &= 1 - 2 - 8 \\ &= -9 \end{aligned}$$

Just like 2×2 systems, a 3×3 system of linear equations in matrix form $\mathbf{AX} = \mathbf{B}$ will have a **unique solution** if $|\mathbf{A}| \neq 0$.

Remember that you can use your graphics calculator or matrix software to find the value of a determinant.

TI-*n*spire
TI-84
Casio

MATRIX OPERATIONS

- We can only find the determinants of square matrices.
- If $\mathbf{A}$ is square and $|\mathbf{A}| \neq 0$, then $\mathbf{A}^{-1}$ exists and $\mathbf{A}$ is called an **invertible** or **non-singular** matrix.
- If $\mathbf{A}$ is square and $|\mathbf{A}| = 0$, then $\mathbf{A}^{-1}$ does not exist and $\mathbf{A}$ is called a **singular** matrix.
- $\det(\mathbf{AB}) = \det\mathbf{A} \det\mathbf{B}$ or $|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|$ for all square matrices $\mathbf{A}$, $\mathbf{B}$ of equal size.

EXERCISE 11D.1

1 Evaluate:

a $\begin{vmatrix} 2 & 3 & 0 \\ -1 & 2 & 1 \\ 2 & 0 & 5 \end{vmatrix}$ **b** $\begin{vmatrix} -1 & 2 & -3 \\ 1 & 0 & 0 \\ -1 & 2 & 1 \end{vmatrix}$ **c** $\begin{vmatrix} 2 & 1 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix}$

d $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$ **e** $\begin{vmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 0 & 0 \end{vmatrix}$ **f** $\begin{vmatrix} 4 & 1 & 3 \\ -1 & 0 & 2 \\ -1 & 1 & 1 \end{vmatrix}$

2 **a** Find the values of x for which the matrix $\begin{pmatrix} x & 2 & 9 \\ 3 & 1 & 2 \\ -1 & 0 & x \end{pmatrix}$ is singular.

b What does your answer to **a** mean?

3 Evaluate:

a $\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$ **b** $\begin{vmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{vmatrix}$ **c** $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

4 For what values of k does $\begin{cases} x + 2y - 3z = 5 \\ 2x - y - z = 8 \\ kx + y + 2z = 14 \end{cases}$ have a unique solution?

5 For what values of k does $\begin{cases} 2x - y - 4z = 8 \\ 3x - ky + z = 1 \\ 5x - y + kz = -2 \end{cases}$ have a unique solution?

6 Find k given that: **a** $\begin{vmatrix} k & 2 & 1 \\ 2 & k & 2 \\ 1 & 2 & 1 \end{vmatrix} = 0$ **b** $\begin{vmatrix} 1 & k & 3 \\ k & 1 & -1 \\ 3 & 4 & 2 \end{vmatrix} = 7$

THE INVERSE OF A 3×3 MATRIX

There is no simple rule for finding the inverse of a 3×3 matrix like there is for a 2×2 matrix. Hence, we use technology. Instructions for doing this can be found in the graphics calculator chapter at the start of the book.

MATRIX OPERATIONS

For example, if $\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \\ 3 & -1 & 2 \end{pmatrix}$ what is $\mathbf{A}^{-1}$?

We obtain $\mathbf{A}^{-1} = \begin{pmatrix} -0.111 & 0.888 & -0.222 \\ 0.111 & 1.111 & -0.777 \\ 0.222 & -0.777 & 0.444 \end{pmatrix}$ which converts to $\begin{pmatrix} -\frac{1}{9} & \frac{8}{9} & -\frac{2}{9} \\ \frac{1}{9} & \frac{10}{9} & -\frac{7}{9} \\ \frac{2}{9} & -\frac{7}{9} & \frac{4}{9} \end{pmatrix}$.

EXERCISE 11D.2

1 Find $\begin{pmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & -3 & 1 \end{pmatrix}\begin{pmatrix} -11 & 9 & 15 \\ -1 & 1 & 1 \\ 8 & -6 & -10 \end{pmatrix}$ and *hence* the inverse of $\begin{pmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & -3 & 1 \end{pmatrix}$.

2 Use technology to find $\mathbf{A}^{-1}$ for: **a** $\mathbf{A} = \begin{pmatrix} 3 & 2 & 3 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & -3 & 1 \end{pmatrix}$.

3 Find $\mathbf{B}^{-1}$ for: **a** $\mathbf{B} = \begin{pmatrix} 13 & 43 & -11 \\ 16 & 9 & 27 \\ -8 & 31 & -13 \end{pmatrix}$ **b** $\mathbf{B} = \begin{pmatrix} 1.61 & 4.32 & 6.18 \\ 0.37 & 6.02 & 9.41 \\ 7.12 & 5.31 & 2.88 \end{pmatrix}$.

4 Check that your answers to **2** and **3** are correct.

E SOLVING SYSTEMS OF LINEAR EQUATIONS

In this course we consider only systems of equations which have a unique solution. The systems can each be written in the form $\mathbf{AX} = \mathbf{B}$ where $|\mathbf{A}| \neq 0$ and $\mathbf{A}$ is invertible.

Given that $\mathbf{A}$ is invertible, the solution to this system is $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.

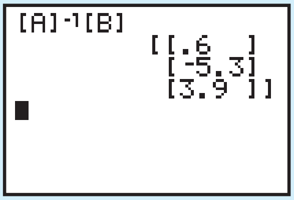
Example 15 **Self Tutor**

Solve the system $\left\{\begin{aligned} x - y - z &= 2 \\ x + y + 3z &= 7 \\ 9x - y - 3z &= -1 \end{aligned}\right.$ using matrix methods and a graphics calculator.

In matrix form $\mathbf{AX} = \mathbf{B}$ the system is:
$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 3 \\ 9 & -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ -1 \end{pmatrix}$$

We enter $\mathbf{A}$ and $\mathbf{B}$ into our graphics calculator and calculate $[\mathbf{A}]^{-1}[\mathbf{B}]$.

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 3 \\ 9 & -1 & -3 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ 7 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.6 \\ -5.3 \\ 3.9 \end{pmatrix}$$



```
[A]⁻¹[B]
               [[.6  ]
                [-5.3]
                [3.9 ]]
```

So, $x = 0.6$, $y = -5.3$, $z = 3.9$.

Example 16 **Self Tutor**

Rent-a-car has three different makes of vehicles, P, Q and R, for hire. These cars are located at yards A and B on either side of a city. Some cars are out (being rented). In total they have 150 cars. At yard A they have 20% of P, 40% of Q and 30% of R, which is 46 cars in total. At yard B they have 40% of P, 20% of Q and 50% of R, which is 54 cars in total. How many of each car type does Rent-a-car have?

Suppose Rent-a-car has x of P, y of Q and z of R.

It has 150 cars in total, so $x + y + z = 150$ ...... (1)

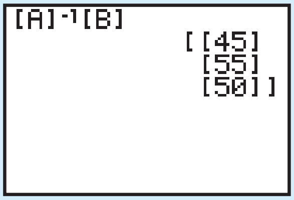
Yard A has 20% of P + 40% of Q + 30% of R, and this is 46 cars.

$$\therefore \quad \tfrac{2}{10}x + \tfrac{4}{10}y + \tfrac{3}{10}z = 46$$
$$\therefore \quad 2x + 4y + 3z = 460 \quad \text{...... (2)}$$

Yard B has 40% of P + 20% of Q + 50% of R, and this is 54 cars.

$$\therefore \quad \tfrac{4}{10}x + \tfrac{2}{10}y + \tfrac{5}{10}z = 54$$
$$\therefore \quad 4x + 2y + 5z = 540 \quad \text{...... (3)}$$

We need to solve the system: $\begin{cases} x + y + z = 150 \\ 2x + 4y + 3z = 460 \\ 4x + 2y + 5z = 540 \end{cases}$



```
[A]-1[B]
              [[45]
               [55]
               [50]]
```

or $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ 4 & 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 150 \\ 460 \\ 540 \end{pmatrix}$

So, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ 4 & 2 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 150 \\ 460 \\ 540 \end{pmatrix} = \begin{pmatrix} 45 \\ 55 \\ 50 \end{pmatrix}$ {using technology}

Thus, Rent-a-car has 45 of P, 55 of Q and 50 of R.

EXERCISE 11E

1 Write as a matrix equation:

a $\begin{aligned} x - y - z &= 2 \\ x + y + 3z &= 7 \\ 9x - y - 3z &= -1 \end{aligned}$

b $\begin{aligned} 2x + y - z &= 3 \\ y + 2z &= 6 \\ x - y + z &= 13 \end{aligned}$

c $\begin{aligned} a + b - c &= 7 \\ a - b + c &= 6 \\ 2a + b - 3c &= -2 \end{aligned}$

2 For $\mathbf{A} = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 2 & 1 \\ 0 & 6 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 7 & -3 \\ -1 & -2 & 1 \\ 6 & 12 & -5 \end{pmatrix}$,

calculate $\mathbf{AB}$ and hence solve the system of equations $\begin{cases} 4a + 7b - 3c = -8 \\ -a - 2b + c = 3 \\ 6a + 12b - 5c = -15. \end{cases}$

3 For $\mathbf{M}=\begin{pmatrix}5 & 3 & -7\\ -1 & -3 & 3\\ -3 & -1 & 5\end{pmatrix}$ and $\mathbf{N}=\begin{pmatrix}3 & 2 & 3\\ 1 & -1 & 2\\ 2 & 1 & 3\end{pmatrix}$,

calculate **MN** and hence solve the system $\begin{cases}3u+2v+3w=18\\ u-v+2w=6\\ 2u+v+3w=16.\end{cases}$

4 Use matrix methods and technology to solve:

a $\begin{aligned}3x+2y-z&=14\\ x-y+2z&=-8\\ 2x+3y-z&=13\end{aligned}$ **b** $\begin{aligned}x-y-2z&=4\\ 5x+y+2z&=-6\\ 3x-4y-z&=17\end{aligned}$ **c** $\begin{aligned}x+3y-z&=15\\ 2x+y+z&=7\\ x-y-2z&=0\end{aligned}$

5 Use your graphics calculator to solve:

a $\begin{aligned}x+y+z&=6\\ 2x+4y+z&=5\\ 2x+3y+z&=6\end{aligned}$ **b** $\begin{aligned}x+4y+11z&=7\\ x+6y+17z&=9\\ x+4y+8z&=4\end{aligned}$ **c** $\begin{aligned}2x-y+3z&=17\\ 2x-2y-5z&=4\\ 3x+2y+2z&=10\end{aligned}$

d $\begin{aligned}x+2y-z&=23\\ x-y+3z&=-23\\ 7x+y-4z&=62\end{aligned}$ **e** $\begin{aligned}10x-y+4z&=-9\\ 7x+3y-5z&=89\\ 13x-17y+23z&=-309\end{aligned}$ **f** $\begin{aligned}1.3x+2.7y-3.1z&=8.2\\ 2.8x-0.9y+5.6z&=17.3\\ 6.1x+1.4y-3.2z&=-0.6\end{aligned}$

6 Westwood School bought two footballs, one baseball, and three basketballs for a total cost of \$90. Sequoia School bought three footballs, two baseballs, and one basketball for \$81. Lamar School bought five footballs and two basketballs for \$104.

a State clearly what the variables x, y and z must represent if this situation is to be described by the set of equations alongside:

$$\begin{cases}2x+y+3z=90,\\ 3x+2y+z=81,\\ 5x+2z=104.\end{cases}$$

b Kato International School needs 4 footballs and 5 baseballs, and wishes to order as many basketballs as they can afford with the remainder of their \$315 budget. How many basketballs will they be able to purchase?

7 Managers, clerks and labourers are paid according to an industry award.
Xenon employs 2 managers, 3 clerks and 8 labourers with a total salary bill of €352 000.
Xanda employs 1 manager, 5 clerks and 4 labourers with a total salary bill of €274 000.
Xylon employs 1 manager, 2 clerks and 11 labourers with a total salary bill of €351 000.

a If x, y and z represent the salaries (in thousands of euros) for managers, clerks and labourers respectively, show how the above information can be represented by a system of three equations.

b Solve the system of equations in **a**.

c Determine the total salary bill for Xulu company which employs 3 managers, 8 clerks and 37 labourers.

8 A mixed nut company uses cashews, macadamias and brazil nuts to make three gourmet mixes. The table alongside indicates the weights in hundreds of grams of each kind of nut required to make a kilogram of mix.

	Mix A	Mix B	Mix C
Cashews	5	2	6
Macas.	3	4	1
Brazils	2	4	3

a If 1 kg of mix A costs \$12.50 to produce, 1 kg of mix B costs \$12.40 and 1 kg of mix C costs \$11.70, determine the cost per kilogram of each kind of nut.

b Hence, find the cost per kilogram to produce a mix containing 400 grams of cashews, 200 grams of macadamias and 400 grams of brazil nuts.

9 Klondike High has a total of 76 students in the three classes P, Q and R. There are p students in P, q in Q and r in R.

One-third of P, one-third of Q and two-fifths of R study Chemistry.

One-half of P, two-thirds of Q and one-fifth of R study Mathematics.

One-quarter of P, one-third of Q and three-fifths of R study Geography.

27 students study Chemistry, 35 study Mathematics and 30 study Geography.

a Find a system of equations which represents this information, making sure that the coefficients of p, q and r are integers.

b Solve for p, q and r.

10 Susan and James opened a new business in 2003. Their annual profit was £160 000 in 2006, £198 000 in 2007, and £240 000 in 2008. Based on the information from these three years they believe that their annual profit can be predicted by the model

$$P(t) = at + b + \frac{c}{t+4} \text{ pounds}$$

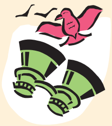
where t is the number of years after 2006. So, $t = 0$ gives the 2006 profit, $t = 1$ gives the 2007 profit, and so on.

a Determine the values of a, b and c which fit the profits for 2006, 2007 and 2008.

b If the profit in 2005 was £130 000, does this profit fit the model in **a**?

c Susan and James believe their profit will continue to grow according to this model. Predict their profit in 2009 and 2011.

INVESTIGATION USING MATRICES IN CRYPTOGRAPHY



Cryptography is the study of encoding and decoding messages.

Cryptography was first developed for the military to send secret messages. However, today it is used to maintain privacy when information is being transmitted via public communication services such as the internet.

Messages are sent in **code** or **cipher** form. The method of converting text to ciphertext is called **enciphering** and the reverse process is called **deciphering**.

The operations of matrix addition and multiplication can be used to create codes and the coded messages are transmitted. Decoding using additive or multiplicative inverses is required by the receiver in order to read the message.

The letters of the alphabet are first assigned integer values.

Notice that Z is assigned 0.

A 1	B 2	C 3	D 4	E 5	F 6	G 7	H 8	I 9	J 10	K 11	L 12	M 13
N 14	O 15	P 16	Q 17	R 18	S 19	T 20	U 21	V 22	W 23	X 24	Y 25	Z 0

The coded form of the word SEND is therefore 19 5 14 4 which we could put in 2×2 matrix form as $\begin{pmatrix} 19 & 5 \\ 14 & 4 \end{pmatrix}$.

An encoding matrix of your choice could be added to this matrix. Suppose it is $\begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix}$.

Adding this matrix we get $\begin{pmatrix} 19 & 5 \\ 14 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix} = \begin{pmatrix} 21 & 12 \\ 27 & 9 \end{pmatrix}$.

Any number not in the range 0 to 25 is adjusted to be in it by adding or subtracting multiples of 26. The 27 becomes $27 - 26 = 1$, and we send the matrix $\begin{pmatrix} 21 & 12 \\ 1 & 9 \end{pmatrix}$ or the string of numbers 21 12 1 9.

The message SEND MONEY PLEASE could be broken into groups of four letters.

SEND|MONE|YPLE|ASEE ← repeat the last letter to make group of 4. This is a dummy letter.

Each group is then encoded separately:

For MONE the matrix required is $\begin{pmatrix} 13 & 15 \\ 14 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix} = \begin{pmatrix} 15 & 22 \\ 27 & 10 \end{pmatrix}$ or $\begin{pmatrix} 15 & 22 \\ 1 & 10 \end{pmatrix}$.

For YPLE the matrix required is $\begin{pmatrix} 25 & 16 \\ 12 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix} = \begin{pmatrix} 27 & 23 \\ 25 & 10 \end{pmatrix}$ or $\begin{pmatrix} 1 & 23 \\ 25 & 10 \end{pmatrix}$.

For ASEE the matrix required is $\begin{pmatrix} 1 & 19 \\ 5 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 26 \\ 18 & 10 \end{pmatrix}$ or $\begin{pmatrix} 3 & 0 \\ 18 & 10 \end{pmatrix}$.

So the whole message is 21 12 1 9 15 22 1 10 1 23 25 10 3 0 18 10

The decoder requires the additive inverse matrix $\begin{pmatrix} -2 & -7 \\ -13 & -5 \end{pmatrix}$ to decode the message.

What to do:

1 Use the decoder matrix to check that the original message is obtained.

2 Use the code given to decode the message:

21 12 1 9 22 15 18 25 20 22 2 21 21 1 2
25 10 12 0 20 23 1 21 20 8 1 21 10 15 2
5 23 3 6 12 4

3 Create your own matrix addition code. Encode a short message. Supply the decoding matrix to a friend so that he or she can decode it.

The problem with encryption by matrix addition is that it is very easy to break. Breaking codes encrypted by matrix multiplication is much more difficult.

A chosen encoder matrix is again required. Suppose it is $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$.

The word SEND is encoded as $\begin{pmatrix} 19 & 5 \\ 14 & 4 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 43 & 67 \\ 32 & 50 \end{pmatrix}$

which is converted to $\begin{pmatrix} 17 & 15 \\ 6 & 24 \end{pmatrix}$.

What to do:

4 What is the coded form of SEND MONEY PLEASE?

5 What decoder matrix needs to be supplied to the receiver so that the message can be read?

6 Check by decoding the message.

7 Create your own code using matrix multiplication using a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $ad - bc = 1$.

8 What are the problems in using a 2 × 2 matrix where $ad - bc \neq 1$?
How can these problems be overcome?

9 Research **Hill ciphers** and explain how they differ from the methods given above.

REVIEW SET 11A NON-CALCULATOR

1 If $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix}$ find:

a $\mathbf{A} + \mathbf{B}$ **b** $3\mathbf{A}$ **c** $-2\mathbf{B}$ **d** $\mathbf{A} - \mathbf{B}$
e $\mathbf{B} - 2\mathbf{A}$ **f** $3\mathbf{A} - 2\mathbf{B}$ **g** $\mathbf{AB}$ **h** $\mathbf{BA}$
i $\mathbf{A}^{-1}$ **j** $\mathbf{A}^2$ **k** $\mathbf{ABA}$ **l** $(\mathbf{AB})^{-1}$

2 Find a, b, c and d if:

a $\begin{pmatrix} a & b-2 \\ c & d \end{pmatrix} = \begin{pmatrix} -a & 3 \\ 2-c & -4 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & 2a \\ b & -2 \end{pmatrix} + \begin{pmatrix} b & -a \\ c & d \end{pmatrix} = \begin{pmatrix} a & 2 \\ 2 & 6 \end{pmatrix}$

3 Make $\mathbf{Y}$ the subject of:

a $\mathbf{B} - \mathbf{Y} = \mathbf{A}$ **b** $2\mathbf{Y} + \mathbf{C} = \mathbf{D}$ **c** $\mathbf{AY} = \mathbf{B}$
d $\mathbf{YB} = \mathbf{C}$ **e** $\mathbf{C} - \mathbf{AY} = \mathbf{B}$ **f** $\mathbf{AY}^{-1} = \mathbf{B}$

4 Suppose $\mathbf{P} = \begin{pmatrix} a & 2 \\ 5 & -3 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} -3 & a-5 \\ -5 & a \end{pmatrix}$.

Find a such that $\mathbf{P} + \mathbf{Q} = \mathbf{O}$.

5 Determine the 2 × 2 matrix which when multiplied by $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ gives an answer of $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. **Hint:** Let the matrix be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

6 $\mathbf{A} = \begin{pmatrix} 4 & 3 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$. Find, if possible:

a $\mathbf{AB}$ **b** $\mathbf{BA}$ **c** $\mathbf{AC}$ **d** $\mathbf{CA}$ **e** $\mathbf{CB}$

7 **a** If $\mathbf{A} = 2\mathbf{A}^{-1}$, show that $\mathbf{A}^2 = 2\mathbf{I}$.

b If $\mathbf{A} = 2\mathbf{A}^{-1}$, simplify $(\mathbf{A} - \mathbf{I})(\mathbf{A} + 3\mathbf{I})$. Give your answer in the form $r\mathbf{A} + s\mathbf{I}$ where r and s are real numbers.

8 If $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix}$, find $\mathbf{AB}$ and $\mathbf{BA}$.

Hence find $\mathbf{A}^{-1}$ in terms of $\mathbf{B}$.

9 For what value(s) of k does $\begin{aligned} kx + 3y &= -6 \\ x + (k+2)y &= 2 \end{aligned}$ have a unique solution?

10 Find $\mathbf{X}$ if $\mathbf{AX} = \mathbf{B}$, $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 5 & 1 \\ -1 & 13 & 18 \end{pmatrix}$.

11 Find m if $\begin{vmatrix} m & 3 \\ m & m \end{vmatrix} = 18$.

12 Prove that $\begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix} = 4abc$.

13 Find $\mathbf{M}$ if $\mathbf{MN} = \begin{pmatrix} -6 & -4 \\ 13 & -3 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$.

14 **a** Find the values of a and b for which $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} a & b & 0 \\ b & 0 & a \\ 0 & 2 & b \end{pmatrix} = \mathbf{I}$, and

hence find $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}^{-1}$.

b Use your answer to **a** to solve the system of equations:

$$\begin{aligned} 2x + y + z &= 1 \\ x + y + z &= 6 \\ 2x + 2y + z &= 5 \end{aligned}$$

REVIEW SET 11B CALCULATOR

1 Solve the system of equations alongside using matrix methods and technology.

$$\begin{aligned} 3x - y + 2z &= 8 \\ 2x + 3y - z &= -3 \\ x - 2y + 3z &= 9 \end{aligned}$$

2 Solve using inverse matrices:

a $\begin{aligned} 3x - 4y &= 2 \\ 5x + 2y &= -1 \end{aligned}$

b $\begin{aligned} 4x - y &= 5 \\ 2x + 3y &= 9 \end{aligned}$

c $\mathbf{X}\begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 0 & -2 \end{pmatrix}$

d $\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}\mathbf{X} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

e $\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

f $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\mathbf{X}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 0 & 3 \end{pmatrix}$

3 **a** Given $\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -7 & 3 \\ 3 & -2 & -1 \end{pmatrix}$, find $\mathbf{A}^{-1}$.

b Hence solve the system of equations: $\begin{cases} 2x + y + z = 8 \\ 4x - 7y + 3z = 10 \\ 3x - 2y - z = 1. \end{cases}$

4 Find x if $\begin{vmatrix} x & 2 & 0 \\ 2 & x+1 & -2 \\ 0 & -2 & x+2 \end{vmatrix} = 0$, given that x is real.

5 If $\mathbf{A} = \begin{pmatrix} -2 & 3 \\ 4 & -1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -7 & 9 \\ 9 & -3 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} -1 & 0 & 3 \\ 0 & 2 & 1 \end{pmatrix}$, evaluate if possible

a $2\mathbf{A} - 2\mathbf{B}$ **b** $\mathbf{AC}$ **c** $\mathbf{CB}$ **d** $\mathbf{D}$, given that $\mathbf{DA} = \mathbf{B}$.

6 **a** If $\mathbf{A}$ and $\mathbf{B}$ are square matrices, under what conditions are the following true?

i If $\mathbf{AB} = \mathbf{B}$ then $\mathbf{A} = \mathbf{I}$ **ii** $(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$

b If $\mathbf{M} = \begin{pmatrix} k & 2 \\ 2 & k \end{pmatrix} \begin{pmatrix} k-1 & -2 \\ -3 & k \end{pmatrix}$ has an inverse $\mathbf{M}^{-1}$, what values can k have?

7 Solve the following system using matrix methods and technology:

$$\begin{aligned} 2x + 3y + z &= 4 \\ 5x + 2y - 4z &= 0 \\ 3x - y - 2z &= -7. \end{aligned}$$

8 A rock is thrown from the top of a cliff. Its distance above sea level is given by $s(t) = at^2 + bt + c$, where t is the time in seconds after the rock is released. After 1 second the rock is 63 m above sea level, after 2 seconds 72 m, and after 7 seconds 27 m.

a Find a, b and c and hence an expression for $s(t)$.

b Find the height of the cliff.

c Find the time taken for the rock to reach sea level.

9 If $\mathbf{A} = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} -12 & -11 \\ -10 & -1 \end{pmatrix}$, and $\mathbf{AXB} = \mathbf{C}$, find $\mathbf{X}$.

10 The cost of producing x hundred bottles of correcting fluid per day is given by the function $C(x) = ax^3 + bx^2 + cx + d$ dollars where a, b, c and d are constants.

a If it costs \$80 before any bottles are produced, find d.

b It costs \$100 to produce 100 bottles, \$148 to produce 200 bottles and \$376 to produce 400 bottles per day. Determine a, b and c.

11 If $\mathbf{A}=\begin{pmatrix}1 & 2 & 1\\ 2 & 4 & 6\\ 3 & 1 & 2\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}-1 & 2 & -3\\ 2 & -1 & 4\\ 3 & 4 & 1\end{pmatrix}$ show by calculation that $\det(\mathbf{AB})=\det\mathbf{A}\times\det\mathbf{B}=80$.

12 Hung, Quan and Ariel bought tickets for three separate performances. The table below shows the number of tickets bought by each person:

	Opera	*Play*	*Concert*
Hung	3	2	5
Quan	2	3	1
Ariel	1	5	4

- **a** If the total cost for Hung was €267, for Quan €145 and for Ariel €230, represent this information in the form of three equations.
- **b** Find the cost per ticket for each of the performances.
- **c** Determine how much it would cost Phuong to purchase 4 opera, 1 play, and 2 concert tickets.

REVIEW SET 11C

1 If $\mathbf{A}$ is $\begin{pmatrix}1 & 2 & 3\end{pmatrix}$ and $\mathbf{B}$ is $\begin{pmatrix}2 & 4\\ 0 & 1\\ 3 & 2\end{pmatrix}$ find, if possible:

a $2\mathbf{B}$ **b** $\frac{1}{2}\mathbf{B}$ **c** $\mathbf{AB}$ **d** $\mathbf{BA}$

2 For $\mathbf{P}=\begin{pmatrix}1 & 2\\ 1 & 0\\ 2 & 3\end{pmatrix}$ and $\mathbf{Q}=\begin{pmatrix}3 & 0\\ 1 & 4\\ 1 & 1\end{pmatrix}$ find:

a $\mathbf{P}+\mathbf{Q}$ **b** $\mathbf{Q}-\mathbf{P}$ **c** $\frac{3}{2}\mathbf{P}-\mathbf{Q}$

3 For what value(s) of k does the system $\begin{matrix}x+4y=2\\ kx+3y=-6\end{matrix}$ have a unique solution?

4 Given $\mathbf{A}=\begin{pmatrix}-1 & 2\\ 3 & 0\end{pmatrix}$ and $\mathbf{A}^2+\mathbf{A}+k\mathbf{I}=\mathbf{O}$, find k.

5 Find, if they exist, the inverse matrices of each of the following:

a $\begin{pmatrix}6 & 8\\ 5 & 7\end{pmatrix}$ **b** $\begin{pmatrix}4 & -3\\ 8 & -6\end{pmatrix}$ **c** $\begin{pmatrix}11 & 5\\ -6 & -3\end{pmatrix}$

6 If $\mathbf{A}=\begin{pmatrix}-3 & 2 & 2\\ 1 & -1 & 0\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}2 & 4\\ -3 & 1\\ 1 & 2\end{pmatrix}$ and $\mathbf{C}=\begin{pmatrix}-2 & 5\\ 1 & 3\end{pmatrix}$ find, if possible

a $3\mathbf{A}$ **b** $\mathbf{AB}$ **c** $\mathbf{BA}$ **d** $\mathbf{AC}$ **e** $\mathbf{BC}$

7 Suppose $\mathbf{A} = \begin{pmatrix} x & 1 \\ 4 & -3x \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\mathbf{AB} = \begin{pmatrix} 9 \\ 23 \end{pmatrix}$. Find x.

8 **a** Write $5\mathbf{A}^2 - 6\mathbf{A} = 3\mathbf{I}$ in the form $\mathbf{AB} = \mathbf{I}$.

b Hence find $\mathbf{A}^{-1}$ in terms of $\mathbf{A}$ and $\mathbf{I}$.

9 Let $\mathbf{A} = \begin{pmatrix} -1 & -2 \\ 0 & -3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ -6 & 7 \end{pmatrix}$.

a Find $\mathbf{A}^3$. **b** Solve the matrix equation $\mathbf{A}^3 + 2\mathbf{X} = \mathbf{B}$.

10 If $\mathbf{A}^2 = 5\mathbf{A} + 2\mathbf{I}$, find $\mathbf{A}^3$, $\mathbf{A}^4$, $\mathbf{A}^5$ and $\mathbf{A}^6$ in the form $r\mathbf{A} + s\mathbf{I}$.

11 A café sells two types of cola drinks. The drinks each come in three sizes: small, medium and large. At the beginning and end of the day the stock in the fridge was counted, the results being:

Start of the day:

$$\begin{array}{lc} & \begin{array}{cc}\text{Brand C} & \text{Brand P}\end{array} \\ \begin{array}{l}\text{small} \rightarrow \\ \text{medium} \rightarrow \\ \text{large} \rightarrow\end{array} & \begin{pmatrix} 42 & 54 \\ 36 & 27 \\ 34 & 30 \end{pmatrix}\end{array}$$

End of the day:

$$\begin{array}{lc} & \begin{array}{cc}\text{Brand C} & \text{Brand P}\end{array} \\ \begin{array}{l}\text{small} \rightarrow \\ \text{medium} \rightarrow \\ \text{large} \rightarrow\end{array} & \begin{pmatrix} 27 & 31 \\ 28 & 15 \\ 28 & 22 \end{pmatrix}\end{array}$$

The profit matrix is:
$$\begin{array}{c} \begin{array}{ccc}\text{small} & \text{medium} & \text{large}\end{array} \\ \begin{pmatrix} \$0.75 & \$0.55 & \$1.20 \end{pmatrix}\end{array}$$

Use matrix methods to calculate the profit made for the day from the sale of these drinks.

12 Suppose $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ a & b \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} c & d \\ 0 & -1 \end{pmatrix}$.

a Find $\mathbf{AB}$. **b** Find $\mathbf{A}^{-1}$. **c** Find c and d given that $\mathbf{B}^{-1} = -\mathbf{I}$.

13 A matrix $\mathbf{A}$ has the property that $\mathbf{A}^2 = \mathbf{A} - \mathbf{I}$.

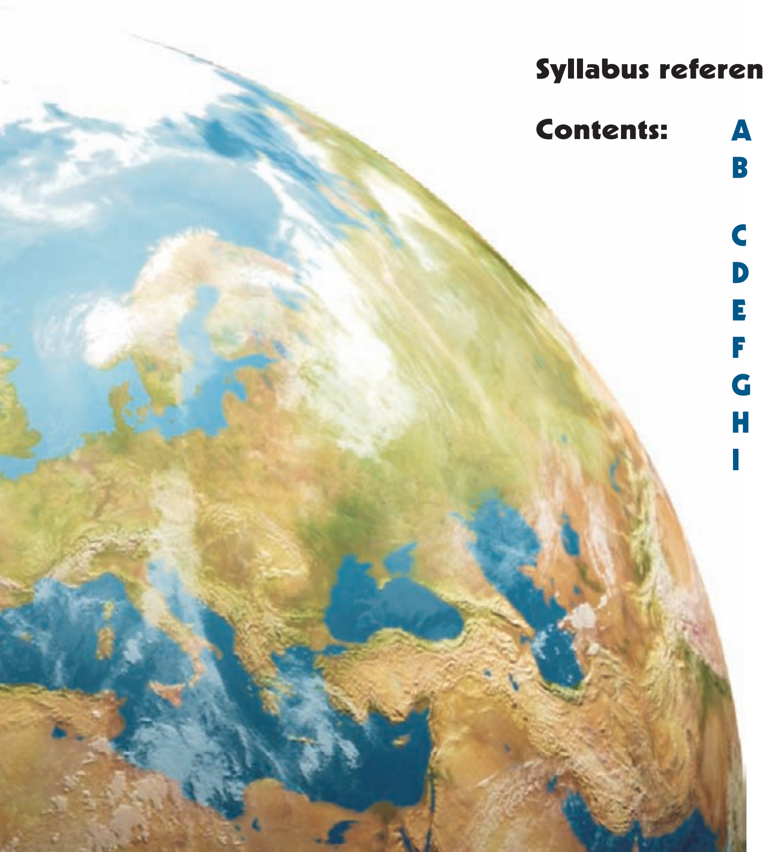
a Find expressions for $\mathbf{A}^n$ for $n = 3, 4, ..., 8$ in terms of $\mathbf{A}$ and $\mathbf{I}$. Hence:

b Deduce simple expressions for $\mathbf{A}^{6n+3}$ and $\mathbf{A}^{6n+5}$.

c Express $\mathbf{A}^{-1}$ in terms of $\mathbf{A}$ and $\mathbf{I}$.

Chapter 12

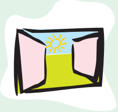
Vectors in 2 and 3 dimensions



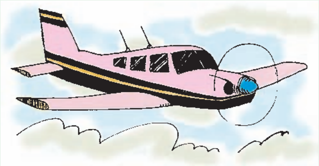
Syllabus reference: 5.1, 5.2

Contents:

OPENING PROBLEM



An aeroplane in calm conditions is flying due east. A cold wind suddenly blows in from the south west. The aeroplane, cruising at 800 km h^{-1}, is blown slightly off course by the 35 km h^{-1} wind.



Things to think about:

- What effect does the wind have on the speed and direction of the aeroplane?
- How can we accurately determine the new speed and direction using mathematics?
- How much of the force of the wind operates in the direction of the aeroplane? How does this affect fuel consumption and the time of the flight?

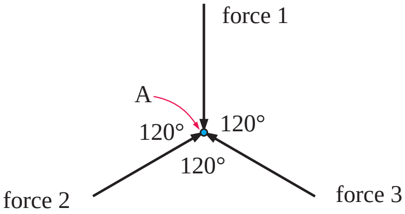
A INTRODUCTION

VECTORS AND SCALARS

To solve questions like those in the **Opening Problem**, we need to examine the **size** or **magnitude** of the quantities under consideration as well as the **direction** in which they are acting.

For example, the effect of the wind on an aeroplane would be different if the wind was blowing from behind the plane rather than against it.

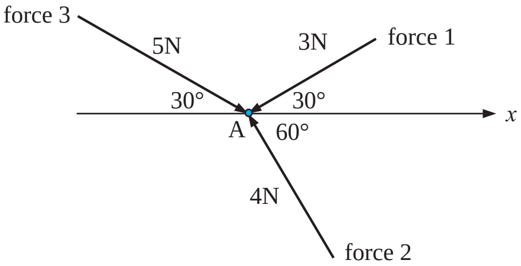
Consider the problem of forces acting at a point.



If three equal forces act on point A and they are from directions 120^o apart then clearly A would not move.

For example, imagine three people 120^o apart around a statue, pushing with equal force.

Now suppose three forces act on the point A as shown opposite. What is the resultant force acting on A? In what direction would A move under these three forces?



To handle these situations we need to consider quantities called **vectors** which have both size (magnitude) and direction.

> Quantities which have only magnitude are called **scalars**.
>
> Quantities which have both magnitude and direction are called **vectors**.

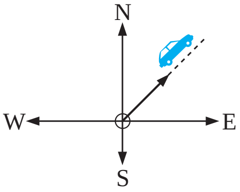
Velocity is a vector since it describes speed (a scalar) in a particular direction.

Other examples of vector quantities are:

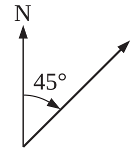
- acceleration
- force
- displacement
- momentum
- weight

DIRECTED LINE SEGMENT REPRESENTATION

Consider a car that is travelling at 80 km h^{-1} in a NE direction.



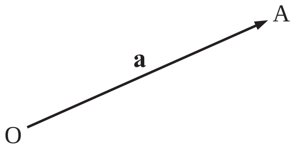
One good way of representing this is to use an arrow on a scale diagram.



Scale: 1 cm represents 40 km h^{-1}

The **length of the arrow** represents the size or magnitude of the quantity, and the **arrowhead** shows the direction of travel.

Consider the vector represented by the line segment from O to A. We call this the **position vector** of A.



- This **position vector** could be represented by

  $\overrightarrow{OA}$ or $\mathbf{a}$ or $\tilde{a}$ or $\vec{a}$

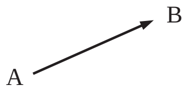
  $\mathbf{a}$: bold used in text books

  $\tilde{a}$, $\vec{a}$: used by students

- The **magnitude** or **length** could be represented by

  $|\overrightarrow{OA}|$ or OA or $|\mathbf{a}|$ or $|\tilde{a}|$ or $|\vec{a}|$

For



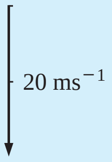
we say that $\overrightarrow{AB}$ is the vector which **emanates** from A and **terminates** at B,

and that $\overrightarrow{AB}$ is the **position vector** of B relative to A.

Example 1 **Self Tutor**

On a scale diagram, sketch the vector which represents a force of 20 m s^{-1} in a southerly direction.

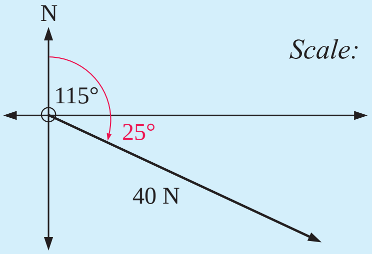
Scale: 1 cm $\equiv$ 10 m s^{-1}



Example 2

Self Tutor

Draw a scaled arrow diagram representing a force of 40 Newtons on the bearing 115^o.



Scale: $1 \text{ cm} \equiv 10 \text{ N}$

EXERCISE 12A.1

1 Using a scale of 1 cm represents 10 units, sketch a vector to represent:

- **a** 30 Newtons in a south-easterly direction
- **b** 25 m s^{-1} in a northerly direction
- **c** a displacement of 35 m in the direction 070^o
- **d** an aeroplane taking off at an angle of 10^o to the runway with a speed of 50 m s^{-1}.

2 If $\longrightarrow$ represents a velocity of 50 m s^{-1} due east, draw a directed line segment representing a velocity of:

- **a** 100 m s^{-1} due west
- **b** 75 m s^{-1} north east.

3 Draw a scaled arrow diagram representing the following vectors:

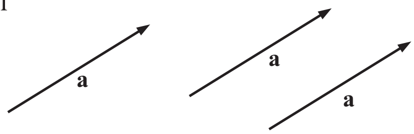
- **a** a force of 30 Newtons in the NW direction
- **b** a velocity of 40 m s^{-1} in the direction 146^o
- **c** a displacement of 25 km in the direction S32^oE
- **d** an aeroplane taking off at an angle of 8^o to the runway at a speed of 150 km h^{-1}.

GEOMETRIC VECTOR EQUALITY

Two vectors are **equal** if they have the same magnitude and direction.

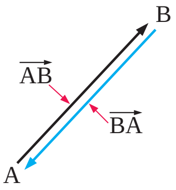
So, if arrows are used to represent vectors, then equal vectors are **parallel** and **equal in length**.

The arrows that represent them are translations of one another.



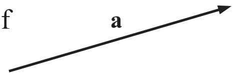
Since we can draw a vector with given magnitude and direction from *any* point, we consider vectors to be **free**. They are sometimes referred to as **free vectors**.

GEOMETRIC NEGATIVE VECTORS



$\overrightarrow{AB}$ and $\overrightarrow{BA}$ have the same length, but they have opposite directions.

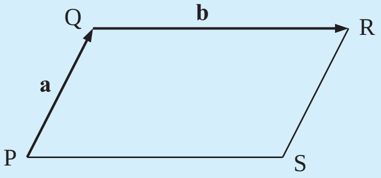
We say that $\overrightarrow{BA}$ is the negative of $\overrightarrow{AB}$ and write $\overrightarrow{BA} = -\overrightarrow{AB}$.

If 

then 

since **a** and −**a** must be parallel and equal in length, but opposite in direction.

Example 3 **Self Tutor**



PQRS is a parallelogram in which $\overrightarrow{PQ} = \mathbf{a}$ and $\overrightarrow{QR} = \mathbf{b}$.

Find vector expressions for:

a $\overrightarrow{QP}$ **b** $\overrightarrow{RQ}$ **c** $\overrightarrow{SR}$ **d** $\overrightarrow{SP}$

a $\overrightarrow{QP} = -\mathbf{a}$ {the negative vector of $\overrightarrow{PQ}$}

b $\overrightarrow{RQ} = -\mathbf{b}$ {the negative vector of $\overrightarrow{QR}$}

c $\overrightarrow{SR} = \mathbf{a}$ {parallel to and the same length as $\overrightarrow{PQ}$}

d $\overrightarrow{SP} = -\mathbf{b}$ {parallel to and the same length as $\overrightarrow{RQ}$}

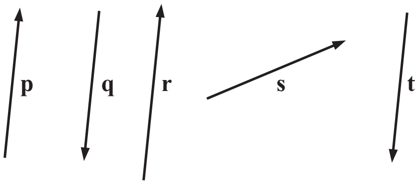
EXERCISE 12A.2

1 State the vectors which are:

a equal in magnitude **b** parallel

c in the same direction **d** equal

e negatives of one another.

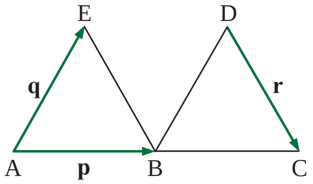


2 The figure alongside consists of 2 equilateral triangles. A, B and C lie on a straight line. $\overrightarrow{AB} = \mathbf{p}$, $\overrightarrow{AE} = \mathbf{q}$ and $\overrightarrow{DC} = \mathbf{r}$.

Which of the following statements are true?

a $\overrightarrow{EB} = \mathbf{r}$ **b** $|\mathbf{p}| = |\mathbf{q}|$ **c** $\overrightarrow{BC} = \mathbf{r}$

d $\overrightarrow{DB} = \mathbf{q}$ **e** $\overrightarrow{ED} = \mathbf{p}$ **f** $\mathbf{p} = \mathbf{q}$



DISCUSSION

- Could we have a zero vector?
- What would its length be?
- What would its direction be?

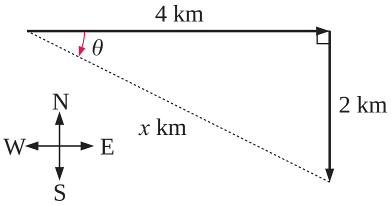
B GEOMETRIC OPERATIONS WITH VECTORS

We have already been operating with vectors without realising it.

Bearing problems are an example of this. The vectors in these cases are **displacements**.

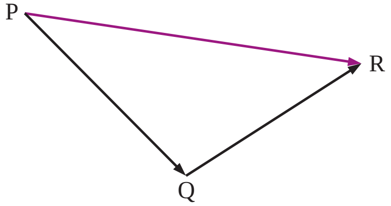
A typical problem could be:

> A runner runs in an easterly direction for 4 km and then in a southerly direction for 2 km.
>
> How far is she from her starting point and in what direction?



Trigonometry and Pythagoras' theorem are used to answer such problems, as we need to find θ and x.

GEOMETRIC VECTOR ADDITION

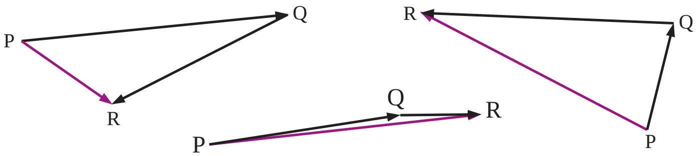


Suppose we have three towns P, Q and R.

A trip from P to Q followed by a trip from Q to R is equivalent to a trip from P to R.

This can be expressed in vector form as the sum $\overrightarrow{PQ}+\overrightarrow{QR}=\overrightarrow{PR}$.

This triangular diagram could take all sorts of shapes, but in each case the sum will be true. For example:



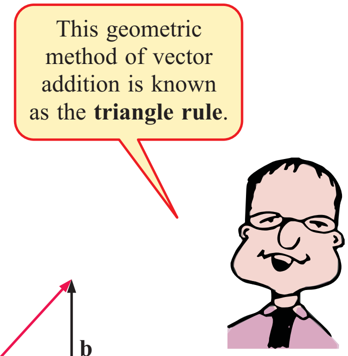
After considering diagrams like those above, we can now define vector addition geometrically:

To add **a** and **b**:

Step 1: Draw **a**.

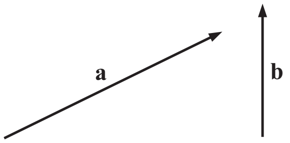
Step 2: At the arrowhead end of **a**, draw **b**.

Step 3: Join the beginning of **a** to the arrowhead end of **b**. This is vector **a** + **b**.

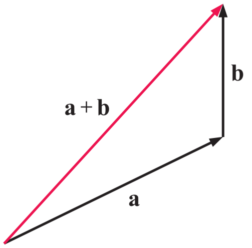


This geometric method of vector addition is known as the **triangle rule**.

So, given

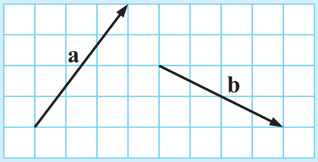


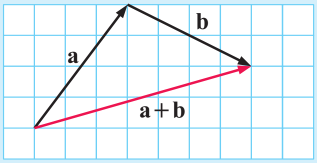
we have



Example 4 **Self Tutor**

Given **a** and **b** as shown, construct **a** + **b**.

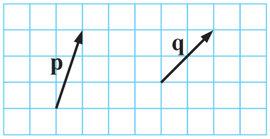




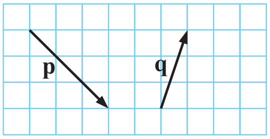
EXERCISE 12B.1

1 Copy the given vectors **p** and **q** and hence show how to find **p** + **q**:

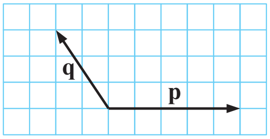
a



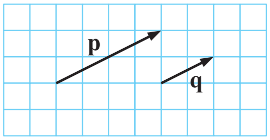
b



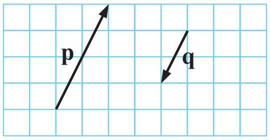
c



d



e



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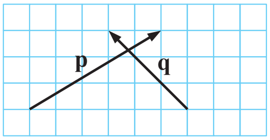
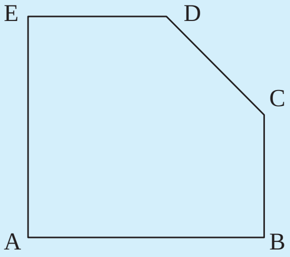


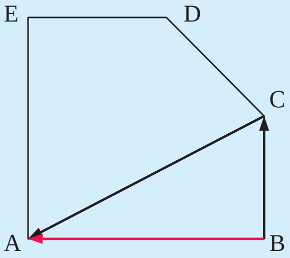
Example 5 **Self Tutor**

Find a single vector which is equal to:

a $\overrightarrow{BC} + \overrightarrow{CA}$
b $\overrightarrow{BA} + \overrightarrow{AE} + \overrightarrow{EC}$
c $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$
d $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$



a $\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$ {as shown}
b $\overrightarrow{BA} + \overrightarrow{AE} + \overrightarrow{EC} = \overrightarrow{BC}$
c $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AA}$
d $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{AE}$

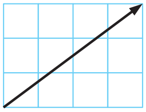


2 Find a single vector which is equal to:

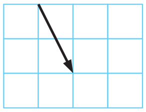
a $\overrightarrow{AB} + \overrightarrow{BC}$ **b** $\overrightarrow{BC} + \overrightarrow{CD}$ **c** $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$ **d** $\overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BD}$

3 **a** Use vector diagrams to find **i** $\mathbf{p}+\mathbf{q}$ **ii** $\mathbf{q}+\mathbf{p}$ given that:

$\mathbf{p}$ is

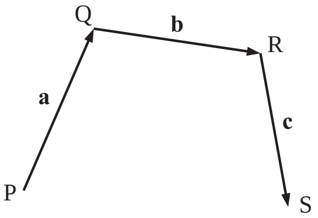


and $\mathbf{q}$ is



b For any two vectors $\mathbf{p}$ and $\mathbf{q}$, is $\mathbf{p}+\mathbf{q}=\mathbf{q}+\mathbf{p}$?

4 Consider:



One way of finding $\overrightarrow{PS}$ is:

$$\begin{aligned}\overrightarrow{PS} &= \overrightarrow{PR}+\overrightarrow{RS}\\ &= (\mathbf{a}+\mathbf{b})+\mathbf{c}.\end{aligned}$$

Use the diagram to show that $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$.

THE ZERO VECTOR

Having defined vector addition, we are now able to state that:

The **zero vector 0** is a vector of length 0.

For any vector **a**: $\mathbf{a}+\mathbf{0}=\mathbf{0}+\mathbf{a}=\mathbf{a}$

$\mathbf{a}+(-\mathbf{a})=(-\mathbf{a})+\mathbf{a}=\mathbf{0}$.

When we write the zero vector by hand, we usually write $\overrightarrow{0}$.

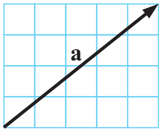
GEOMETRIC VECTOR SUBTRACTION

To subtract one vector from another, we simply **add its negative**.

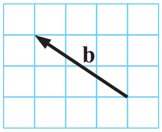
$\mathbf{a}-\mathbf{b}=\mathbf{a}+(-\mathbf{b})$

For example,

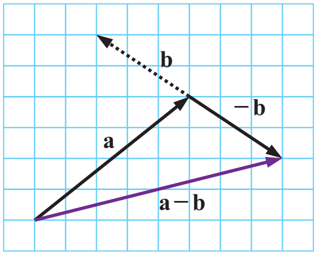
given



and



then

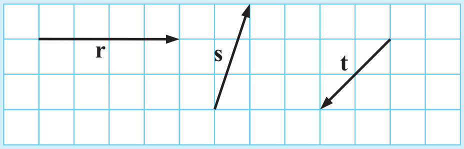


Example 6 **Self Tutor**

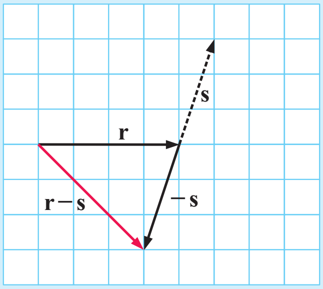
For **r**, **s** and **t** as shown find geometrically:

a $\mathbf{r}-\mathbf{s}$

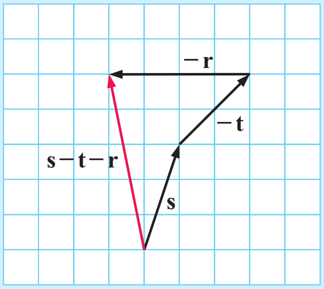
b $\mathbf{s}-\mathbf{t}-\mathbf{r}$



a



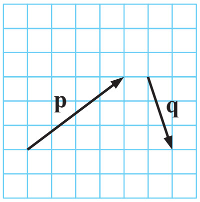
b



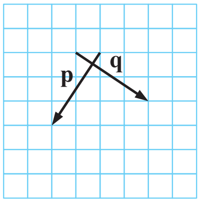
EXERCISE 12B.2

1 For the following vectors **p** and **q**, show how to construct **p** − **q**:

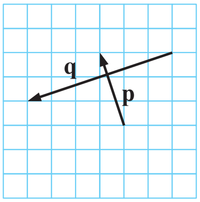
a



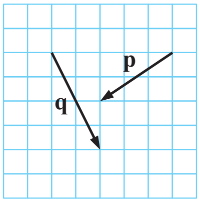
b



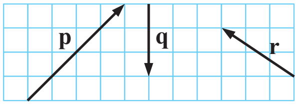
c



d



2 For the vectors illustrated, show how to construct:



a **p** + **q** − **r** **b** **p** − **q** − **r** **c** **r** − **q** − **p**

Example 7 Self Tutor

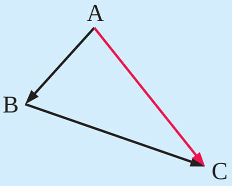
For points A, B, C and D, simplify the following vector expressions:

a $\overrightarrow{AB} - \overrightarrow{CB}$ **b** $\overrightarrow{AC} - \overrightarrow{BC} - \overrightarrow{DB}$

a $\overrightarrow{AB} - \overrightarrow{CB}$

$= \overrightarrow{AB} + \overrightarrow{BC}$ $\{$as $\overrightarrow{BC} = -\overrightarrow{CB}\}$

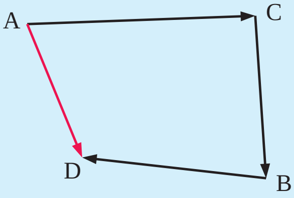
$= \overrightarrow{AC}$



b $\overrightarrow{AC} - \overrightarrow{BC} - \overrightarrow{DB}$

$= \overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BD}$

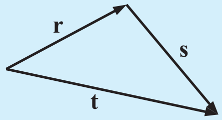
$= \overrightarrow{AD}$



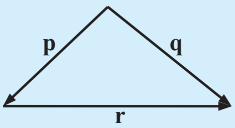
Example 8 **Self Tutor**

Construct vector equations for:

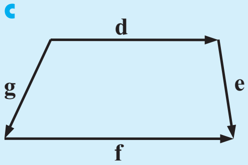
a



b



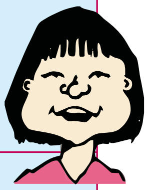
c



a $\mathbf{t} = \mathbf{r} + \mathbf{s}$

b $\mathbf{r} = -\mathbf{p} + \mathbf{q}$

c $\mathbf{f} = -\mathbf{g} + \mathbf{d} + \mathbf{e}$

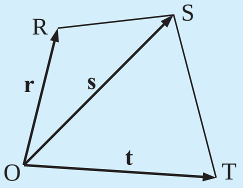


We select any vector for the LHS and then take another path from its starting point to its finishing point.

Example 9 **Self Tutor**

Find, in terms of $\mathbf{r}$, $\mathbf{s}$ and $\mathbf{t}$:

a $\overrightarrow{RS}$ **b** $\overrightarrow{SR}$ **c** $\overrightarrow{ST}$



a
$\overrightarrow{RS}$
$= \overrightarrow{RO} + \overrightarrow{OS}$
$= -\overrightarrow{OR} + \overrightarrow{OS}$
$= -\mathbf{r} + \mathbf{s}$
$= \mathbf{s} - \mathbf{r}$

b
$\overrightarrow{SR}$
$= \overrightarrow{SO} + \overrightarrow{OR}$
$= -\overrightarrow{OS} + \overrightarrow{OR}$
$= -\mathbf{s} + \mathbf{r}$
$= \mathbf{r} - \mathbf{s}$

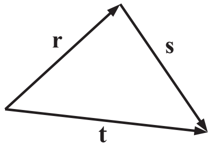
c
$\overrightarrow{ST}$
$= \overrightarrow{SO} + \overrightarrow{OT}$
$= -\overrightarrow{OS} + \overrightarrow{OT}$
$= -\mathbf{s} + \mathbf{t}$
$= \mathbf{t} - \mathbf{s}$

3 For points A, B, C and D, simplify the following vector expressions:

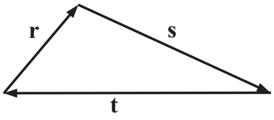
a $\overrightarrow{AC} + \overrightarrow{CB}$
b $\overrightarrow{AD} - \overrightarrow{BD}$
c $\overrightarrow{AC} + \overrightarrow{CA}$
d $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$
e $\overrightarrow{BA} - \overrightarrow{CA} + \overrightarrow{CB}$
f $\overrightarrow{AB} - \overrightarrow{CB} - \overrightarrow{DC}$

4 Construct vector equations for:

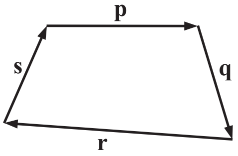
a



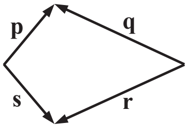
b



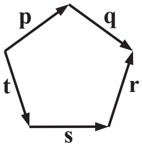
c



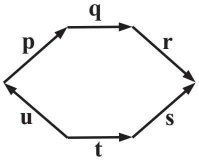
d



e

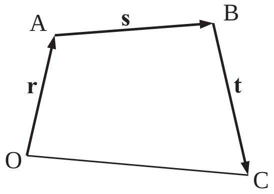


f



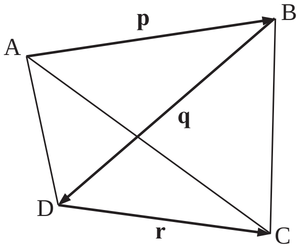
5 **a** Find, in terms of **r**, **s** and **t**:

i $\overrightarrow{OB}$ **ii** $\overrightarrow{CA}$ **iii** $\overrightarrow{OC}$



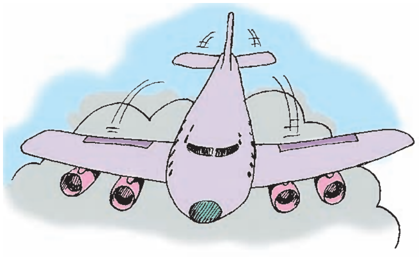
b Find, in terms of **p**, **q** and **r**:

i $\overrightarrow{AD}$ **ii** $\overrightarrow{BC}$ **iii** $\overrightarrow{AC}$



AN APPLICATION OF VECTOR SUBTRACTION

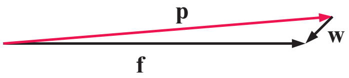
Consider the following velocity application:



An aeroplane needs to fly due east from one city to another at a speed of 400 km h^{-1}. However, a 50 km h^{-1} wind blows constantly from the north-east.

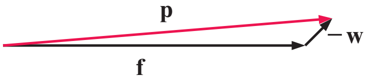
In what direction must the aeroplane head to compensate for the wind, and how does the wind affect its speed?

On this occasion we are given:



We also know that the aeroplane will have to head a little north of its final destination so the north-easterly wind will blow it back to the correct direction.

In order to move in the **f** direction, the aeroplane must actually head in the **p** direction.



Notice that $\mathbf{p} + \mathbf{w} = \mathbf{f}$

$\therefore \quad \mathbf{p} + \mathbf{w} + (-\mathbf{w}) = \mathbf{f} + (-\mathbf{w})$

$\therefore \quad \mathbf{p} + \mathbf{0} = \mathbf{f} - \mathbf{w}$

$\therefore \quad \mathbf{p} = \mathbf{f} - \mathbf{w}$

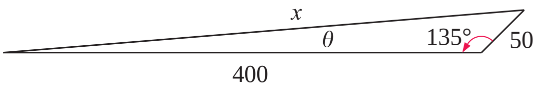
By the cosine rule,

$$x^2 = 50^2 + 400^2 - 2 \times 50 \times 400 \cos 135^o$$

$\therefore \quad x \approx 437$

By the sine rule, $\dfrac{\sin\theta}{50} = \dfrac{\sin 135^o}{436.8}$

$\therefore \quad \theta \approx 4.64^o$



Consequently, the aeroplane must head 4.64° north of east. It needs to fly so that its speed in still air would be 437 km h^{-1}. The wind slows the aeroplane down to 400 km h^{-1}.

EXERCISE 12B.3

1 A boat needs to travel south at a speed of 20 km h^{-1}. However a constant current of 6 km h^{-1} is flowing from the south-east. Use vector subtraction to find:

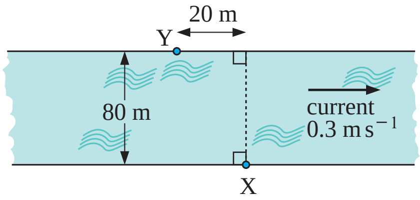
- **a** the equivalent speed in still water for the boat to achieve the actual speed of 20 km h^{-1}
- **b** the direction in which the boat must head to compensate for the current.

2 As part of an endurance race, Stephanie needs to swim from X to Y across a wide river.

Stephanie swims at 1.8 m s^{-1} in still water.

If the river flows with a consistent current of 0.3 m s^{-1} as shown, find:

- **a** the distance from X to Y
- **b** the direction in which Stephanie should head
- **c** the time Stephanie will take to cross the river.



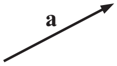
GEOMETRIC SCALAR MULTIPLICATION

A **scalar** is a non-vector quantity. It has a size but no direction.

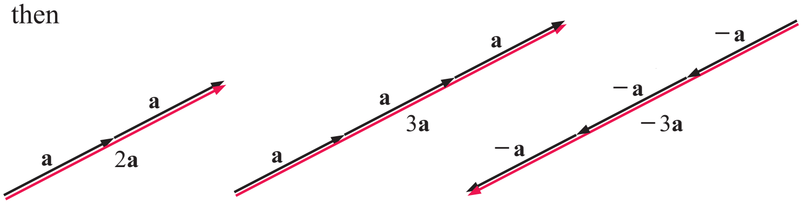
We can multiply vectors by scalars such as 2 and -3. If **a** is a vector, we define

$$2\mathbf{a} = \mathbf{a} + \mathbf{a} \quad \text{and} \quad 3\mathbf{a} = \mathbf{a} + \mathbf{a} + \mathbf{a}$$
$$\text{so} \quad -3\mathbf{a} = 3(-\mathbf{a}) = (-\mathbf{a}) + (-\mathbf{a}) + (-\mathbf{a}).$$

If **a** is



then



So, **2a** is in the same direction as **a** but is twice as long as **a**
3a is in the same direction as **a** but is three times longer than **a**
$-$3a has the opposite direction to **a** and is three times longer than **a**.

If **a** is a vector and k is a scalar, then $k\mathbf{a}$ is also a vector and we are performing **scalar multiplication**.

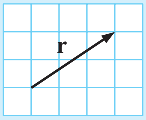
If $k > 0$, $k\mathbf{a}$ and $\mathbf{a}$ have the same direction.

If $k < 0$, $k\mathbf{a}$ and $\mathbf{a}$ have opposite directions.

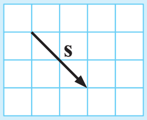
If $k = 0$, $k\mathbf{a} = \mathbf{0}$ the zero vector.

Example 10 Self Tutor

Given vectors

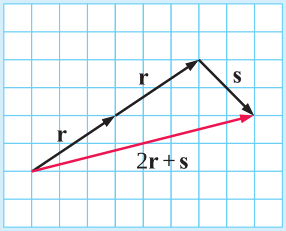


and

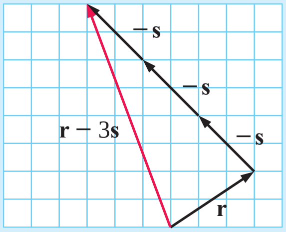


show how to find **a** $2\mathbf{r} + \mathbf{s}$ **b** $\mathbf{r} - 3\mathbf{s}$ geometrically.

a



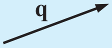
b



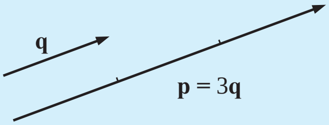
Example 11 Self Tutor

Sketch vectors **p** and **q** if **a** $\mathbf{p} = 3\mathbf{q}$ **b** $\mathbf{p} = -\frac{1}{2}\mathbf{q}$.

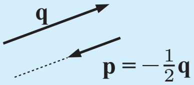
Let **q** be



a

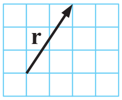


b

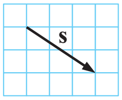


EXERCISE 12B.4

1 Given vectors



and



, show how to find geometrically:

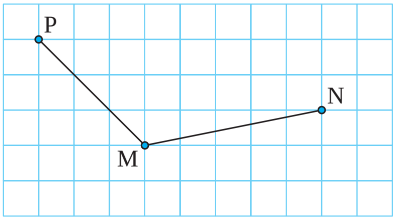
a $-\mathbf{r}$ **b** $2\mathbf{s}$ **c** $\frac{1}{2}\mathbf{r}$ **d** $-\frac{3}{2}\mathbf{s}$

e $2\mathbf{r} - \mathbf{s}$ **f** $2\mathbf{r} + 3\mathbf{s}$ **g** $\frac{1}{2}\mathbf{r} + 2\mathbf{s}$ **h** $\frac{1}{2}(\mathbf{r} + 3\mathbf{s})$

2 Draw sketches of **p** and **q** if:

a $\mathbf{p} = \mathbf{q}$ **b** $\mathbf{p} = -\mathbf{q}$ **c** $\mathbf{p} = 2\mathbf{q}$ **d** $\mathbf{p} = \frac{1}{3}\mathbf{q}$ **e** $\mathbf{p} = -3\mathbf{q}$

3 **a** Copy this diagram and on it mark the points:



i X such that $\overrightarrow{MX} = \overrightarrow{MN} + \overrightarrow{MP}$

ii Y such that $\overrightarrow{MY} = \overrightarrow{MN} - \overrightarrow{MP}$

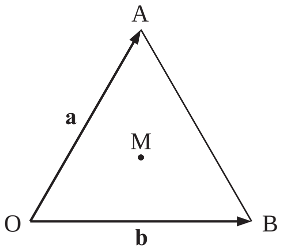
iii Z such that $\overrightarrow{PZ} = 2\overrightarrow{PM}$

b What type of figure is MNYZ?

4 OAB is an equilateral triangle with M its **orthocentre**. This means that the lines from each vertex which are drawn perpendicular to the opposite sides, all pass through M.

a Find $\overrightarrow{AB}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

b Show that $\overrightarrow{OM} = \frac{1}{3}(\mathbf{a} + \mathbf{b})$.

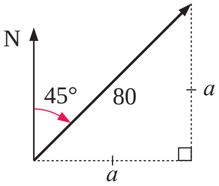


C 2-D VECTORS IN COMPONENT FORM

So far we have examined vectors from their geometric representation.

We have used arrows where:

- the **length** of the arrow represents size or magnitude
- the **arrowhead** indicates the direction of the vector.

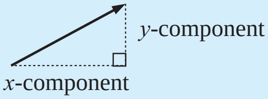


Consider a car travelling at 80 km h^{-1} in a NE direction.

The velocity vector could be represented using the x and y steps which are necessary to go from the start to the finish.

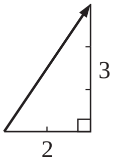
In this case the column vector $\begin{pmatrix} 56.6 \\ 56.6 \end{pmatrix}$ gives the x and y steps.

$$a^2 + a^2 = 80^2$$
$$\therefore \quad 2a^2 = 6400$$
$$\therefore \quad a^2 = 3200$$
$$\therefore \quad a \approx 56.6$$



$\begin{pmatrix} x \\ y \end{pmatrix}$ is the **component form** of a vector.

For example, given $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ we could draw



where 2 is the horizontal step and 3 is the vertical step.

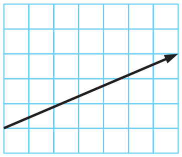
EXERCISE 12C.1

1 Draw arrow diagrams to represent the vectors:

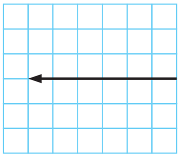
a $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ **c** $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$ **d** $\begin{pmatrix} -1 \\ -3 \end{pmatrix}$

2 Write the illustrated vectors in component form:

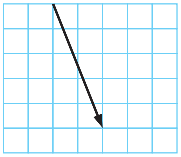
a



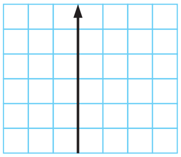
b



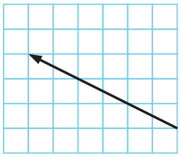
c



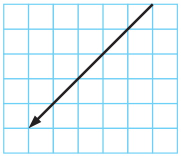
d



e



f

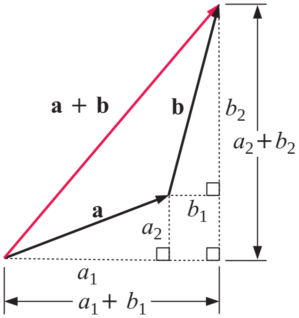


ALGEBRAIC VECTOR ADDITION

Consider adding vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

Notice that:

- the horizontal step for $\mathbf{a} + \mathbf{b}$ is $a_1 + b_1$
- the vertical step for $\mathbf{a} + \mathbf{b}$ is $a_2 + b_2$.



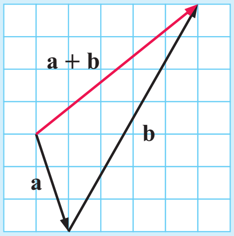
If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ then $\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$.

Example 12 **Self Tutor**

If $\mathbf{a} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$, find $\mathbf{a} + \mathbf{b}$. Check your answer graphically.

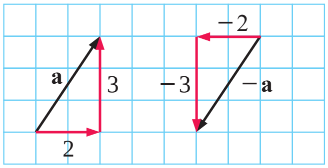
$$\begin{aligned} \mathbf{a} + \mathbf{b} &= \begin{pmatrix} 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 4 \\ 7 \end{pmatrix} \\ &= \begin{pmatrix} 1+4 \\ -3+7 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 4 \end{pmatrix} \end{aligned}$$

Graphical check:



ALGEBRAIC NEGATIVE VECTORS

Consider the vector $\mathbf{a} = \binom{2}{3}$. Notice that $-\mathbf{a} = \binom{-2}{-3}$.



If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ then $-\mathbf{a} = \begin{pmatrix} -a_1 \\ -a_2 \end{pmatrix}$.

ZERO VECTOR

The zero vector is $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

For any vector $\mathbf{a}$: $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$
$\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$.

ALGEBRAIC VECTOR SUBTRACTION

To subtract one vector from another, we simply **add its negative**. So, $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$.

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

then $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} -b_1 \\ -b_2 \end{pmatrix} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix}$

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ then $\mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix}$.

Example 13 **Self Tutor**

Given $\mathbf{p} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$

find: **a** $\mathbf{q} - \mathbf{p}$ **b** $\mathbf{p} - \mathbf{q} - \mathbf{r}$

a $\mathbf{q} - \mathbf{p}$

$= \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} 1 - 3 \\ 4 + 2 \end{pmatrix}$

$= \begin{pmatrix} -2 \\ 6 \end{pmatrix}$

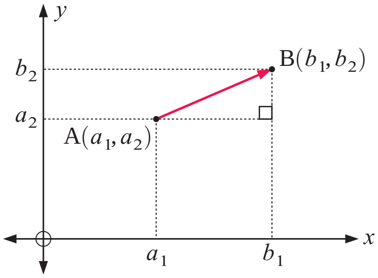
b $\mathbf{p} - \mathbf{q} - \mathbf{r}$

$= \begin{pmatrix} 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ -5 \end{pmatrix}$

$= \begin{pmatrix} 3 - 1 + 2 \\ -2 - 4 + 5 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

VECTORS BETWEEN TWO POINTS



$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \mathbf{b} - \mathbf{a}$

$\therefore \quad \overrightarrow{AB} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$

The position vector of $B(b_1, b_2)$ relative to $A(a_1, a_2)$ is $\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$.

EXERCISE 12C.2

VECTOR RACE GAME

1 If $\mathbf{a} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$ find:

- **a** $\mathbf{a} + \mathbf{b}$
- **b** $\mathbf{b} + \mathbf{a}$
- **c** $\mathbf{b} + \mathbf{c}$
- **d** $\mathbf{c} + \mathbf{b}$
- **e** $\mathbf{a} + \mathbf{c}$
- **f** $\mathbf{c} + \mathbf{a}$
- **g** $\mathbf{a} + \mathbf{a}$
- **h** $\mathbf{b} + \mathbf{a} + \mathbf{c}$

2 Given $\mathbf{p} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ find:

- **a** $\mathbf{p} - \mathbf{q}$
- **b** $\mathbf{q} - \mathbf{r}$
- **c** $\mathbf{p} + \mathbf{q} - \mathbf{r}$
- **d** $\mathbf{p} - \mathbf{q} - \mathbf{r}$
- **e** $\mathbf{q} - \mathbf{r} - \mathbf{p}$
- **f** $\mathbf{r} + \mathbf{q} - \mathbf{p}$

3 **a** Given $\overrightarrow{BA} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ find $\overrightarrow{AC}$.

b If $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and $\overrightarrow{CA} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, find $\overrightarrow{CB}$.

c If $\overrightarrow{PQ} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, $\overrightarrow{RQ} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\overrightarrow{RS} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, find $\overrightarrow{SP}$.

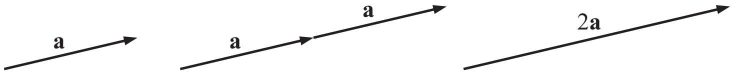
4 Find $\overrightarrow{AB}$ given:

a A(2, 3) and B(4, 7) **b** A(3, −1) and B(1, 4) **c** A(−2, 7) and B(1, 4)

d B(3, 0) and A(2, 5) **e** B(6, −1) and A(0, 4) **f** B(0, 0) and A(−1, −3).

ALGEBRAIC SCALAR MULTIPLICATION

We have already seen a geometric approach for scalar multiplication:



Consider $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

$$2\mathbf{a} = \mathbf{a} + \mathbf{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \quad \text{and}$$

$$3\mathbf{a} = \mathbf{a} + \mathbf{a} + \mathbf{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$$

Examples like these suggest the following algebraic definition for **scalar multiplication**:

If k is a scalar and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, then $k\mathbf{v} = \begin{pmatrix} kv_1 \\ kv_2 \end{pmatrix}$.

Notice that:

- $(-1)\mathbf{v} = \begin{pmatrix} (-1)\,v_1 \\ (-1)\,v_2 \end{pmatrix} = \begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix} = -\mathbf{v}$
- $(0)\mathbf{v} = \begin{pmatrix} (0)\,v_1 \\ (0)\,v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}$

Example 14 **Self Tutor**

For $\mathbf{p} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ find: **a** $3\mathbf{q}$ **b** $\mathbf{p} + 2\mathbf{q}$ **c** $\frac{1}{2}\mathbf{p} - 3\mathbf{q}$

a $3\mathbf{q}$

$$= 3\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ -9 \end{pmatrix}$$

b $\mathbf{p} + 2\mathbf{q}$

$$= \begin{pmatrix} 4 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 + 2(2) \\ 1 + 2(-3) \end{pmatrix}$$

$$= \begin{pmatrix} 8 \\ -5 \end{pmatrix}$$

c $\frac{1}{2}\mathbf{p} - 3\mathbf{q}$

$$= \tfrac{1}{2}\begin{pmatrix} 4 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}(4) - 3(2) \\ \frac{1}{2}(1) - 3(-3) \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ 9\frac{1}{2} \end{pmatrix}$$

EXERCISE 12C.3

1 For $\mathbf{p}=\begin{pmatrix}1\\5\end{pmatrix}$, $\mathbf{q}=\begin{pmatrix}-2\\4\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}-3\\-1\end{pmatrix}$ find:

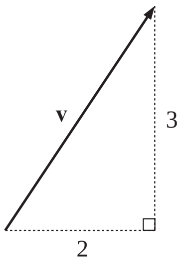
a $-3\mathbf{p}$ **b** $\frac{1}{2}\mathbf{q}$ **c** $2\mathbf{p}+\mathbf{q}$ **d** $\mathbf{p}-2\mathbf{q}$

e $\mathbf{p}-\frac{1}{2}\mathbf{r}$ **f** $2\mathbf{p}+3\mathbf{r}$ **g** $2\mathbf{q}-3\mathbf{r}$ **h** $2\mathbf{p}-\mathbf{q}+\frac{1}{3}\mathbf{r}$

2 If $\mathbf{p}=\begin{pmatrix}1\\1\end{pmatrix}$ and $\mathbf{q}=\begin{pmatrix}2\\-1\end{pmatrix}$ find by diagram and then comment on the results:

a $\mathbf{p}+\mathbf{p}+\mathbf{q}+\mathbf{q}+\mathbf{q}$ **b** $\mathbf{p}+\mathbf{q}+\mathbf{p}+\mathbf{q}+\mathbf{q}$ **c** $\mathbf{q}+\mathbf{p}+\mathbf{q}+\mathbf{p}+\mathbf{q}$

ALGEBRAIC LENGTH OF A VECTOR



Consider vector $\mathbf{v}=\begin{pmatrix}2\\3\end{pmatrix}$ as illustrated.

Recall that $|\mathbf{v}|$ represents the length of $\mathbf{v}$.

By Pythagoras, $|\mathbf{v}|^2=2^2+3^2=4+9=13$

$\therefore\ |\mathbf{v}|=\sqrt{13}$ units $\{\text{since } |\mathbf{v}|>0\}$

If $\mathbf{v}=\begin{pmatrix}v_1\\v_2\end{pmatrix}$, then $|\mathbf{v}|=\sqrt{{v_1}^2+{v_2}^2}$.

Example 15 **Self Tutor**

If $\mathbf{p}=\begin{pmatrix}3\\-5\end{pmatrix}$ and $\mathbf{q}=\begin{pmatrix}-1\\-2\end{pmatrix}$ find: **a** $|\mathbf{p}|$ **b** $|\mathbf{q}|$ **c** $|\mathbf{p}-2\mathbf{q}|$

a $\mathbf{p}=\begin{pmatrix}3\\-5\end{pmatrix}$ $\therefore\ |\mathbf{p}|=\sqrt{9+25}$
$=\sqrt{34}$ units

b $\mathbf{q}=\begin{pmatrix}-1\\-2\end{pmatrix}$ $\therefore\ |\mathbf{q}|=\sqrt{1+4}$
$=\sqrt{5}$ units

c $\mathbf{p}-2\mathbf{q}=\begin{pmatrix}3\\-5\end{pmatrix}-2\begin{pmatrix}-1\\-2\end{pmatrix}=\begin{pmatrix}5\\-1\end{pmatrix}$

$\therefore\ |\mathbf{p}-2\mathbf{q}|=\sqrt{5^2+(-1)^2}$
$=\sqrt{26}$ units

EXERCISE 12C.4

1 For $\mathbf{r}=\begin{pmatrix}2\\3\end{pmatrix}$ and $\mathbf{s}=\begin{pmatrix}-1\\4\end{pmatrix}$ find:

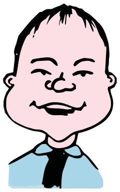
a $|\mathbf{r}|$ **b** $|\mathbf{s}|$ **c** $|\mathbf{r}+\mathbf{s}|$ **d** $|\mathbf{r}-\mathbf{s}|$ **e** $|\mathbf{s}-2\mathbf{r}|$

2 If $\mathbf{p}=\begin{pmatrix}1\\3\end{pmatrix}$ and $\mathbf{q}=\begin{pmatrix}-2\\4\end{pmatrix}$ find:

a $|\mathbf{p}|$ **b** $|2\mathbf{p}|$ **c** $|-2\mathbf{p}|$ **d** $|3\mathbf{p}|$ **e** $|-3\mathbf{p}|$

f $|\mathbf{q}|$ **g** $|4\mathbf{q}|$ **h** $|-4\mathbf{q}|$ **i** $|\frac{1}{2}\mathbf{q}|$ **j** $|-\frac{1}{2}\mathbf{q}|$

3 From your answers in **2**, you should have noticed that $|k\mathbf{v}| = |k|\,|\mathbf{v}|$.
So, (the length of $k\mathbf{v}$) = (the modulus of k) $\times$ (the length of $\mathbf{v}$).
By letting $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, prove that $|k\mathbf{v}| = |k|\,|\mathbf{v}|$.



The *modulus* of k is its *size*.

4 The length of the vector between A and B is denoted $|\overrightarrow{AB}|$ or simply AB.
Given A(2, -1), B(3, 5), C(-1, 4) and D(-4, -3), find:

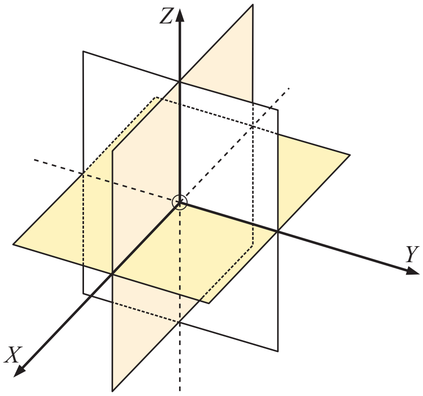
a $\overrightarrow{AB}$ and AB **b** $\overrightarrow{BA}$ and BA **c** $\overrightarrow{BC}$ and BC
d $\overrightarrow{DC}$ and DC **e** $\overrightarrow{CA}$ and CA **f** $\overrightarrow{DA}$ and DA.

D 3-D COORDINATE GEOMETRY

To specify points in **3-dimensional space** we need a point of reference, O, called the **origin**.

Through O we draw 3 **mutually perpendicular** lines and call them the X, Y and Z-axes. We often think of the YZ-plane as the plane of the page, with the X-axis coming directly out of the page. However, we cannot of course draw this.

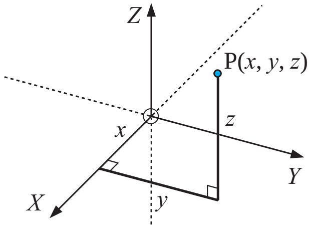
In the diagram alongside the **coordinate planes** divide space into 8 regions, each pair of planes intersecting on the axes.



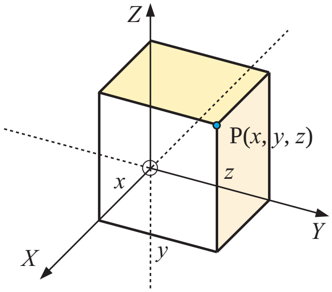
The **positive direction** of each axis is a solid line whereas the **negative direction** is ‘dashed’.

Any point P in space can be specified by an **ordered triple** of numbers (x, y, z) where x, y and z are the **steps** in the X, Y and Z directions from the origin O, to P.

The **position vector** of P is $\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.



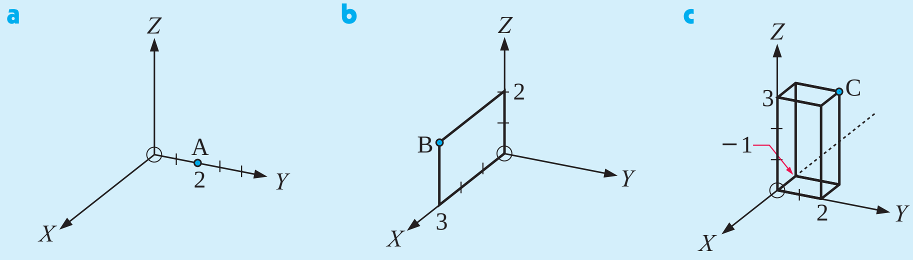
To help us visualise the 3-D position of a point on our 2-D paper, it is useful to complete a rectangular prism or box with the origin O as one vertex, the axes as sides adjacent to it, and P being the vertex opposite O.



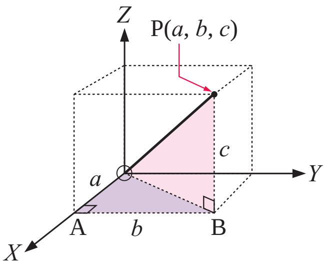
3-D POINT PLOTTER

Example 16 **Self Tutor**

Illustrate the points: **a** A(0, 2, 0) **b** B(3, 0, 2) **c** C(−1, 2, 3)



DISTANCE AND MIDPOINTS



Triangle OAB is right angled at A

$\therefore \quad OB^2 = a^2 + b^2$ ...... (1) {Pythagoras}

Triangle OBP is right angled at B

$\therefore \quad OP^2 = OB^2 + c^2$ {Pythagoras}

$\therefore \quad OP^2 = a^2 + b^2 + c^2$ {from (1)}

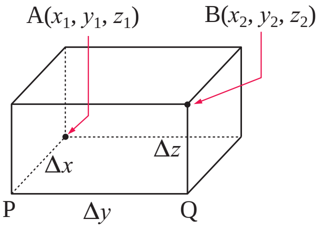
$\therefore \quad OP = \sqrt{a^2 + b^2 + c^2}$

In general, the magnitude or length of the vector

$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ is $|\mathbf{v}| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2}$.

For two general points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$:

- the x-step from A to B $= x_2 - x_1 = \Delta x$
- the y-step from A to B $= y_2 - y_1 = \Delta y$
- the z-step from A to B $= z_2 - z_1 = \Delta z$



The distance between the points A and B is

$$\mathbf{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\,.$$

A simple extension from 2-D to 3-D geometry also gives

the **midpoint** of [AB] is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}, \dfrac{z_1 + z_2}{2}\right)$.

As with the 2-D case, a proof of this rule can be made using similar triangles.

Example 17 Self Tutor

If A$(-1, 2, 4)$ and B$(1, 0, -1)$ are two points in space, find:

a the distance from A to B **b** the coordinates of the midpoint of [AB].

a AB

$$= \sqrt{(1 - -1)^2 + (0-2)^2 + (-1-4)^2}$$
$$= \sqrt{4+4+25}$$
$$= \sqrt{33} \text{ units}$$

b The midpoint is

$$\left(\tfrac{-1+1}{2}, \tfrac{2+0}{2}, \tfrac{4+(-1)}{2}\right)$$

which is $(0, 1, \frac{3}{2})$.

EXERCISE 12D

1 Illustrate P and find its distance from the origin O if P is:

a $(0, 0, -3)$ **b** $(0, -1, 2)$ **c** $(3, 1, 4)$ **d** $(-1, -2, 3)$

2 For each of the following:

i find the distance AB **ii** find the midpoint of [AB]

a A$(-1, 2, 3)$ and B$(0, -1, 1)$ **b** A$(0, 0, 0)$ and B$(2, -1, 3)$

c A$(3, -1, -1)$ and B$(-1, 0, 1)$ **d** A$(2, 0, -3)$ and B$(0, 1, 0)$.

3 Show that P$(0, 4, 4)$, Q$(2, 6, 5)$ and R$(1, 4, 3)$ are vertices of an isosceles triangle.

4 Determine the nature of triangle ABC using distances:

a A$(2, -1, 7)$, B$(3, 1, 4)$ and C$(5, 4, 5)$

b A$(0, 0, 3)$, B$(2, 8, 1)$ and C$(-9, 6, 18)$

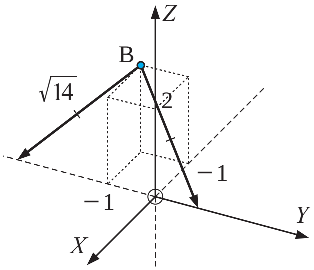
c A$(5, 6, -2)$, B$(6, 12, 9)$ and C$(2, 4, 2)$

d A$(1, 0, -3)$, B$(2, 2, 0)$ and C$(4, 6, 6)$.

5 A sphere has centre C$(-1, 2, 4)$ and diameter [AB] where A is $(-2, 1, 3)$. Find the coordinates of B and the radius of the sphere.

6 **a** State the coordinates of any point on the Y-axis.

b Use **a** and the diagram opposite to find the coordinates of two points on the Y-axis which are $\sqrt{14}$ units from B$(-1, -1, 2)$.

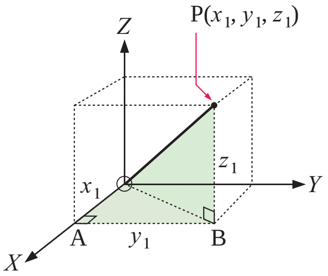


E 3-D VECTORS IN COMPONENT FORM

Consider a point $\mathrm{P}(x_1, y_1, z_1)$.

The x, y and z-steps from the origin to P are x_1, y_1 and z_1 respectively.

So, $\overrightarrow{\mathrm{OP}} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ is the vector which emanates from O and terminates at P.



If $\mathrm{A}(x_1, y_1, z_1)$ and $\mathrm{B}(x_2, y_2, z_2)$ are two points in space then:

$$\overrightarrow{\mathrm{AB}} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} \begin{matrix} \longleftarrow x\text{-step} \\ \longleftarrow y\text{-step} \\ \longleftarrow z\text{-step} \end{matrix}$$

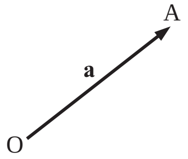
$\overrightarrow{\mathrm{AB}}$ is called the 'vector AB' or the '**position vector of B relative to A**'.

$\overrightarrow{\mathrm{OP}}$ is called the **position vector of point P** in both 2-D and 3-D. Its usefulness is marked by the fact that its components are exactly the same as the coordinates of the point P.

GEOMETRIC REPRESENTATION

As for 2-D vectors, 3-D vectors are represented by **directed line segments** or **arrows**.

Consider the vector $\overrightarrow{\mathrm{OA}}$ or **a** represented by the directed line segment from O to A.



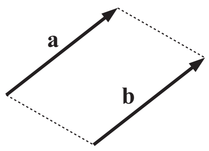
The **magnitude** of the vector is $|\overrightarrow{\mathrm{OA}}|$, OA or $|\mathbf{a}|$.

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ then $|\mathbf{a}| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$.

VECTOR EQUALITY

Two vectors are **equal** if they have the same magnitude and direction.

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then $\mathbf{a} = \mathbf{b} \Leftrightarrow a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3$.



$\mathbf{a} = \mathbf{b}$ implies that vector **a** is parallel to vector **b**, and $|\mathbf{a}| = |\mathbf{b}|$.

Consequently, **a** and **b** are opposite sides of a parallelogram and lie in the same plane.

DISCUSSION

- Do any three points in space define a plane? What about four points? Illustrate.
- What simple tests on four points in space enable us to deduce that the points are vertices of a parallelogram? Consider using vectors and not using vectors.

Example 18

Self Tutor

If A is $(3, -1, 2)$ and B is $(1, 0, -2)$ find: **a** $\overrightarrow{OA}$ **b** $\overrightarrow{AB}$

a $\overrightarrow{OA} = \begin{pmatrix} 3-0 \\ -1-0 \\ 2-0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

b $\overrightarrow{AB} = \begin{pmatrix} 1-3 \\ 0-(-1) \\ -2-2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$

Example 19

Self Tutor

If P is $(-3, 1, 2)$ and Q is $(1, -1, 3)$, find $|\overrightarrow{PQ}|$.

$\overrightarrow{PQ} = \begin{pmatrix} 1-(-3) \\ -1-1 \\ 3-2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$

$$\therefore\ |\overrightarrow{PQ}| = \sqrt{4^2+(-2)^2+1^2}$$
$$= \sqrt{21} \text{ units}$$

Example 20

Self Tutor

If A is $(-1, 3, 2)$ and B is $(2, 1, -4)$, find:

a the position vector of A from B **b** the distance between A and B.

a The position vector of A from B is

$$\overrightarrow{BA} = \begin{pmatrix} -1-2 \\ 3-1 \\ 2-(-4) \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix}$$

b $|\overrightarrow{AB}| = |\overrightarrow{BA}|$

$$= \sqrt{9+4+36}$$
$$= 7 \text{ units}$$

Example 21

Self Tutor

Find a, b, and c if $\begin{pmatrix} a-3 \\ b-2 \\ c-1 \end{pmatrix} = \begin{pmatrix} 1-a \\ -b \\ -3-c \end{pmatrix}$.

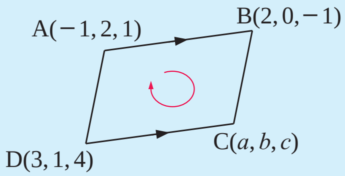
Equating components, $a-3=1-a$, $b-2=-b$ and $c-1=-3-c$

$\therefore\ 2a=4$, $2b=2$ and $2c=-2$

$\therefore\ a=2$, $b=1$ and $c=-1$

Example 22 **Self Tutor**

ABCD is a parallelogram. A is $(-1, 2, 1)$, B is $(2, 0, -1)$ and D is $(3, 1, 4)$.
Find the coordinates of C.



First we sketch the points as shown:

Let C be (a, b, c).

Now [AB] is parallel to [DC] and has the same length, so $\overrightarrow{DC} = \overrightarrow{AB}$

$$\therefore \begin{pmatrix} a-3 \\ b-1 \\ c-4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix}$$

$\therefore \quad a-3=3, \quad b-1=-2, \quad c-4=-2$

$\therefore \quad a=6, \quad b=-1, \quad c=2$ So, C is $(6, -1, 2)$.

Check: Midpoint of [DB] is

$$\left(\frac{3+2}{2}, \frac{1+0}{2}, \frac{4+-1}{2}\right)$$

or $\left(\frac{5}{2}, \frac{1}{2}, \frac{3}{2}\right)$

Midpoint of [AC] is

$$\left(\frac{-1+6}{2}, \frac{2+-1}{2}, \frac{1+2}{2}\right)$$

or $\left(\frac{5}{2}, \frac{1}{2}, \frac{3}{2}\right)$

Since the midpoints are the same, the diagonals of the parallelogram bisect. ✓

EXERCISE 12E

1 Consider the point $T(3, -1, 4)$.

- **a** Draw a diagram to locate the position of T in space.
- **b** Find $\overrightarrow{OT}$.
- **c** How far is it from O to T?

2 Given $A(-3, 1, 2)$ and $B(1, 0, -1)$ find:

- **a** $\overrightarrow{AB}$ and $\overrightarrow{BA}$
- **b** the lengths of $\overrightarrow{AB}$ and $\overrightarrow{BA}$.

3 Given $A(3, 1, 0)$ and $B(-1, 1, 2)$ find $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{AB}$.

4 Given $M(4, -2, -1)$ and $N(-1, 2, 0)$ find:

- **a** the position vector of M from N
- **b** the position vector of N from M
- **c** the distance between M and N.

5 For $A(-1, 2, 5)$, $B(2, 0, 3)$ and $C(-3, 1, 0)$ find the position vector of:

- **a** A from O and the distance from O to A
- **b** C from A and the distance from A to C
- **c** B from C and the distance from C to B.

6 Find the distance from $Q(3, 1, -2)$ to:

- **a** the Y-axis
- **b** the origin
- **c** the YOZ plane.

7 Find a, b and c if: **a** $\begin{pmatrix} a-4 \\ b-3 \\ c+2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$ **b** $\begin{pmatrix} a-5 \\ b-2 \\ c+3 \end{pmatrix} = \begin{pmatrix} 3-a \\ 2-b \\ 5-c \end{pmatrix}$

8 Find scalars a, b and c if:

a $2\begin{pmatrix} 1 \\ 0 \\ 3a \end{pmatrix} = \begin{pmatrix} b \\ c-1 \\ 2 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ a \\ 3 \end{pmatrix} = \begin{pmatrix} b \\ a^2 \\ a+b \end{pmatrix}$

c $a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$

9 A$(-1, 3, 4)$, B$(2, 5, -1)$, C$(-1, 2, -2)$ and D (r, s, t) are four points in space.
Find r, s and t if: **a** $\overrightarrow{AC} = \overrightarrow{BD}$ **b** $\overrightarrow{AB} = \overrightarrow{DC}$.

10 A quadrilateral has vertices A$(1, 2, 3)$, B$(3, -3, 2)$, C$(7, -4, 5)$ and D$(5, 1, 6)$.

a Find $\overrightarrow{AB}$ and $\overrightarrow{DC}$.

b What can be deduced about the quadrilateral ABCD?

11 PQRS is a parallelogram. P is $(-1, 2, 3)$, Q is $(1, -2, 5)$ and R is $(0, 4, -1)$.

a Use vectors to find the coordinates of S.

b Use midpoints of diagonals to check your answer.

F ALGEBRAIC OPERATIONS WITH VECTORS

The rules for algebra with vectors readily extend from 2-D to 3-D:

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then $\mathbf{a}+\mathbf{b} = \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{pmatrix}$, $\mathbf{a}-\mathbf{b} = \begin{pmatrix} a_1-b_1 \\ a_2-b_2 \\ a_3-b_3 \end{pmatrix}$,

and $k\mathbf{a} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}$ for some scalar k.

SOME PROPERTIES OF VECTORS

- $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$
- $(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c})$
- $\mathbf{a}+\mathbf{0} = \mathbf{0}+\mathbf{a} = \mathbf{a}$
- $\mathbf{a}+(-\mathbf{a}) = (-\mathbf{a})+\mathbf{a} = \mathbf{0}$
- $\underbrace{|k\mathbf{a}|}_{\text{length of } k\mathbf{a}} = \underbrace{|k|}_{\text{modulus of } k}\,\underbrace{|\mathbf{a}|}_{\text{length of } \mathbf{a}}$ where $k\mathbf{a}$ is parallel to $\mathbf{a}$

The rules for solving vector equations are similar to those for solving real number equations. However, there is no such thing as dividing a vector by a scalar. Instead, we multiply by reciprocals.

For example, if $2\mathbf{x} = \mathbf{a}$ then $\mathbf{x} = \frac{1}{2}\mathbf{a}$ and *not* $\dfrac{\mathbf{a}}{2}$.

$\dfrac{\mathbf{a}}{2}$ has no meaning in vector algebra.

Two useful rules are:

- if $\mathbf{x} + \mathbf{a} = \mathbf{b}$ then $\mathbf{x} = \mathbf{b} - \mathbf{a}$
- if $k\mathbf{x} = \mathbf{a}$ then $\mathbf{x} = \frac{1}{k}\mathbf{a}$ $\quad (k \neq 0)$

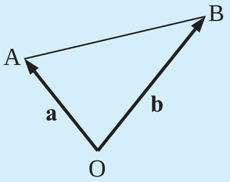
To establish these notice that:

$$\begin{aligned} \text{if} \quad \mathbf{x} + \mathbf{a} &= \mathbf{b} \\ \text{then} \quad \mathbf{x} + \mathbf{a} + (-\mathbf{a}) &= \mathbf{b} + (-\mathbf{a}) \\ \therefore \quad \mathbf{x} + \mathbf{0} &= \mathbf{b} - \mathbf{a} \\ \therefore \quad \mathbf{x} &= \mathbf{b} - \mathbf{a} \end{aligned}$$

$$\begin{aligned} \text{and if} \quad k\mathbf{x} &= \mathbf{a} \\ \text{then} \quad \tfrac{1}{k}(k\mathbf{x}) &= \tfrac{1}{k}\mathbf{a} \\ \therefore \quad 1\mathbf{x} &= \tfrac{1}{k}\mathbf{a} \\ \therefore \quad \mathbf{x} &= \tfrac{1}{k}\mathbf{a} \end{aligned}$$

Another useful property is:

If $\overrightarrow{\mathrm{OA}} = \mathbf{a}$ and $\overrightarrow{\mathrm{OB}} = \mathbf{b}$ where O is the origin then $\overrightarrow{\mathrm{AB}} = \mathbf{b} - \mathbf{a}$ and $\overrightarrow{\mathrm{BA}} = \mathbf{a} - \mathbf{b}$.



Example 23 Self Tutor

Solve for $\mathbf{x}$:

a $3\mathbf{x} - \mathbf{r} = \mathbf{s}$

b $\mathbf{c} - 2\mathbf{x} = \mathbf{d}$

a
$$\begin{aligned} 3\mathbf{x} - \mathbf{r} &= \mathbf{s} \\ \therefore \quad 3\mathbf{x} &= \mathbf{s} + \mathbf{r} \\ \therefore \quad \mathbf{x} &= \tfrac{1}{3}(\mathbf{s} + \mathbf{r}) \end{aligned}$$

b
$$\begin{aligned} \mathbf{c} - 2\mathbf{x} &= \mathbf{d} \\ \therefore \quad \mathbf{c} - \mathbf{d} &= 2\mathbf{x} \\ \therefore \quad \tfrac{1}{2}(\mathbf{c} - \mathbf{d}) &= \mathbf{x} \end{aligned}$$

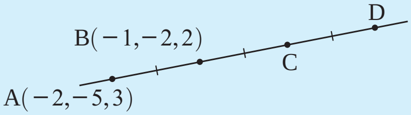
Example 24 Self Tutor

If $\mathbf{a} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$, find $|\,2\mathbf{a}\,|$.

$$\begin{aligned} |\,2\mathbf{a}\,| &= 2\,|\,\mathbf{a}\,| \\ &= 2\sqrt{(-1)^2 + 3^2 + 2^2} \\ &= 2\sqrt{1 + 9 + 4} \\ &= 2\sqrt{14} \quad \text{units} \end{aligned}$$

Example 25 Self Tutor

Find the coordinates of C and D in:



$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$$

$$\begin{aligned} \overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AC} \\ &= \overrightarrow{OA} + 2\overrightarrow{AB} \\ &= \begin{pmatrix} -2 \\ -5 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

$\therefore$ C is $(0, 1, 1)$

$$\begin{aligned} \overrightarrow{OD} &= \overrightarrow{OA} + \overrightarrow{AD} \\ &= \overrightarrow{OA} + 3\overrightarrow{AB} \\ &= \begin{pmatrix} -2 \\ -5 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 9 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} \end{aligned}$$

$\therefore$ D is $(1, 4, 0)$

EXERCISE 12F

1 Solve the following vector equations for $\mathbf{x}$:

a $2\mathbf{x} = \mathbf{q}$
b $\frac{1}{2}\mathbf{x} = \mathbf{n}$
c $-3\mathbf{x} = \mathbf{p}$
d $\mathbf{q} + 2\mathbf{x} = \mathbf{r}$
e $4\mathbf{s} - 5\mathbf{x} = \mathbf{t}$
f $4\mathbf{m} - \frac{1}{3}\mathbf{x} = \mathbf{n}$

2 Suppose $\mathbf{r} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Find $\mathbf{y}$ if:

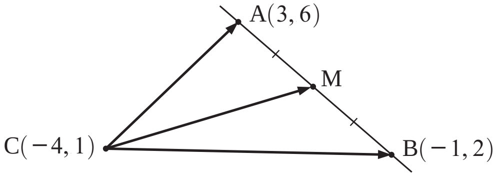
a $2\mathbf{y} = \mathbf{r}$
b $\frac{1}{2}\mathbf{y} = \mathbf{s}$
c $\mathbf{r} + 2\mathbf{y} = \mathbf{s}$
d $3\mathbf{s} - 4\mathbf{y} = \mathbf{r}$

3 Show by equating components, that if $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $k\mathbf{x} = \mathbf{a}$, then $\mathbf{x} = \frac{1}{k}\mathbf{a}$.

4 Suppose that C is the centre of a circle with diameter [AB]. Find B given the following coordinates:

a A(3, -2) and C(1, 4)
b A(0, 5) and C(-1, -2)
c A(-1, -4) and C(3, 0).

5



a Find the coordinates of M.
b Find vectors $\overrightarrow{CA}$, $\overrightarrow{CM}$ and $\overrightarrow{CB}$.
c Verify that $\overrightarrow{CM} = \frac{1}{2}\overrightarrow{CA} + \frac{1}{2}\overrightarrow{CB}$.

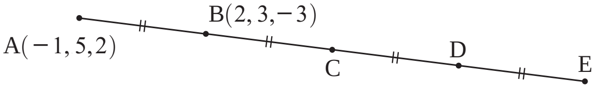
6 Suppose $\mathbf{a} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$. Find $\mathbf{x}$ if:

a $2\mathbf{a} + \mathbf{x} = \mathbf{b}$
b $3\mathbf{x} - \mathbf{a} = 2\mathbf{b}$
c $2\mathbf{b} - 2\mathbf{x} = -\mathbf{a}$

7 If $\overrightarrow{OA} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$ find $\overrightarrow{AB}$ and hence the distance from A to B.

8 The position vectors of A, B, C and D from O are $\begin{pmatrix}2\\1\\-2\end{pmatrix}$, $\begin{pmatrix}0\\3\\-4\end{pmatrix}$, $\begin{pmatrix}1\\-2\\1\end{pmatrix}$ and $\begin{pmatrix}-2\\-3\\2\end{pmatrix}$ respectively. Deduce that $\overrightarrow{BD} = 2\overrightarrow{AC}$.

9 Find the coordinates of C, D and E.



10 Use vectors to determine whether ABCD is a parallelogram:

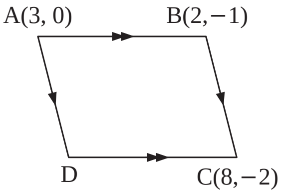
a A(3, −1), B(4, 2), C(−1, 4) and D(−2, 1)

b A(5, 0, 3), B(−1, 2, 4), C(4, −3, 6) and D(10, −5, 5)

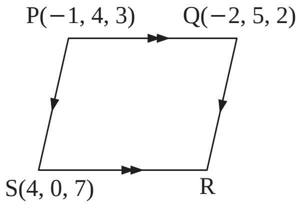
c A(2, −3, 2), B(1, 4, −1), C(−2, 6, −2) and D(−1, −1, 2).

11 Use vector methods to find the remaining vertex of:

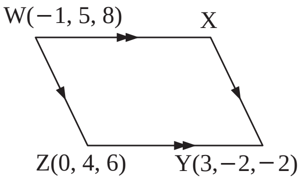
a



b



c



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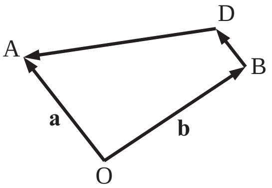


In the given figure [BD] is parallel to [OA] and half its length. Find, in terms of **a** and **b**, vector expressions for:

a $\overrightarrow{BD}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{BA}$

d $\overrightarrow{OD}$ **e** $\overrightarrow{AD}$ **f** $\overrightarrow{DA}$

13 If $\overrightarrow{AB} = \begin{pmatrix}-1\\3\\2\end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix}2\\-1\\4\end{pmatrix}$ and $\overrightarrow{BD} = \begin{pmatrix}0\\2\\-3\end{pmatrix}$ find:

a $\overrightarrow{AD}$ **b** $\overrightarrow{CB}$ **c** $\overrightarrow{CD}$

14 For $\mathbf{a} = \begin{pmatrix}2\\-1\\1\end{pmatrix}$, $\mathbf{b} = \begin{pmatrix}1\\2\\-3\end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix}0\\1\\-3\end{pmatrix}$, find:

a $\mathbf{a}+\mathbf{b}$ **b** $\mathbf{a}-\mathbf{b}$ **c** $\mathbf{b}+2\mathbf{c}$ **d** $\mathbf{a}-3\mathbf{c}$

e $\mathbf{a}+\mathbf{b}+\mathbf{c}$ **f** $\mathbf{c}-\frac{1}{2}\mathbf{a}$ **g** $\mathbf{a}-\mathbf{b}-\mathbf{c}$ **h** $2\mathbf{b}-\mathbf{c}+\mathbf{a}$

15 If $\mathbf{a} = \begin{pmatrix}-1\\1\\3\end{pmatrix}$, $\mathbf{b} = \begin{pmatrix}1\\-3\\2\end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix}-2\\2\\4\end{pmatrix}$ find:

a $|\mathbf{a}|$ **b** $|\mathbf{b}|$ **c** $|\mathbf{b}+\mathbf{c}|$ **d** $|\mathbf{a}-\mathbf{c}|$ **e** $|\mathbf{a}|\,\mathbf{b}$ **f** $\frac{1}{|\mathbf{a}|}\,\mathbf{a}$

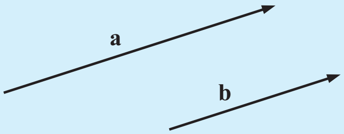
16 Find scalars r and s such that:

a $r\begin{pmatrix}1\\-1\end{pmatrix}+s\begin{pmatrix}2\\5\end{pmatrix}=\begin{pmatrix}-8\\-27\end{pmatrix}$

b $r\begin{pmatrix}2\\-3\\1\end{pmatrix}+s\begin{pmatrix}1\\7\\2\end{pmatrix}=\begin{pmatrix}7\\-19\\2\end{pmatrix}$

G PARALLELISM

Two non-zero vectors are **parallel** if and only if one is a scalar multiple of the other.



- If **a** is parallel to **b**, then there exists a scalar k such that $\mathbf{a}=k\mathbf{b}$.
- If $\mathbf{a}=k\mathbf{b}$ for some scalar k, then
  - **a** is parallel to **b**, and
  - $|\mathbf{a}|=|k|\,|\mathbf{b}|$.

For example:

$\mathbf{a}=\begin{pmatrix}2\\6\\-4\end{pmatrix}$ is parallel to $\mathbf{b}=\begin{pmatrix}1\\3\\-2\end{pmatrix}$ and $\mathbf{c}=\begin{pmatrix}4\\12\\-8\end{pmatrix}$ as $\mathbf{a}=2\mathbf{b}$ and $\mathbf{a}=\frac{1}{2}\mathbf{c}$.

$\mathbf{a}=\begin{pmatrix}2\\6\\-4\end{pmatrix}$ is also parallel to $\mathbf{d}=\begin{pmatrix}-3\\-9\\6\end{pmatrix}$ as $\mathbf{a}=-\frac{2}{3}\mathbf{d}$.

Example 26 **Self Tutor**

Find r and s given that $\mathbf{a}=\begin{pmatrix}2\\-1\\r\end{pmatrix}$ is parallel to $\mathbf{b}=\begin{pmatrix}s\\2\\-3\end{pmatrix}$.

Since **a** and **b** are parallel, $\mathbf{a}=k\mathbf{b}$ for some scalar k.

$\therefore \begin{pmatrix}2\\-1\\r\end{pmatrix}=k\begin{pmatrix}s\\2\\-3\end{pmatrix}$ $\quad\therefore\ 2=ks,\ \ -1=2k$ and $r=-3k$

Consequently, $k=-\frac{1}{2}$ and $\therefore\ 2=-\frac{1}{2}s$ and $r=-3\left(-\frac{1}{2}\right)$

$\therefore\ r=\frac{3}{2}$ and $s=-4$

EXERCISE 12G

1 $\mathbf{a}=\begin{pmatrix}2\\-1\\3\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}-6\\r\\s\end{pmatrix}$ are parallel. Find r and s.

2 Find scalars a and b given that $\begin{pmatrix}3\\-1\\2\end{pmatrix}$ and $\begin{pmatrix}a\\2\\b\end{pmatrix}$ are parallel.

3 **a** Find a vector of length 1 unit which is parallel to $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$.

Hint: Let the vector be $k\mathbf{a}$.

b Find a vector of length 2 units which is parallel to $\mathbf{b} = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$.

4 What can be deduced from the following?

a $\overrightarrow{AB} = 3\overrightarrow{CD}$ **b** $\overrightarrow{RS} = -\frac{1}{2}\overrightarrow{KL}$ **c** $\overrightarrow{AB} = 2\overrightarrow{BC}$

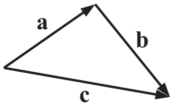
5 The position vectors of P, Q, R and S from O are $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$ respectively.

a Deduce that [PR] and [QS] are parallel.

b What is the relationship between the lengths of [PR] and [QS]?

6 **Triangle inequality**

In any triangle, the sum of any two sides must always be greater than the third side. This is based on the well known result: "the shortest distance between two points is a straight line".



Prove that $|\mathbf{a} + \mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|$ using a geometrical argument.

Hint: Consider:
- $\mathbf{a}$ is not parallel to $\mathbf{b}$ and use the triangle inequality
- $\mathbf{a}$ and $\mathbf{b}$ parallel
- any other cases.

H UNIT VECTORS

A **unit vector** is any vector which has a length of one unit.

For example:

- $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a unit vector as its length is $\sqrt{1^2 + 0^2} = 1$

$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are 2-dimensional unit vectors in the positive x and y-directions respectively.

- $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$ is a unit vector as its length is $\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + 0^2 + \left(-\frac{1}{\sqrt{2}}\right)^2} = 1$

$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are 3-dimensional unit vectors in the directions of the positive X, Y and Z-axes respectively.

Notice that $\mathbf{v} = \underset{\text{component form}}{\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}} \Leftrightarrow \mathbf{v} = \underbrace{v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}}_{\text{unit vector form}}$.

Thus, $\mathbf{v} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$ can be written as $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ and vice versa.

We call **i**, **j** and **k** the **base** vectors as any vector can be written as a linear combination of the vectors **i**, **j** and **k**.

Example 27 **Self Tutor**

Find the length of the 2-D vector $2\mathbf{i} - 5\mathbf{j}$.

As $2\mathbf{i} - 5\mathbf{j} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$, its length is $\sqrt{2^2 + (-5)^2}$

$= \sqrt{29}$ units.

Example 28 **Self Tutor**

Find k given that $\begin{pmatrix} -\frac{1}{3} \\ k \end{pmatrix}$ is a unit vector.

Since $\begin{pmatrix} -\frac{1}{3} \\ k \end{pmatrix}$ is a unit vector,

$$\begin{aligned} \sqrt{(-\tfrac{1}{3})^2 + k^2} &= 1 \\ \therefore \quad \sqrt{\tfrac{1}{9} + k^2} &= 1 \\ \therefore \quad \tfrac{1}{9} + k^2 &= 1 \\ \therefore \quad k^2 &= \tfrac{8}{9} \\ \therefore \quad k &= \pm\tfrac{\sqrt{8}}{3} \end{aligned}$$

- A unit vector in the direction of $\mathbf{v}$ is $\dfrac{1}{|\mathbf{v}|}\mathbf{v}$.
- A vector $\mathbf{b}$ of length k in the same direction as $\mathbf{a}$ is $\mathbf{b} = \dfrac{k}{|\mathbf{a}|}\mathbf{a}$.
- A vector $\mathbf{b}$ of length k which is *parallel to* $\mathbf{a}$ could be $\mathbf{b} = \pm\dfrac{k}{|\mathbf{a}|}\mathbf{a}$.

Example 29 **Self Tutor**

Find a vector $\mathbf{b}$ of length 7 in the opposite direction to the vector $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

The unit vector in the direction of $\mathbf{a}$ is $\dfrac{1}{|\mathbf{a}|}\mathbf{a} = \dfrac{1}{\sqrt{4+1+1}}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$$= \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

We now multiply this unit vector by -7. The negative reverses the direction and the 7 gives the required length.

Thus $\mathbf{b} = -\frac{7}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. Check that $|\mathbf{b}| = 7$.

EXERCISE 12H

1 Which of the following are unit vectors?

a $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ **b** $\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ **c** $\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$ **d** $\begin{pmatrix} -\frac{3}{5} \\ -\frac{4}{5} \end{pmatrix}$ **e** $\begin{pmatrix} \frac{2}{7} \\ -\frac{5}{7} \end{pmatrix}$

2 Write in terms of **i** and **j**:

a $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ **b** $\begin{pmatrix} -3 \\ -4 \end{pmatrix}$ **c** $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ **d** $\begin{pmatrix} 0 \\ 7 \end{pmatrix}$ **e** $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

3 Write in component form:

a $3\mathbf{i} + 5\mathbf{j}$ **b** $5\mathbf{i} - 4\mathbf{j}$ **c** $-4\mathbf{i}$ **d** $3\mathbf{j}$ **e** $\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$

4 Express the following vectors in component form and find their lengths:

a $\mathbf{i} - \mathbf{j} + \mathbf{k}$ **b** $3\mathbf{i} - \mathbf{j} + \mathbf{k}$ **c** $\mathbf{i} - 5\mathbf{k}$ **d** $\frac{1}{2}(\mathbf{j} + \mathbf{k})$

5 Find k for the unit vectors:

a $\begin{pmatrix} 0 \\ k \end{pmatrix}$ **b** $\begin{pmatrix} k \\ 0 \end{pmatrix}$ **c** $\begin{pmatrix} k \\ 1 \end{pmatrix}$ **d** $\begin{pmatrix} -\frac{1}{2} \\ k \\ \frac{1}{4} \end{pmatrix}$ **e** $\begin{pmatrix} k \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$

6 Find the lengths of the vectors:

a $3\mathbf{i} + 4\mathbf{j}$ **b** $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ **c** $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ **d** $-2.36\mathbf{i} + 5.65\mathbf{j}$

7 Find the unit vector in the direction of: **a** $\mathbf{i} + 2\mathbf{j}$ **b** $2\mathbf{i} - 3\mathbf{k}$ **c** $-2\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$

8 Find a vector **b** in:

a the same direction as $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and with length 3 units

b the opposite direction to $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$ and with length 2 units

c the same direction as $\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$ and with length 6 units

d the opposite direction to $\begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$ and with length 5 units.

I THE SCALAR PRODUCT OF TWO VECTORS

We have learned how to add, subtract and multiply vectors by a scalar. These operations have all been demonstrated to have practical uses. For example, **scalar multiplication** is used in the concept of **parallelism** and finding **unit vectors**.

There are two different forms of the **product of two vectors**, both of which have practical applications.

VECTOR PRODUCTS

For ordinary numbers numbers a and b we can write the product of a and b as ab or $a \times b$. There is only one interpretation for this product, and so we have developed power notation as a shorthand: $a^2 = a \times a$, $a^3 = a \times a \times a$, and so on.

However, there are *two* different types of product involving two vectors. These are:

- The **scalar product** of 2 vectors, which results in a **scalar** answer and has the notation $\mathbf{v} \bullet \mathbf{w}$ (read "**v** dot **w**").
- The **vector product** of 2 vectors, which results in a **vector** answer and has the notation $\mathbf{v} \times \mathbf{w}$ (read "**v** cross **w**").

Consequently, for vector **v**, $\mathbf{v}^2$ or $(\mathbf{v})^2$ has no meaning, as it not clear which of the vector products we mean. More generally, we should **never** write $\mathbf{v}^n$ or $(\mathbf{v})^n$.

In this course we consider only the scalar product.

SCALAR DOT PRODUCT

The **scalar product** of two vectors is also known as the **dot product** or **inner product**.

If $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$, the **scalar product** of **v** and **w** is defined as

$\mathbf{v} \bullet \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3$.

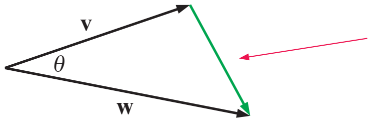
ANGLE BETWEEN VECTORS

Consider vectors:

$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and

$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

We translate one of the vectors so that they both emanate from the same point.



This vector is $-\mathbf{v} + \mathbf{w} = \mathbf{w} - \mathbf{v}$ and has length $|\,\mathbf{w} - \mathbf{v}\,|$.

Using the cosine rule, $|\mathbf{w} - \mathbf{v}|^2 = |\mathbf{v}|^2 + |\mathbf{w}|^2 - 2|\mathbf{v}||\mathbf{w}|\cos\theta$

$$\text{But} \quad \mathbf{w} - \mathbf{v} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} - \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} w_1 - v_1 \\ w_2 - v_2 \\ w_3 - v_3 \end{pmatrix}$$

$$\therefore \quad (w_1 - v_1)^2 + (w_2 - v_2)^2 + (w_3 - v_3)^2 = v_1^2 + v_2^2 + v_3^2 + w_1^2 + w_2^2 + w_3^2 - 2|\mathbf{v}||\mathbf{w}|\cos\theta$$

which simplifies to $v_1w_1 + v_2w_2 + v_3w_3 = |\mathbf{v}||\mathbf{w}|\cos\theta$

$$\therefore \quad \mathbf{v} \bullet \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos\theta$$

So, $\cos\theta = \dfrac{\mathbf{v} \bullet \mathbf{w}}{|\mathbf{v}||\mathbf{w}|}$ can be used to find the angle between two vectors **v** and **w**.

ALGEBRAIC PROPERTIES OF THE SCALAR PRODUCT

The scalar product has the following algebraic properties for both 2-D and 3-D vectors:

- $\mathbf{v} \bullet \mathbf{w} = \mathbf{w} \bullet \mathbf{v}$ and $\mathbf{v} \bullet \mathbf{v} = |\mathbf{v}|^2$
- $\mathbf{v} \bullet (\mathbf{w} + \mathbf{x}) = \mathbf{v} \bullet \mathbf{w} + \mathbf{v} \bullet \mathbf{x}$
- $(\mathbf{v} + \mathbf{w}) \bullet (\mathbf{x} + \mathbf{y}) = \mathbf{v} \bullet \mathbf{x} + \mathbf{v} \bullet \mathbf{y} + \mathbf{w} \bullet \mathbf{x} + \mathbf{w} \bullet \mathbf{y}$

These properties are proven in general by using vectors such as: $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$.

Be careful not to confuse the **scalar product**, which is the product of two vectors to give a scalar answer, with **scalar multiplication**, which is the product of a scalar and a vector to give a parallel vector. They are quite different.

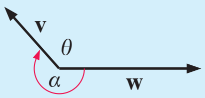
GEOMETRIC PROPERTIES OF THE SCALAR PRODUCT

- If θ is the angle between vectors **v** and **w** then: $\mathbf{v} \bullet \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos\theta$

  So, if θ is acute, $\cos\theta > 0$ and $\therefore$ $\mathbf{v} \bullet \mathbf{w} > \mathbf{0}$

  if θ is obtuse, $\cos\theta < 0$ and $\therefore$ $\mathbf{v} \bullet \mathbf{w} < \mathbf{0}$.

Two vectors form two angles θ and α as in the diagram drawn. The angle between two vectors is always taken as the smaller angle, so we take θ to be the angle between the two vectors with $0^o \leqslant \theta \leqslant 180^o$.



- For non-zero vectors **v** and **w**:

  $\mathbf{v} \bullet \mathbf{w} = 0 \quad \Leftrightarrow$ **v** and **w** are **perpendicular** or **orthogonal**.
- $\mathbf{v} \bullet \mathbf{w} = \pm|\mathbf{v}||\mathbf{w}| \quad \Leftrightarrow$ **v** and **w** are non-zero **parallel vectors**.

We can demonstrate these results as follows:

If **v** is perpendicular to **w** then $\theta = 90^o$.

$$\begin{aligned} \therefore\quad \mathbf{v} \bullet \mathbf{w} &= |\mathbf{v}|\,|\mathbf{w}|\cos\theta \\ &= |\mathbf{v}|\,|\mathbf{w}|\cos 90^o \\ &= 0 \end{aligned}$$

If **v** is parallel to **w** then $\theta = 0^o$ or 180^o.

$$\begin{aligned} \therefore\quad \mathbf{v} \bullet \mathbf{w} &= |\mathbf{v}|\,|\mathbf{w}|\cos\theta \\ &= |\mathbf{v}|\,|\mathbf{w}|\cos 0^o \quad \text{or} \quad |\mathbf{v}|\,|\mathbf{w}|\cos 180^o \\ &= \pm\,|\mathbf{v}|\,|\mathbf{w}| \end{aligned}$$

To formally prove these results we must also show that their converses are true.

Example 30 **Self Tutor**

If $\mathbf{p} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, find:

a $\mathbf{p} \bullet \mathbf{q}$

b the angle between **p** and **q**.

a $\mathbf{p} \bullet \mathbf{q}$

$$\begin{aligned} &= \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \\ &= 2(-1) + 3(0) + (-1)2 \\ &= -2 + 0 - 2 \\ &= -4 \end{aligned}$$

b

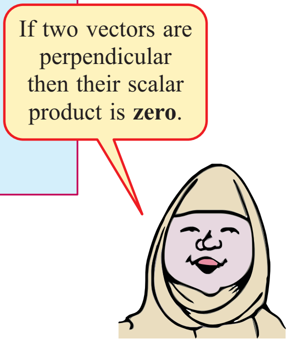
$$\begin{aligned} \mathbf{p} \bullet \mathbf{q} &= |\,\mathbf{p}\,|\,|\,\mathbf{q}\,|\cos\theta \\ \therefore\quad \cos\theta &= \frac{\mathbf{p} \bullet \mathbf{q}}{|\,\mathbf{p}\,|\,|\,\mathbf{q}\,|} \\ &= \frac{-4}{\sqrt{4+9+1}\,\sqrt{1+0+4}} \\ &= \frac{-4}{\sqrt{70}} \\ \therefore\quad \theta &= \cos^{-1}\left(\frac{-4}{\sqrt{70}}\right) \approx 119^o \end{aligned}$$

Example 31 **Self Tutor**

Find t such that $\mathbf{a} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ t \end{pmatrix}$ are perpendicular.

Since **a** and **b** are perpendicular, $\mathbf{a} \bullet \mathbf{b} = 0$

$$\begin{aligned} \therefore\quad \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ t \end{pmatrix} &= 0 \\ \therefore\quad (-1)(2) + 5t &= 0 \\ \therefore\quad -2 + 5t &= 0 \\ \therefore\quad 5t = 2 \quad \text{and so} \quad t &= \tfrac{2}{5} \end{aligned}$$



Example 32

Self Tutor

Consider the points A(2, 1), B(6, −1), and C(5, −3). Use a scalar product to check if triangle ABC is right angled. If it is, state the right angle.

$\overrightarrow{AB} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

We notice that $\overrightarrow{AB} \bullet \overrightarrow{BC} = 4(-1) + (-2)(-2) = -4 + 4 = 0$.

$\therefore$ $\overrightarrow{AB} \perp \overrightarrow{BC}$ and so triangle ABC is right angled at B.

Example 33

Self Tutor

Find the measure of the angle between the lines $2x + y = 5$ and $3x - 2y = 8$.

$2x + y = 5$ has gradient $-\frac{2}{1}$ and $\therefore$ direction vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ which we call **a**.

$3x - 2y = 8$ has gradient $\frac{3}{2}$ and $\therefore$ direction vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ which we call **b**.

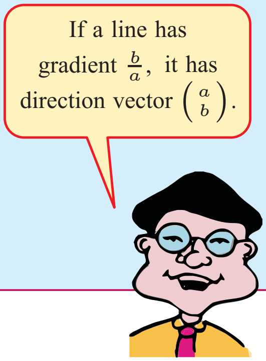
If the angle between the lines is θ, then

$$\cos\theta = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{(1 \times 2) + (-2 \times 3)}{\sqrt{1+4}\sqrt{4+9}} = \frac{-4}{\sqrt{5}\sqrt{13}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\tfrac{-4}{\sqrt{65}}\right) \approx 119.7^o$$

When finding the angle between two lines we choose the acute angle, in this case $180^o - \theta$.

$\therefore$ the angle is about 60.3^o.



If a line has gradient $\frac{b}{a}$, it has direction vector $\begin{pmatrix} a \\ b \end{pmatrix}$.

EXERCISE 12I

1 For $\mathbf{p} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$, find:

- **a** $\mathbf{q} \bullet \mathbf{p}$
- **b** $\mathbf{q} \bullet \mathbf{r}$
- **c** $\mathbf{q} \bullet (\mathbf{p} + \mathbf{r})$
- **d** $3\mathbf{r} \bullet \mathbf{q}$
- **e** $2\mathbf{p} \bullet 2\mathbf{p}$
- **f** $\mathbf{i} \bullet \mathbf{p}$
- **g** $\mathbf{q} \bullet \mathbf{j}$
- **h** $\mathbf{i} \bullet \mathbf{i}$

2 For $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ find:

- **a** $\mathbf{a} \bullet \mathbf{b}$
- **b** $\mathbf{b} \bullet \mathbf{a}$
- **c** $|\mathbf{a}|^2$
- **d** $\mathbf{a} \bullet \mathbf{a}$
- **e** $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c})$
- **f** $\mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$

3 If $\mathbf{p} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$, find: **a** $\mathbf{p} \bullet \mathbf{q}$ **b** the angle between **p** and **q**.

4 Find: **a** $(\mathbf{i}+\mathbf{j}-\mathbf{k})\bullet(2\mathbf{j}+\mathbf{k})$ **b** $\mathbf{i}\bullet\mathbf{i}$ **c** $\mathbf{i}\bullet\mathbf{j}$

5 Find $\mathbf{p}\bullet\mathbf{q}$ if: **a** $|\mathbf{p}|=2,\ |\mathbf{q}|=5,\ \theta=60^o$ **b** $|\mathbf{p}|=6,\ |\mathbf{q}|=3,\ \theta=120^o$

6 Using $\mathbf{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ and $\mathbf{c}=\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$ prove that

$\mathbf{a}\bullet(\mathbf{b}+\mathbf{c})=\mathbf{a}\bullet\mathbf{b}+\mathbf{a}\bullet\mathbf{c}$.

Hence, prove that $(\mathbf{a}+\mathbf{b})\bullet(\mathbf{c}+\mathbf{d})=\mathbf{a}\bullet\mathbf{c}+\mathbf{a}\bullet\mathbf{d}+\mathbf{b}\bullet\mathbf{c}+\mathbf{b}\bullet\mathbf{d}$.

7 Find t given that these vectors are perpendicular:

a $\mathbf{p}=\begin{pmatrix}3\\t\end{pmatrix}$ and $\mathbf{q}=\begin{pmatrix}-2\\1\end{pmatrix}$ **b** $\mathbf{r}=\begin{pmatrix}t\\t+2\end{pmatrix}$ and $\mathbf{s}=\begin{pmatrix}3\\-4\end{pmatrix}$

c $\mathbf{a}=\begin{pmatrix}t\\t+2\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}2-3t\\t\end{pmatrix}$ **d** $\mathbf{a}=\begin{pmatrix}3\\-1\\t\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}2t\\-3\\-4\end{pmatrix}$

8 For question **7** find, where possible, the value(s) of t for which the given vectors are *parallel*. Explain why in some cases the vectors can never be parallel.

9 Show that $\mathbf{a}=\begin{pmatrix}3\\1\\2\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}-1\\1\\1\end{pmatrix}$ and $\mathbf{c}=\begin{pmatrix}1\\5\\-4\end{pmatrix}$ are mutually perpendicular.

10 **a** Show that $\begin{pmatrix}1\\1\\5\end{pmatrix}$ and $\begin{pmatrix}2\\3\\-1\end{pmatrix}$ are perpendicular.

b Find t if $\begin{pmatrix}3\\t\\-2\end{pmatrix}$ is perpendicular to $\begin{pmatrix}1-t\\-3\\4\end{pmatrix}$.

11 Use a scalar product to check if triangle ABC is right angled. If it is, state the right angle.

a A$(-2, 1)$, B$(-2, 5)$ and C$(3, 1)$ **b** A$(4, 7)$, B$(1, 2)$ and C$(-1, 6)$

c A$(2, -2)$, B$(5, 7)$ and C$(-1, -1)$ **d** A$(10, 1)$, B$(5, 2)$ and C$(7, 4)$

12 Consider triangle ABC in which A is $(5, 1, 2)$, B is $(6, -1, 0)$ and C is $(3, 2, 0)$. Using scalar product only, show that the triangle is right angled.

13 A$(2, 4, 2)$, B$(-1, 2, 3)$, C$(-3, 3, 6)$ and D$(0, 5, 5)$ are vertices of a quadrilateral.

a Prove that ABCD is a parallelogram.

b Find $|\overrightarrow{AB}|$ and $|\overrightarrow{BC}|$. What can be said about ABCD?

c Find $\overrightarrow{AC}\bullet\overrightarrow{BD}$. What property of figure ABCD has been found to be valid?

14 Find the measure of the angle between the lines:

a $x-y=3$ and $3x+2y=11$ **b** $y=x+2$ and $y=1-3x$

c $y+x=7$ and $x-3y+2=0$ **d** $y=2-x$ and $x-2y=7$

Example 34 **Self Tutor**

Find the form of all vectors which are perpendicular to $\begin{pmatrix}3\\4\end{pmatrix}$.

$$\begin{pmatrix}3\\4\end{pmatrix} \bullet \begin{pmatrix}-4\\3\end{pmatrix} = -12 + 12 = 0$$

So, $\begin{pmatrix}-4\\3\end{pmatrix}$ is one such vector

The required vectors have the form $k\begin{pmatrix}-4\\3\end{pmatrix}$, $k \neq 0$.

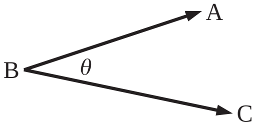
15 Find the form of all vectors which are perpendicular to:

a $\begin{pmatrix}5\\2\end{pmatrix}$ **b** $\begin{pmatrix}-1\\-2\end{pmatrix}$ **c** $\begin{pmatrix}3\\-1\end{pmatrix}$ **d** $\begin{pmatrix}-4\\3\end{pmatrix}$ **e** $\begin{pmatrix}2\\0\end{pmatrix}$

16 Find the angle ABC of triangle ABC for A(3, 0, 1), B(−3, 1, 2) and C(−2, 1, −1).

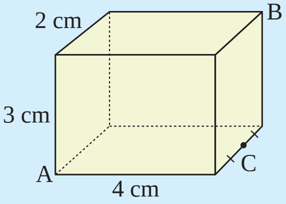
Hint: To find the angle at B, use $\overrightarrow{\text{BA}}$ and $\overrightarrow{\text{BC}}$.

What angle is found if $\overrightarrow{\text{BA}}$ and $\overrightarrow{\text{CB}}$ are used?



Example 35 **Self Tutor**

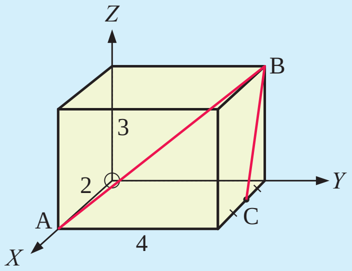
Use vector methods to determine the measure of angle ABC.



The vectors used must both be away from B (or towards B). If this is not done you will be finding the exterior angle at B.

Placing the coordinate axes as illustrated,
A is (2, 0, 0), B is (0, 4, 3) and C is (1, 4, 0)

$$\therefore \quad \overrightarrow{\text{BA}} \text{ is } \begin{pmatrix}2\\-4\\-3\end{pmatrix} \text{ and } \overrightarrow{\text{BC}} \text{ is } \begin{pmatrix}1\\0\\-3\end{pmatrix}$$



$$\cos(\text{A}\widehat{\text{B}}\text{C}) = \frac{\overrightarrow{\text{BA}} \bullet \overrightarrow{\text{BC}}}{|\overrightarrow{\text{BA}}|\,|\overrightarrow{\text{BC}}|}$$

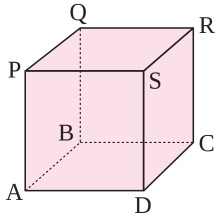
$$= \frac{2(1) + (-4)(0) + (-3)(-3)}{\sqrt{4+16+9}\sqrt{1+0+9}}$$

$$= \frac{11}{\sqrt{290}}$$

$$\therefore \quad \text{A}\widehat{\text{B}}\text{C} = \cos^{-1}\left(\frac{11}{\sqrt{290}}\right) \approx 49.8^o$$

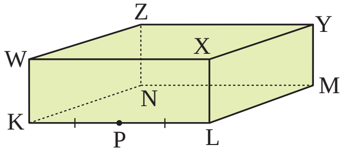
17 For the cube alongside with sides of length 2 cm, find using vector methods:

- **a** the measure of angle ABS
- **b** the measure of angle RBP
- **c** the measure of angle PBS.



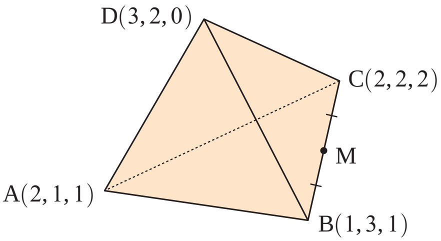
18 [KL], [LM] and [LX] are 8, 5 and 3 units long respectively. P is the midpoint of [KL]. Find, using vector methods:

- **a** the measure of angle YNX
- **b** the measure of angle YNP.



19 For the tetrahedron ABCD:

- **a** find the coordinates of M
- **b** find the measure of angle DMA.



20 **a** Find t if $2\mathbf{i} + t\mathbf{j} + (t-2)\mathbf{k}$ and $t\mathbf{i} + 3\mathbf{j} + t\mathbf{k}$ are perpendicular.

b Find r, s and t if $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ r \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} s \\ t \\ 1 \end{pmatrix}$ are mutually perpendicular.

21 Find the angle made by:

a the X-axis and $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ **b** a line parallel to the Y-axis and $\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$.

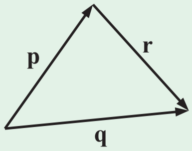
REVIEW SET 12A NON-CALCULATOR

1 Using a scale of 1 cm represents 10 units, sketch a vector to represent:

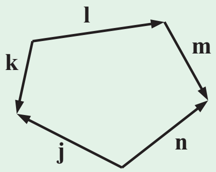
- **a** an aeroplane taking off at an angle of 8^o to the runway with a speed of 60 m s^{-1}
- **b** a displacement of 45 m in a direction of 060^o.

2 Simplify: **a** $\overrightarrow{AB} - \overrightarrow{CB}$ **b** $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{DC}$.

3 Construct vector equations for: **a**



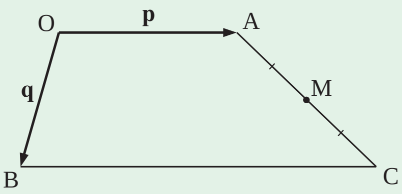
b



4 Draw arrow diagrams to represent: **a** $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ **b** $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ **c** $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$

5 If $\overrightarrow{PQ} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$, $\overrightarrow{RQ} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\overrightarrow{RS} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, find $\overrightarrow{SP}$.

6



[BC] is parallel to [OA] and is twice its length. Find, in terms of **p** and **q**, vector expressions for: **a** $\overrightarrow{AC}$ **b** $\overrightarrow{OM}$.

7 Find m and n if $\begin{pmatrix} 3 \\ m \\ n \end{pmatrix}$ and $\begin{pmatrix} -12 \\ -20 \\ 2 \end{pmatrix}$ are parallel vectors.

8 If $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -7 \\ 4 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} -6 \\ 1 \\ -3 \end{pmatrix}$, find $\overrightarrow{CB}$.

9 If $\mathbf{p} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ find: **a** $\mathbf{p} \bullet \mathbf{q}$ **b** $\mathbf{q} \bullet (\mathbf{p} - \mathbf{r})$

10 Use vectors to show that WYZX is a parallelogram given X(−2, 5), Y(3, 4), W(−3, −1), and Z(4, 10).

11 Suppose $\mathbf{a} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ -1 \\ 4 \end{pmatrix}$. Find **x** if:

a $\mathbf{a} - \mathbf{x} = 2\mathbf{b}$

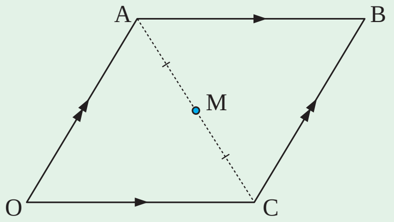
b $\mathbf{b} - 2\mathbf{x} = -\mathbf{a}$

12 Consider the points A(2, 3), B(−1, 4) and C(3, k). Find k if $B\widehat{A}C$ is a right angle.

13 Find all vectors which are perpendicular to the vector $\begin{pmatrix} -4 \\ 5 \end{pmatrix}$.

14 In this question you may **not** assume any diagonal properties of parallelograms.

OABC is a parallelogram with $\overrightarrow{OA} = \mathbf{p}$ and $\overrightarrow{OC} = \mathbf{q}$. M is the midpoint of [AC].

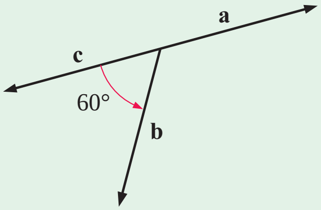


a Find in terms of **p** and **q**:

i $\overrightarrow{OB}$ **ii** $\overrightarrow{OM}$

b Using **a** only, show that O, M and B are collinear, and that M is the midpoint of [OB].

15 Suppose $|\mathbf{a}| = 2$, $|\mathbf{b}| = 4$, and $|\mathbf{c}| = 5$.
Find $\mathbf{a} \bullet \mathbf{b}$, $\mathbf{b} \bullet \mathbf{c}$ and $\mathbf{a} \bullet \mathbf{c}$.



16 Find a and b if J$(-4, 1, 3)$, K$(2, -2, 0)$ and L$(a, b, 2)$ are collinear.

Hint: $\overrightarrow{\text{JK}}$ is parallel to $\overrightarrow{\text{KL}}$.

17 [AB] and [CD] are diameters of a circle with centre O. If $\overrightarrow{\text{OC}} = \mathbf{q}$ and $\overrightarrow{\text{OB}} = \mathbf{r}$, find:

a $\overrightarrow{\text{DB}}$ in terms of $\mathbf{q}$ and $\mathbf{r}$ **b** $\overrightarrow{\text{AC}}$ in terms of $\mathbf{q}$ and $\mathbf{r}$.

What can be deduced about [DB] and [AC]?

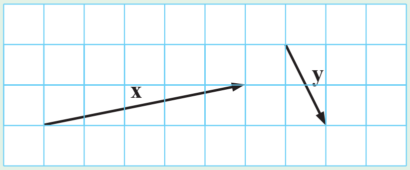
18 **a** Find t given that $\begin{pmatrix} 2-t \\ 3 \\ t \end{pmatrix}$ and $\begin{pmatrix} t \\ 4 \\ t+1 \end{pmatrix}$ are perpendicular.

b Show that K$(4, 3, -1)$, L$(-3, 4, 2)$ and M$(2, 1, -2)$ are vertices of a right angled triangle.

REVIEW SET 12B CALCULATOR

1 Copy the given vectors and find geometrically:

a $\mathbf{x} + \mathbf{y}$ **b** $\mathbf{y} - 2\mathbf{x}$



2 Dino walks for 9 km in the direction 246^o and then for 6 km in the direction 096^o. Find his displacement from his starting point.

3 Show that A$(-2, -1, 3)$, B$(4, 0, -1)$ and C$(-2, 1, -4)$ are vertices of an isosceles triangle.

4 If $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ find: **a** $|\mathbf{r}|$ **b** $|\mathbf{s}|$ **c** $|\mathbf{r} + \mathbf{s}|$ **d** $|2\mathbf{s} - \mathbf{r}|$

5 Determine the angle between $\begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$.

6 Find scalars r and s such that $r\begin{pmatrix} -2 \\ 1 \end{pmatrix} + s\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 13 \\ -24 \end{pmatrix}$.

7 For P$(2, 3, -1)$ and Q$(-4, 4, 2)$ find:

a $\overrightarrow{\text{PQ}}$ **b** the distance between P and Q **c** the midpoint of [PQ].

8 For $\mathbf{p} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ find:

a $\mathbf{p} \bullet \mathbf{q}$ **b** $\mathbf{p} + 2\mathbf{q} - \mathbf{r}$ **c** the angle between $\mathbf{p}$ and $\mathbf{r}$.

9 If A$(4,\ 2,\ -1)$, B$(-1,\ 5,\ 2)$, C$(3,\ -3,\ c)$ are vertices of triangle ABC which is right angled at B, find the value of c.

10 Suppose $\mathbf{a} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$. Find: **a** $2\mathbf{a} - 3\mathbf{b}$

b $\mathbf{x}$ if $\mathbf{a} - 3\mathbf{x} = \mathbf{b}$.

11 Find the angle between the vectors $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 5\mathbf{j} + \mathbf{k}$.

12 Find two points on the Z-axis which are 6 units from P$(-4,\ 2,\ 5)$.

13 Determine the value of t if $\begin{pmatrix} 3 \\ 3-2t \end{pmatrix}$ and $\begin{pmatrix} t^2+t \\ -2 \end{pmatrix}$ are perpendicular.

14 Find the angle between the two lines with equations $4x - 5y = 11$ and $2x + 3y = 7$.

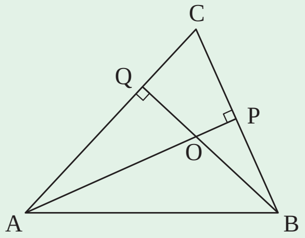
15 [AP] and [BQ] are altitudes of triangle ABC.

Let $\overrightarrow{OA} = \mathbf{p}$, $\overrightarrow{OB} = \mathbf{q}$ and $\overrightarrow{OC} = \mathbf{r}$.

a Find vector expressions for $\overrightarrow{AC}$ and $\overrightarrow{BC}$ in terms of $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$.

b Deduce that $\mathbf{q} \bullet \mathbf{r} = \mathbf{p} \bullet \mathbf{q} = \mathbf{p} \bullet \mathbf{r}$.

c Hence prove that [OC] is perpendicular to [AB].



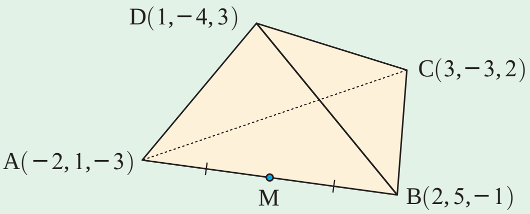
16 If $\mathbf{u} = \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$, find:

a $\mathbf{u} \bullet \mathbf{v}$ **b** the angle between $\mathbf{u}$ and $\mathbf{v}$.

17 **a** Find r and s if $\begin{pmatrix} r \\ 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 10 \\ s \end{pmatrix}$ are parallel.

b Find a vector of length 4 units which is parallel to $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

18 For the given tetrahedron, find the measure of angle DMC.



REVIEW SET 12C

1 Find a single vector which is equal to: **a** $\overrightarrow{PR}+\overrightarrow{RQ}$ **b** $\overrightarrow{PS}+\overrightarrow{SQ}+\overrightarrow{QR}$

2 For $\mathbf{m}=\begin{pmatrix}6\\-3\\1\end{pmatrix}$, $\mathbf{n}=\begin{pmatrix}2\\3\\-4\end{pmatrix}$ and $\mathbf{p}=\begin{pmatrix}-1\\3\\6\end{pmatrix}$, find:

a $\mathbf{m}-\mathbf{n}+\mathbf{p}$ **b** $2\mathbf{n}-3\mathbf{p}$ **c** $|\mathbf{m}+\mathbf{p}|$

3 What geometrical facts can be deduced from the equations:

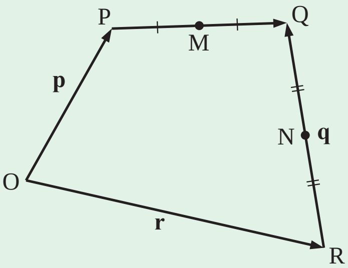
a $\overrightarrow{AB}=\frac{1}{2}\overrightarrow{CD}$ **b** $\overrightarrow{AB}=2\overrightarrow{AC}$?

4 Given P(2, −5, 6) and Q(−1, 7, 9), find:

a the position vector of Q from P **b** the distance from P to Q

c the distance from P to the X-axis.

5



In the figure alongside, $\overrightarrow{OP}=\mathbf{p}$, $\overrightarrow{OR}=\mathbf{r}$ and $\overrightarrow{RQ}=\mathbf{q}$.

If M and N are midpoints of the sides as shown, find in terms of $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$:

a $\overrightarrow{OQ}$ **b** $\overrightarrow{PQ}$ **c** $\overrightarrow{ON}$ **d** $\overrightarrow{MN}$

6 If $\mathbf{p}=\begin{pmatrix}-3\\1\end{pmatrix}$, $\mathbf{q}=\begin{pmatrix}2\\-4\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}1\\3\end{pmatrix}$ find:

a $2\mathbf{p}+\mathbf{q}$ **b** $\mathbf{q}-3\mathbf{r}$ **c** $\mathbf{p}-\mathbf{q}+\mathbf{r}$

7 Suppose $\mathbf{p}=\begin{pmatrix}-3\\1\end{pmatrix}$, $\mathbf{q}=\begin{pmatrix}2\\-4\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}3\\2\end{pmatrix}$. Find $\mathbf{x}$ if: **a** $\mathbf{p}-3\mathbf{x}=\mathbf{0}$
b $2\mathbf{q}-\mathbf{x}=\mathbf{r}$

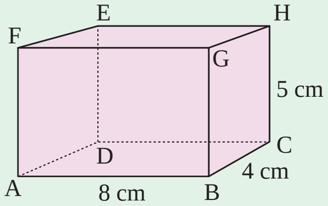
8 Suppose $|\mathbf{v}|=3$ and $|\mathbf{w}|=2$. If $\mathbf{v}$ is parallel to $\mathbf{w}$, what values might $\mathbf{v}\bullet\mathbf{w}$ take?

9 Find t if $\begin{pmatrix}-4\\t+2\\t\end{pmatrix}$ and $\begin{pmatrix}t\\1+t\\-3\end{pmatrix}$ are perpendicular vectors.

10 Find all angles of the triangle with vertices K(3, 1, 4), L(−2, 1, 3) and M(4, 1, 3).

11 Find k if the following are unit vectors: **a** $\begin{pmatrix}\frac{4}{7}\\\frac{1}{k}\end{pmatrix}$ **b** $\begin{pmatrix}k\\k\\k\end{pmatrix}$

12 Find the measure of angle GAC in the rectangular box alongside. Use vector methods.



13 Using $\mathbf{p}=\begin{pmatrix}3\\-2\end{pmatrix}$, $\mathbf{q}=\begin{pmatrix}-2\\5\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}1\\-3\end{pmatrix}$ verify that:

$\mathbf{p}\bullet(\mathbf{q}-\mathbf{r})=\mathbf{p}\bullet\mathbf{q}-\mathbf{p}\bullet\mathbf{r}$.

14 Find the measure of all angles of triangle KLM for K$(-2, 1)$, L$(3, 2)$ and M$(1, -3)$.

15 Suppose $\overrightarrow{\text{OM}}=\begin{pmatrix}-2\\4\end{pmatrix}$, $\overrightarrow{\text{MP}}=\begin{pmatrix}5\\-1\end{pmatrix}$, $\overrightarrow{\text{MP}}\bullet\overrightarrow{\text{PT}}=0$ and $|\overrightarrow{\text{MP}}|=|\overrightarrow{\text{PT}}|$.

Write down the two possible position vectors $\overrightarrow{\text{OT}}$.

16 **a** Find k given that $\begin{pmatrix}k\\\frac{1}{\sqrt{2}}\\-k\end{pmatrix}$ is a unit vector.

b Find the vector which is 5 units long and has the opposite direction to $\begin{pmatrix}3\\2\\-1\end{pmatrix}$.

17 Given $\mathbf{p}=2\mathbf{i}-\mathbf{j}+4\mathbf{k}$ and $\mathbf{q}=-\mathbf{i}-4\mathbf{j}+2\mathbf{k}$, find:

a $\mathbf{p}\bullet\mathbf{q}$

b the angle between $\mathbf{p}$ and $\mathbf{q}$.

18 Find scalars r, s and t such that $2\begin{pmatrix}s-1\\r+1\\t\end{pmatrix}=\begin{pmatrix}4s\\3r\\r\end{pmatrix}+\begin{pmatrix}r\\-1\\s\end{pmatrix}$.

19 Suppose $\mathbf{u}=2\mathbf{i}+\mathbf{j}$, $\mathbf{v}=3\mathbf{j}$, and θ is the acute angle between $\mathbf{u}$ and $\mathbf{v}$. Find the exact value of $\sin\theta$.

Chapter 13

Lines and planes in space

Syllabus reference: 5.3, 5.4

Contents:

INTRODUCTION

Suppose the vector $\begin{pmatrix}1\\0\end{pmatrix}$ represents a displacement of 1 km due east and

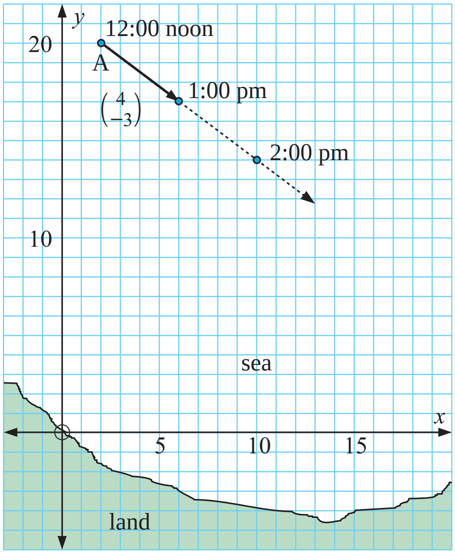
$\begin{pmatrix}0\\1\end{pmatrix}$ represents a displacement of 1 km due north.

The diagram shows the path of a yacht relative to a yacht club which is situated at $(0, 0)$. At 12:00 noon the yacht is at the point A$(2, 20)$.

The yacht is travelling in the direction $\begin{pmatrix}4\\-3\end{pmatrix}$ with a constant speed of 5 km h^{-1}.

Since $\left|\begin{pmatrix}4\\-3\end{pmatrix}\right| = 5$ we can see that $\begin{pmatrix}4\\-3\end{pmatrix}$ gives both the direction and speed of travel.

So, $\begin{pmatrix}4\\-3\end{pmatrix}$ is called the **velocity vector** of the yacht.



In order to define the position of the yacht at any time t hours after 12 noon, we can use the **parametric equations** $x = 2 + 4t$ and $y = 20 - 3t$ where t is called the **parameter**.

We can see that: When $t = 0$, $x = 2$ and $y = 20$, so the yacht is at $(2, 20)$.
When $t = 1$, $x = 6$ and $y = 17$, so the yacht is at $(6, 17)$.
When $t = 2$, $x = 10$ and $y = 14$, so the yacht is at $(10, 14)$.

We can also find a **vector equation** for the yacht's path as follows:

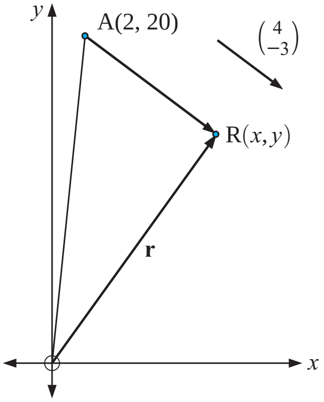
Suppose the yacht is at R(x, y) at time t hours after 12:00 noon.

$$\therefore \quad \overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$$

$$\therefore \quad \mathbf{r} = \begin{pmatrix}2\\20\end{pmatrix} + t\begin{pmatrix}4\\-3\end{pmatrix} \quad \text{for } t \geqslant 0$$

$$\therefore \quad \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2\\20\end{pmatrix} + t\begin{pmatrix}4\\-3\end{pmatrix}$$

which is the **vector equation** of the yacht's path.



Notice how the parametric equations are easily found from the vector equation:

If $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2\\20\end{pmatrix} + t\begin{pmatrix}4\\-3\end{pmatrix}$ then $x = 2 + 4t$ and $y = 20 - 3t$.

We can also find a **Cartesian equation** of the yacht's path. This equation does not contain the parameter t.

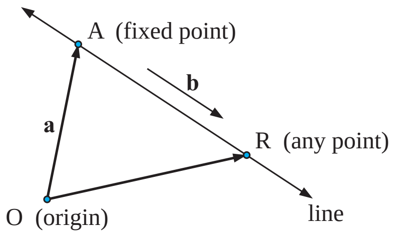
As $x = 2 + 4t$, $4t = x - 2$ and so $t = \dfrac{x-2}{4}$.

Substituting into $y = 20 - 3t$ gives $y = 20 - 3\left(\dfrac{x-2}{4}\right)$

$$\therefore \quad 4y = 80 - 3(x-2)$$
$$\therefore \quad 4y = 80 - 3x + 6$$
$$\therefore \quad 3x + 4y = 86 \text{ where } x \geqslant 2 \text{ because } t \geqslant 0.$$

A LINES IN 2-D AND 3-D

In both 2-D and 3-D geometry we can determine the **equation of a line** by its **direction** and a **fixed point** through which it passes.



Suppose a line passes through a fixed point A such that $\overrightarrow{OA} = \mathbf{a}$, and its direction is given by vector $\mathbf{b}$. This means the line is parallel to $\mathbf{b}$.

Consider a point R on the line so that $\overrightarrow{OR} = \mathbf{r}$.

By vector addition, $\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$.

Since $\overrightarrow{AR} \parallel \mathbf{b}$, $\overrightarrow{AR} = t\mathbf{b}$ for some $t \in \mathbb{R}$

$$\therefore \quad \mathbf{r} = \mathbf{a} + t\mathbf{b}$$

So, $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, $t \in \mathbb{R}$ is the **vector equation** of the line.

LINES IN 2-D

- In 2-D we are dealing with a **line in a plane**.
- $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + t\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is the **vector equation** of the line
  where R(x, y) is any point on the line
  A(a_1, a_2) is a known fixed point on the line
  $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is the **direction vector** of the line.

  Note: $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ also enables us to calculate the gradient of the line, as

  $$m = \frac{b_2}{b_1} = \frac{\text{rise}}{\text{run}}.$$

- $\left.\begin{array}{l} x = a_1 + b_1 t \\ y = a_2 + b_2 t \end{array}\right\}$ are the **parametric equations** of the line where $t \in \mathbb{R}$ is called the **parameter**.

  With these equations each point on the line corresponds to exactly one value of t.

- We can convert these equations into Cartesian form by equating t values.

  Using $t = \dfrac{x-a_1}{b_1} = \dfrac{y-a_2}{b_2}$ we obtain $b_2x - b_1y = b_2a_1 - b_1a_2$ which is the **Cartesian equation** of the line.

Example 1 **Self Tutor**

Find **a** the vector **b** the parametric **c** the Cartesian equation of the line passing through the point A(1, 5) with direction $\begin{pmatrix}3\\2\end{pmatrix}$.

a $\mathbf{a} = \overrightarrow{\text{OA}} = \begin{pmatrix}1\\5\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}3\\2\end{pmatrix}$

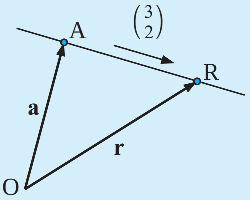
But $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ $\therefore$ $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\5\end{pmatrix} + t\begin{pmatrix}3\\2\end{pmatrix}$, $t \in \mathbb{R}$

b From **a**, $x = 1+3t$ and $y = 5+2t$, $t \in \mathbb{R}$

c Now $t = \dfrac{x-1}{3} = \dfrac{y-5}{2}$

$\therefore$ $2x - 2 = 3y - 15$

$\therefore$ $2x - 3y = -13$ {general form}



Example 2 **Self Tutor**

A particle at $P(x(t), y(t))$ moves such that $x(t) = 2-3t$ and $y(t) = 2t+4$, $t \geqslant 0$. The distance units are metres and t is in seconds.

a Find the initial position of P.

b Illustrate the motion showing points where $t = 0$, 1, 2 and 3.

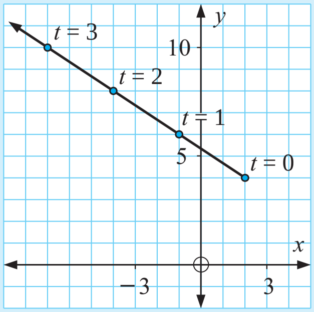
c Find the speed of P.

a $x(0) = 2$, $y(0) = 4$ $\therefore$ the initial position of P is (2, 4)

b $x(1) = -1$, $y(1) = 6$

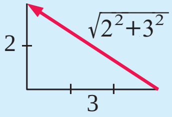
$x(2) = -4$, $y(2) = 8$

$x(3) = -7$, $y(3) = 10$



c Every second P moves with x-step -3 and y-step 2, which is a distance of $\sqrt{13}$ m.

$\therefore$ the speed is constant and is $\sqrt{13}$ m s^{-1}.



EXERCISE 13A.1

1 Find **i** the vector equation **ii** the Cartesian equation of the line:

- **a** passing through $(3, -4)$ with direction $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$
- **b** passing through $(5, 2)$ with direction perpendicular to $\begin{pmatrix} -8 \\ 2 \end{pmatrix}$
- **c** cutting the x-axis at -6 with direction $3\mathbf{i} + 7\mathbf{j}$
- **d** passing through $(-1, 11)$ and $(-3, 12)$.

2 Find the parametric equations of the line passing through $(-1, 4)$ with direction vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and parameter t. Find the points on the line when $t = 0, 1, 3, -1, -4$.

3 **a** Does $(3, -2)$ lie on the line with vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \end{pmatrix}$?

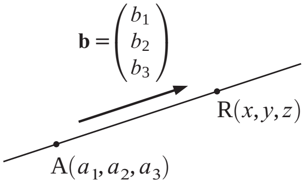
- **b** $(k, 4)$ lies on the line with parametric equations $x = 1 - 2t$, $y = 1 + t$. Find k.

4 A particle at $\text{P}(x(t), y(t))$ moves such that $x(t) = 1 + 2t$ and $y(t) = 2 - 5t$, $t \geqslant 0$. The distances are in centimetres and t is in seconds.

- **a** Find the initial position of P.
- **b** Illustrate the initial part of the motion of P where $t = 0, 1, 2, 3$.
- **c** Find the speed of P.

5 Line L has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} + t\begin{pmatrix} -1 \\ 3 \end{pmatrix}$. Write two other equations to represent line L.

LINES IN 3-D



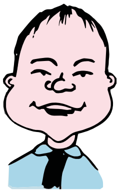
- In 3-D we are dealing with a **line in space**.
- $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is the **vector equation** of the line

  where R(x, y, z) is any point on the line
  A(a_1, a_2, a_3) is the known or fixed point on the line
  $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is the **direction vector** of the line.

- $\left.\begin{array}{l} x = a_1 + tb_1 \\ y = a_2 + tb_2 \\ z = a_3 + tb_3 \end{array}\right\}$ are the **parametric equations** of the line where $t \in \mathbb{R}$ is called the **parameter**.

  Every point on the line corresponds to exactly one value of t.

We do not talk about the **gradient** of a line in 3-D.
We describe its direction only by its **direction vector**.



Example 3 Self Tutor

Find the vector equation and the parametric equations of the line through $(1, -2, 3)$ in the direction $4\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}$.

The vector equation is $\mathbf{r} = \mathbf{a} + t\mathbf{b}$

$$\therefore \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 4 \\ 5 \\ -6 \end{pmatrix}, \quad t \in \mathbb{R}.$$

The parametric equations are: $x = 1 + 4t,\ y = -2 + 5t,\ z = 3 - 6t,\ t \in \mathbb{R}$.

Example 4 Self Tutor

Find the parametric equations of the line through A$(2, -1, 4)$ and B$(-1, 0, 2)$.

We require a direction vector for the line, either $\overrightarrow{\text{AB}}$ or $\overrightarrow{\text{BA}}$.

$$\overrightarrow{\text{AB}} = \begin{pmatrix} -1-2 \\ 0--1 \\ 2-4 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix}$$

Using the point A, the equations are: $x = 2 - 3t,\ y = -1 + t,\ z = 4 - 2t,\ t \in \mathbb{R}$.

Note: Using the point B, the equations are: $x = -1 - 3s,\ y = s,\ z = 2 - 2s,\ s \in \mathbb{R}$.

These sets of equations are actually equivalent, and generate the same set of points. They are related by $s = t - 1$.

EXERCISE 13A.2

1 Find the vector equation of the line:

- **a** parallel to $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and through the point $(1, 3, -7)$
- **b** through $(0, 1, 2)$ and with direction vector $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
- **c** parallel to the X-axis and through the point $(-2, 2, 1)$.

2 Find the parametric equations of the line:

- **a** parallel to $\begin{pmatrix} -1 \\ 2 \\ 6 \end{pmatrix}$ and through the point $(5, 2, -1)$
- **b** parallel to $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and through the point $(0, 2, -1)$
- **c** perpendicular to the XOY plane and through $(3, 2, -1)$.

3 Find the vector equation of the line through:

- **a** A$(1, 2, 1)$ and B$(-1, 3, 2)$
- **b** C$(0, 1, 3)$ and D$(3, 1, -1)$
- **c** E$(1, 2, 5)$ and F$(1, -1, 5)$
- **d** G$(0, 1, -1)$ and H$(5, -1, 3)$

4 Find the coordinates of the point where the line with parametric equations $x = 1 - t$, $y = 3 + t$ and $z = 3 - 2t$ meets:

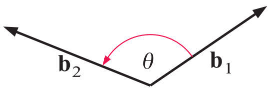
a the XOY plane **b** the YOZ plane **c** the XOZ plane.

5 Find points on the line with parametric equations $x = 2 - t$, $y = 3 + 2t$ and $z = 1 + t$ which are $5\sqrt{3}$ units from the point $(1, 0, -2)$.

THE ANGLE BETWEEN TWO LINES (2-D AND 3-D)

In **Chapter 12** we saw that the angle between two vectors is measured in the range $0^o \leqslant \theta \leqslant 180^o$.

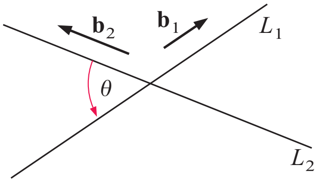
We used the formula $\cos\theta = \dfrac{\mathbf{b}_1 \bullet \mathbf{b}_2}{|\mathbf{b}_1||\mathbf{b}_2|}$.



In the case of lines which continue infinitely in both directions, we agree to talk about the *acute* angle between them. We therefore use the formula

$$\cos\theta = \frac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1||\mathbf{b}_2|}$$

where $\mathbf{b}_1$ and $\mathbf{b}_2$ are the direction vectors of the given lines L_1 and L_2 respectively.



Example 5 Self Tutor

Find the angle between the lines L_1: $x = 2 - 3t$, $y = -1 + t$ and
L_2: $x = 1 + 2s$, $y = -4 + 3s$.

$\mathbf{b}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ $\mathbf{b}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\therefore$ $\cos\theta = \dfrac{|-6+3|}{\sqrt{10}\sqrt{13}}$

$\therefore$ $\cos\theta \approx 0.2631$

and so $\theta \approx 74.7^o$ (1.30 radians)

Example 6 Self Tutor

Find the angle between the lines:

$$L_1: \ \mathbf{r}_1 = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + s\begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix} \quad \text{and} \quad L_2: \ \mathbf{r}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix}.$$

$\mathbf{b}_1 = \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix}$ and $\mathbf{b}_2 = \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix}$

Since $\mathbf{b}_1 = \mathbf{b}_2$, the lines are parallel and the angle between them is 0^o.

EXERCISE 13A.3

1 Find the angle between the lines: L_1 passing through $(-6, 3)$ parallel to $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and L_2 cutting the y-axis at $(0, 8)$ with direction $5\mathbf{i} + 4\mathbf{j}$.

2 Find the angle between the lines: L_1: $x = -4 + 12t$, $y = 3 + 5t$ and
L_2: $x = 3s$, $y = -6 - 4s$

3 Show that the lines: $x = 2 + 5p$, $y = 19 - 2p$ and
$x = 3 + 4r$, $y = 7 + 10r$ are perpendicular.

4 **a** Find the angle between the lines:

$$L_1 : \mathbf{r}_1 = \begin{pmatrix} 8 \\ 9 \\ 10 \end{pmatrix} + s\begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix} \quad \text{and} \quad L_2 : \mathbf{r}_2 = \begin{pmatrix} 15 \\ 29 \\ 5 \end{pmatrix} + t\begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}.$$

b A third line L_3 is perpendicular to L_1 and has direction vector $\begin{pmatrix} 0 \\ -3 \\ x \end{pmatrix}$.

Find the value of x.

B APPLICATIONS OF A LINE IN A PLANE

THE VELOCITY VECTOR OF A MOVING OBJECT

In **Example 2** we considered a particle which moves $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ every second.

$\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ is called the **velocity vector** of the particle.

Since $\left|\begin{pmatrix} -3 \\ 2 \end{pmatrix}\right| = \sqrt{(-3)^2 + 2^2} = \sqrt{13}$,

- the **velocity** of the particle is $\sqrt{13}$ metres per second in the direction $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$
- the **speed** of the particle is $\sqrt{13}$ metres per second.

If $\begin{pmatrix} a \\ b \end{pmatrix}$ is the **velocity vector** of a moving object, then it is travelling at a speed of $\left|\begin{pmatrix} a \\ b \end{pmatrix}\right| = \sqrt{a^2 + b^2}$ in the direction $\begin{pmatrix} a \\ b \end{pmatrix}$.

Example 7 **Self Tutor**

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} + t\begin{pmatrix} 6 \\ -8 \end{pmatrix}$ is the vector equation of the path of an object.

t is the time in seconds, $t \geqslant 0$. The distance units are metres.

a Find the object's initial position **b** Find the velocity vector of the object

c Find the object's speed.

d If the object continues in the same direction but increases its speed to 30 m s^{-1}, state its new velocity vector.

a At $t = 0$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$ $\therefore$ the object is at (7, 5).

b The velocity vector is $\begin{pmatrix} 6 \\ -8 \end{pmatrix}$ because the object moves $\begin{pmatrix} 6 \\ -8 \end{pmatrix}$ every second.

c The speed is $\left|\begin{pmatrix} 6 \\ -8 \end{pmatrix}\right| = \sqrt{36+64} = \sqrt{100} = 10$ m s^{-1}.

d Previously, the speed was 10 m s^{-1} and the velocity vector was $\begin{pmatrix} 6 \\ -8 \end{pmatrix}$.

$\therefore$ the new velocity vector is $3\begin{pmatrix} 6 \\ -8 \end{pmatrix} = \begin{pmatrix} 18 \\ -24 \end{pmatrix}$.

EXERCISE 13B.1

1 Each of the following vector equations represents the path of a moving object. t is measured in seconds and $t \geqslant 0$. Distances are measured in metres. In each case, find:

i the initial position **ii** the velocity vector **iii** the speed of the object.

a $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} + t\begin{pmatrix} 12 \\ 5 \end{pmatrix}$ **b** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \end{pmatrix} + t\begin{pmatrix} 3 \\ -4 \end{pmatrix}$

c $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ -7 \end{pmatrix} + t\begin{pmatrix} -6 \\ -4 \end{pmatrix}$ **d** $x = 5 + 8t$ and $y = -5 + 4t$

2 Find the velocity vector of a speed boat moving parallel to:

a $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ with a speed of 150 km h^{-1}

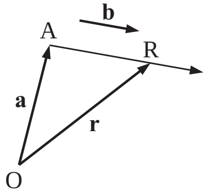
b $\begin{pmatrix} 24 \\ 7 \end{pmatrix}$ with a speed of 12.5 km h^{-1}

c $2\mathbf{i} + \mathbf{j}$ with a speed of 50 km h^{-1}

3 Find the velocity vector of a swooping eagle moving in the direction $-2\mathbf{i} + 5\mathbf{j} - 14\mathbf{k}$ with a speed of 90 km h^{-1}.

CONSTANT VELOCITY PROBLEMS

Suppose an object moves with constant velocity **b**. If the object is initially at A (when time $t = 0$) and at time t it is at R, then



$$\overrightarrow{\text{AR}} = t\mathbf{b} \qquad \{\text{displacement} = \text{time} \times \text{velocity}\}$$

$$\text{Now} \quad \mathbf{r} = \overrightarrow{\text{OA}} + \overrightarrow{\text{AR}}$$

$$\therefore \quad \mathbf{r} = \mathbf{a} + t\mathbf{b}$$

If a body has initial position vector **a** and moves with constant velocity **b**, its position at time t is given by

$$\mathbf{r} = \mathbf{a} + t\mathbf{b} \quad \text{for} \quad t \geqslant 0.$$

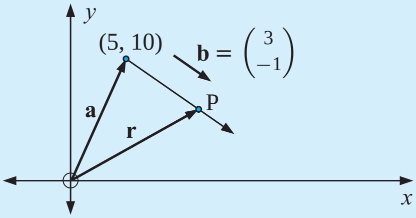
Example 8 **Self Tutor**

An object is initially at $(5, 10)$ and moves with velocity vector $3\mathbf{i} - \mathbf{j}$. Find:

a the position of the object at any time t where t is in minutes

b the position at $t = 3$ **c** the time when the object is due east of $(0, 0)$.

a



$$\mathbf{r} = \mathbf{a} + t\mathbf{b}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \end{pmatrix}, \quad t \in \mathbb{R}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5+3t \\ 10-t \end{pmatrix}$$

$\therefore$ P is at $(5 + 3t, 10 - t)$

b At $t = 3$, $5 + 3t = 14$ and $10 - t = 7$, so the object is at $(14, 7)$.

c When the object is due east of $(0, 0)$, y must be zero

$\therefore \quad 10 - t = 0$

$\therefore \quad t = 10$ The object is due east of $(0, 0)$ after 10 minutes.

EXERCISE 13B.2

1 A remote controlled toy car is initially at the point $(-3, -2)$. It moves with constant velocity $2\mathbf{i} + 4\mathbf{j}$. The distance units are centimetres, and the time is in seconds.

- **a** Write an expression for the position vector of the car at any time $t \geqslant 0$.
- **b** Hence find the position vector of the car at time $t = 2.5$.
- **c** Find when the car is **i** due north **ii** due west of the observation point $(0, 0)$.
- **d** Plot the car's positions at times $t = 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, ...$

2 Yacht A moves according to $x(t) = 4 + t$, $y(t) = 5 - 2t$ where the distance units are kilometres and the time units are hours. Yacht B moves according to $x(t) = 1 + 2t$, $y(t) = -8 + t$, $t \geqslant 0$.

- **a** Find the initial position of each yacht.
- **b** Find the velocity vector of each yacht.
- **c** Show that the speed of each yacht is constant and state the speeds.
- **d** Prove that the paths of the yachts are at right angles to each other.

3 Submarine P is at $(-5, 4)$ and fires a torpedo with velocity vector $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ at 1:34 pm.

Submarine Q is at $(15, 7)$ and a minutes later fires a torpedo in the direction $\begin{pmatrix} -4 \\ -3 \end{pmatrix}$.

Distances are measured in kilometres and time is in minutes.

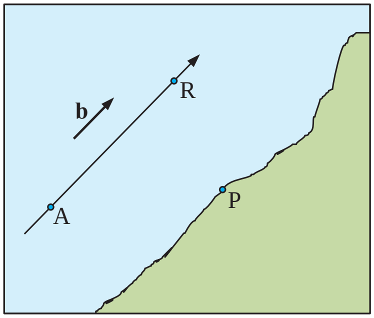
- **a** Show that the position of P's torpedo can be written as $\text{P}(x_1(t), y_1(t))$ where $x_1(t) = -5 + 3t$ and $y_1(t) = 4 - t$.
- **b** What is the speed of P's torpedo?
- **c** Show that the position of Q's torpedo can be written in the form $x_2(t) = 15 - 4(t - a)$, $y_2(t) = 7 - 3(t - a)$.
- **d** Q's torpedo is successful in knocking out P's torpedo. At what time did Q fire its torpedo and at what time did the explosion occur?

4 A helicopter at A(6, 9, 3) moves with constant velocity in a straight line. 10 minutes later it is at B(3, 10, 2.5). If distances are in kilometres, find:

- **a** $\overrightarrow{\text{AB}}$
- **b** the helicopter's speed
- **c** the equation of the straight line
- **d** the time taken until the helicopter lands on the helipad where $z = 0$.

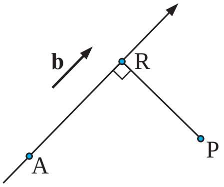
THE CLOSEST DISTANCE FROM A POINT TO A LINE

A ship sails through point A in the direction **b** and continues past a port P. At what time will the ship R be closest to the port?



The ship is closest when [PR] is perpendicular to [AR].

$\therefore \quad \overrightarrow{\text{PR}} \bullet \mathbf{b} = 0 \quad$ {the scalar product is zero}

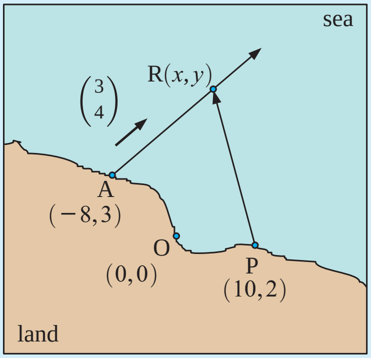


Example 9 **Self Tutor**

On the map shown, distances are measured in kilometres. Ship R moves in the direction $\begin{pmatrix}3\\4\end{pmatrix}$ at a speed of 10 km h^{-1}.

a Find an expression for the position of the ship in terms of t, the number of hours after leaving port A.

b Find the time when the ship is closest to port P(10, 2).



a $\left|\begin{pmatrix}3\\4\end{pmatrix}\right| = \sqrt{3^2+4^2} = 5$ Since the speed is 10 km h^{-1}, the ship's velocity vector must be $2\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}6\\8\end{pmatrix}$.

Now $\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$

$\therefore \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-8\\3\end{pmatrix} + t\begin{pmatrix}6\\8\end{pmatrix}$

$\therefore$ R is at $(-8+6t, 3+8t)$

b The ship is closest to P when $\overrightarrow{PR} \perp \begin{pmatrix}3\\4\end{pmatrix}$

$$\therefore \overrightarrow{PR} \bullet \begin{pmatrix}3\\4\end{pmatrix} = 0$$

$$\therefore \begin{pmatrix}-8+6t-10\\3+8t-2\end{pmatrix} \bullet \begin{pmatrix}3\\4\end{pmatrix} = 0$$

$$\therefore 3(6t-18)+4(1+8t) = 0$$

$$\therefore 18t-54+4+32t = 0$$

$$\therefore 50t-50 = 0$$

$$\therefore t = 1$$

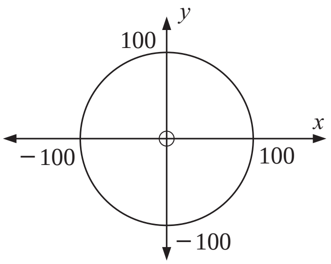
So, the ship is closest to port P 1 hour after leaving A.

EXERCISE 13B.3

1 An ocean liner is at $(6, -6)$, cruising at 10 km h^{-1} in the direction $\begin{pmatrix}-3\\4\end{pmatrix}$.

A fishing boat is anchored at $(0, 0)$. Distances are in kilometres.

- **a** Find, in terms of **i** and **j**, the original position vector of the liner from the fishing boat.
- **b** Write an expression for the position vector of the liner at any time t hours after it has sailed from $(6, -6)$.
- **c** Find when the liner is due east of the fishing boat.
- **d** Find the time and position of the liner when it is nearest to the fishing boat.

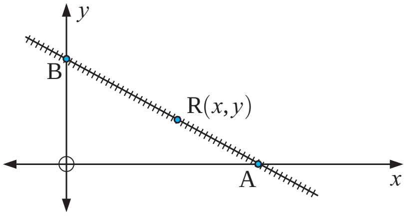


2 Let $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ represent a 1 km displacement due east and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represent a 1 km displacement due north.

The control tower of an airport is at $(0, 0)$. Aircraft within 100 km of $(0, 0)$ will become visible on the radar screen at the control tower.

At 12:00 noon an aircraft is 200 km east and 100 km north of the control tower. It is flying parallel to the vector $\mathbf{b} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$ with a speed of $40\sqrt{10}\ \text{km h}^{-1}$.

- **a** Write down the velocity vector of the aircraft.
- **b** Write a vector equation for the path of the aircraft using t to represent the time in hours that have elapsed since 12:00 noon.
- **c** Find the position of the aircraft at 1:00 pm.
- **d** Show that the aircraft first becomes visible on the radar screen at 1:00 pm.
- **e** Find the time when the aircraft is closest to the control tower and find the distance between the aircraft and the control tower at this time.
- **f** At what time will the aircraft disappear from the radar screen?



3 The diagram shows a railway track that has equation $2x + 3y = 36$.
The axes represent two long country roads.
All distances are in kilometres.

- **a** Find the coordinates of A and B.
- **b** R(x, y) is any point on the railway track. Express the coordinates of point R in terms of x only.
- **c** Some railway workers have set up a base camp at P$(4, 0)$. Find $\overrightarrow{PR}$ and $\overrightarrow{AB}$.
- **d** Find the coordinates of the point on the railway track that would be closest to P. Find this distance.

4 Boat A's position is given by
$x(t) = 3 - t,\ \ y(t) = 2t - 4$ where the distance units are kilometres and the time units are hours.
Boat B's position is given by
$x(t) = 4 - 3t,\ \ y(t) = 3 - 2t.$

- **a** Find the initial position of each boat.
- **b** Find the velocity vector of each boat.
- **c** What is the angle between the paths of the boats?
- **d** At what time are the boats closest to each other?

Example 10

Self Tutor

Consider the point P$(-1, 2, 3)$ and the line with parametric equations $x = 1 + 2t$, $y = -4 + 3t$, $z = 3 + t$.

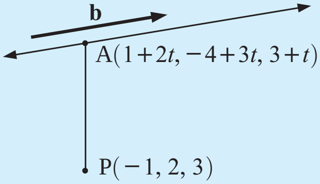
a Find the coordinates of the foot of the perpendicular from P to the line.

b Hence find the shortest distance from P to the line.

a The line has direction vector $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

Let $\text{A}(1 + 2t, -4 + 3t, 3 + t)$ be any point on the given line.

$$\therefore \quad \overrightarrow{\text{PA}} = \begin{pmatrix} 1 + 2t - (-1) \\ -4 + 3t - 2 \\ 3 + t - 3 \end{pmatrix} = \begin{pmatrix} 2 + 2t \\ -6 + 3t \\ t \end{pmatrix}$$



Now $\overrightarrow{\text{PA}}$ and $\mathbf{b}$ are perpendicular, so $\overrightarrow{\text{PA}} \bullet \mathbf{b} = 0$

$$\therefore \quad \begin{pmatrix} 2 + 2t \\ -6 + 3t \\ t \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 0$$

$$\therefore \quad 2(2 + 2t) + 3(-6 + 3t) + 1(t) = 0$$

$$\therefore \quad 4 + 4t - 18 + 9t + t = 0$$

$$\therefore \quad 14t = 14$$

$$\therefore \quad t = 1$$

Substituting $t = 1$ into the parametric equations, we obtain the foot of the perpendicular $(3, -1, 4)$.

b When $t = 1$, $\overrightarrow{\text{PA}} = \begin{pmatrix} 2 + 2 \\ -6 + 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$

$$\therefore \quad \text{PA} = \sqrt{16 + 9 + 1} = \sqrt{26} \text{ units}$$

$\therefore$ the shortest distance from P to the line is $\sqrt{26}$ units.

5 **a** Find the coordinates of the foot of the perpendicular from $(3, 0, -1)$ to the line with parametric equations $x = 2 + 3t$, $y = -1 + 2t$, $z = 4 + t$.

b Hence find the shortest distance from the point to the line.

6 **a** Find the coordinates of the foot of the perpendicular from $(1, 1, 3)$ to the line with vector equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

b Hence find the shortest distance from the point to the line.

GEOMETRIC APPLICATIONS OF $r = a + tb$

Vector equations of two intersecting lines can be **solved simultaneously** to find the point where the lines meet.

Example 11

Self Tutor

Line 1 has vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + s\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and

line 2 has vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \end{pmatrix} + t\begin{pmatrix} -4 \\ 1 \end{pmatrix}$, where s and t are scalars.

Use vector methods to find where the two lines meet.

The lines meet where $\begin{pmatrix} -2 \\ 1 \end{pmatrix} + s\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \end{pmatrix} + t\begin{pmatrix} -4 \\ 1 \end{pmatrix}$

$\therefore \quad -2 + 3s = 15 - 4t \quad$ and $\quad 1 + 2s = 5 + t$

$\therefore \quad 3s + 4t = 17 \quad ......\ (1) \quad$ and $\quad 2s - t = 4 \quad ......\ (2)$

$$\begin{aligned} 3s + 4t &= 17 \\ 8s - 4t &= 16 \quad \{(2) \times 4\} \\ \hline \therefore \quad 11s \qquad &= 33 \end{aligned}$$

So, $s = 3$ and in (2): $\quad 2(3) - t = 4$ and so $t = 2$.

Using line 1, $\quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$

Checking in line 2, $\quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \end{pmatrix} + 2\begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix} \qquad \therefore$ they meet at $(7, 7)$.

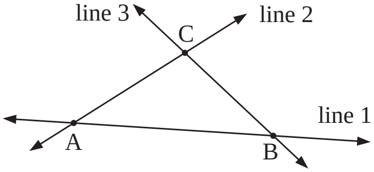
EXERCISE 13B.4

1 Triangle ABC is formed by three lines:

Line 1 (AB) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix} + r\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, line 2 (AC) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

and line 3 (BC) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ -3 \end{pmatrix} + t\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ where r, s and t are scalars.

- **a** Draw the three lines accurately on a grid.
- **b** Hence, find the coordinates of A, B and C.
- **c** Prove that ΔABC is isosceles.
- **d** Use vector methods to *check* your answers to **b**.



2 A parallelogram is defined by four lines as follows:

Line 1 (AB) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} + r\begin{pmatrix} 7 \\ 3 \end{pmatrix}$, line 2 (AD) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \end{pmatrix}$,

line 3 (CD) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 22 \\ 25 \end{pmatrix} + t\begin{pmatrix} -7 \\ -3 \end{pmatrix}$, line 4 (CB) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 22 \\ 25 \end{pmatrix} + u\begin{pmatrix} -1 \\ -2 \end{pmatrix}$,

where r, s, t and u are scalars.

- **a** Draw an accurate sketch of the four lines and the parallelogram formed by them. Label the vertices.

b From your diagram find the coordinates of A, B, C and D.

c Use vector methods to confirm your answers to **b**.

3 An isosceles triangle ABC is formed by these lines:

Line 1 (AB) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + r\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, line 2 (BC) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix} + s\begin{pmatrix} -1 \\ -2 \end{pmatrix}$

and line 3 (AC) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ where r, s and t are scalars.

a Use vector methods to find the coordinates of A, B and C.

b Which two sides of the triangle are equal in length? Find their lengths.

4 Line QP is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + r\begin{pmatrix} 14 \\ 10 \end{pmatrix}$, line QR is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + s\begin{pmatrix} 17 \\ -9 \end{pmatrix}$ and

line PR is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 18 \end{pmatrix} + t\begin{pmatrix} 5 \\ -7 \end{pmatrix}$ where r, s and t are scalars.

Triangle PQR is formed by these lines.

a Use vector methods to find the coordinates of P, Q and R.

b Find vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and evaluate $\overrightarrow{PQ} \bullet \overrightarrow{PR}$.

c Hence, find the size of $Q\widehat{P}R$.

d Find the area of ΔPQR.

5 Quadrilateral ABCD is formed by these lines:

Line 1 (AB) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + r\begin{pmatrix} 4 \\ 1 \end{pmatrix}$, line 2 (BC) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ 9 \end{pmatrix} + s\begin{pmatrix} -8 \\ 32 \end{pmatrix}$,

line 3 (CD) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 25 \end{pmatrix} + t\begin{pmatrix} -8 \\ -2 \end{pmatrix}$ and line 4 (AD) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + u\begin{pmatrix} -3 \\ 12 \end{pmatrix}$

where r, s, t and u are scalars.

a Use vector methods to find the coordinates of A, B, C and D.

b Write down vectors $\overrightarrow{AC}$ and $\overrightarrow{DB}$ and hence find:

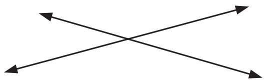
- **i** $|\overrightarrow{AC}|$
- **ii** $|\overrightarrow{DB}|$
- **iii** $\overrightarrow{AC} \bullet \overrightarrow{DB}$

c What do the answers to **b** tell you about quadrilateral ABCD?

C RELATIONSHIPS BETWEEN LINES

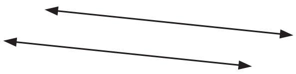
CLASSIFICATION IN 2 DIMENSIONS

intersecting



one point of intersection
unique solution

parallel



lines do not meet
no solutions

coincident

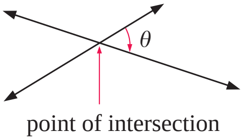
the same line
infinitely many solutions

LINE CLASSIFICATION IN 3 DIMENSIONS

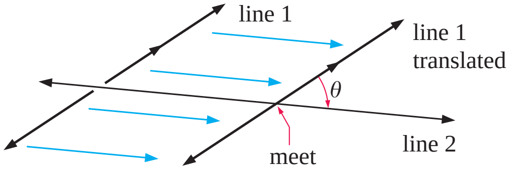
- If the lines are **coplanar**, or lie in the same plane, they may be:
  - intersecting
  - parallel
  - coincident
- If the lines are not coplanar then they are **skew**.

Skew lines are lines which are neither parallel nor intersecting.

- If the lines are **parallel**, the angle between them is 0^o.
- If the lines are **intersecting**, the angle between them is θ, as shown.



- If the lines are **skew**, there is still an angle that one line makes with the other. If we translate one line to intersect the other, the angle between the original lines is defined as the angle between the intersecting lines, which is the angle θ.



lines 1 and 2 are skew

Example 12 **Self Tutor**

Line 1 has equations $x = -1 + 2s,\ y = 1 - 2s$ and $z = 1 + 4s$.

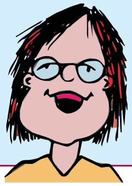
Line 2 has equations $x = 1 - t,\ y = t$ and $z = 3 - 2t$.

Show that the lines are parallel.

Line 1 is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + s\begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}$ with direction vector $\begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}$.

Likewise, line 2 has direction vector $\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$.

Since $\begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix} = -2\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$, the lines are parallel.



If $\mathbf{a} = k\mathbf{b}$ for some scalar k, then $\mathbf{a} \parallel \mathbf{b}$.

Example 13 **Self Tutor**

Line 1 has equations $x = -1 + 2s,\ y = 1 - 2s$ and $z = 1 + 4s$.

Line 2 has equations $x = 1 - t,\ y = t$ and $z = 3 - 2t$.

Line 3 has equations $x = 1 + 2u,\ y = -1 - u,\ z = 4 + 3u$.

a Show that line 2 and line 3 intersect and find the angle between them.

b Show that line 1 and line 3 are skew.

a Equating x, y and z values in lines 2 and 3 gives

$$\underbrace{\begin{array}{ll} 1-t=1+2u & t=-1-u \\ \therefore\ \ t=-2u & \therefore\ \ t=-1-u \end{array}} \quad \begin{array}{ll} \text{and} & 3-2t=4+3u \\ \text{and} & 3u+2t=-1 \text{ ...... (1)} \end{array}$$

Solving these we get

$$\begin{aligned} -2u &= -1-u \\ \therefore\ \ -u &= -1 \\ \therefore\ \ u &= 1 \ \text{ and so } \ t=-2 \end{aligned}$$

Checking in (1): $3u+2t=3(1)+2(-2)=3-4=-1$ ✓

$\therefore$ $u=1,\ t=-2$ satisfies all three equations, a *common solution.*

Using $u=1$, lines 2 and 3 meet at $(1+2(1),\ -1-(1),\ 4+3(1))$ which is $(3,\ -2,\ 7)$.

Direction vectors for lines 2 and 3 are $\mathbf{a}=\begin{pmatrix}-1\\1\\-2\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}2\\-1\\3\end{pmatrix}$ respectively.

Now $\cos\theta=\dfrac{|\,\mathbf{a}\bullet\mathbf{b}\,|}{|\,\mathbf{a}\,|\,|\,\mathbf{b}\,|}=\dfrac{|-2-1-6|}{\sqrt{1+1+4}\sqrt{4+1+9}}$, where θ is the acute angle between **a** and **b**.

$\therefore$ $\cos\theta=\dfrac{9}{\sqrt{84}}$ and so $\theta\approx 10.89^o$

$\therefore$ the angle between lines 2 and 3 is about 10.9^o.

b Equating x, y and z values in lines 1 and 3 gives

$$\underbrace{\begin{array}{ll} -1+2s=1+2u & 1-2s=-1-u \\ \therefore\ \ 2s-2u=2 & \therefore\ \ -2s+u=-2 \end{array}} \quad \begin{array}{ll} \text{and} & 1+4s=4+3u \\ \text{and} & 4s-3u=3 \text{ ...... (2)} \end{array}$$

Solving these we get

$$\begin{aligned} 2s-2u &= 2 \\ \therefore\ \ -2s+u &= -2 \\ \hline \therefore\ \ -u &= 0 \quad \{\text{adding them}\} \end{aligned}$$

$\therefore$ $2s=2$ and so $s=1$

Checking in (2), $4s-3u=4(1)-3(0)=4\neq 3$

So, there is no simultaneous solution to all 3 equations.

$\therefore$ the lines cannot meet, and as they are not parallel they must be skew.

EXERCISE 13C

1 Classify the following line pairs as either parallel, intersecting or skew, and in each case find the measure of the acute angle between them:

a $x=1+2t,\ y=2-t,\ z=3+t$ and $x=-2+3s,\ y=3-s,\ z=1+2s$

b $x=-1+2t,\ y=2-12t,\ z=4+12t$ and $x=4s-3,\ y=3s+2,\ z=-s-1$

c $x=6t,\ y=3+8t,\ z=-1+2t$ and $x=2+3s,\ y=4s,\ z=1+s$

d $x=2-y=z+2$ and $x=1+3s,\ y=-2-2s,\ z=2s+\frac{1}{2}$

e $x = 1 + t,\ y = 2 - t,\ z = 3 + 2t$ and $x = 2 + 3s,\ y = 3 - 2s,\ z = s - 5$

f $x = 1 - 2t,\ y = 8 + t,\ z = 5$ and $x = 2 + 4s,\ y = -1 - 2s,\ z = 3$

REVIEW SET 13A NON-CALCULATOR

1 Find **a** the vector equation **b** the parametric equations

of the line that passes through $(-6, 3)$ with direction $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$.

2 $(-3, m)$ lies on the line with vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ -2 \end{pmatrix} + t\begin{pmatrix} -7 \\ 4 \end{pmatrix}$. Find m.

3 Line L has equation $\mathbf{r} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 5 \end{pmatrix}$. Write two other equations to represent line L.

4 P(2, 0, 1), Q(3, 4, -2) and R(-1, 3, 2) are three points in space. Find:

a $\overrightarrow{PQ}$, $|\overrightarrow{PQ}|$ and $\overrightarrow{QR}$ **b** the parametric equations of line (PQ).

5 Find a unit vector parallel to $\mathbf{i} + r\mathbf{j} + 3\mathbf{k}$ and perpendicular to $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

6 Triangle ABC is formed by three lines:

Line AB is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, line BC is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \end{pmatrix}$,

line AC is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + u\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

a Use vector methods to find the coordinates of A, B and C.

b Find $|\overrightarrow{AB}|$, $|\overrightarrow{BC}|$ and $|\overrightarrow{AC}|$. **c** Classify triangle ABC.

7 **a** Consider two unit vectors **a** and **b**. Prove that the vector $\mathbf{a} + \mathbf{b}$ bisects the angle between vector **a** and vector **b**.

b Consider the points H(9, 5, -5), J(7, 3, -4) and K(1, 0, 2).
Find the equation of the line L that passes through J and bisects angle HJK.

c Find the coordinates of the point where L meets (HK).

8 $x^2 + y^2 + z^2 = 26$ is the equation of a sphere with centre (0, 0, 0) and radius $\sqrt{26}$ units.
Find the point(s) where the line through (3, -1, -2) and (5, 3, -4) meets the sphere.

REVIEW SET 13B CALCULATOR

1 Find the vector equation of the line which cuts the y-axis at (0, 8) and has direction $5\mathbf{i} + 4\mathbf{j}$.

2 P(-1, 2, 3) and Q(4, 0, -1) are two points in space. Find:

a $\overrightarrow{PQ}$ **b** the angle that $\overrightarrow{PQ}$ makes with the X-axis.

3 A yacht is sailing at a constant speed of $5\sqrt{10}$ km h^{-1} in the direction $-\mathbf{i} - 3\mathbf{j}$. Initially it is at point $(-6, 10)$. A beacon is at $(0, 0)$ at the centre of a tiny atoll.

- **a** Find in terms of **i** and **j**:
  - **i** the initial position vector of the yacht
  - **ii** the direction vector of the yacht
  - **iii** the position vector of the yacht at any time t hours, $t \geqslant 0$.
- **b** Find the time when the yacht is closest to the beacon.
- **c** If there is a reef of radius 8 km around the atoll, will the yacht hit the reef?

4 Find the **i** vector **ii** Cartesian equation of the line:

- **a** passing through $(2, -3)$ with direction $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$
- **b** passing through $(-1, 6)$ and $(5, -2)$.

5 Find the coordinates of the point where the line with parametric equations $x = 2 + 3t$, $y = 1 - 2t$, $z = 5 - t$ meets:

a the XOY plane **b** the YOZ plane **c** the XOZ plane.

6 Find the angle between the lines L_1, passing through $(0, 3)$ and $(5, -2)$, and L_2, passing through $(-2, 4)$ and $(-6, 7)$.

7 Submarine X23 is at $(2, 4)$. It fires a torpedo with velocity vector $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ at exactly 2:17 pm. Submarine Y18 is at $(11, 3)$. It fires a torpedo with velocity vector $\begin{pmatrix} -1 \\ a \end{pmatrix}$ at 2:19 pm to intercept the torpedo from X23.

- **a** Find $x_1(t)$ and $y_1(t)$ for the torpedo fired from submarine X23.
- **b** Find $x_2(t)$ and $y_2(t)$ for the torpedo fired from submarine Y18.
- **c** At what time does the interception occur?
- **d** What was the direction and speed of the interception torpedo?

8 Classify the following line pairs as either parallel, intersecting or skew, and in each case find the measure of the acute angle between them:

- **a** $x = 2 + t$, $y = -1 + 2t$, $z = 3 - t$ and $x = -8 + 4s$, $y = s$, $z = 7 - 2s$
- **b** $x = 3 + t$, $y = 5 - 2t$, $z = -1 + 3t$ and $x = 2 - s$, $y = 1 + 3s$, $z = 4 + s$

REVIEW SET 13C

1 Find the velocity vector of an object that is moving in the direction $3\mathbf{i} - \mathbf{j}$ with a speed of 20 km h^{-1}.

2 Suppose A is $(3, 2, -1)$ and B is $(-1, 2, 4)$.

- **a** Write down the vector equation of the line through A and B.
- **b** Find *two* points on the line (AB) which are $2\sqrt{41}$ units from A.

3 A particle at $P(x(t),\ y(t))$ moves such that $x(t) = -4 + 8t$ and $y(t) = 3 + 6t$, where $t \geqslant 0$ is the time in seconds. The distance units are metres. Find the:

a initial position of P
b position of P after 4 seconds
c speed of P
d velocity vector of P.

4 Trapezium KLMN is formed by the lines:

$$(\text{KL}) \text{ is } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 19 \end{pmatrix} + p\begin{pmatrix} 5 \\ -2 \end{pmatrix}; \quad (\text{ML}) \text{ is } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 33 \\ -5 \end{pmatrix} + q\begin{pmatrix} -11 \\ 16 \end{pmatrix};$$

$$(\text{NK}) \text{ is } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} + r\begin{pmatrix} 4 \\ 10 \end{pmatrix}; \quad (\text{MN}) \text{ is } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 43 \\ -9 \end{pmatrix} + s\begin{pmatrix} -5 \\ 2 \end{pmatrix}$$

where p, q, r and s are scalars.

a Which two lines are parallel? Why?
b Which lines are perpendicular? Why?
c Use vector methods to find the coordinates of K, L, M and N.
d Calculate the area of trapezium KLMN.

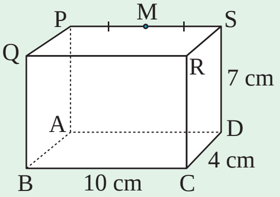
5 Find the angle between the lines:

$$L_1 : \ x = 1 - 4t, \quad y = 3t \qquad \text{and} \qquad L_2 : \ x = 2 + 5s, \quad y = 5 - 12s.$$

6 Consider $A(3,\ -1,\ 1)$ and $B(0,\ 2,\ -1)$.

a Find $|\overrightarrow{AB}|$.
b Find the vector equation of the line passing through A and B.

7

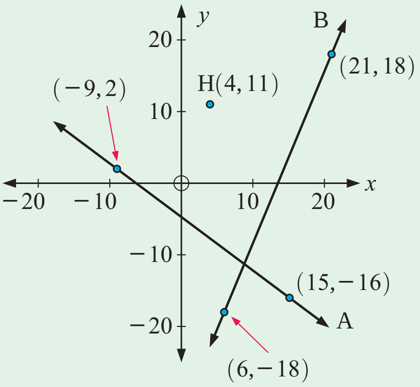


Use vector methods to determine the measure of angle QDM given that M is the midpoint of [PS].

8 Let $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ represent a 1 km displacement due east and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represent a 1 km displacement due north.

Road A passes through $(-9,\ 2)$ and $(15,\ -16)$.
Road B passes through $(6,\ -18)$ and $(21,\ 18)$.

a Find vector equations for each of the roads.
b An injured hiker is at $(4,\ 11)$, and needs to get to a road in the shortest possible distance. Towards which road should he head, and how far will he need to walk to reach the road?



9 Given the points $A(4,\ 2,\ -1)$, $B(2,\ 1,\ 5)$, and $C(9,\ 4,\ 1)$:

- **a** Show that $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{AC}$.
- **b** Find the equation of the line through A and B.
- **c** Find the equation of (AC).

Chapter 14

Descriptive statistics

Syllabus reference: 6.1, 6.2, 6.3, 6.4

Contents:

OPENING PROBLEM

A farmer is investigating the effect of a new organic fertiliser on his crops of peas. He has divided a small garden into two equal plots and planted many peas in each. Both plots have been treated the same except the fertiliser have been used on one but not the other.

A random sample of 150 pods is harvested from each plot at the same time, and the number of peas in each pod is counted. The results are:

© iStockphoto

Without fertiliser

4 6 5 6 5 6 4 6 4 9 5 3 6 8 5 4 6 8 6 5 6 7 4 6 5 2 8 6 5 6 5 5 5 4 4 4 6 7 5 6 7 5 5 6
4 8 5 3 7 5 3 6 4 7 5 6 5 7 5 7 6 7 5 4 7 5 5 5 6 6 5 6 7 5 8 6 8 6 7 6 6 3 7 6 8 3 3 4
4 7 6 5 6 4 5 7 3 7 7 6 7 7 4 6 6 5 6 7 6 3 4 6 6 3 7 6 7 6 8 6 6 6 6 4 7 6 6 5 3 8 6 7
6 8 6 7 6 6 6 8 4 4 8 6 6 2 6 5 7 3

With fertiliser

6 7 7 4 9 5 5 5 8 9 8 9 7 7 5 8 7 6 6 7 9 7 7 7 8 9 3 7 4 8 5 10 8 6 7 6 7 5 6
8 7 9 4 4 9 6 8 5 8 7 7 4 7 8 10 6 10 7 7 7 9 7 7 8 6 8 6 8 7 4 8 6 8 7 3 8 7 6
9 7 6 9 7 6 8 3 9 5 7 6 8 7 9 7 8 4 8 7 7 7 6 6 8 6 3 8 5 8 7 6 7 4 9 6 6 6 8 4
7 8 9 7 7 4 7 5 7 4 7 6 4 6 7 7 6 7 8 7 6 6 7 8 6 7 10 5 13 4 7 11

Things to think about:

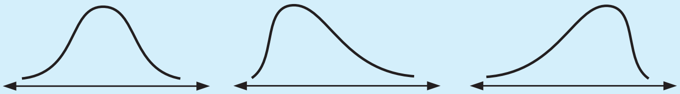
- Can you state clearly the problem that the farmer wants to solve?
- How has the farmer tried to make a fair comparison?
- How could the farmer make sure that his selection was at random?
- What is the best way of organising this data?
- What are suitable methods of displaying the data?
- Are there any abnormally high or low results and how should they be treated?
- How can we best describe the most typical pod size?
- How can we best describe the spread of possible pod sizes?
- Can the farmer make a reasonable conclusion from his investigation?

A KEY STATISTICAL CONCEPTS

In statistics we collect information about a group of individuals, then analyse this information to draw conclusions about those individuals.

You should already be familiar with these words which are commonly used in statistics:

• **Population**	A collection of individuals about which we want to draw conclusions.
• **Census**	The collection of information from the **whole population**.

Sample	A subset of the population. It is important to choose a sample at **random** to avoid **bias** in the results.
Survey	The collection of information from a **sample**.
Data (singular **datum**)	Information about individuals in a population.
Parameter	A numerical quantity measuring some aspect of a population.
Statistic	A quantity calculated from data gathered from a sample. It is usually used to estimate a population parameter.
Distribution	The pattern of variation of data. The distribution may be described as:  symmetrical positively skewed negatively skewed
Outliers	Data values that are either much larger or much smaller than the general body of data. They should be included in an analysis *unless* they are the result of human or other error.

The data we collect may be **categorical** or **numerical**.

There are two types of numerical variable we will deal with in this course:

A **discrete numerical variable** takes exact number values and is often a result of **counting**.

Examples of discrete numerical variables are:

The number of people in a car:	the variable could take the values 1, 2, 3, ...
The score out of 20 for a test:	the variable could take the values 0, 1, 2, 3, ..., 20.

A **continuous variable** takes numerical values within a certain continuous range. It is usually a result of **measuring**.

Examples of continuous numerical variables are:

The height of Year 11 students:	the variable can take any value from about 140 cm to 200 cm.
The speed of cars on a stretch of highway:	the variable can take any value from 0 km h^{-1} to the fastest speed that a car can travel, but is most likely to be in the range 60 km h^{-1} to 160 km h^{-1}.

ORGANISATION AND DISPLAY OF NUMERICAL DATA

From previous courses you should be familiar with **column graphs** used to display **discrete** numerical variables.

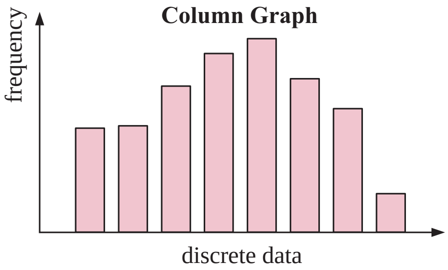
When data is recorded for a **continuous** variable there are likely to be many different values. We organise the data by grouping it into **class intervals** of **equal width**.

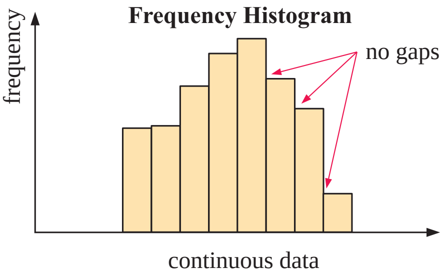
A special type of graph called a **frequency histogram** or just **histogram** is used to display the data. This is similar to a column graph but, to account for the continuous nature of the variable, a number line is used for the horizontal axis and the 'columns' are joined together.

Column graphs and frequency histograms both have the following features:

- The frequency of occurrence is on the vertical axis.
- The range of scores is on the horizontal axis.
- The column widths are equal and the column height represents frequency.

The **modal class**, or class of values that appears most often, is easy to identify from a frequency histogram.





CASE STUDY DRIVING A GOLF BALL

While Ethan Moreau was here for the golf championship, I measured how far he hit 30 drives on the practice fairway. The results are given below in metres:

244.6	245.1	248.0	248.8	250.0
251.1	251.2	253.9	254.5	254.6
255.9	257.0	260.6	262.8	262.9
263.1	263.2	264.3	264.4	265.0
265.5	265.6	266.5	267.4	269.7
270.5	270.7	272.9	275.6	277.5

This type of data must be **grouped** before a frequency histogram can be drawn.

When forming groups, we find the lowest and highest values, and then choose a group width so that there are about 6 to 12 groups. In this case the lowest value is 244.6 m and the largest is 277.5 m. This gives a range of approximately 35 m, so a group width of 5 m will give eight groups of equal width.

We will use the following method of grouping. We let d be the length of a drive. The first group $240 \leqslant d < 245$ includes any data value which is at least 240 m but less than 245 m. The group $260 \leqslant d < 265$ includes data which is at least 260 m but < 265 m. We use this technique to create eight groups into which all data values will fall.

A tally is used to count the data that falls in each group. Do not try to determine the number of data values in the $240 \leqslant d < 245$ group first off. Simply place a vertical stroke in the tally column to register an entry as you work through the data from start to finish. Every fifth entry in a group is marked with a diagonal line through the previous four so groups of five can be counted quickly.

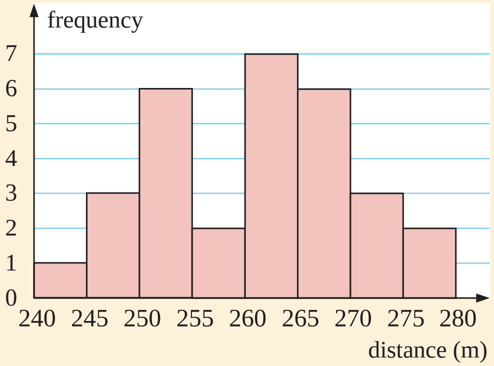
A frequency column summarises the number of data values in each group. The relative frequency column measures the percentage of the total number of data values that are in each group.

Ethan Moreau's 30 drives

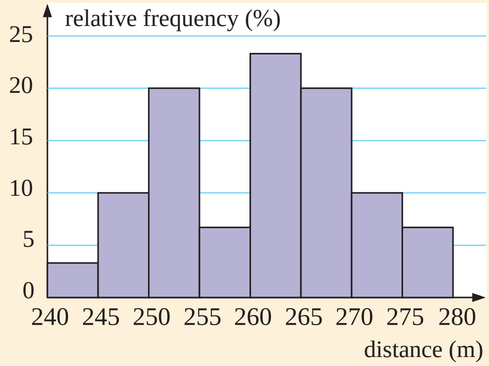
Distance (m)	*Tally*	*Frequency* (f)	*% Relative Frequency*
$240 \leqslant d < 245$	|	1	3.3
$245 \leqslant d < 250$	|||	3	10.0
$250 \leqslant d < 255$	𝍸 |	6	20.0
$255 \leqslant d < 260$	||	2	6.7
$260 \leqslant d < 265$	𝍸 ||	7	23.3
$265 \leqslant d < 270$	𝍸 |	6	20.0
$270 \leqslant d < 275$	|||	3	10.0
$275 \leqslant d < 280$	||	2	6.7
	Totals	30	100.0

From this table we can draw both a **frequency histogram** and a **relative frequency histogram**:

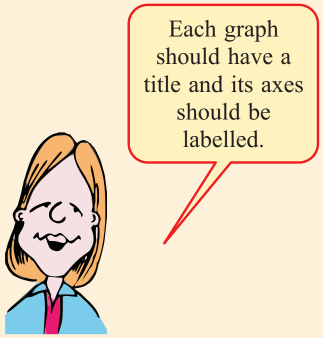
Frequency histogram



Relative frequency histogram



The advantage of the relative frequency histogram is that we can easily compare it with other distributions with different numbers of data values. Using percentages allows for a fair comparison.



Example 1 **Self Tutor**

A sample of 20 juvenile lobsters is randomly selected from a tank containing several hundred. Each lobster is measured for length (in cm) and the results were:

4.9, 5.6, 7.2, 6.7, 3.1, 4.6, 6.0, 5.0, 3.7, 7.3,
6.0, 5.4, 4.2, 6.6, 4.7, 5.8, 4.4, 3.6, 4.2, 5.4

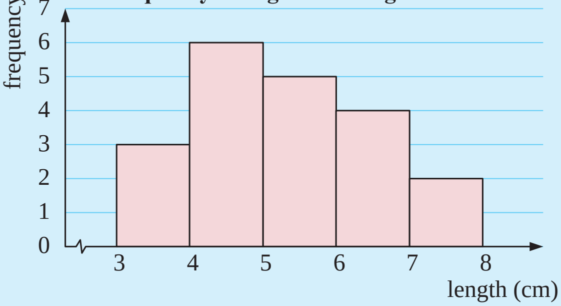
Organise the data using a frequency table, and hence graph the data.

The variable 'the length of a lobster' is *continuous* even though lengths have been rounded to the nearest mm.

The shortest length is 3.1 cm and the longest is 7.3 cm, so we will use class intervals of length 1 cm.

Length (cm)	*Frequency*
$3 \leqslant l < 4$	3
$4 \leqslant l < 5$	6
$5 \leqslant l < 6$	5
$6 \leqslant l < 7$	4
$7 \leqslant l < 8$	2

Frequency histogram of lengths of lobsters

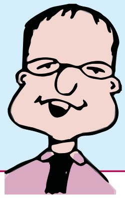


A stemplot could also be used to organise the data:

Stem	*Leaf*
3	1 6 7
4	2 2 4 6 7 9
5	0 4 4 6 8
6	0 0 6 7
7	2 3

Scale: 3 | 1 means 3.1 cm

The modal class is $4 \leqslant l < 5$ cm as this occurred most frequently.



EXERCISE 14A

1 A frequency table for the heights of a basketball squad is given alongside.

a Explain why 'height' is a continuous variable.

b Construct a frequency histogram for the data. The axes should be carefully marked and labelled, and you should include a heading for the graph.

c What is the modal class? Explain what this means.

d Describe the distribution of the data.

Height (cm)	*Frequency*
$170 \leqslant H < 175$	1
$175 \leqslant H < 180$	8
$180 \leqslant H < 185$	9
$185 \leqslant H < 190$	11
$190 \leqslant H < 195$	9
$195 \leqslant H < 200$	3
$200 \leqslant H < 205$	3

2 A school has conducted a survey of 60 students to investigate the time it takes for them to travel to school. The following data gives the travel times to the nearest minute:

12	15	16	8	10	17	25	34	42	18	24	18	45	33	38
45	40	3	20	12	10	10	27	16	37	45	15	16	26	32
35	8	14	18	15	27	19	32	6	12	14	20	10	16	14
28	31	21	25	8	32	46	14	15	20	18	8	10	25	22

a Is travel time a discrete or continuous variable?

b Construct an ordered stemplot for the data using stems 0, 1, 2, ...

c Describe the distribution of the data.

d What is the modal travelling time?

3 For the following data, state whether a frequency histogram or a column graph should be used and draw the appropriate graph.

a The number of matches in 30 match boxes:

Number of matches per box	47	49	50	51	52	53	55
Frequency	1	1	9	12	4	2	1

b The heights of 25 hockey players (to the nearest cm):

Height (cm)	120 - 129	130 - 139	140 - 149	150 - 159	160 - 169
Frequency	1	2	7	14	1

4 A plant inspector takes a random sample of six month old seedlings from a nursery and measures their heights to the nearest mm.

The results are shown in the table alongside.

Height (mm)	*frequency*
300 - 324	12
325 - 349	18
350 - 374	42
375 - 399	28
400 - 424	14
425 - 449	6

a Represent the data on a frequency histogram.

b How many of the seedlings are 400 mm or more?

c What percentage of the seedlings are between 349 and 400 mm?

d The total number of seedlings in the nursery is 1462. Estimate the number of seedlings which measure: **i** less than 400 mm **ii** between 374 and 425 mm.

B MEASURING THE CENTRE OF DATA

We can get a better understanding of a data set if we can locate the **middle** or **centre** of the data and get an indication of its **spread** or dispersion. Knowing one of these without the other is often of little use.

There are *three statistics* that are used to measure the **centre** of a data set. These are the **mode**, the **mean**, and the **median**.

THE MODE

For discrete numerical data, the **mode** is the most frequently occurring value in the data set.

For continuous numerical data, we cannot talk about a mode in this way because no two data values will be *exactly* equal. Instead we talk about a **modal class**, which is the class that occurs most frequently.

THE MEAN

The **mean** of a data set is the statistical name for the arithmetic average.

$$\textbf{mean} = \frac{\textbf{sum of all data values}}{\textbf{the number of data values}}$$

The mean gives us a single number which indicates a centre of the data set. It is usually not a member of the data set.

For example, a mean test mark of 73% tells us that there are several marks below 73% and several above it. 73% is at the centre, but it does not necessarily mean that one of the students scored 73%.

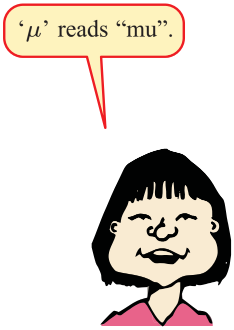
Suppose x is a numerical variable. We let:

x_i be the ith data value

n be the number of data values in the sample or population

$\overline{x}$ represent the mean of a **sample**, so $\overline{x} = \dfrac{\sum_{i=1}^{n} x_i}{n}$

μ represent the mean of a **population**, so $\mu = \dfrac{\sum_{i=1}^{n} x_i}{n}$.



THE MEDIAN

The **median** is the *middle value* of an ordered data set.

An ordered data set is obtained by listing the data, usually from smallest to largest.

The median splits the data in halves. Half of the data are less than or equal to the median and half are greater than or equal to it.

For example, if the median mark for a test is 73% then you know that half the class scored less than or equal to 73% and half scored greater than or equal to 73%.

For an **odd number** of data, the median is one of the original data values.

For an **even number** of data, the median is the average of the two middle values, and may not be in the original data set.

If there are n data values, find $\dfrac{n+1}{2}$. The median is the $\left(\dfrac{n+1}{2}\right)$th data value.

For example:

If $n = 13$, $\frac{13+1}{2} = 7$, so the median = 7th ordered data value.

If $n = 14$, $\frac{14+1}{2} = 7.5$, so the median = average of 7th and 8th ordered data values.

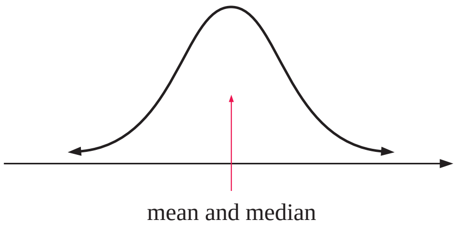
THE MERITS OF THE MEAN AND MEDIAN AS MEASURES OF CENTRE

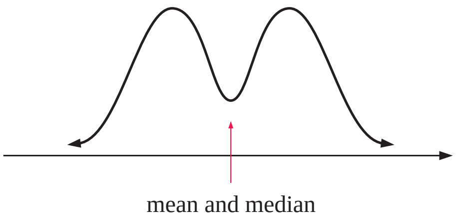
The **median** is the only measure of centre that will locate the true centre regardless of the data set's features. It is unaffected by the presence of extreme values. It is called a *resistant* measure of centre.

The **mean** is an accurate measure of centre if the distribution is symmetrical or approximately symmetrical. If it is not, then unbalanced high or low values will *drag* the mean toward them and cause it to be an inaccurate measure of the centre. It is called a *non-resistant* measure of centre because it is influenced by all data values in the set. *If it is considered inaccurate, it should not be used in discussion.*

THE RELATIONSHIP BETWEEN THE MEAN AND THE MEDIAN FOR DIFFERENT DISTRIBUTIONS

For distributions that are **symmetric**, the mean or median will be approximately equal.

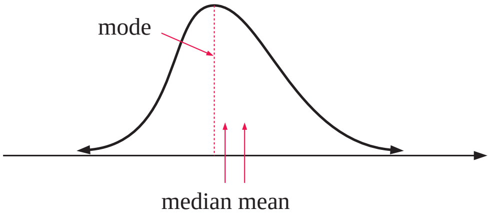




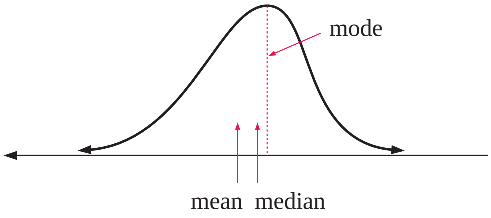
If the data set has symmetry, both the mean and the median should accurately measure the centre of the distribution.

If the data set is not symmetric, it may be positively or negatively skewed:

positively skewed distribution



negatively skewed distribution



Notice that the mean and median are clearly different for these skewed distributions.

INVESTIGATION MERITS OF THE MEAN AND MEDIAN

For this investigation you can either use the software provided on the CD or your graphics calculator. Instructions for using your calculator are found at the front of the book.

Consider the data gained from Ethan Moreau while he was here for the golf championship. The data was as follows:

244.6	245.1	248.0	248.8	250.0	270.5	251.1	251.2
253.9	254.5	254.6	270.7	255.9	257.0	260.6	262.8
262.9	272.9	263.1	263.2	264.3	264.4	265.0	275.6
265.5	265.6	266.5	267.4	269.7	277.5		

STATISTICS PACKAGE

What to do:

1 Enter the data into your graphics calculator as a list, or use the **statistics package** supplied.

- **a** Produce a frequency histogram of the data. Set the X values from 240 to 280 with an increment of 5. Set the Y values from 0 to 30.
- **b** Comment on the shape of the distribution.
- **c** Find: **i** the median **ii** the mean
- **d** Compare the mean and the median. Is the mean an accurate measure of the centre?

2 Since we have continuous numerical data, we have a modal class rather than an individual mode.

- **a** What is the modal class?
- **b** What would the modal class be if our intervals were 2 m wide starting at 240 m?

3 Now suppose Ethan had hit a few very bad drives. Let us say that his three shortest drives were very short!

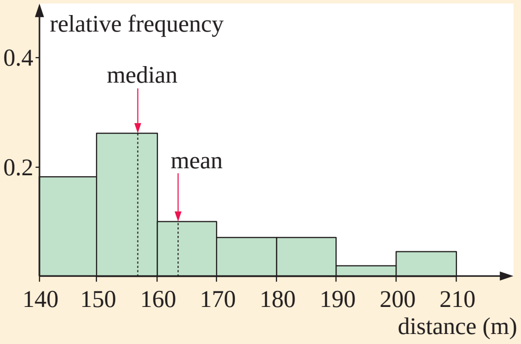
- **a** Change the three shortest drives to 82.1 m, 103.2 m and 111.1 m.
- **b** Repeat **1 a**, **b**, **c** and **d** but set the X values from 75 to 300 with an increment of 25 for the frequency histogram.
- **c** Describe the distribution as symmetric, positively skewed, or negatively skewed. What effect have the changed values had on the mean and median as measures of the centre of the data?

4 What would have happened if Ethan had hit a few really long balls in addition to the very bad ones? Let us imagine that the longest balls he hit were very long indeed!

- **a** Change the three longest drives to 403.9 m, 415.5 m and 420.0 m.
- **b** Repeat **1 a**, **b**, **c** and **d** but set the X values from 50 to 450 with an increment of 50 for the frequency histogram.
- **c** Describe the distribution as symmetric, positively skewed, or negatively skewed. What effect have the changed values had on the mean and median as measures of the centre of the data?

5 While collecting the data from Ethan, I decided to have a hit as well. I hit 30 golf balls with my driver. The relative frequency histogram alongside reveals the results. The distribution is clearly positively skewed.

Discuss the merits of the median and mean as measures of the centre of this distribution.



UNGROUPED DATA

Example 2 **Self Tutor**

The number of faulty products returned to an electrical goods store over a 21 day period is:

3 4 4 9 8 8 6 4 7 9 1 3 5 3 5 9 8 6 3 7 1

For this data set, find: **a** the mean **b** the median **c** the mode.

a $$\text{mean} = \frac{3+4+4+ \ldots\ldots +3+7+1}{21} \begin{array}{l} \leftarrow \text{sum of the data} \\ \leftarrow \text{21 data values} \end{array}$$

$$= \tfrac{113}{21}$$

$$\approx 5.38 \text{ faulty products}$$

b The ordered data set is: ~~1 1 3 3 3 3 4 4 4 5~~ 5 ~~6 6 7 7 8 8 8 9 9 9~~

{as $n = 21$, $\frac{n+1}{2} = 11$}

$\therefore$ median = 5 faulty products

c 3 is the score which occurs the most often, so the mode is 3 faulty products.

For the faulty products data in **Example 2**, how are the measures of the middle affected if on the 22nd day the number of faulty products was 9?

We expect the mean to rise as the new data value is greater than the old mean.

In fact, the new mean $= \dfrac{113+9}{22} = \frac{122}{22} \approx 5.55$ faulty products.

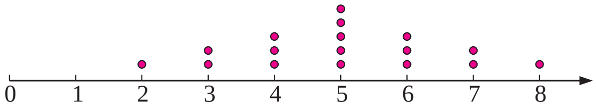
The new ordered data set would be: $1\ 1\ 3\ 3\ 3\ 3\ 4\ 4\ 4\ 5\ \underbrace{5\ 6}_{\text{two middle scores}}\ 6\ 7\ 7\ 8\ 8\ 8\ 9\ 9\ 9\ 9$

$\therefore$ median $= \dfrac{5+6}{2} = 5.5$ faulty products {as $n = 22$, $\frac{n+1}{2} = 11.5$}

This new data set has two modes. The modes are 3 and 9 faulty products and we say that the data set is **bimodal**.

Note:

- If a data set has three or more modes, we do not use the mode as a measure of the middle.
- Consider the data: 4 2 5 6 7 4 5 3 5 4 7 6 3 5 8 6 5.
  The dot plot of this data is:



For this data the mean, median and mode are all 5.
Equal or approximately equal values of the mean, mode and median indicate a *symmetrical distribution* of data.

Example 3 **Self Tutor**

If 6 people have a mean mass of 53.7 kg, find their total mass.

$$\frac{\text{sum of masses}}{6} = 53.7 \text{ kg}$$

$$\therefore \quad \text{sum of masses} = 53.7 \times 6$$

So the total mass = 322.2 kg.

EXERCISE 14B.1

1 Find the **i** mean **ii** median **iii** mode for each of the following data sets:

a 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 5, 6, 6, 6, 6, 6, 7, 7, 8, 8, 8, 9, 9

b 10, 12, 12, 15, 15, 16, 16, 17, 18, 18, 18, 18, 19, 20, 21

c 22.4, 24.6, 21.8, 26.4, 24.9, 25.0, 23.5, 26.1, 25.3, 29.5, 23.5

2 Consider the following two data sets:

Data set A: 3, 4, 4, 5, 6, 6, 7, 7, 7, 8, 8, 9, 10
Data set B: 3, 4, 4, 5, 6, 6, 7, 7, 7, 8, 8, 9, 15

a Find the mean for both data set A and data set B.

b Find the median of both data set A and data set B.

c Explain why the mean of data set A is less than the mean of data set B.

d Explain why the median of data set A is the same as the median of data set B.

3 The annual salaries of ten office workers are:

$23 000, $46 000, $23 000, $38 000, $24 000,
$23 000, $23 000, $38 000, $23 000, $32 000

a Find the mean, median and modal salaries of this group.

b Explain why the mode is an unsatisfactory measure of the middle in this case.

c Is the median a satisfactory measure of the middle of this data set?

4 The following raw data is the daily rainfall (to the nearest millimetre) for the month of July 2007 in the desert:

3, 1, 0, 0, 0, 0, 0, 2, 0, 0, 3, 0, 0, 0, 7, 1, 1, 0, 3, 8, 0, 0, 0, 42, 21, 3, 0, 3, 1, 0, 0

a Find the mean, median and mode for the data.

b Give a reason why the median is not the most suitable measure of centre for this set of data.

- **c** Give a reason why the mode is not the most suitable measure of centre for this set of data.
- **d** Are there any outliers in this data set?
- **e** On some occasions outliers are removed because they must be due to errors in observation or calculation. If the outliers in the data set were accurately found, should they be removed before finding the measures of the middle?

5 A basketball team scored 43, 55, 41 and 37 points in their first four matches.

- **a** What is the mean number of points scored for the first four matches?
- **b** What score will the team need to shoot in the next match so that they maintain the same mean score?
- **c** The team scores only 25 points in the fifth match. What is the mean number of points scored for the five matches?
- **d** The team scores 41 points in their sixth and final match. Will this increase or decrease their previous mean score? What is the mean score for all six matches?

6 The mean monthly sales for a clothing store total \$15 467. Calculate the total sales for the store for the year.

7 While on an outback safari, Bill drove an average of 262 km per day for a period of 12 days. How far did Bill drive in total while on safari?

8 Given $\overline{x} = 11.6$ and $n = 10$, calculate $\sum\limits_{i=1}^{10} x_i$.

9 Towards the end of season, a netballer had played 14 matches and had an average of 16.5 goals per game. In the final two matches of the season she threw 21 goals and 24 goals. Find the netballer's new average.

10 The selling prices of the last 10 houses sold in a certain district were as follows:

\$146 400, \$127 600, \$211 000, \$192 500,
\$256 400, \$132 400, \$148 000, \$129 500,
\$131 400, \$162 500

- **a** Calculate the mean and median selling prices and comment on the results.
- **b** Which measure would you use if you were:
  - **i** a vendor wanting to sell your house
  - **ii** looking to buy a house in the district?

11 Find x if 5, 9, 11, 12, 13, 14, 17 and x have a mean of 12.

12 Find a given that 3, 0, a, a, 4, a, 6, a and 3 have a mean of 4.

13 Over the complete assessment period, Aruna averaged 35 out of a possible 40 marks for her maths tests. However, when checking her files, she could only find 7 of the 8 tests. For these she scored 29, 36, 32, 38, 35, 34 and 39. Determine how many marks out of 40 she scored for the eighth test.

14 A sample of 10 measurements has a mean of 15.7 and a sample of 20 measurements has a mean of 14.3. Find the mean of all 30 measurements.

15 The mean and median of a set of 9 measurements are both 12. If 7 of the measurements are 7, 9, 11, 13, 14, 17 and 19, find the other two measurements.

16 Jana took seven spelling tests, each with twelve words, but she could only find the results of five of them. These were: 9, 5, 7, 9 and 10. She asked her teacher for the other two results and the teacher said that the mode of her scores was 9 and the mean was 8. Given that Jana knows her worst result was a 5, find the two missing results.

MEASURES OF THE CENTRE FROM OTHER SOURCES

When the same data appears several times we often summarise the data in table form.

Consider the data in the given table:

Data value (x)	*Frequency* (f)	*Product* (fx)
3	1	$1 \times 3 = 3$
4	1	$1 \times 4 = 4$
5	3	$3 \times 5 = 15$
6	7	$7 \times 6 = 42$
7	15	$15 \times 7 = 105$
8	8	$8 \times 8 = 64$
9	5	$5 \times 9 = 45$
Total	$\sum f = 40$	$\sum fx = 278$

We can find the measures of the centre directly from the table.

The mode

There are 15 of data value 7, which is more than for any other data value.

The mode is therefore 7.

The mean

Adding a 'Product' column to the table helps to add all scores.

For example, there are 15 data of value 7 and these add to $7 \times 15 = 105$.

Remembering that the mean $= \dfrac{\text{sum of all data values}}{\text{the number of data values}}$, we find

$$\overline{x} = \frac{\sum_{i=1}^{k} f_i x_i}{\sum_{i=1}^{k} f_i}$$ where k is the number of *different* data values.

This formula is often abbreviated as $\overline{x} = \dfrac{\sum fx}{\sum f}$.

In this case the mean $= \dfrac{278}{40} = 6.95$.

The median

There are 40 data values, an even number, so there are *two middle* data values.

As the sample size $n = 40$,

$$\frac{n+1}{2} = \frac{41}{2} = 20.5\,.$$

So, the median is the average of the 20th and 21st data values.

In the table, the blue numbers show us accumulated values, or the **cumulative frequency**.

Data Value	*Frequency*	*Cumulative Frequency*
3	1	1 ← one number is 3
4	1	2 ← two numbers are 4 or less
5	3	5 ← five numbers are 5 or less
6	7	12 ← 12 numbers are 6 or less
7	15	27 ← 27 numbers are 7 or less
8	8	35 ← 35 numbers are 8 or less
9	5	40 ← all numbers are 9 or less
Total	40	

We can see that the 20th and 21st data values (in order) are both 7s.

$$\therefore \quad \text{median} = \frac{7+7}{2} = 7$$

Notice that we have a skewed distribution even though the mean, median and mode are nearly equal. So, we must be careful if we use measures of the middle to call a distribution symmetric.

Example 4 **Self Tutor**

The table below shows the number of aces served by tennis players in their first sets of a tournament.

Number of aces	1	2	3	4	5	6
Frequency	4	11	18	13	7	2

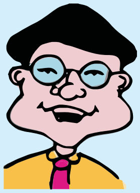
Determine the:
- **a** mean number of aces for these sets.
- **b** median number of aces for these sets.
- **c** mode.

a

No. of aces (x)	*Freq.* (f)	*Product* (fx)
1	4	4
2	11	22
3	18	54
4	13	52
5	7	35
6	2	12
Total	$\sum f = 55$	$\sum fx = 179$

$$\overline{x} = \frac{\sum fx}{\sum f} = \frac{179}{55} \approx 3.25 \text{ aces}$$

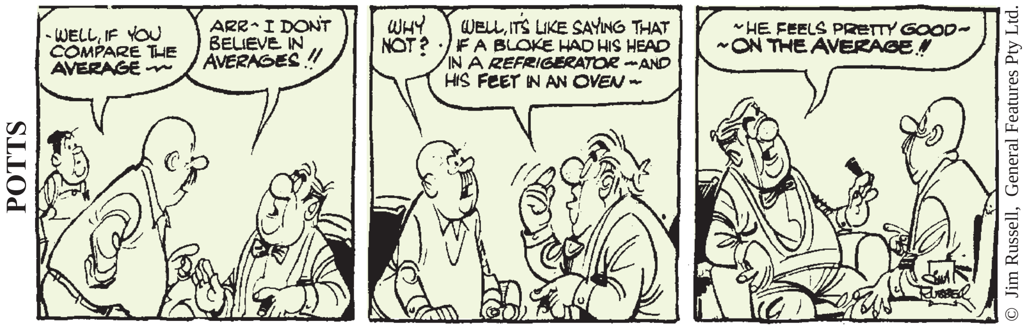
In this case $\dfrac{\sum fx}{\sum f}$ is short for $\dfrac{\sum_{i=1}^{6} f_i x_i}{\sum_{i=1}^{6} f_i}$.



b There are 55 data values, so $n = 55$.
$\frac{n+1}{2} = 28$, so the median is the 28th data value. We add a cumulative frequency column to the table to help us find this data value.
Data values 16 to 33 are 3 aces. So the 28th data value is 3 aces.
$\therefore$ the median is 3 aces.

No. of aces (x)	*Freq.* (f)	*Cumul. Freq.*
1	4	4
2	11	15
3	18	33
4	13	46
5	7	53
6	2	55

c Looking down the frequency column, the highest frequency is 18. This corresponds to 3 aces, so the mode is 3 aces.



The publishers acknowledge the late Mr Jim Russell, General Features for the reproduction of this cartoon.

EXERCISE 14B.2

1 The table alongside shows the results when 3 coins were tossed simultaneously 30 times.
Calculate the:

a mode **b** median
c mean.

Number of heads	*Number of times occurred*
0	4
1	12
2	11
3	3
Total	30

2 The following frequency table records the number of phone calls made in a day by 50 fifteen-year-olds.

a For this data, find the:
i mean **ii** median **iii** mode.

b Construct a column graph for the data and show the position of the mean, median and mode on the horizontal axis.

c Describe the distribution of the data.

d Why is the mean larger than the median for this data?

e Which measure of centre would be the most suitable for this data set?

No. of phone calls	*Frequency*
0	5
1	8
2	13
3	8
4	6
5	3
6	3
7	2
8	1
11	1

3 A company claims that their match boxes contain, on average, 50 matches per box. On doing a survey, the Consumer Protection Society recorded the following results:

Number in a box	*Frequency*
47	5
48	4
49	11
50	6
51	3
52	1
Total	30

a For the data calculate the:

i mode **ii** median **iii** mean

b Do the results of this survey support the company's claim?

c In a court for 'false advertising', the company won their case against the Consumer Protection Society. Suggest how they did this.

4 Families at a school in Australia were surveyed, and the number of children in each family recorded. The results of the survey are shown alongside.

Number of children	*Frequency*
1	5
2	28
3	15
4	8
5	2
6	1
Total	59

a Calculate the:

i mean **ii** mode **iii** median.

b The average Australian family has 2.2 children. How does this school compare to the national average?

c The data set is skewed. Is the skewness positive or negative?

d How has the skewness of the data affected the measures of the centre of the data set?

5 For the data displayed in the following stem-and-leaf plots find the:

i mean **ii** median **iii** mode.

a

Stem	*Leaf*
5	3 5 6
6	0 1 2 4 6 7 9
7	3 3 6 8
8	4 7
9	1

where 5 | 3 means 53

b

Stem	*Leaf*
3	7
4	0 4 8 8
5	0 0 1 3 6 7 8 9
6	0 3 6 7 7 7
7	0 6 9
8	1

where 3 | 7 means 3.7

6 The table alongside shows the IB mathematics scores for a class. A pass is considered to be a score of 4 or more.

Score	1	2	3	4	5	6	7
Number of students	0	2	3	5	x	4	1

a Given the mean score was 4.45, find the value of x.

b Find the percentage of students who passed.

7 Revisit the **Opening Problem** on page **376**.

a Use a frequency table for the *Without fertiliser* data to find the:

i mean **ii** mode **iii** median number of peas per pod.

b Use a frequency table for the *With fertiliser* data to find the:

i mean **ii** mode **iii** median number of peas per pod.

- **c** Which of the measures of the centre is appropriate to use in a report on this data?
- **d** Has the application of fertiliser significantly improved the number of peas per pod?

8 The table alongside compares the mass at birth of some guinea pigs with their mass when they were two weeks old.

- **a** What was the mean birth mass?
- **b** What was the mean mass after two weeks?
- **c** What was the mean increase over the two weeks?

Guinea Pig	*Mass (g) at birth*	*Mass (g) at 2 weeks*
A	75	210
B	70	200
C	80	200
D	70	220
E	74	215
F	60	200
G	55	206
H	83	230

9 Out of 31 measurements, 15 are below 10 cm and 12 are above 11 cm. Find the median if the other 4 measurements are 10.1 cm, 10.4 cm, 10.7 cm and 10.9 cm.

10 Two brands of toothpicks claim that their boxes contain an average of 50 toothpicks per box. In a survey the Consumer Protection Society (C.P.S.) recorded the following results:

Brand A

Number in a box	46	47	48	49	50	51	52	53	55
Frequency	1	2	3	7	10	20	15	3	1

Brand B

Number in a box	48	49	50	51	52	53	54
Frequency	3	17	30	7	2	1	1

- **a** Find the average contents of Brands A and B.
- **b** Would it be fair for the C.P.S. to prosecute the manufacturers of either brand, based on these statistics?

11 In an office of 20 people there are only 4 salary levels paid:
€50 000 (1 person), €42 000 (3 people), €35 000 (6 people), €28 000 (10 people).

- **a** Calculate: **i** the median salary **ii** the modal salary **iii** the mean salary.
- **b** Which measure of central tendency might be used by the boss who is against a pay rise for the other employees?

DATA IN CLASSES

When information has been gathered in classes we use the **midpoint** of the class to represent all scores within that interval.

We are assuming that the scores within each class are evenly distributed throughout that interval. The mean calculated is an **approximation** of the true value, and we cannot do better than this without knowing each individual data value.

Example 5 **Self Tutor**

Find the approximate mean of the following *ages of bus drivers* data, to the nearest year:

Age (yrs)	21-25	26-30	31-35	36-40	41-45	46-50	51-55
Frequency	11	14	32	27	29	17	7

Age (yrs)	*Frequency (f)*	*Midpoint (x)*	$f\,x$
21-25	11	23	253
26-30	14	28	392
31-35	32	33	1056
36-40	27	38	1026
41-45	29	43	1247
46-50	17	48	816
51-55	7	53	371
Total	$\sum f = 137$		$\sum fx = 5161$

$$\bar{x} = \frac{\sum f\,x}{\sum f}$$

$$= \frac{5161}{137}$$

$$\approx 37.7$$

$\therefore$ the mean age of the drivers is about 38 years.

EXERCISE 14B.3

1 50 students sit a mathematics test. Given the results in the table alongside, estimate the mean score.

Score	0-9	10-19	20-29	30-39	40-49
Frequency	2	5	7	27	9

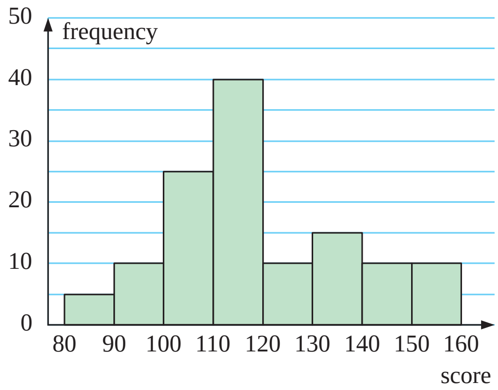
2 The table shows the petrol sales in one day by a number of city service stations.

Litres (L)	*Frequency*
2000 to 2999	4
3000 to 3999	4
4000 to 4999	9
5000 to 5999	14
6000 to 6999	23
7000 to 7999	16

- **a** How many service stations were involved in the survey?
- **b** Estimate the total amount of petrol sold for the day by the service stations.
- **c** Find the approximate mean sales of petrol for the day.

3 This frequency histogram illustrates the results of an aptitude test given to a group of people seeking positions in a company.

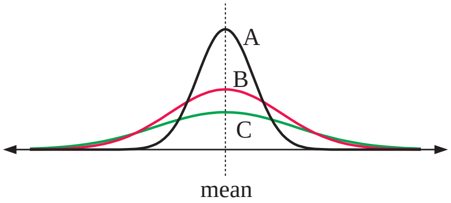
- **a** How many people sat for the test?
- **b** Estimate the mean score for the test.
- **c** What fraction of the people scored less than 100 for the test?
- **d** If the top 20% of the people are offered positions in the company, estimate the minimum mark required.



C MEASURING THE SPREAD OF DATA

To accurately describe a distribution we need to measure both its **centre** and its **spread** or **dispersion**.

The given distributions have the same mean, but clearly they have different spreads. The A distribution has most scores close to the mean whereas the C distribution has the greatest spread.



We will examine four different measures of spread: the **range**, the **interquartile range** (**IQR**), the **variance**, and the **standard deviation**.

THE RANGE

The **range** is the difference between the maximum (largest) and the minimum (smallest) data value.

Example 6 **Self Tutor**

A library surveys 20 borrowers each day from Monday to Friday, and records the number who are not satisfied with the range of reading material. The results are:
3 7 6 8 11.

The following year the library receives a grant that enables the purchase of a large number of books. The survey is then repeated and the results are: 2 3 5 4 6.

Find the range of data in each survey.

The range is the maximum minus the minimum data value.

So, for the first survey, the range is $11 - 3 = 8$.

For the second survey, the range is $6 - 2 = 4$.

The **range** is not considered to be a particularly reliable measure of spread as it uses only two data values. It may be influenced by extreme values or outliers.

THE QUARTILES AND THE INTERQUARTILE RANGE

The median divides the ordered data set into two halves and these halves are divided in half again by the **quartiles**.

The middle value of the lower half is called the **lower quartile** or **25th percentile**. One quarter or 25% of the data have values less than or equal to the lower quartile. 75% of the data have values greater than or equal to the lower quartile.

The middle value of the upper half is called the **upper quartile** or **75th percentile**. One quarter or 25% of the data have values greater than or equal to the upper quartile. 75% of the data have values less than or equal to the upper quartile.

The **interquartile range** is the range of the middle half or 50% of the data.

interquartile range = upper quartile − lower quartile

The data set is thus divided into quarters by the lower quartile (Q_1), the median (Q_2), and the upper quartile (Q_3).

So, the interquartile range, $\mathbf{IQR = Q_3 - Q_1}$.

Example 7 **Self Tutor**

For the data set: 7, 3, 1, 7, 6, 9, 3, 8, 5, 8, 6, 3, 7, 1, 9

a median **b** lower quartile **c** upper quartile **d** interquartile range

The ordered data set is:

~~1, 1, 3, 3, 3, 5, 6,~~ 6, ~~7, 7, 7, 8, 8, 9, 9~~ (15 of them)

a As $n = 15$, $\dfrac{n+1}{2} = 8$ $\therefore$ the median = 8th data value = 6

b/c As the median is a data value we now ignore it and split the remaining data into two:

$$\overbrace{1\ 1\ 3\ 3\ 3\ 5\ 6}^{\text{lower}}\quad \overbrace{7\ 7\ 7\ 8\ 8\ 9\ 9}^{\text{upper}}$$

Q_1 = median of lower half = 3

Q_3 = median of upper half = 8

d $\text{IQR} = Q_3 - Q_1 = 8 - 3 = 5$

Example 8 **Self Tutor**

For the data set: 6, 4, 9, 15, 5, 13, 7, 12, 8, 10, 4, 1, 13, 1, 6, 4, 5, 2, 8, 2 find:

a the median **b** Q_1 **c** Q_3 **d** the interquartile range

The ordered data set is:

~~1 1 2 2 4 4 4 5 5~~ 6 6 ~~7 8 8 9 10 12 13 13 15~~ (20 of them)

a As $n = 20$, $\dfrac{n+1}{2} = 10.5$

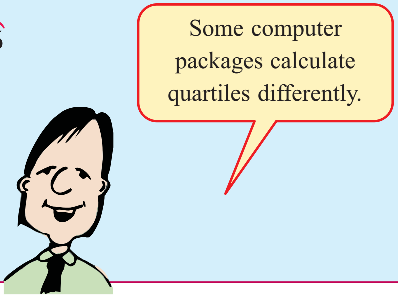
$$\therefore \text{ median} = \frac{\text{10th value} + \text{11th value}}{2} = \frac{6+6}{2} = 6$$

b/c As we have an even number of data values, we split the data into two:

$$\overbrace{1\ 1\ 2\ 2\ 4\ 4\ 4\ 5\ 5\ 6}^{\text{lower}}\quad \overbrace{6\ 7\ 8\ 8\ 9\ 10\ 12\ 13\ 13\ 15}^{\text{upper}}$$

$$\therefore\ Q_1 = \frac{4+4}{2} = 4, \quad Q_3 = \frac{9+10}{2} = 9.5$$

d
$$\begin{aligned}\text{IQR} &= Q_3 - Q_1\\ &= 9.5 - 4\\ &= 5.5\end{aligned}$$



EXERCISE 14C.1

1 For each of the following data sets, make sure the data is ordered and then find:

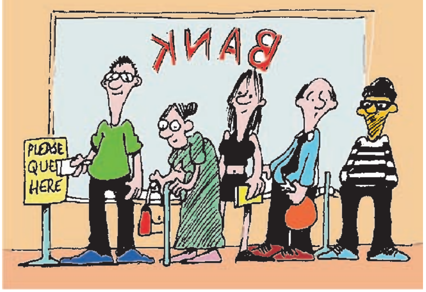
i the median **ii** the upper and lower quartiles
iii the range **iv** the interquartile range.

a 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 5, 6, 6, 6, 6, 6, 7, 7, 8, 8, 8, 9, 9

b 10, 12, 15, 12, 24, 18, 19, 18, 18, 15, 16, 20, 21, 17, 18, 16, 22, 14

c 21.8, 22.4, 23.5, 23.5, 24.6, 24.9, 25, 25.3, 26.1, 26.4, 29.5

2 The times spent (in minutes) by 20 people waiting in a queue at a bank for a teller were:



3.4 2.1 3.8 2.2 4.5 1.4 0 0 1.6 4.8
1.5 1.9 0 3.6 5.2 2.7 3.0 0.8 3.8 5.2

a Find the median waiting time and the upper and lower quartiles.

b Find the range and interquartile range of the waiting times.

c Copy and complete the following statements:

i "50% of the waiting times were greater than .... minutes."

ii "75% of the waiting times were less than ...... minutes."

iii "The minimum waiting time was .... minutes and the maximum waiting time was ..... minutes. The waiting times were spread over ...... minutes."

3 For the data set given, find:

a the minimum value **b** the maximum value
c the median **d** the lower quartile
e the upper quartile **f** the range
g the interquartile range.

Stem	*Leaf*
0	3 4 7 9
1	0 3 4 6 7 8
2	0 0 3 5 6 9 9 9
3	1 3 7 8
4	2

3 | 7 means 37

4 The heights of 20 ten year olds are recorded in the following stem-and-leaf plot:

Stem	*Leaf*
10	9
11	1 3 4 4 8 9
12	2 2 4 4 6 8 9 9
13	1 2 5 8 8

10 | 9 means 109 cm

a Find the: **i** median height
ii upper and lower quartiles of the data.

b Copy and complete the following statements:

i "Half of the children are no more than ..... cm tall."

ii "75% of the children are no more than ..... cm tall."

c Find the: **i** range **ii** interquartile range for the height of the ten year olds.

d Copy and complete:
"The middle 50% of the children have heights spread over ..... cm."

5 Revisit the **Opening Problem** on page **376**.

a For the *Without fertiliser* data, find:

i the range **ii** the median **iii** the lower quartile
iv the upper quartile **v** the interquartile range.

b Repeat **a** for the *With fertiliser* data.

c Consider again the questions posed in the **Opening Problem**. Amend your solutions where appropriate.

BOX-AND-WHISKER PLOTS

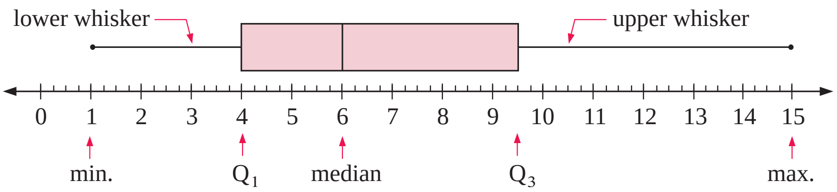
A **box-and-whisker plot** (or simply a **boxplot**) is a visual display of some of the descriptive statistics of a data set. It shows:

- the minimum value
- the lower quartile (Q_1)
- the median (Q_2)
- the upper quartile (Q_3)
- the maximum value

These five numbers form the **five-number summary** of the data set.

For the data set in **Example 8** on page **395**, the five-number summary and boxplot are:

minimum $= 1$
$Q_1 = 4$
median $= 6$
$Q_3 = 9.5$
maximum $= 15$



The rectangular box represents the 'middle' half of the data set.

The lower whisker represents the 25% of the data with smallest values.

The upper whisker represents the 25% of the data with greatest values.

Example 9

Self Tutor

Consider the data set: 8 2 3 9 6 5 3 2 2 6 2 5 4 5 5 6

a Construct the five-number summary for this data.

b Draw a boxplot.

c Find the: **i** range **ii** interquartile range of the data.

d Find the percentage of data values less than 3.

---

a The ordered data set is:

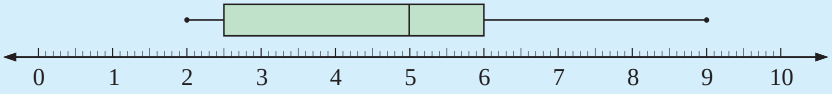
2 2 2 2 3 3 4 5 5 5 5 6 6 6 8 9 (16 of them)

$Q_1 = 2.5$ median $= 5$ $Q_3 = 6$

So the 5-number summary is:

minimum $= 2$ $Q_1 = 2.5$
median $= 5$ $Q_3 = 6$
maximum $= 9$

b



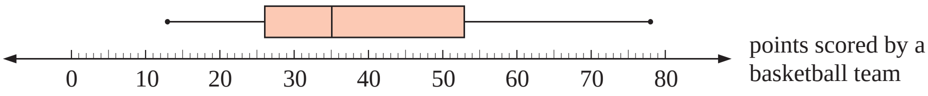
c **i** range = maximum − minimum
$= 9 - 2$
$= 7$

ii $\text{IQR} = Q_3 - Q_1$
$= 6 - 2.5$
$= 3.5$

d 25% of the data values are less than 3.

EXERCISE 14C.2

1

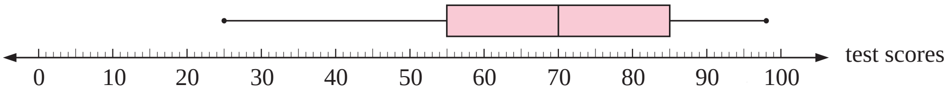


a The boxplot given summarises the goals scored by a basketball team. Locate:

- **i** the median
- **ii** the maximum value
- **iii** the minimum value
- **iv** the upper quartile
- **v** the lower quartile

b Calculate: **i** the range **ii** the interquartile range.

2



The boxplot above summarises the class results for a test out of 100 marks. Copy and complete the following statements about the test results:

- **a** The highest mark scored for the test was ..... , and the lowest mark was .....
- **b** Half of the class scored a mark greater than or equal to .....
- **c** The top 25% of the class scored at least ..... marks for the test.
- **d** The middle half of the class had scores between ..... and ..... for this test.
- **e** Find the range of the data set.
- **f** Find the interquartile range of the data set.
- **g** Estimate the mean mark for these test scores.

3 For the following data sets:

- **i** construct a 5-number summary
- **ii** draw a boxplot
- **iii** find the range
- **iv** find the interquartile range.

a 3, 5, 5, 7, 10, 9, 4, 7, 8, 6, 6, 5, 8, 6

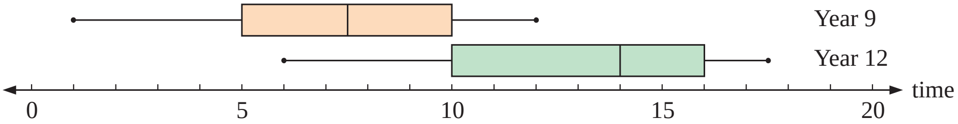
b 3, 7, 0, 1, 4, 6, 8, 8, 8, 9, 7, 5, 6, 8, 7, 8, 8, 2, 9

c

Stem	*Leaf*
11	7
12	0 3 6 6 8
13	0 1 1 1 3 5 5 7
14	4 7 7 9 9
15	1

11 | 7 represents 117

4 The following side-by-side boxplots compare the times students in years 9 and 12 spend on homework.



a Copy and complete:

Statistic	*Year 9*	*Year 12*
minimum		
Q_1		
median		
Q_3		
maximum		

b Determine the: **i** range **ii** interquartile range for each group.

c True or false:

i On average, Year 12 students spend about twice as much time on homework than Year 9 students.

ii Over 25% of Year 9 students spend less time on homework than all Year 12 students.

5 Enid examines a new variety of bean and counts the number of beans in 33 pods. Her results were:

5, 8, 10, 4, 2, 12, 6, 5, 7, 7, 5, 5, 5, 13, 9, 3, 4, 4, 7, 8, 9, 5, 5, 4, 3, 6, 6, 6, 6, 9, 8, 7, 6

a Find the median, lower quartile and upper quartile of the data set.

b Find the interquartile range of the data set.

c Draw a boxplot of the data set.

6 Ranji counts the number of bolts in several boxes and tabulates the data as follows:

Number of bolts	33	34	35	36	37	38	39	40
Frequency	1	5	7	13	12	8	0	1

a Find the five-number summary for this data set.

b Find the **i** range **ii** IQR for this data set.

c Construct a boxplot for the data set.

D CUMULATIVE FREQUENCY GRAPHS

Sometimes, in addition to finding the median, it is useful to know the number of scores that lie above or below a particular value. In such situations we can construct a **cumulative frequency distribution table** and use a graph called a **cumulative frequency graph** or **ogive** to represent the data.

PERCENTILES

A **percentile** is the score below which a certain percentage of the data lies.

For example:
- the 85th percentile is the score below which 85% of the data lies.
- If your score in a test is the 95th percentile, then 95% of the class have scored less than you.

Notice that:
- the **lower quartile** ($\mathbf{Q}_1$) is the 25th percentile
- the **median** ($\mathbf{Q}_2$) is the 50th percentile
- the **upper quartile** ($\mathbf{Q}_3$) is the 75th percentile.

A cumulative frequency graph provides a convenient way to find percentiles.

Example 10 **Self Tutor**

The data shows the results of the women's marathon at the 2008 Olympics, for all competitors who finished the race.

Finishing time (t hours & mins)	*Frequency*
2 h 26 $\leqslant t <$ 2 h 28	8
2 h 28 $\leqslant t <$ 2 h 30	3
2 h 30 $\leqslant t <$ 2 h 32	9
2 h 32 $\leqslant t <$ 2 h 34	11
2 h 34 $\leqslant t <$ 2 h 36	12
2 h 36 $\leqslant t <$ 2 h 38	7
2 h 38 $\leqslant t <$ 2 h 40	5
2 h 40 $\leqslant t <$ 2 h 48	8
2 h 48 $\leqslant t <$ 2 h 56	6

a Construct a cumulative frequency distribution table.

b Represent the data on a cumulative frequency graph.

c Use your graph to estimate the:
  - **i** median finishing time
  - **ii** number of competitors who finished in a time less than 2 hours 35 minutes
  - **iii** percentage of competitors who took more than 2 hours 39 minutes to finish
  - **iv** time taken by a competitor who finished in the top 20% of those who completed the marathon.

---

a

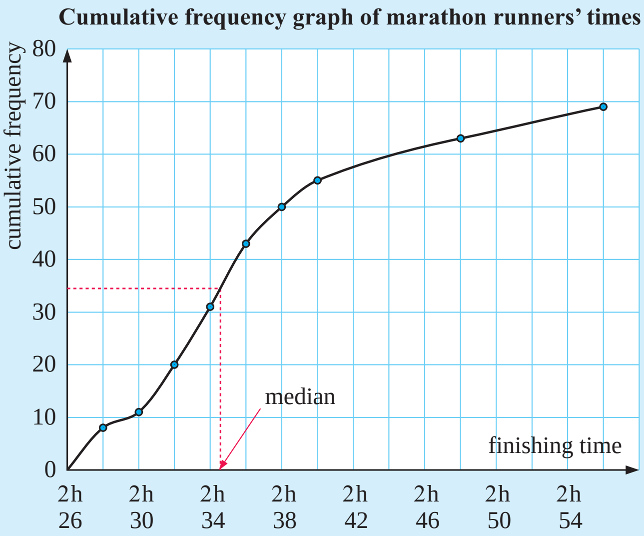
Finishing time (t hours & mins)	*Frequency*	*Cumulative Frequency*
2 h 26 $\leqslant t <$ 2 h 28	8	8
2 h 28 $\leqslant t <$ 2 h 30	3	11
2 h 30 $\leqslant t <$ 2 h 32	9	20
2 h 32 $\leqslant t <$ 2 h 34	11	31
2 h 34 $\leqslant t <$ 2 h 36	12	43
2 h 36 $\leqslant t <$ 2 h 38	7	50
2 h 38 $\leqslant t <$ 2 h 40	5	55
2 h 40 $\leqslant t <$ 2 h 48	8	63
2 h 48 $\leqslant t <$ 2 h 56	6	69

This is $8+3$.

This is $8+3+9$.

This 50 means that 50 competitors completed the marathon in less than 2 hours 38 minutes.

b


Cumulative frequency graph of marathon runners' times

c **i** The median is estimated using the 50th percentile. As 50% of 69 is 34.5, we start with the cumulative frequency of 34.5 and find the corresponding time. So the median is approximately 2 hours 34.5 min.

ii There are approximately 37 competitors who took less than 2 h 35 min to complete the race.

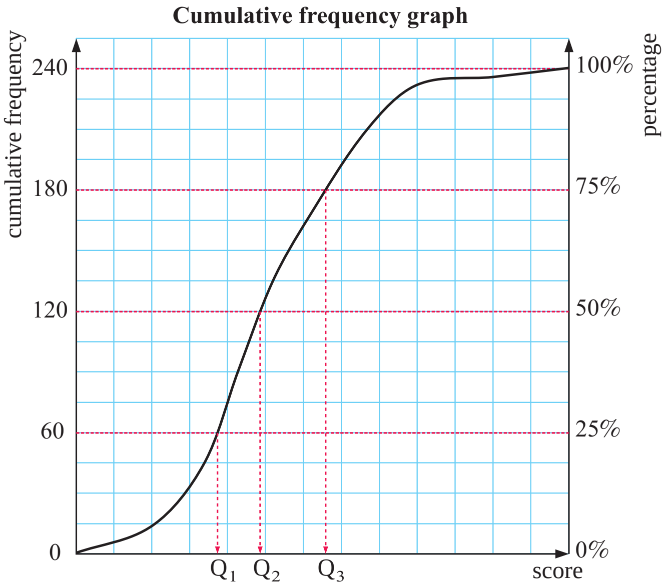
iii There are $69 - 52 = 17$ competitors who took more than 2 hours 39 min.

So $\frac{17}{69} \approx 26.4\%$ took more than 2 hours 39 min.

iv The time taken is estimated using the 20th percentile. As 20% of 69 is 13.8, we find the time corresponding to a cumulative frequency of approximately 14. So, about 20% of the competitors took less than 2 hours 30 minutes and 40 seconds.

Another way to calculate percentiles is to add a separate scale to a cumulative frequency graph. On the graph alongside, the cumulative frequency is read from the axis on the left side, and each value corresponds to a percentage on the right side.

We can use the percentage scale to help find the quartiles and other percentiles.


Cumulative frequency graph

EXERCISE 14D

1 The following frequency distribution was obtained by asking 50 randomly selected people the lengths of their feet. Their answers were given to the nearest centimetre.

Foot length (cm)	20	21	22	23	24	25	26	27	28	29	30
Frequency	1	1	0	3	5	13	17	7	2	0	1

Draw a cumulative frequency graph for the data and use it to find:

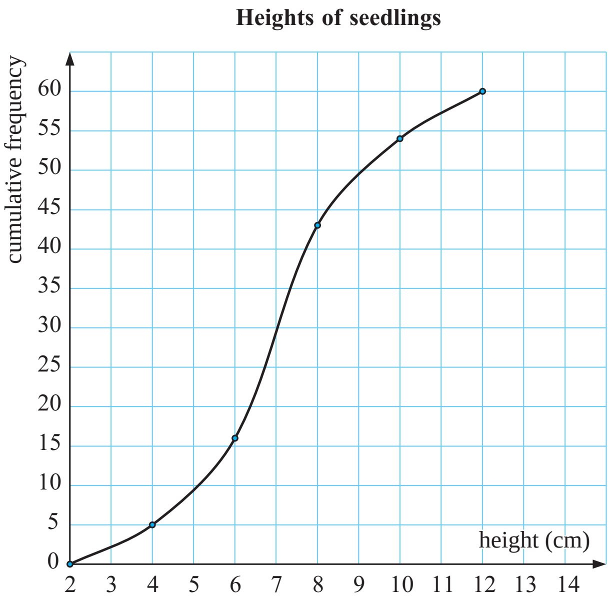
- **a** the median foot length
- **b** how many people had a foot length of: **i** 25 cm or more **ii** 26 cm or less.

2 The following data shows the lengths of 30 trout caught in a lake during a fishing competition. The measurements were rounded down to the next centimetre.

31	38	34	40	24	33	30	36	38	32	35	32	36	27	35
40	34	37	44	38	36	34	33	31	38	35	36	33	33	28

- **a** Construct a cumulative frequency table for trout lengths, x cm, using the intervals $24 \leqslant x < 27$, $27 \leqslant x < 30$, and so on.
- **b** Draw a cumulative frequency graph for the data.
- **c** Use **b** to find the median length.
- **d** Use the original data to find its median and compare your answer with **c**. Comment on your results.

3 A botanist has measured the heights of 60 seedlings and has presented her findings on the cumulative frequency graph below.


Heights of seedlings

- **a** How many seedlings have heights of 5 cm or less?
- **b** What percentage of seedlings are taller than 8 cm?
- **c** What is the median height?
- **d** What is the interquartile range for the heights?
- **e** Find the 90th percentile for the data and explain what your answer means.

4 In an examination the following scores were achieved by a group of students:
Draw a cumulative frequency graph for the data and use it to find:

- **a** the median examination mark
- **b** how many students scored less than 65 marks
- **c** how many students scored between 50 and 70 marks
- **d** how many students failed, given that the pass mark was 45
- **e** the credit mark, given that the top 16% of students were awarded credits.

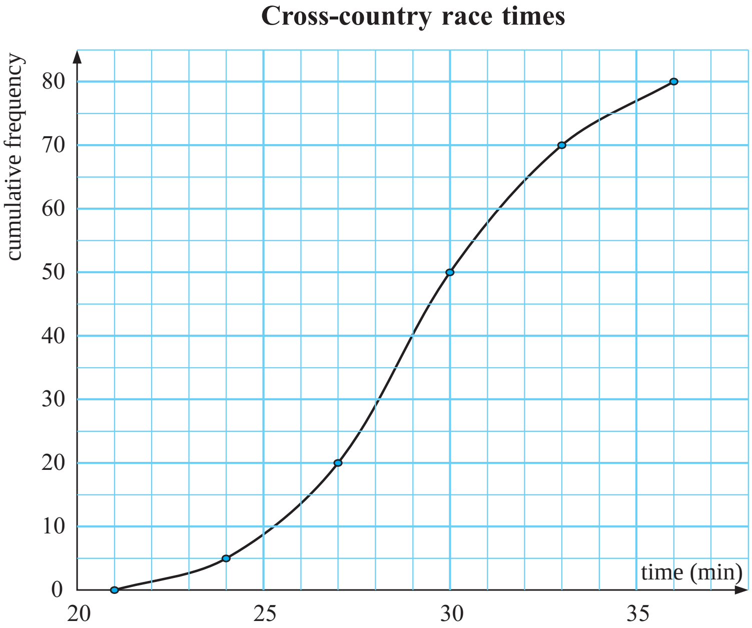
Score	*Frequency*
$10 \leqslant x < 20$	2
$20 \leqslant x < 30$	5
$30 \leqslant x < 40$	7
$40 \leqslant x < 50$	21
$50 \leqslant x < 60$	36
$60 \leqslant x < 70$	40
$70 \leqslant x < 80$	27
$80 \leqslant x < 90$	9
$90 \leqslant x < 100$	3

5 The following table gives the age groups of car drivers involved in an accident in a city for a given year. Draw a cumulative frequency graph for the data and use it to find:

- **a** the median age of the drivers involved in the accidents
- **b** the percentage of drivers involved in accidents who had an age of 23 or less.
- **c** Estimate the probability that a driver involved in an accident is:
  - **i** aged less than or equal to 27 years
  - **ii** aged 27 years.

Age (in years)	*No. of accidents*
$16 \leqslant x < 20$	59
$20 \leqslant x < 25$	82
$25 \leqslant x < 30$	43
$30 \leqslant x < 35$	21
$35 \leqslant x < 40$	19
$40 \leqslant x < 50$	11
$50 \leqslant x < 60$	24
$60 \leqslant x < 80$	41

6 The following cumulative frequency graph displays the performance of 80 competitors in a cross-country race.



Find:

- **a** the lower quartile time
- **b** the median
- **c** the upper quartile
- **d** the interquartile range
- **e** an estimate of the 40th percentile.

7 The table below gives the distribution of the life of electric light globes.

Draw a cumulative frequency graph for the data and use it to estimate:

- **a** the median life of a globe
- **b** the percentage of globes which had a life of 2700 hours or less
- **c** the number of globes which had a life between 1500 and 2500 hours.

Life (hours)	*Number of globes*
$0 \leqslant l < 500$	5
$500 \leqslant l < 1000$	17
$1000 \leqslant l < 2000$	46
$2000 \leqslant l < 3000$	79
$3000 \leqslant l < 4000$	27
$4000 \leqslant l < 5000$	4

E STATISTICS USING TECHNOLOGY

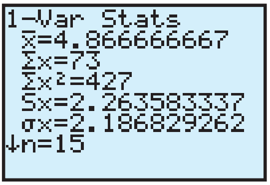
GRAPHICS CALCULATOR

A **graphics calculator** can be used to find descriptive statistics and to draw some types of graphs. For instructions on how to do this, consult the graphics calculator instructions at the start of the book.

Consider the data set: 5 2 3 3 6 4 5 3 7 5 7 1 8 9 5

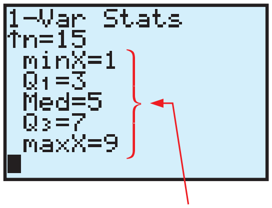
You should be able to:

- Enter the data as a **list**.
- Enter the **statistics calculation** part of the menu and obtain the descriptive statistics like these shown.



```
1-Var Stats
 x̄=4.866666667
 Σx=73
 Σx²=427
 Sx=2.263583337
 σx=2.186829262
↓n=15
```

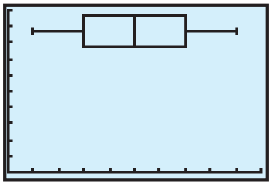
$\overline{x}$ is the mean

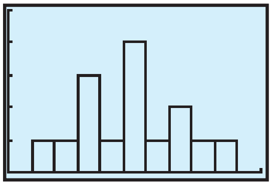


```
1-Var Stats
↑n=15
 minX=1
 Q₁=3
 Med=5
 Q₃=7
 maxX=9
```

five number summary

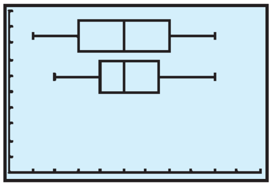
- Obtain a box-and-whisker plot, column graph, or histogram.





- Enter a second data set into another list and obtain a side-by-side boxplot for comparison with the first one.
  Use the data: 9 6 2 3 5 5 7 5 6 7 6 3 4 4 5 8 4

You will need to change the **viewing window** as appropriate.



STATISTICS FROM A COMPUTER PACKAGE

Click on the icon to load our **statistics package**.

STATISTICS PACKAGE

Enter data set 1: 5 2 3 3 6 4 5 3 7 5 7 1 8 9 5
Enter data set 2: 9 6 2 3 5 5 7 5 6 7 6 3 4 4 5 8 4

Examine the side-by-side column graphs.

Click on the Box-and-Whisker tab to view the side-by-side boxplots.

Click on the Statistics tab to obtain the descriptive statistics.

EXERCISE 14E

Use technology to answer the following questions:

1 **a** Enter the data set: 5 2 3 3 6 4 5 3 7 5 7 1 8 9 5 and obtain the mean and the 5-number summary. This is the data used in the screendumps above, so you can use them to check your results.

b Obtain the boxplot for the data in **a**.

c Obtain the vertical bar chart for the data in **a**.

d Enter the data set: 9 6 2 3 5 5 7 5 6 7 6 3 4 4 5 8 4 into a second list. Find the mean and 5-number summary. Create a side-by-side boxplot comparing the data sets.

2 Shane and Brett play in the same cricket team and are fierce but friendly rivals when it comes to bowling. During a season the number of wickets taken in each innings by the bowlers were:

Shane:	1	6	2	0	3	4	1	4	2	3	0	3	2	4	3	4	3	3
	3	4	2	4	3	2	3	3	0	5	3	5	3	2	4	3	4	3
Brett:	7	2	4	8	1	3	4	2	3	0	5	3	5	2	3	1	2	0
	4	3	4	0	3	3	0	2	5	1	1	2	2	5	1	4	0	1

a Is the variable discrete or continuous?

b Enter the data into a graphics calculator or statistics package.

c Produce a vertical column graph for each data set.

d Are there any outliers? Should they be deleted before we start to analyse the data?

e Describe the shape of each distribution.

f Compare the measures of the centre of each distribution.

g Compare the spreads of each distribution.

h Obtain a side-by-side boxplot.

i What conclusions, if any, can be drawn from the data?

3 A manufacturer of light globes claims that their new design has a 20% longer life than those they are presently selling. Forty of each globe are randomly selected and tested. Here are the results to the nearest hour:

	103	96	113	111	126	100	122	110	84	117	103	113	104	104
Old type:	111	87	90	121	99	114	105	121	93	109	87	118	75	111
	87	127	117	131	115	116	82	130	113	95	108	112		
	146	131	132	160	128	119	133	117	139	123	109	129	109	131
New type:	191	117	132	107	141	136	146	142	123	144	145	125	164	125
	133	124	153	129	118	130	134	151	145	131	133	135		

a Is the variable discrete or continuous?

b Enter the data into a graphics calculator or statistics package.

c Are there any outliers? Should they be deleted before we start to analyse the data?

d Compare the measures of centre and spread.

e Obtain a side-by-side boxplot.

f Describe the shape of each distribution.

g What conclusions, if any, can be drawn from the data?

F VARIANCE AND STANDARD DEVIATION

The problem with using the range and the IQR as measures of spread or dispersion of scores is that both of them only use two values in their calculation. Some data sets can therefore have their spread characteristics hidden when the range or IQR are quoted, and so we need a better way of describing spread.

Consider a data set of n values: $x_1, x_2, x_3, x_4, \ldots\ldots, x_n$, with mean $\overline{x}$.

$x_i - \overline{x}$ measures how far x_i deviates from the mean, so one might suspect that the mean of the deviations $\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})$ would give a good measure of spread. However, this value turns out to always be zero. Instead we define:

The **variance** of the n data values is $\boldsymbol{s_n}^2 = \dfrac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n}$.

Notice in this formula that:

- $(x_i - \overline{x})^2$ is also a measure of how far x_i deviates from $\overline{x}$. However, the square ensures that each of these is positive, which is why the sum turns out not to be zero.
- If $\sum_{i=1}^{n}(x_i - \overline{x})^2$ is small, it will indicate that most of the data values are close to $\overline{x}$.
- Dividing by n gives an indication of how far, on average, the data is from the mean.

For a data set of n values, $\boldsymbol{s_n} = \sqrt{\dfrac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n}}$ is called the **standard deviation**.

The square root in the standard deviation is used to correct the units.

For example, if x_i is the weight of a student in kg, s_n^2 would be in kg^2.

For this reason the standard deviation is more frequently quoted than the variance.

The standard deviation is a **non-resistant** measure of spread. This is due to its dependence on the mean of the sample and because extreme data values will give large values for $(x_i - \overline{x})^2$.

It is only a useful measure if the distribution is approximately symmetrical. However, the standard deviation is particularly useful when the data from which it came is **normally distributed**. This will be discussed later.

The IQR and percentiles are more appropriate tools for measuring spread if the distribution is considerably skewed.

Example 11 **Self Tutor**

Find the means and standard deviations for the library surveys of **Example 6**.

What do these statistics tell us?

Survey 1

x	$x-\overline{x}$	$(x-\overline{x})^2$
3	-4	16
7	0	0
6	-1	1
8	1	1
11	4	16
35	*Total*	34

$$\therefore \quad \overline{x} = \frac{35}{5} = 7 \qquad s_n = \sqrt{\frac{\sum(x-\overline{x})^2}{n}} = \sqrt{\frac{34}{5}} \approx 2.61$$

Survey 2

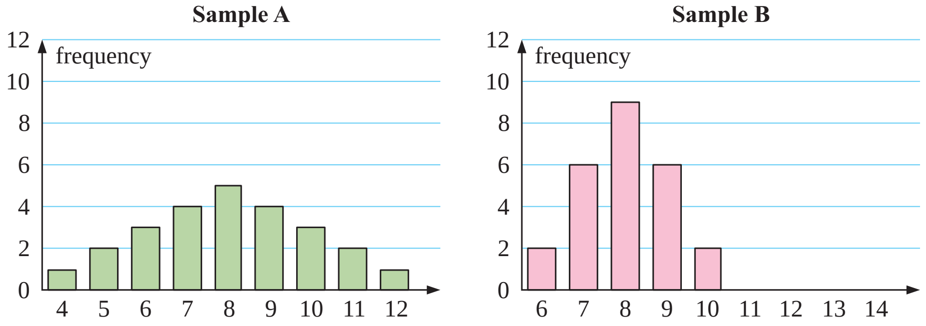
x	$x-\overline{x}$	$(x-\overline{x})^2$
2	-2	4
3	-1	1
5	1	1
4	0	0
6	2	4
20	*Total*	10

$$\therefore \quad \overline{x} = \frac{20}{5} = 4 \qquad s_n = \sqrt{\frac{\sum(x-\overline{x})^2}{n}} = \sqrt{\frac{10}{5}} \approx 1.41$$

The second survey shows that the number of dissatisfied borrowers has almost halved and there is less variability in the number of dissatisfied borrowers.

EXERCISE 14F.1

1 The column graphs show two distributions:



- **a** By looking at the graphs, which distribution appears to have wider spread?
- **b** Find the mean of each sample.
- **c** Find the standard deviation for each sample. Comment on your answers.

2 The number of points scored by Andrew and Brad in the last 8 basketball matches are tabulated below.

Points by Andrew	23	17	31	25	25	19	28	32
Points by Brad	9	29	41	26	14	44	38	43

a Find the mean and standard deviation of the number of points scored by each player.

b Which of the two players is more consistent?

3 Two baseball coaches compare the number of runs scored by their teams in their last ten matches:

Rockets	0	10	1	9	11	0	8	5	6	7
Bullets	4	3	4	1	4	11	7	6	12	5

a Show that each team has the same mean and range of runs scored.

b Which team's performance do you suspect is more variable over the period?

c Check your answer to **b** by finding the standard deviation for each distribution.

d Does the range or the standard deviation give a better indication of variability?

4 A manufacturer of soft drinks employs a statistician for quality control. He needs to check that 375 mL of drink goes into each can, but realises the machine which fills the cans will slightly vary each delivery.

a Would you expect the standard deviation for the whole production run to be the same for one day as it is for one week? Explain your answer.

b If samples of 125 cans are taken each day, what measure would be used to:

- **i** check that an average of 375 mL of drink goes into each can
- **ii** check the variability of the volume of drink going into each can?

c What is the significance of a low standard deviation in this case?

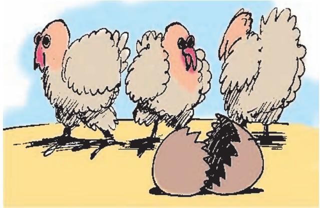
5 The weights in kg of seven footballers are: 79, 64, 59, 71, 68, 68 and 74.

a Find the mean and standard deviation for this group.

b Surprisingly, each footballer's weight had increased by exactly 10 kg when measured five years later. Find the new mean and standard deviation.

c Comment on your findings from **a** and **b** in general terms.

6 The weights of ten young turkeys to the nearest 0.1 kg are:
0.8, 1.1, 1.2, 0.9, 1.2, 1.2, 0.9, 0.7, 1.0, 1.1



a Find the mean and standard deviation for the turkeys.

b After being fed a special diet for one month, the weights of the turkeys doubled. Find the new mean and standard deviation.

c Comment, in general terms, on your findings from **a** and **b**.

7 A sample of 8 integers has a mean of 5 and a variance of 5.25 .
The integers are: 1, 3, 5, 7, 4, 5, p, q. Find p and q given that $p < q$.

8 A sample of 10 integers has a mean of 6 and a variance of 3.2 .
The integers are: 3, 9, 5, 5, 6, 4, a, 6, b, 8. Find a and b given that $a > b$.

9 The following table shows the decrease in cholesterol levels in 6 volunteers after a two week trial of special diet and exercise.

Volunteer	A	B	C	D	E	F
Decrease in cholesterol	0.8	0.6	0.7	0.8	0.4	2.8

- **a** Find the standard deviation of the data.
- **b** Recalculate the standard deviation with the outlier removed.
- **c** Discuss the effect of an extreme value on the standard deviation.

SAMPLING FROM A POPULATION

Populations are often huge, and gathering data from every individual is impossible due to time constraints and cost.

Consequently, a **random sample** is taken from the population with the hope that it will truly reflect the characteristics of the population. To ensure this, the sample must be sufficiently large, and be taken in such a way that the results are unbiased.

To help distinguish between a sample and the whole population, we use different notation for the mean, variance, and standard deviation. This is shown in the table opposite.

In general, the population mean μ and standard deviation σ will be unknown.

	sample	*population*
mean	$\overline{x}$	μ
variance	${s_n}^2$	σ^2
standard deviation	s_n	σ

However, given statistics from a sample, we can make **inferences** about the population using the following results which are assumed without proof:

When a sample of size n is used to draw inference about a population:

- the mean of the sample $\overline{x}$ is an unbiased estimate of μ
- s_n is an estimate of the standard deviation σ.

NOTE ON PARAMETERS AND STATISTICS

A **parameter** is a numerical characteristic of a *population*.

A **statistic** is a numerical characteristic of a *sample*.

Parameter
Population

Statistic
Sample

For example, if we are examining the mean age of people in retirement villages throughout Canada, the mean age found would be a *parameter*. If we take a random sample of 300 people from the population of all retirement village persons, then the mean age would be a *statistic*.

Example 12 **Self Tutor**

A random sample of 48 sheep was taken from a flock of over 2000 sheep.
The sample mean of their weights was 48.6 kg with variance 17.5 kg^2.

a Find the standard deviation of the sample. Hence estimate the standard deviation of the population from which the sample was taken.

b Find an unbiased estimate of the mean weight of sheep in the flock.

a $s_n = \sqrt{\text{variance}} = \sqrt{17.5} \approx 4.18$ kg

σ is estimated by s_n, so we estimate the standard deviation to be 4.18 kg.

b μ is estimated by $\overline{x} = 48.6$ kg.

EXERCISE 14F.2

1 A random sample of 87 deer from a huge herd had a mean weight of 93.8 kg with a variance of 45.9 kg^2.

a Find the standard deviation of the sample. Hence estimate the standard deviation of the whole herd.

b Find an unbiased estimate of the mean of the whole herd.

2 The weights (in grams) of a random sample of sparrows are as follows:

87 75 68 69 81 89 73 66 91 77 84 83 77 74 80 76 67

a Find the mean and standard deviation of the sample.

b Estimate the mean and standard deviation of the population from which the sample was taken.

STANDARD DEVIATION FOR GROUPED DATA

For grouped data $$s_n = \sqrt{\frac{\sum f_i(x_i - \overline{x})^2}{\sum f_i}}$$

where s_n is the **standard deviation,** x_i is the ith score, $\overline{x}$ is the **mean,**
and f_i is the **frequency** of the ith score.

Example 13 **Self Tutor**

Find the standard deviation of the distribution:

Score	1	2	3	4	5
Frequency	1	2	4	2	1

x	f	$f\,x$	$x - \overline{x}$	$(x-\overline{x})^2$	$f(x-\overline{x})^2$
1	1	1	−2	4	4
2	2	4	−1	1	2
3	4	12	0	0	0
4	2	8	1	1	2
5	1	5	2	4	4
Total	10	30			12

$$\overline{x} = \frac{\sum f\,x}{\sum f} = \frac{30}{10} = 3$$

$$s_n = \sqrt{\frac{\sum f(x-\overline{x})^2}{\sum f}}$$

$$= \sqrt{\frac{12}{10}} \approx 1.10$$

For continuous data, or data that has been grouped in classes, we use the midpoint of the interval to represent all data in that interval.

Instructions for finding the mean and standard deviation using your calculator are found in the chapter at the front of the book. You should use your calculator to check your answers to the following questions.

EXERCISE 14F.3

1 Below is a sample of family sizes taken at random from people in a city.

Number of children, x	0	1	2	3	4	5	6	7
Frequency, f	14	18	13	5	3	2	2	1

a Find the sample mean and standard deviation.

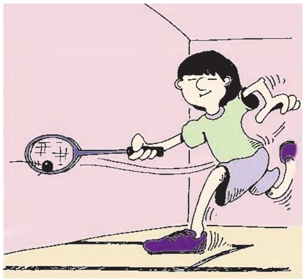
b Estimate the mean and standard deviation of the population from which the sample was taken.

2 Below is a random sample of the ages of squash players at the Junior National Squash Championship.

Age	11	12	13	14	15	16	17	18
Frequency	2	1	4	5	6	4	2	1

a Find the mean and standard deviation of the ages.

b Estimate the mean and standard deviation of the population from which the sample was taken.



3 The number of toothpicks in a random sample of 48 boxes was counted and the results tabulated.

Number of toothpicks	33	35	36	37	38	39	40
Frequency	1	5	7	13	12	8	2

a Find the mean and standard deviation of the number of toothpicks in the boxes.

b Estimate the mean and standard deviation of the population from which the sample was taken.

4 The lengths of 30 randomly selected 12-day old babies were measured and the following data obtained:

a Estimate the mean length and the standard deviation of the lengths.

b Estimate the mean and standard deviation of the population from which the sample was taken.

Length (cm)	*Frequency*
$40 \leqslant L < 42$	1
$42 \leqslant L < 44$	1
$44 \leqslant L < 46$	3
$46 \leqslant L < 48$	7
$48 \leqslant L < 50$	11
$50 \leqslant L < 52$	5
$52 \leqslant L < 54$	2

5 The weekly wages (in dollars) of 200 randomly selected steel workers are given alongside:

a Estimate the mean and the standard deviation of the wages.

b Estimate the mean and standard deviation of the population from which the sample was taken.

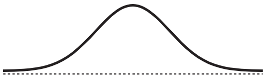
Wage ($)	*Number of workers*
360 - 369.99	17
370 - 379.99	38
380 - 389.99	47
390 - 399.99	57
400 - 409.99	18
410 - 419.99	10
420 - 429.99	10
430 - 439.99	3

G THE SIGNIFICANCE OF STANDARD DEVIATION

Consider the volumes of liquid in different cans of a particular brand of soft drink. The distribution of volumes is symmetrical and bell-shaped. This is due to natural variation produced by the machine which has been set to produce a particular volume. Random or chance factors cause roughly the same number of cans to be overfilled as underfilled.

The resulting bell-shaped distribution is called the **normal distribution**. It will be discussed in more detail in **Chapter 24**, but it is worth noting here how it relates to standard deviation.

If a large sample from a typical bell-shaped data distribution is taken, what percentage of the data values would lie between $\overline{x} - s$ and $\overline{x} + s$?



bell-shaped distribution

Click on the icon and try to answer this question. Repeat the sampling many times.

DEMO

Now try to determine the percentage of data values which would lie between $\overline{x} - 2s$ and $\overline{x} + 2s$, and then between $\overline{x} - 3s$ and $\overline{x} + 3s$.

It can be shown that for any measured variable from any population that is normally distributed, no matter the values of the mean and standard deviation:

- approximately **68%** of the population will measure between **1** standard deviation either side of the mean
- approximately **95%** of the population will measure between **2** standard deviations either side of the mean
- approximately **99.7%** of the population will measure between **3** standard deviations either side of the mean.

Example 14 **Self Tutor**

A sample of 200 boxes of nails was taken from a manufacturer and the contents of each box measured for net weight. The sample mean was 486 g with standard deviation 6.2 g.

a What proportion of the boxes are expected to weigh between 479.8 g and 492.2 g?

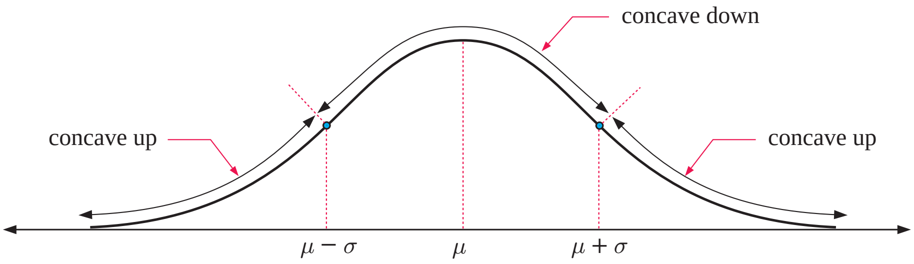
b How many boxes in the sample are expected to be in this weight range?

a $486 - 6.2 = 479.8$ and $486 + 6.2 = 492.2$.
So, the weight range is 1 standard deviation either side of the mean.
$\therefore$ 68% of the boxes are expected to weigh between 479.8 g and 492.2 g.

b About 68% of 200 $=$ 136 boxes are expected to weigh between 479.8 g and 492.2 g.

THE NORMAL CURVE

The smooth curve that models normally distributed data is asymptotic to the horizontal axis, so in theory there are no limits within which all the members of the population will fall.



In practice, however, it is rare to find data outside of 3 standard deviations from the mean, and *exceptionally* rare to find data beyond 5 standard deviations from the mean.

You may also notice that the position of 1 standard deviation either side of the mean corresponds to the inflection point where the normal curve changes from a concave down to a concave up curve.

In the following exercise you should assume that the data is normally distributed.

EXERCISE 14G

1 The mean height of players in a basketball competition is 184 cm. If the standard deviation is 5 cm, what percentage of them are likely to be:

a taller than 189 cm
b taller than 179 cm
c between 174 cm and 199 cm
d over 199 cm tall?

2 The mean average rainfall of Claudona for August is 48 mm with a standard deviation of 6 mm. Over a 20 year period, how many times would you expect there to be less than 42 mm of rainfall during August in Claudona?

3 Two hundred lifesavers competed in a swimming race. The mean time was 10 minutes 30 seconds, and the standard deviation was 15 seconds. Estimate the number of competitors who:

a took longer than 11 minutes
b took less than 10 minutes 15 seconds
c completed the race in a time between 10 min 15 s and 10 min 45 s.

4 The weights of babies born at Prince Louis Maternity Hospital last year averaged 3.0 kg with a standard deviation of 200 grams. If there were 545 babies born at this hospital last year, estimate the number that weighed:

a less than 3.2 kg
b between 2.8 kg and 3.4 kg.

REVIEW SET 14A NON-CALCULATOR

1 The data supplied below is the diameter (in cm) of a number of bacteria colonies as measured by a microbiologist 12 hours after seeding.

0.4 2.1 3.4 3.9 4.7 3.7 0.8 3.6 4.1 4.9 2.5 3.1 1.5 2.6 4.0
1.3 3.5 0.9 1.5 4.2 3.5 2.1 3.0 1.7 3.6 2.8 3.7 2.8 3.2 3.3

a Produce a stemplot for this data.

b Find the **i** median **ii** range of the data.

c Comment on the skewness of the data.

2 The data set 5, 6, 8, a, 3, b has a mean of 6 and a variance of 3. Find the values of a and b.

3 The back-to-back stemplot alongside represents the times for the 100 metre freestyle recorded by members of a swimming squad.

Girls		*Boys*
	32	1
4	33	0227
763	34	13448
87430	35	024799
8833	36	788
7666	37	0
6	38	
0	39	
	40	
1	41	

leaf unit: 0.1 s

a Copy and complete the following table:

Distribution	*Girls*	*Boys*
shape		
median		
range		

b Discuss the distributions of times for the boys and girls. What conclusion can you make?

4 The average height of 17-year-old boys was found to be normally distributed with a mean of 179 cm and a standard deviation of 8 cm. Calculate the percentage of 17-year-old boys whose heights are:

a more than 195 cm

b between 163 cm and 195 cm

c between 171 cm and 187 cm.

5 Draw a box and whisker plot for the following data:

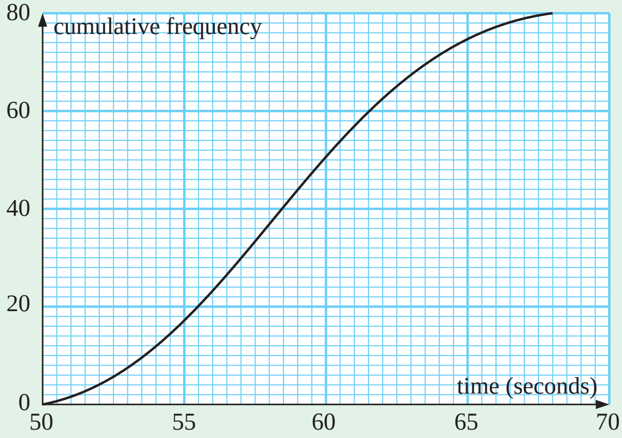
11, 12, 12, 13, 14, 14, 15, 15, 15, 16, 17, 17, 18.

6 80 senior students needed to run 400 metres in a Physical Education program. Their times were recorded and the results were used to produce the cumulative frequency graph shown.

Estimate:

a the median

b the interquartile range.



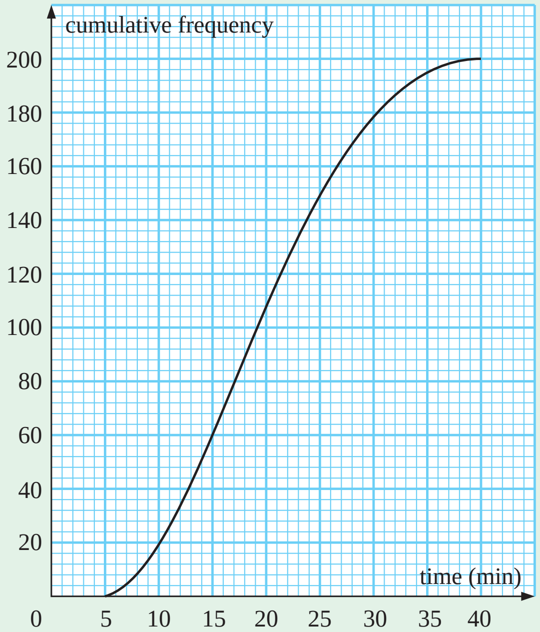
7 

A bottle shop sells on average 2500 bottles per day with a standard deviation of 300 bottles. Assuming that the number of bottles is normally distributed, calculate the percentage of days when:

- **a** less than 1900 bottles are sold
- **b** more than 2200 bottles are sold
- **c** between 2200 and 3100 bottles are sold.

8 This cumulative frequency curve shows the times taken for 200 students to travel to school by bus.

- **a** Estimate how many of the students spent between 10 and 20 minutes travelling to school.
- **b** If 30% of the students spent more than m minutes travelling to school, estimate the value of m.



REVIEW SET 14B CALCULATOR

1 The data below shows the distance in metres that Thabiso threw a baseball:

71.2	65.1	68.0	71.1	74.6	68.8	83.2	85.0	74.5	87.4
84.3	77.0	82.8	84.4	80.6	75.9	89.7	83.2	97.5	82.9
90.5	85.5	90.7	92.9	95.6	85.5	64.6	73.9	80.0	86.5

- **a** Determine the highest and lowest value for the data set.
- **b** Choose between 6 and 12 groups into which all the data values can be placed.
- **c** Prepare a frequency distribution table.
- **d** Draw a frequency histogram for the data.
- **e** Determine: **i** the mean **ii** the median.

2 Consider the following distribution of continuous grouped data:

Scores	0 to 9.9	10 to 19.9	20 to 29.9	30 to 39.9	40 to 49.9
Frequency	1	13	27	17	2

a Construct a cumulative frequency graph for the data.

b Find the median of the data. **c** Find the interquartile range.

d Find the mean and standard deviation.

3 Katja's golf scores for her last 20 rounds were:

90	106	84	103	112	100	105	81	104	98
107	95	104	108	99	101	106	102	98	101

a Find the **i** median **ii** lower quartile **iii** upper quartile.

b Find the interquartile range of the data set.

c Find the mean and standard deviation of her scores.

4 The number of litres of petrol purchased by a random sample of motor vehicle drivers is shown alongside.

Litres	*Number of vehicles*
$15 \leqslant l < 20$	5
$20 \leqslant l < 25$	13
$25 \leqslant l < 30$	17
$30 \leqslant l < 35$	29
$35 \leqslant l < 40$	27
$40 \leqslant l < 45$	18
$45 \leqslant l < 50$	7

a Find the mean and standard deviation of the number of litres purchased.

b Estimate the mean and standard deviation for the population this sample comes from.

5 The table alongside shows the number of matches in a sample of boxes.

Number	47	48	49	50	51	52
Frequency	21	29	35	42	18	31

a Find the mean and standard deviation for this data.

b Does this result justify a claim that the average number of matches per box is 50?

6 Find the range, lower quartile, upper quartile and standard deviation for the following data: 120, 118, 132, 127, 135, 116, 122, 128.

7 3, a, 6, b and 13 have mean 6.8 and standard deviation $\sqrt{12.56}$.
Find a and b given that $a > b$.

REVIEW SET 14C

1 The winning margin in 100 basketball games was recorded. The results are given alongside:
Draw a column graph to represent this information.

Margin (points)	*Frequency*
1 - 10	13
11 - 20	35
21 - 30	27
31 - 40	18
41 - 50	7

2 The mean and standard deviation of a normal distribution are 150 and 12 respectively. What percentage of values lie between:

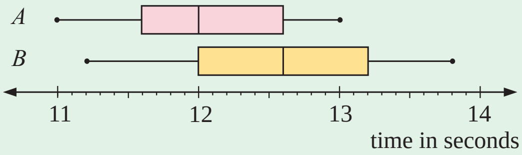
a 138 and 162 **b** 126 and 174 **c** 126 and 162 **d** 162 and 174?

3 The table alongside shows the number of customers visiting a supermarket on various days.

Find the mean number of customers per day.

No. of customers	*Frequency*
250 - 299	14
300 - 349	34
350 - 399	68
400 - 449	72
450 - 499	54
500 - 549	23
550 - 599	7

4 The given parallel boxplots represent the 100-metre sprint times for the members of two athletics squads.



a Determine the 5 number summaries for both A and B.

b Determine the **i** range **ii** interquartile range for each group.

c Copy and complete:

i We know the members of squad ...... generally ran faster because ......

ii We know the times in squad ...... are more varied because ......

5 A random sample of the weekly supermarket bills for a number of families was observed and recorded in the table given.

a Find the mean bill and the standard deviation of the bills.

b Estimate the mean and standard deviation of the population from which the data was taken.

Bill (€)	*No. of families*
70 - 79.99	27
80 - 89.99	32
90 - 99.99	48
100 - 109.99	25
110 - 119.99	37
120 - 129.99	21
130 - 139.99	18
140 - 149.99	7

6 The middle 68% of a normal distribution lies between 16.2 and 21.4.

a What is the mean and standard deviation of the distribution?

b Over what range of values would you expect the middle 95% of the data to spread?

7 An examination worth 100 marks was given to 800 biology students. The cumulative frequency graph for the students' results is shown on the following page.

a Find the number of students who scored 45 marks or less for the test.

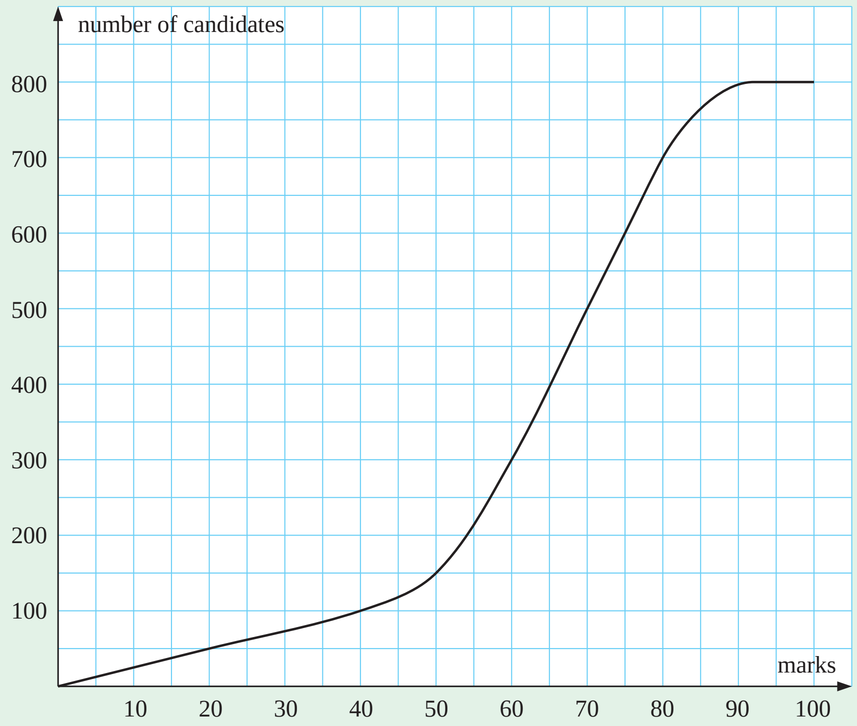
b Find the median score.

c Between what values do the middle 50% of test results lie?

d Find the interquartile range of the data.

e What percentage of students obtained a mark of 55 or more?

f If a 'distinction' is awarded to the top 10% of students, what score is required to receive this honour?


number of candidates
800
700
600
500
400
300
200
100
marks
10
20
30
40
50
60
70
80
90
100

Chapter 15

Probability

Syllabus reference: 6.5, 6.6, 6.7, 6.8

Contents:

In the field of **probability theory** we use a mathematical method to describe the **chance** or **likelihood** of an event happening.

This theory has important applications in physical and biological sciences, economics, politics, sport, life insurance, quality control, production planning, and a host of other areas.

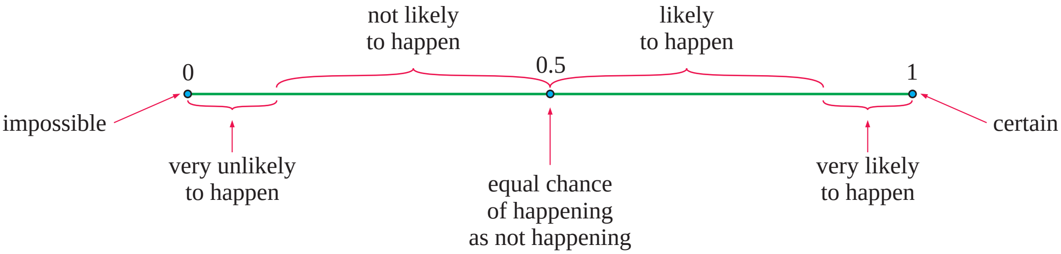
We assign to every event a number which lies between 0 and 1 inclusive. We call this number a **probability**.

An **impossible** event which has 0% chance of happening is assigned a probability of 0.

A **certain** event which has 100% chance of happening is assigned a probability of 1.

All other events can be assigned a probability between 0 and 1.

The number line below shows how we could interpret different probabilities:



The assigning of probabilities is usually based on either:

- observing the results of an experiment (experimental probability), or
- using arguments of symmetry (theoretical probability).

HISTORICAL NOTE

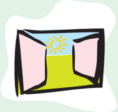
The development of modern probability theory began in 1653 when gambler Chevalier de Mere contacted mathematician **Blaise Pascal** with a problem on how to divide the stakes when a gambling game is interrupted during play. Pascal involved **Pierre de Fermat**, a lawyer and amateur mathematician, and together they solved the problem. In the process they laid the foundations upon which the laws of probability were formed.

Blaise Pascal

Pierre de Fermat

In the late 17th century, English mathematicians compiled and analysed mortality tables which showed the number of people who died at different ages. From these tables they could estimate the probability that a person would be alive at a future date. This led to the establishment of the first life-insurance company in 1699.

OPENING PROBLEM



Life Insurance Companies use statistics on **life expectancy** and **death rates** to work out the premiums to charge people who insure with them.

The **life table** shown is from Australia. It shows the number of people out of 100 000 births who survive to different ages, and the expected years of remaining life at each age.

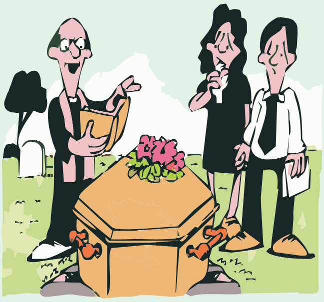
LIFE TABLE

Male			Female		
Age	*Number surviving*	*Expected remaining life*	*Age*	*Number surviving*	*Expected remaining life*
0	100 000	73.03	0	100 000	79.46
5	98 809	68.90	5	99 307	75.15
10	98 698	63.97	10	99 125	70.22
15	98 555	59.06	15	98 956	65.27
20	98 052	54.35	20	98 758	60.40
25	97 325	49.74	25	98 516	55.54
30	96 688	45.05	30	98 278	50.67
35	96 080	40.32	35	98 002	45.80
40	95 366	35.60	40	97 615	40.97
45	94 323	30.95	45	96 997	36.22
50	92 709	26.45	50	95 945	31.59
55	89 891	22.20	55	94 285	27.10
60	85 198	18.27	60	91 774	22.76
65	78 123	14.69	65	87 923	18.64
70	67 798	11.52	70	81 924	14.81
75	53 942	8.82	75	72 656	11.36
80	37 532	6.56	80	58 966	8.38
85	20 998	4.79	85	40 842	5.97
90	8416	3.49	90	21 404	4.12
95	2098	2.68	95	7004	3.00
99	482	2.23	99	1953	2.36

For example, we can see that out of 100 000 births, 98 052 males are expected to survive to the age of 20, and from that age the survivors are expected to live a further 54.35 years.

Things to think about:

- Can you use the life table to estimate how many years you can expect to live?
- Can you estimate the probability of a new-born boy or girl reaching the age of 15?
- Can the table be used to estimate the probability that:
  - a 15 year old boy *will* reach age 75
  - a 15 year old girl *will not* reach age 75?
- An insurance company sells policies to people to insure them against death over a 30-year period. If the person dies during this period, the beneficiaries receive the agreed payout figure. Why are such policies cheaper to take out for a 20 year old than for a 50 year old?
- How many of your classmates would you expect to be alive and able to attend a 30 year class reunion?



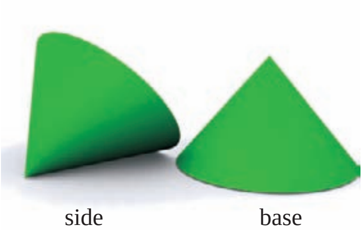
A EXPERIMENTAL PROBABILITY

In experiments involving chance we use the following terms to talk about what we are doing and the results we obtain:

- The **number of trials** is the total number of times the experiment is repeated.
- The **outcomes** are the different results possible for one trial of the experiment.
- The **frequency** of a particular outcome is the number of times that this outcome is observed.
- The **relative frequency** of an outcome is the frequency of that outcome expressed as a fraction or percentage of the total number of trials.

When a small plastic cone was tossed into the air 279 times it fell on its *side* 183 times and on its *base* 96 times.

The relative frequencies of *side* and *base* are $\frac{183}{279} \approx 0.656$ and $\frac{96}{279} \approx 0.344$ respectively.



In the absence of any further data, the relative frequency of each event is our best estimate of the probability of that event occurring.

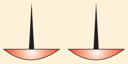
Experimental probability $=$ relative frequency.

In this case: Experimental P(side) $= 0.656$ and Experimental P(base) $= 0.344$.

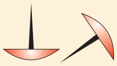
INVESTIGATION 1 TOSSING DRAWING PINS

If a drawing pin tossed in the air finishes  we say it has finished on its *back*. If it finishes  we say it has finished on its *side*.

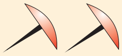
If two drawing pins are tossed simultaneously the possible results are:



two backs



back and side



two sides

What to do:

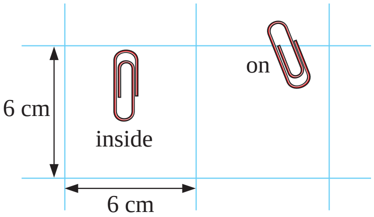
1. Obtain two drawing pins of the same shape and size. Toss the pair 80 times and record the outcomes in a table.
2. Obtain relative frequencies (experimental probabilities) for each of the three events.
3. Pool your results with four other people and so obtain experimental probabilities from 400 tosses. The other people must have pins with the same shape.
4. Which gives the more reliable probability estimates, your results or the group's? Why?
5. Keep your results as they may be useful later in this chapter.

In some situations, for example in the investigation above, experimentation is the only way of obtaining probabilities.

EXERCISE 15A

1 When a batch of 145 paper clips was dropped onto 6 cm by 6 cm squared paper it was observed that 113 fell completely inside squares and 32 finished up on the grid lines. Find, to 2 decimal places, the experimental probability of a clip falling:

a inside a square **b** on a line.



2

Length	*Frequency*
0 - 19	17
20 - 39	38
40 - 59	19
60+	4

Jose surveyed the length of TV commercials (in seconds). Find, to 3 decimal places, the experimental probability that a randomly chosen TV commercial will last:

a 20 to 39 seconds **b** more than a minute

c between 20 and 59 seconds (inclusive).

3 Betul records the number of phone calls she receives over a period of consecutive days.

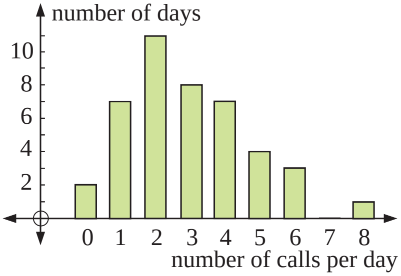
a For how many days did the survey last?

b Estimate Betul's chance of receiving:

  i no phone calls on one day

  ii 5 or more phone calls on a day

  iii less than 3 phone calls on a day.



4 Pat does a lot of travelling in her car and she keeps records on how often she fills her car with petrol. The table alongside shows the frequencies of the number of days between refills. Estimate the likelihood that:

a there is a four day gap between refills

b there is at least a four day gap between refills.

Days between refills	*Frequency*
1	37
2	81
3	48
4	17
5	6
6	1

INVESTIGATION 2 COIN TOSSING EXPERIMENTS

The coins of most currencies have two distinct faces, usually referred to as 'heads' and 'tails'. When we toss a coin in the air, we expect it to finish on a head or tail with equal likelihood.

In this investigation the coins do not have to be all the same type.

What to do:

1 Toss *one coin* 40 times. Record the number of heads resulting in a table:

Result	*Tally*	*Frequency*	*Relative frequency*
1 head 0 head			

2 Toss *two coins* 60 times. Record the number of heads resulting in a table.

Result	*Tally*	*Frequency*	*Relative frequency*
2 heads			
1 head			
0 head			

3 Toss *three coins* 80 times. Record the number of heads resulting in a table.

Result	*Tally*	*Frequency*	*Relative frequency*
3 heads			
2 heads			
1 head			
0 head			

4 Share your results to **1**, **2** and **3** with several other students. Comment on any similarities and differences.

5 Pool your results and find new relative frequencies for tossing one coin, two coins, and three coins.

6 Click on the icon to examine a coin tossing simulation.
Set it to toss one coin 10 000 times.
Run the simulation ten times, each time recording the % frequency for each possible result. Comment on these results. Do your results agree with what you expected?

COIN TOSSING

7 Repeat **6** but this time with *two coins* and then with *three coins*.

From the previous investigation you should have observed that, when tossing two coins, there are roughly twice as many ‘one head’ results as there are ‘no heads’ or ‘two heads’.

The explanation for this is best seen using two different coins where you could get:

This shows that we should expect the ratio two heads : one head : no heads to be 1 : 2 : 1. However, due to chance, there will be variations from this when we look at experimental results.

INVESTIGATION 3 DICE ROLLING EXPERIMENTS

You will need: At least one normal six-sided die with numbers 1 to 6 on its faces. Several dice would be useful to speed up the experimentation.

WORKSHEET

What to do:

1 List the possible outcomes for the uppermost face when the die is rolled.

2 Consider the possible outcomes when the die is rolled 60 times.
Copy and complete the following table of your **expected results**:

Outcomes	*Expected frequency*	*Expected rel. frequency*
⋮		

3 Roll the die 60 times. Record the results in a table like the one shown:

Outcome	*Tally*	*Frequency*	*Relative frequency*
1			
2			
⋮			
6			
	Total	60	

4 Pool as much data as you can with other students.

a Look at similarities and differences from one set to another.

b Summarise the overall pooled data in one table.

c Compare your results with your expectation in **2**.

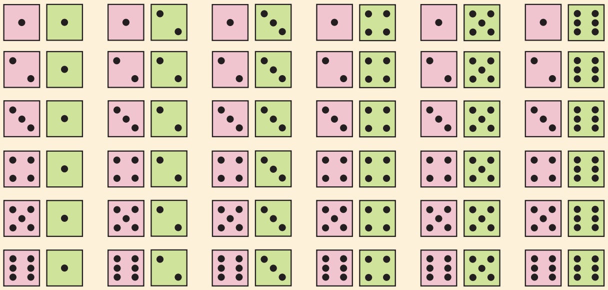
SIMULATION

5 Use the die rolling simulation on the CD to roll the die 10 000 times. Repeat this 10 times. On each occasion, record your results in a table like that in **3**. Do your results further confirm your expected results?

6 The different possible results when a pair of dice is rolled are shown alongside.

There are 36 possible outcomes.

Notice that three of the outcomes, {1, 3}, {2, 2} and {3, 1}, give a sum of 4.



Using the illustration above, copy and complete the table of **expected results**:

Sum	2	3	4	5	···	12
Fraction of total			$\frac{3}{36}$			
Fraction as decimal			0.083			

7 If a pair of dice is rolled 360 times, how many of each result (2, 3, 4, ..., 12) would you expect to get? Extend the table in **6** by adding another row and writing your **expected frequencies** within it.

8 Toss two dice 360 times. Record the *sum of the two numbers* for each toss in a table.

WORKSHEET

Sum	*Tally*	*Frequency*	*Rel. Frequency*
2			
3			
4			
⋮			
12			
	Total	360	1

9 Pool as much data as you can with other students and find the overall relative frequency of each sum.

10 Use the two dice simulation on the CD to roll the pair of dice 10 000 times. Repeat this 10 times and on each occasion record your results in a table like that in **8**. Are your results consistent with your expectations?

SIMULATION

B SAMPLE SPACE

A **sample space** U is the set of all possible outcomes of an experiment.
It is also referred to as the **universal set** U.

There are a variety of ways of representing or illustrating sample spaces, including:

- lists
- tables of outcomes
- 2-dimensional grids
- Venn diagrams
- tree diagrams

We will use tables and Venn diagrams later in the chapter.

LISTING OUTCOMES

Example 1 **Self Tutor**

List the sample space of possible outcomes for:

a tossing a coin **b** rolling a die.

a When a coin is tossed, there are two possible outcomes.
$\therefore$ sample space = {H, T}

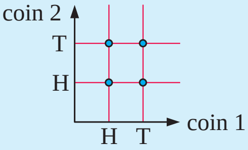
b When a die is rolled, there are 6 possible outcomes.
$\therefore$ sample space = {1, 2, 3, 4, 5, 6}

2-DIMENSIONAL GRIDS

When an experiment involves more than one operation we can still use listing to illustrate the sample space. However, a grid can often be more efficient.

Example 2 **Self Tutor**

Illustrate the possible outcomes when 2 coins are tossed by using a 2-dimensional grid.



Each of the points on the grid represents one of the possible outcomes:
{HH, HT, TH, TT}

TREE DIAGRAMS

The sample space in **Example 2** could also be represented by a tree diagram. The advantage of tree diagrams is that they can be used when more than two operations are involved.

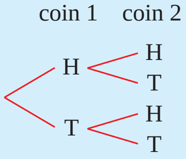
Example 3 **Self Tutor**

Illustrate, using a tree diagram, the possible outcomes when:

a tossing two coins

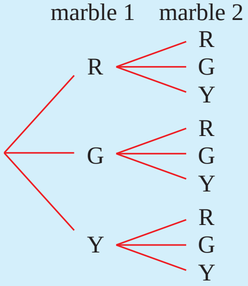
b drawing two marbles from a bag containing many red, green, and yellow marbles.

a



Each 'branch' gives a different outcome. The sample space is seen to be {HH, HT, TH, TT}.

b



EXERCISE 15B

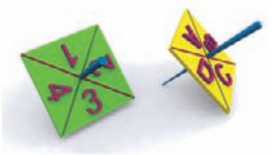
1 List the sample space for the following:

  a twirling a square spinner labelled A, B, C, D

  b the sexes of a 2-child family

  c the order in which 4 blocks A, B, C and D can be lined up

  d the 8 different 3-child families.



2 Illustrate on a 2-dimensional grid the sample space for:

  a rolling a die and tossing a coin simultaneously

  b rolling two dice

  c rolling a die and spinning a spinner with sides A, B, C, D

  d twirling two square spinners, one labelled A, B, C, D and the other 1, 2, 3, 4.

3 Illustrate on a tree diagram the sample space for:

  a tossing a 5-cent and a 10-cent coin simultaneously

  b tossing a coin and twirling an equilateral triangular spinner labelled A, B and C

  c twirling two equilateral triangular spinners labelled 1, 2 and 3 and X, Y and Z

  d drawing two tickets from a hat containing a number of pink, blue and white tickets.

C THEORETICAL PROBABILITY

Consider the **octagonal spinner** alongside.

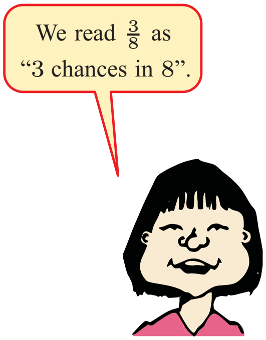
Since the spinner is symmetrical, when it is spun the arrowed marker could finish with equal likelihood on each of the sections marked 1 to 8.

The likelihood of obtaining a particular number, for example 4, would be:

1 chance in 8, $\frac{1}{8}$, $12\frac{1}{2}\%$ or 0.125.

This is a **mathematical** or **theoretical** probability and is based on what we theoretically expect to occur. It is the chance of that event occurring in any trial of the experiment.

If we are interested in the event of getting a result of *6 or more* from one spin of the octagonal spinner, there are three favourable results (6, 7 or 8) out of the eight possible results. Since each of these is equally likely to occur, P(6 or more) $= \frac{3}{8}$.



In general, for an event A containing **equally likely** possible results, the probability of A occurring is

$$\mathbf{P}(A) = \frac{\textbf{the number of members of the event } A}{\textbf{the total number of possible outcomes}} = \frac{n(A)}{n(U)}.$$

Example 4 **Self Tutor**

A ticket is *randomly selected* from a basket containing 3 green, 4 yellow and 5 blue tickets. Determine the probability of getting:

a a green ticket
b a green or yellow ticket
c an orange ticket
d a green, yellow or blue ticket.

The sample space is $\{G_1, G_2, G_3, Y_1, Y_2, Y_3, Y_4, B_1, B_2, B_3, B_4, B_5\}$ which has $3+4+5=12$ outcomes.

a P(G) $= \frac{3}{12}$ $= \frac{1}{4}$

b P(G or Y) $= \frac{3+4}{12}$ $= \frac{7}{12}$

c P(O) $= \frac{0}{12}$ $= 0$

d P(G, Y or B) $= \frac{3+4+5}{12}$ $= 1$

In **Example 4** notice that in **c** an orange result cannot occur. The calculated probability is 0 because the event has *no chance of occurring*.

In **d** we see that a green, yellow or blue result is certain to occur. It is 100% likely so the theoretical probability is 1.

Events which have *no chance of occurring* or probability 0, or are *certain to occur* or probability 1, are two extremes.

Consequently, for any event A, $\mathbf{0 \leqslant P(A) \leqslant 1}$.

Example 5 **Self Tutor**

An ordinary 6-sided die is rolled once. Determine the chance of:

a getting a 6
b not getting a 6
c getting a 1 or 2
d not getting a 1 or 2

The sample space of possible outcomes is $\{1, 2, 3, 4, 5, 6\}$

a P(6) $= \frac{1}{6}$

b P(not a 6) $=$ P(1, 2, 3, 4 or 5) $= \frac{5}{6}$

c P(1 or 2) $= \frac{2}{6}$

d P(not a 1 or 2) $=$ P(3, 4, 5, or 6) $= \frac{4}{6}$

COMPLEMENTARY EVENTS

In **Example 5** notice that $\text{P}(6) + \text{P}(\text{not getting a } 6) = 1$ and that
$$\text{P}(1 \text{ or } 2) + \text{P}\,(\text{not getting a } 1 \text{ or } 2) = 1.$$

This is no surprise as *getting a* 6 and *not getting a* 6 are **complementary events** where one of them **must occur**.

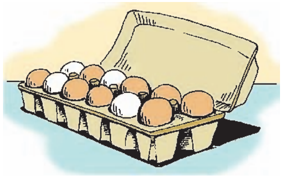
> Two events are **complementary** if exactly one of the events must occur.
> If A is an event, then A' is the complementary event of A, or 'not A'.
>
> $$\mathbf{P}(A) + \mathbf{P}(A') = \mathbf{1}$$

EXERCISE 15C.1

1 A marble is randomly selected from a box containing 5 green, 3 red and 7 blue marbles. Determine the probability that the marble is:

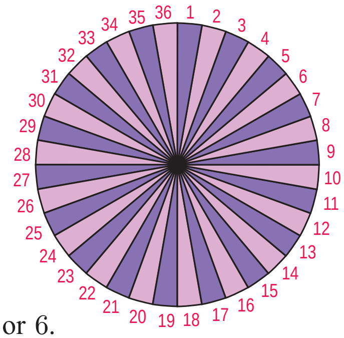
- **a** red
- **b** green
- **c** blue
- **d** not red
- **e** neither green nor blue
- **f** green or red.

2 A carton of a dozen eggs contains eight brown eggs. The rest are white.



- **a** How many white eggs are there in the carton?
- **b** What is the probability that an egg selected at random is: **i** brown **ii** white?

3 A dart board has 36 sectors labelled 1 to 36. Determine the probability that a dart thrown at the centre of the board hits:



- **a** a multiple of 4
- **b** a number between 6 and 9 inclusive
- **c** a number greater than 20
- **d** 9
- **e** a multiple of 13
- **f** an odd number that is a multiple of 3
- **g** a multiple of 4 and 6
- **h** a multiple of 4 or 6.

4 What is the probability that a randomly chosen person has his or her next birthday:

- **a** on a Tuesday
- **b** on a weekend
- **c** in July
- **d** in January or February?

5 List the six different orders in which Antti, Kai and Neda may sit in a row. If the three of them sit randomly in a row, determine the probability that:

- **a** Antti sits in the middle
- **b** Antti sits at the left end
- **c** Antti sits at the right end
- **d** Kai and Neda are seated together

6 **a** List the 8 possible 3-child families according to the gender of the children. For example, GGB means *"the first is a girl, the second is a girl, the third is a boy"*.

- **b** Assuming that each of these is equally likely to occur, determine the probability that a randomly selected 3-child family consists of:
  - **i** all boys
  - **ii** all girls
  - **iii** boy then girl then girl
  - **iv** two girls and a boy
  - **v** a girl for the eldest
  - **vi** at least one boy.

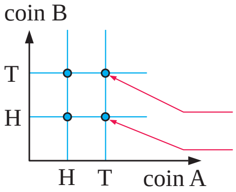
7 **a** List, in systematic order, the 24 different orders in which four people A, B, C and D may sit in a row.

b Determine the probability that when the four people sit at random in a row:

- **i** A sits on one of the end seats
- **ii** B sits on one of the two middle seats
- **iii** A and B are seated together
- **iv** A, B and C are seated together, not necessarily in that order.

USING GRIDS TO FIND PROBABILITIES

Two-dimensional grids can give us excellent visual displays of sample spaces. We can use them to count favourable outcomes and so calculate probabilities.



This point represents 'a tail from coin A' and 'a tail from coin B'.

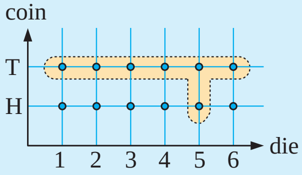
This point represents 'a tail from coin A' and 'a head from coin B'.

There are four members of the sample space.

Example 6 **Self Tutor**

Use a two-dimensional grid to illustrate the sample space for tossing a coin and rolling a die simultaneously. From this grid determine the probability of:

a tossing a head **b** getting a tail and a 5 **c** getting a tail or a 5.



There are 12 members in the sample space.

a P(head) $= \frac{6}{12} = \frac{1}{2}$ **b** P(tail and a '5') $= \frac{1}{12}$

c P(tail or a '5') $= \frac{7}{12}$ {the enclosed points}

EXERCISE 15C.2

1 Draw the grid of the sample space when a 5-cent and a 10-cent coin are tossed simultaneously. Hence determine the probability of getting:

a two heads **b** two tails

c exactly one head **d** at least one head.

2 A coin and a pentagonal spinner with sectors 1, 2, 3, 4 and 5 are tossed and spun respectively.

a Draw a grid to illustrate the sample space of possible outcomes.

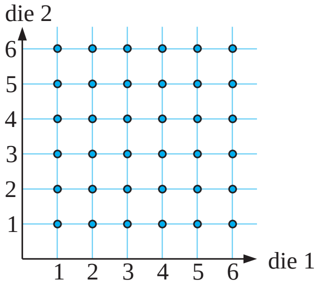
b How many outcomes are possible?

c Use your grid to determine the chance of getting:

- **i** a tail and a 3
- **ii** a head and an even number
- **iii** an odd number
- **iv** a head or a 5.

3 A pair of dice is rolled. The 36 different possible results are illustrated in the 2-dimensional grid.
Use the grid to determine the probability of getting:

- **a** two 3s
- **b** a 5 and a 6
- **c** a 5 or a 6
- **d** at least one 6
- **e** exactly one 6
- **f** no sixes
- **g** a sum of 7
- **h** a sum greater than 8
- **i** a sum of 7 or 11
- **j** a sum of no more than 8.



DISCUSSION

Read and discuss:

Three children have been experimenting with a coin, tossing it in the air and recording the outcomes. They have done this 10 times and have recorded 10 tails. Before the next toss they make the following statements:

Jackson: "It's got to be a head next time!"

Sally: "No, it always has an equal chance of being a head or a tail. The coin cannot remember what the outcomes have been."

Amy: "Actually, I think it will probably be a tail again, because I think the coin must be biased. It might be weighted so it is more likely to give a tail."

D TABLES OF OUTCOMES

Tables of outcomes are tables which compare two categorical variables. They usually result from a survey.

For example, a group of teachers were asked which mode of transport they used to travel to school. Their responses are summarised in the table below.

	Car	Bicycle	Bus
Male	37	10	10
Female	30	5	13

13 female teachers catch the bus to school.

In this case the variables are *gender* and *mode of transport*.

In the following example we will see how these tables can be used to estimate probabilities. To help us, we extend the table to include totals in each row and column.

Example 7 **Self Tutor**

People exiting a new ride at a theme park were asked whether they liked or disliked the ride. The results are shown in the table alongside.

	Child	Adult
Liked the ride	55	28
Disliked the ride	17	30

Use this table to estimate the probability that a randomly selected person who went on the ride:

a liked the ride
b is a child *and* disliked the ride
c is an adult *or* disliked the ride
d liked the ride, given that he or she is a child
e is an adult, given that he or she disliked the ride.

We extend the table to include totals:

	Child	Adult	Total
Liked the ride	55	28	83
Disliked the ride	17	30	47
Total	72	58	130

a 83 out of the 130 people surveyed liked the ride.

$\therefore$ P(liked the ride) $\approx \frac{83}{130} \approx 0.638$

b 17 of the 130 people surveyed are children who disliked the ride.

$\therefore$ P(child *and* disliked the ride) $\approx \frac{17}{130} \approx 0.131$

c $28 + 30 + 17 = 75$ of the 130 people are adults or disliked the ride.

$\therefore$ P(adults *or* disliked the ride) $\approx \frac{75}{130} \approx 0.577$

d Of the 72 children, 55 liked the ride.

$\therefore$ P(liked the ride given that he or she is a child) $\approx \frac{55}{72} \approx 0.764$

e Of the 47 people who disliked the ride, 30 were adults.

$\therefore$ P(adult given that he or she disliked the ride) $\approx \frac{30}{47} \approx 0.638$

EXERCISE 15D

1 A sample of adults in a suburb were surveyed about their current employment status and their level of education. The results are summarised in the table below.

	Employed	Unemployed
Attended university	225	164
Did not attend university	197	231

Estimate the probability that the next randomly chosen adult:

a attended university
b did not attend university and is currently employed
c is unemployed
d is employed, given that the adult attended university
e attended university, given that the adult is unemployed.

2 The types of ticket used to gain access to a basketball match were recorded as people entered the stadium. The results are shown alongside.

	Adult	Child
Season ticket holder	1824	779
Not a season ticket holder	3247	1660

a What was the total attendance for the match?

b One person is randomly selected to sit on the home team's bench. Find the probability that the person selected:

i is a child **ii** is not a season ticket holder

iii is an adult season ticket holder.

3 A motel in London has kept a record of all the room bookings made for the year, categorised by season and booking type. Find the probability that a randomly selected booking:

	Single	Couple	Family
Peak season	125	220	98
Off-peak season	248	192	152

a was in the peak season **b** was a single room in the off-peak season

c was a single or a couple room

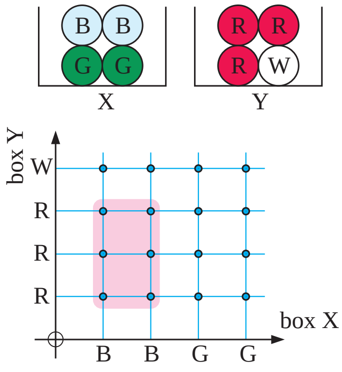
d was a family room, given that it was in the off-peak season

e was in the peak season, given that it was not a single room.

E COMPOUND EVENTS

Consider the following problem:

Box X contains 2 blue and 2 green balls. Box Y contains 1 white and 3 red balls. A ball is randomly selected from each of the boxes. Determine the probability of getting "a blue ball from X and a red ball from Y".



By illustrating the sample space on the two-dimensional grid shown, we can see that 6 of the 16 possibilities are blue from X and red from Y. Each of the outcomes is equally likely, so

P(blue from X **and** red from Y) $= \frac{6}{16}$.

In this section we look for a quicker method for finding the probability of two events both occurring.

INVESTIGATION 4 PROBABILITIES OF COMPOUND EVENTS

The purpose of this investigation is to find a rule for calculating P(A *and* B) for two events A and B.

Suppose a coin is tossed and a die is rolled at the same time. The result of the coin toss will be called outcome A, and the result of the die roll will be outcome B.

What to do:

1 Copy and complete, using a 2-dimensional grid if necessary:

	P(A and B)	P(A)	P(B)
P(a head and a 4)			
P(a head and an odd number)			
P(a tail and a number larger than 1)			
P(a tail and a number less than 3)			

2 What is the connection between P(A and B), P(A), and P(B)?

INVESTIGATION 5 REVISITING DRAWING PINS

We cannot find by theoretical argument the probability that a drawing pin will land on its back  . We can only find this probability by experimentation.

So, when tossing *two* drawing pins can we use the rule for compound events:

$$\text{P(back } \textit{and} \text{ back)} = \text{P(back)} \times \text{P(back)?}$$

What to do:

1 From **Investigation 1** on page **422**, what is your estimate of P(back *and* back)?

2 **a** Count the number of drawing pins in a full packet. They must be identical to each other and the same ones that you used in **Investigation 1**.

b Drop the whole packet onto a solid surface and count the number of *backs* and *sides*. Repeat this several times. Pool results with others and finally estimate P(back).

3 Find P(back) × P(back) using **2 b**.

4 Is P(back *and* back) $\approx$ P(back) × P(back)?

From **Investigations 4** and **5**, it seems that:

If A and B are two events for which the occurrence of each one does not affect the occurrence of the other, then P(A **and** B) = P(A) × P(B).

Before we can formalise this as a rule, however, we need to distinguish between **independent** and **dependent** events.

INDEPENDENT EVENTS

Events are independent if the occurrence of each of them does not affect the probability that the others occur.

Consider again the example on the previous page. Suppose we happen to choose a blue ball from box X. This does not affect the outcome when we choose a ball from box Y. So, the two events "a blue ball from X" and "a red ball from Y" are independent.

If A and B are **independent events** then P(A **and** B) = P(A) × P(B).

This rule can be extended for any number of independent events.

For example: If A, B and C are all **independent events**, then $\mathrm{P}(A \textbf{ and } B \textbf{ and } C) = \mathrm{P}(A) \times \mathrm{P}(B) \times \mathrm{P}(C)$.

Example 8 **Self Tutor**

A coin and a die are tossed simultaneously. Determine the probability of getting a head and a 3 without using a grid.

$$\begin{aligned}\text{P(a head and a 3)} &= \text{P(H)} \times \text{P(3)} \qquad \{\text{events are clearly physically independent}\}\\ &= \tfrac{1}{2} \times \tfrac{1}{6} = \tfrac{1}{12}\end{aligned}$$

EXERCISE 15E.1

1 At a mountain village in Papua New Guinea it rains on average 6 days a week. Determine the probability that it rains on:

a any one day **b** two successive days **c** three successive days.

2 A coin is tossed 3 times. Determine the probability of getting the following sequences of results: **a** head then head then head **b** tail then head then tail.

3 A school has two photocopiers. On any one day, machine A has an 8% chance of malfunctioning and machine B has a 12% chance of malfunctioning. Determine the probability that on any one day both machines will:

a malfunction **b** work effectively.

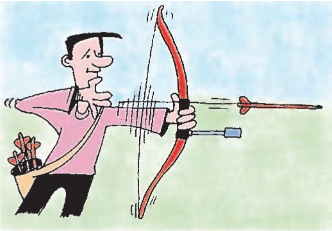
4 A couple decide that they want 4 children, none of whom will be adopted. They will be disappointed if the children are not born in the order boy, girl, boy, girl. Determine the probability that they will be:

a happy with the order of arrival **b** unhappy with the order of arrival.

5 Two marksmen fire at a target simultaneously. Jiri hits the target 70% of the time and Benita hits it 80% of the time. Determine the probability that:

a they both hit the target
b they both miss the target
c Jiri hits but Benita misses
d Benita hits but Jiri misses.

6 An archer always hits a circular target with each arrow shot, and hits the bullseye on average 2 out of every 5 shots. If 3 arrows are shot at the target, determine the probability that the bullseye is hit:



a every time
b the first two times, but not on the third shot
c on no occasion.

DEPENDENT EVENTS

Suppose a hat contains 5 red and 3 blue tickets. One ticket is randomly chosen, its colour is noted, and it is then put aside. A second ticket is then randomly selected. What is the chance that it is red?

If the first ticket was red, P(second is red) $= \frac{4}{7}$ ← 4 reds remaining
← 7 to choose from

If the first ticket was blue, P(second is red) $= \frac{5}{7}$ ← 5 reds remaining
← 7 to choose from

So, the probability of the second ticket being red depends on what colour the first ticket was. We therefore have **dependent events**.

> Two or more events are **dependent** if they are **not independent**.
>
> **Dependent** events are events for which the occurrence of one of the events *does affect* the occurrence of the other event.

For compound events which are dependent, a similar product rule applies to that for independent events:

> If A and B are dependent events then
>
> $$\mathrm{P}(A \textbf{ then } B) = \mathrm{P}(A) \times \mathrm{P}(B \text{ given that } A \text{ has occurred}).$$

Example 9 **Self Tutor**

A box contains 4 red and 2 yellow tickets. Two tickets are randomly selected from the box one by one *without* replacement. Find the probability that:

a both are red **b** the first is red and the second is yellow.

a P(both red)

= P(first selected is red *and* second is red)

= P(first selected is red) × P(second is red given that the first is red)

$= \frac{4}{6} \times \frac{3}{5}$ ← If a red is drawn first, 3 reds remain out of a total of 5.

$= \frac{2}{5}$ ← 4 reds out of a total of 6 tickets

b P(first is red *and* second is yellow)

= P(first is red) × P(second is yellow given that the first is red)

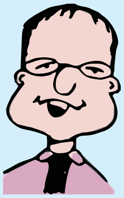
$= \frac{4}{6} \times \frac{2}{5}$ ← If a red is drawn first, 2 yellows remain out of a total of 5.

$= \frac{4}{15}$ ← 4 reds out of a total of 6 tickets

Example 10 **Self Tutor**

A hat contains tickets with the numbers 1, 2, 3, ..., 19, 20 printed on them. If 3 tickets are drawn from the hat, without replacement, determine the probability that all are prime numbers.

In each fraction the numerator is the number of outcomes in the event. The denominator is the total number of possible outcomes.



$\{2, 3, 5, 7, 11, 13, 17, 19\}$ are primes.

$\therefore$ there are 20 numbers of which 8 are primes.

$\therefore$ P(3 primes)

= P(1st drawn is prime *and* 2nd is prime *and* 3rd is prime)

$= \frac{8}{20} \times \frac{7}{19} \times \frac{6}{18}$

8 primes out of 20 numbers

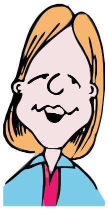
7 primes out of 19 numbers after a successful first draw

6 primes out of 18 numbers after two successful draws

≈ 0.0491

EXERCISE 15E.2

Drawing three chocolates *simultaneously* implies there is no replacement.



1 A bin contains 12 identically shaped chocolates of which 8 are strawberry creams. If 3 chocolates are selected simultaneously from the bin, determine the probability that:

  a they are all strawberry creams
  b none of them are strawberry creams.

2 A box contains 7 red and 3 green balls. Two balls are drawn one after another from the box without replacement. Determine the probability that:

  a both are red
  b the first is green and the second is red
  c a green and a red are obtained.

3 A lottery has 100 tickets which are placed in a barrel. Three tickets are drawn at random from the barrel, without replacement, to decide 3 prizes. If John has 3 tickets in the lottery, determine his probability of winning:

  a first prize
  b first and second prize
  c all 3 prizes
  d none of the prizes.

4 A hat contains 7 names of players in a tennis squad including the captain and the vice captain. If a team of 3 is chosen at random by drawing the names from the hat, determine the probability that it does *not* contain:

  a the captain
  b the captain or the vice captain.

F USING TREE DIAGRAMS

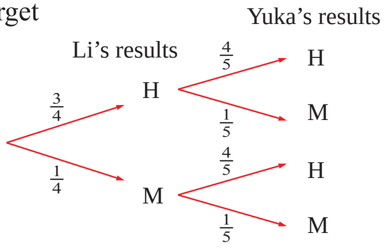
Tree diagrams can be used to illustrate sample spaces if the alternatives are not too numerous. Once the sample space is illustrated, the tree diagram can be used for determining probabilities.

Consider two archers firing simultaneously at a target.

Li has probability $\frac{3}{4}$ of hitting a target and Yuka has probability $\frac{4}{5}$.

The tree diagram for this information is:

H = hit M = miss



outcome	probability
H and H	$\frac{3}{4}\times\frac{4}{5}=\frac{12}{20}$
H and M	$\frac{3}{4}\times\frac{1}{5}=\frac{3}{20}$
M and H	$\frac{1}{4}\times\frac{4}{5}=\frac{4}{20}$
M and M	$\frac{1}{4}\times\frac{1}{5}=\frac{1}{20}$
	total 1

Notice that:

- The probabilities for hitting and missing are marked on the branches.
- There are *four* alternative branches, each showing a particular outcome.
- All outcomes are represented.
- The probability of each outcome is obtained by **multiplying** the probabilities along its branch.

Example 11 **Self Tutor**

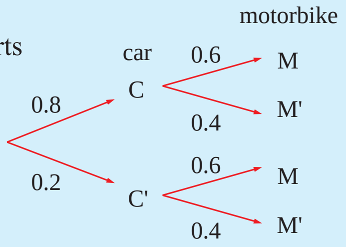
Carl is not having much luck lately. His car will only start 80% of the time and his motorbike will only start 60% of the time.

a Draw a tree diagram to illustrate this situation.

b Use the tree diagram to determine the chance that:

i both will start **ii** Carl can only use his car.

a C = car starts
M = motorbike starts



outcome	probability
C and M	$0.8\times0.6 = 0.48$
C and M'	$0.8\times0.4 = 0.32$
C' and M	$0.2\times0.6 = 0.12$
C' and M'	$0.2\times0.4 = 0.08$
	total 1.00

b **i** P(both start)
= P(C and M)
= 0.8×0.6
= 0.48

ii P(car starts but motorbike does not)
= P(C and M$'$)
= 0.8×0.4
= 0.32

If there is more than one outcome in an event then we need to **add** the probabilities of these outcomes.

Example 12 **Self Tutor**

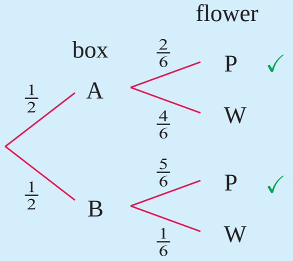
Two boxes each contain 6 petunia plants that are not yet flowering. Box A contains 2 plants that will have purple flowers and 4 plants that will have white flowers. Box B contains 5 plants that will have purple flowers and 1 plant that will have white flowers. A box is selected by tossing a coin, and one plant is removed at random from it. Determine the probability that it will have purple flowers.

Box A

P	W	W
W	P	W

Box B

P	P	P
W	P	P



P(purple flowers)

$= \text{P}(A \text{ and } P) + \text{P}(B \text{ and } P)$

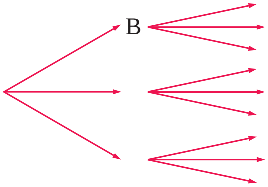
$= \frac{1}{2} \times \frac{2}{6} + \frac{1}{2} \times \frac{5}{6}$ {branches marked ✓}

$= \frac{7}{12}$

EXERCISE 15F

1 Suppose this spinner is spun twice.

a Copy and complete the branches on the tree diagram shown.



b Find the probability that black appears on both spins.

c Find the probability that yellow appears on both spins.

d Find the probability that different colours appear on the two spins.

e Find the probability that black appears on either spin.

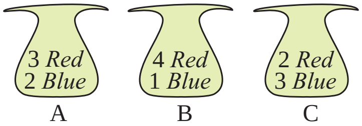
2 The probability of rain tomorrow is estimated to be $\frac{1}{5}$. If it does rain, Mudlark will start favourite in the horse race, with probability $\frac{1}{2}$ of winning. If it is fine, he only has a 1 in 20 chance of winning. Display the sample space of possible results of the horse race on a tree diagram. Hence determine the probability that Mudlark will win tomorrow.

3 Machine A makes 40% of the bottles produced at a factory. Machine B makes the rest. Machine A spoils 5% of its product, while Machine B spoils only 2%. Using an appropriate tree diagram, determine the probability that the next bottle inspected at this factory is spoiled.

4 Jar A contains 2 white and 3 red discs. Jar B contains 3 white and 1 red disc. A jar is chosen at random by the flip of a coin, and one disc is taken at random from it. Determine the probability that the disc is red.

5 The English Premier League consists of 20 teams. Tottenham is currently in 8th place on the table. It has 20% chance of winning and 50% chance of losing against any team placed above it. If a team is placed below it, Tottenham has a 50% chance of winning and a 30% chance of losing. Find the probability that Tottenham will draw its next game.

6 Three bags contain different numbers of blue and red marbles. A bag is selected using a die which has three A faces, two B faces, and one C face.



One marble is then selected randomly from the bag. Determine the probability that it is:

a blue **b** red.

G SAMPLING WITH AND WITHOUT REPLACEMENT

Suppose we have a large group of objects. If we select one of the objects at random and inspect it for particular features, then this process is known as **sampling**.

If the object is put back in the group before another is chosen, we call it **sampling with replacement**.

If the object is put to one side, we call it **sampling without replacement**.

Sampling is commonly used in the quality control of industrial processes.

Sometimes the inspection process makes it impossible to return the object to the large group.

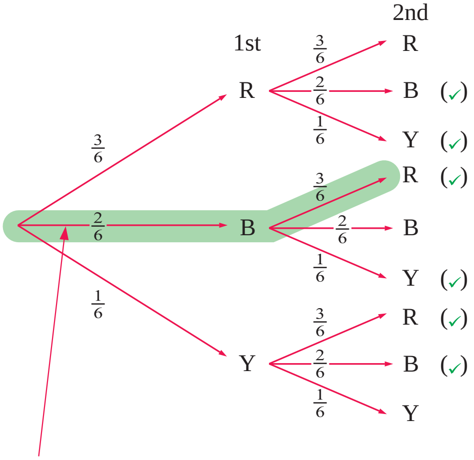
For example:
- To see if a chocolate is hard or soft-centred, we need to bite it or squeeze it.
- To see if an egg contains one or two yolks, we need to break it open.
- To see if an object is correctly made, we may need to pull it apart.

Consider a box containing 3 red, 2 blue and 1 yellow marble. If we sample two marbles, we can do this either:

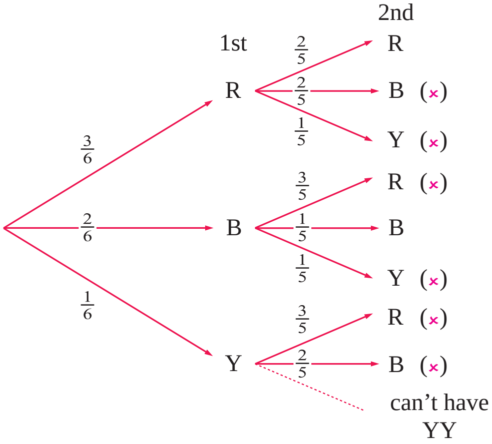
- **with replacement** of the first before the second is drawn, or
- **without replacement** of the first before the second is drawn.

Examine how the tree diagrams differ:

With replacement



Without replacement



This branch represents a blue marble with the first draw and a red marble with the second draw. We write this as BR.

Notice that:
- with replacement P(two reds) $= \frac{3}{6} \times \frac{3}{6} = \frac{1}{4}$
- without replacement P(two reds) $= \frac{3}{6} \times \frac{2}{5} = \frac{1}{5}$

Example 13 **Self Tutor**

A box contains 3 red, 2 blue and 1 yellow marble. Find the probability of getting two different colours:

a if replacement occurs **b** if replacement does not occur.

To answer this question we use the tree diagram on page **440**.

a P(two different colours)

$= $ P(RB or RY or BR or BY or YR or YB) {ticked ones ✓ }

$= \frac{3}{6} \times \frac{2}{6} + \frac{3}{6} \times \frac{1}{6} + \frac{2}{6} \times \frac{3}{6} + \frac{2}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{3}{6} + \frac{1}{6} \times \frac{2}{6}$

$= \frac{11}{18}$

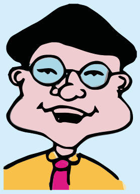
b P(two different colours)

$= $ P(RB or RY or BR or BY or YR or YB) {crossed ones ✗ }

$= \frac{3}{6} \times \frac{2}{5} + \frac{3}{6} \times \frac{1}{5} + \frac{2}{6} \times \frac{3}{5} + \frac{2}{6} \times \frac{1}{5} + \frac{1}{6} \times \frac{3}{5} + \frac{1}{6} \times \frac{2}{5}$

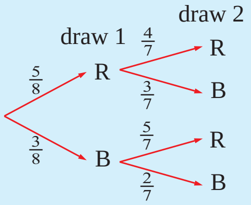
$= \frac{11}{15}$

Notice that in **b**
P(2 different colours)
$= 1 -$ P(2 the same)
$= 1 -$ P(RR or BB)
$= 1 - (\frac{3}{6} \times \frac{2}{5} + \frac{2}{6} \times \frac{1}{5})$
$= \frac{11}{15}$



Example 14 **Self Tutor**

A bag contains 5 red and 3 blue marbles. Two marbles are drawn simultaneously from the bag. Determine the probability that at least one is red.



P(at least one red)

$= $ P(RR or RB or BR)

$= \frac{5}{8} \times \frac{4}{7} + \frac{5}{8} \times \frac{3}{7} + \frac{3}{8} \times \frac{5}{7}$

$= \frac{20+15+15}{56}$

$= \frac{25}{28}$

Drawing *simultaneously* is the same as sampling *without* replacement.

Alternatively, P(at least one red)

$= 1 -$ P(no reds) {complementary events}

$= 1 -$ P(BB) and so on.

EXERCISE 15G

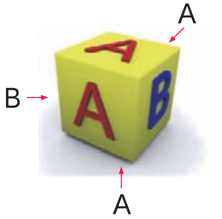
1 Two marbles are drawn in succession from a box containing 2 purple and 5 green marbles. Determine the probability that the two marbles are different colours if:

a the first is replaced **b** the first is *not* replaced.

2 5 tickets numbered 1, 2, 3, 4 and 5 are placed in a bag. Two are taken from the bag without replacement. Determine the probability that:

a both are odd **b** both are even **c** one is odd and the other even.

3 Jar A contains 3 red and 2 green tickets. Jar B contains 3 red and 7 green tickets. A die has 4 faces showing A, and 2 faces showing B. When rolled, the die is used to select either jar A or jar B. Once a jar has been selected, two tickets are randomly selected from it without replacement. Determine the probability that:



a both are green **b** they are different in colour.

4 Marie has a bag of sweets which are all identical in shape. The bag contains 6 orange drops and 4 lemon drops. She selects one sweet at random, eats it, and then takes another at random. Determine the probability that:

a both sweets were orange drops

b both sweets were lemon drops

c the first was an orange drop and the second was a lemon drop

d the first was a lemon drop and the second was an orange drop.

Add your answers to **a**, **b**, **c** and **d**. Explain why the total must be 1.

5 A bag contains four red and two blue marbles. Three marbles are selected simultaneously. Determine the probablity that:

a all are red **b** only two are red **c** at least two are red.

6 Bag A contains 3 red and 2 white marbles. Bag B contains 4 red and 3 white marbles. One marble is randomly selected from A and its colour noted. If it is red, 2 reds are added to B. If it is white, 2 whites are added to B. A marble is then selected from B. What is the chance that the marble selected from B is white?

7 A man holds two tickets in a 100-ticket lottery in which there are two winning tickets. If no replacement occurs, determine the probability that he will win:

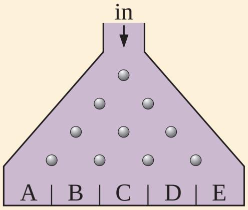
a both prizes **b** neither prize **c** at least one prize.

8 A container holds 3 red, 7 white, and 2 black balls. A ball is chosen at random from the container and is not replaced. A second ball is then chosen. Find the probability of choosing one white and one black ball in any order.

9 A bag contains 7 yellow and n blue markers.
The probability of choosing 2 yellow markers, without replacement after the first choice, is $\frac{3}{13}$. How many blue markers are there in the bag?

INVESTIGATION 6 SAMPLING SIMULATION

When balls enter the 'sorting' chamber shown they hit a metal rod and may go left or right. This movement continues as the balls fall from one level of rods to the next. The balls finally come to rest in collection chambers at the bottom of the sorter.

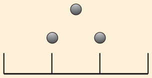


This sorter looks very much like a tree diagram rotated through 90^o.

Click on the icon to open the simulation. Notice that the sliding bar will alter the probabilities of balls going to the left or right at each rod.

What to do:

1 To simulate the results of tossing two coins, set the bar to 50% and the sorter to show:



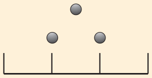
Run the simulation 200 times and repeat this four more times. Record each set of results.

SIMULATION

2 A bag contains 7 blue and 3 red marbles. Two marbles are randomly selected from the bag, the first being *replaced* before the second is drawn.

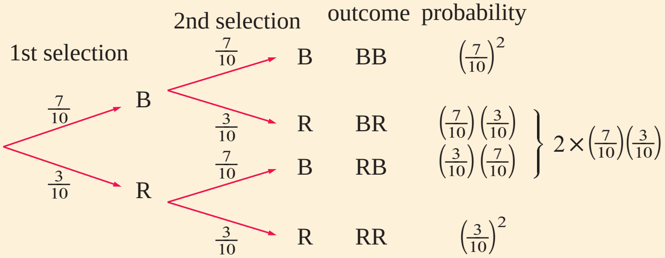
Since P(blue) $= \frac{7}{10} = 70\%$, set the bar to 70%.

The sorter should show:



Run the simulation a large number of times. Use the results to estimate the probability of getting: **a** two blues **b** one blue **c** no blues.

3 The tree diagram representation of the marble selection in **2** gives us theoretical probabilities for the different outcomes:



a Do the theoretical probabilities agree with the experimental results obtained in **2**?

b Write down the algebraic expansion of $(a+b)^2$.

c Substitute $a = \frac{7}{10}$ and $b = \frac{3}{10}$ in the $(a+b)^2$ expansion. What do you notice?

4 From the bag of 7 blue and 3 red marbles, *three* marbles are randomly selected *with replacement*. Set the sorter to 3 levels and the bar to 70%.
Run the simulation a large number of times to obtain the experimental probabilities of getting:

a three blues **b** two blues **c** one blue **d** no blues.

5 **a** Use a tree diagram showing 1st selection, 2nd selection and 3rd selection to find theoretical probabilities for getting the outcomes in **4**.

b Show that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ and use this expansion with $a = \frac{7}{10}$ and $b = \frac{3}{10}$ to also check the results of **4** and **5 a**.

6 Consider the sampling simulator with the bar at 50% to explain why many distributions are symmetrical and bell-shaped.

H BINOMIAL PROBABILITIES

Consider a die which has 2 red faces and 4 black faces. We roll the die three times and record the results.

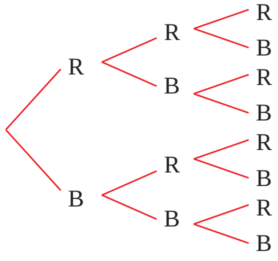
If R represents "the result is red" and B represents "the result is black", the possible outcomes are as shown alongside:

All red	*2 red 1 black*	*1 red 2 black*	*All black*
RRR	BRR	RBB	BBB
	RBR	BRB	
	RRB	BBR	

Notice that the ratio of possible outcomes is 1 : 3 : 3 : 1.

Now for each die, $\text{P(R)} = \frac{1}{3}$ and $\text{P(B)} = \frac{2}{3}$.

So, for rolling the die 3 times we have the following events and probabilities:



Event	*Outcome*	*Probabilities*	*Total Probability*
all red	RRR	$\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$	$\left(\frac{1}{3}\right)^3 = \frac{1}{27}$
2 red and 1 black	BRR RBR RRB	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$ $\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)$ $\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)$	$3\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) = \frac{6}{27}$
1 red and 2 black	RBB BRB BBR	$\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)$ $\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)$ $\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)$	$3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 = \frac{12}{27}$
all black	BBB	$\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)$	$\left(\frac{2}{3}\right)^3 = \frac{8}{27}$

Notice that $\left(\frac{1}{3}\right)^3 + 3\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) + 3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3$ is the binomial expansion for $\left(\frac{1}{3} + \frac{2}{3}\right)^3$.

In general,

If A is an event with probability p of occurring and its complement A' has probability $q = 1 - p$ of occurring, then the **probability generator** for the various outcomes over n **independent trials** is $(p+q)^n$.

For example:

Suppose A is the event of a randomly chosen light globe being faulty, with $\text{P}(A) = p = 0.03$ and $\text{P}(A') = q = 0.97$.

If four independent samples are taken, the probability generator is $(0.03 + 0.97)^4$

$$= \underset{4A\text{s}}{(0.03)^4} + \underset{3A\text{s and }1A'}{4(0.03)^3(0.97)} + \underset{2A\text{s and }2A'\text{s}}{6(0.03)^2(0.97)^2} + \underset{1A\text{ and }3A'\text{s}}{4(0.03)(0.97)^3} + \underset{4A'\text{s}}{(0.97)^4}$$

Notice that

$\text{P}(A$ occurs x times **and** A' occurs $n - x$ times$) = \binom{n}{x} p^x q^{n-x}$.

Example 15 **Self Tutor**

An archer has a 90% chance of hitting the target with each arrow. If 5 arrows are fired, determine the probability generator and hence the chance of hitting the target:

a twice only **b** at most 3 times.

Let H be the event of '*hitting the target*', so $P(H) = 0.9$ and $P(H') = 0.1$

The probability generator is $(0.9 + 0.1)^5$

$$= \underbrace{(0.9)^5}_{\text{5 hits}} + \underbrace{5(0.9)^4(0.1)}_{\substack{\text{4 hits and} \\ \text{1 miss}}} + \underbrace{10(0.9)^3(0.1)^2}_{\substack{\text{3 hits and} \\ \text{2 misses}}} + \underbrace{10(0.9)^2(0.1)^3}_{\substack{\text{2 hits and} \\ \text{3 misses}}} + \underbrace{5(0.9)(0.1)^4}_{\substack{\text{1 hit and} \\ \text{4 misses}}} + \underbrace{(0.1)^5}_{\text{5 misses}}$$

Let X be the number of arrows that hit the target.

a $$\begin{aligned} P(\text{hits twice only}) &= P(X = 2) \\ &= 10(0.9)^2(0.1)^3 \\ &= 0.0081 \end{aligned}$$

b $$\begin{aligned} P(\text{hits at most 3 times}) &= P(X = 0,\ 1,\ 2 \text{ or } 3) \\ &= (0.1)^5 + 5(0.9)(0.1)^4 + 10(0.9)^2(0.1)^3 + 10(0.9)^3(0.1)^2 \\ &\approx 0.0815 \end{aligned}$$

A graphics calculator can be used to find **binomial probabilities**.

For example, to find the probabilities in **Example 15** on a TI-84+ we can use:

a $P(X = 2) = \text{binompdf}(\underset{\overset{\uparrow}{n}}{5},\ \underset{\overset{\uparrow}{p}}{0.9},\ \underset{\overset{\uparrow}{x}}{2})$ **b** $P(X \leqslant 3) = \text{binomcdf}(\underset{\overset{\uparrow}{n}}{5},\ \underset{\overset{\uparrow}{p}}{0.9},\ \underset{\overset{\uparrow}{x}}{3})$

Use your calculator to check the answers given above.

We will learn more about binomial probabilities in **Chapter 23**.

EXERCISE 15H

1 **a** Expand $(p + q)^4$.

b If a coin is tossed *four* times, what is the probability of getting 3 heads?

2 **a** Expand $(p + q)^5$.

b If *five* coins are tossed simultaneously, what is the probability of getting:

i 4 heads and 1 tail in any order **ii** 2 heads and 3 tails

iii 4 heads and 1 tail in that order?

3 **a** Expand $(\frac{2}{3} + \frac{1}{3})^4$.

b Four chocolates are selected at random, with replacement, from a box which contains strawberry creams and almond centres in the ratio 2 : 1. What is the probability of getting:

i all strawberry creams **ii** two of each type

iii at least 2 strawberry creams?

4 **a** Expand $(\frac{3}{4}+\frac{1}{4})^5$.

b In New Zealand in 1946 there were two different coins of value two shillings. These were 'normal' kiwis and 'flat back' kiwis, in the ratio 3 : 1. From a batch of 1946 two shilling coins, five were selected at random with replacement. What is the probability that:

i two were 'flat backs'
ii at least 3 were 'flat backs'
iii at most 3 were 'normal' kiwis?

5 When rifle shooter Huy fires a shot, he hits the target 80% of the time. If Huy fires 4 shots at the target, determine the probability that he has:

a 2 hits and 2 misses in any order
b at least 2 hits.

6 5% of electric light bulbs are defective at manufacture. If 6 bulbs are tested at random with each one being replaced before the next is chosen, determine the probability that:

a two are defective
b at least one is defective.

7 In a multiple choice test there are 10 questions. Each question has 5 choices, one of which is correct. If 70% is the pass mark and Raj, who knows absolutely nothing about the subject, guesses each answer at random, determine the probability that he will pass.

8 Martina beats Jelena in 2 games out of 3 at tennis. What is the probability that Jelena wins a set of tennis 6 games to 4?
Hint: What is the score after 9 games?

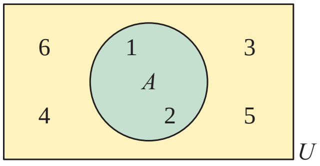
9 How many ordinary dice are needed for there to be a better than an even chance of at least one six when they are thrown together?

I SETS AND VENN DIAGRAMS

Venn diagrams are a useful way of representing the events in a sample space. These diagrams usually consist of a rectangle which represents the complete sample space U, and circles within it which represent particular events.

Venn diagrams can be used to solve certain types of probability questions and also to establish a number of probability laws.

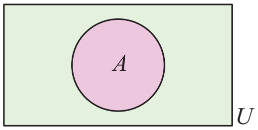
The Venn diagram alongside shows the sample space for rolling a die.



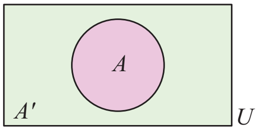
We can write the universal set $U=\{1, 2, 3, 4, 5, 6\}$ since the sample space consists of the numbers from 1 to 6.

The event A is "*a number less than 3*". There are two outcomes which satisfy event A, and we can write $A=\{1, 2\}$.

SET NOTATION

- 

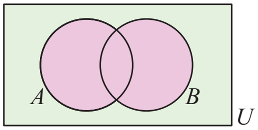
  The **universal set** or **sample space** U is represented by a rectangle.
  An **event** A is usually represented by a circle.

- 

  A' (shaded green) is the **complement** of A (shaded purple).
  A' represents the non-occurrence of A,
  so $\text{P}(A) + \text{P}(A') = 1$.

  If $U = \{1, 2, 3, 4, 5, 6, 7\}$ and $A = \{2, 4, 6\}$ then $A' = \{1, 3, 5, 7\}$.

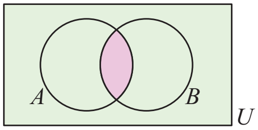
- $x \in A$ reads 'x is in A' and means that x is an element of the set A.
- $n(A)$ reads 'the number of elements in set A'.
- $A \cup B$ denotes the **union** of sets A and B. This set contains all elements belonging to A **or** B **or both** A and B.



$A \cup B$ is shaded in purple.

$A \cup B = \{x \mid x \in A \ \textbf{or} \ x \in B\}$

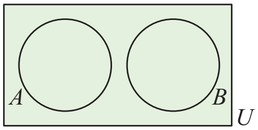
- $A \cap B$ denotes the **intersection** of sets A and B. This set contains all elements common to **both** sets.



$A \cap B$ is shaded in purple.

$A \cap B = \{x \mid x \in A \ \textbf{and} \ x \in B\}$

- **Disjoint sets** are sets which do not have elements in common.



These two sets are disjoint.
$A \cap B = \varnothing$ where $\varnothing$ represents an **empty set**.
A and B are said to be **mutually exclusive**.

Example 16 Self Tutor

If A is the set of all factors of 36 and B is the set of all factors of 54, find:

a $A \cup B$ **b** $A \cap B$

$A = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$ and $B = \{1, 2, 3, 6, 9, 18, 27, 54\}$

a $A \cup B =$ the set of factors of 36 **or** 54
$= \{1, 2, 3, 4, 6, 9, 12, 18, 27, 36, 54\}$

b $A \cap B =$ the set of factors of both 36 **and** 54 $= \{1, 2, 3, 6, 9, 18\}$

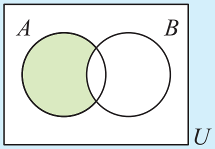
Example 17 **Self Tutor**

On separate Venn diagrams containing two events A and B that intersect, shade the region representing:

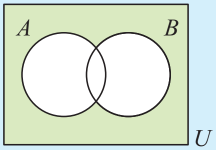
a in A but not in B

b neither in A nor B.

a

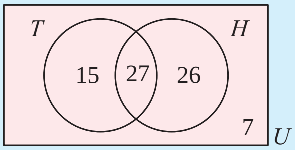


b



Example 18 **Self Tutor**

If the Venn diagram alongside illustrates the number of people in a sporting club who play tennis (T) and hockey (H), determine the number of people:



a in the club

b who play hockey

c who play both sports

d who play neither sport

e who play at least one sport.

a Number in the club
$= 15 + 27 + 26 + 7 = 75$

b Number who play hockey
$= 27 + 26 = 53$

c Number who play both sports $= 27$

d Number who play neither sport
$= 7$

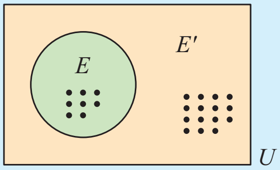
e Number who play at least one sport
$= 15 + 27 + 26 = 68$

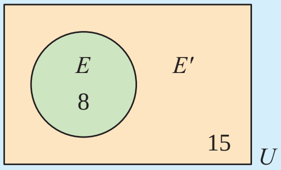
Example 19 **Self Tutor**

The Venn diagram alongside represents the set U of all children in a class. Each dot represents a student. The event E shows all those students with blue eyes. Determine the probability that a randomly selected child:

a has blue eyes

b does not have blue eyes.





$n(U) = 23, \quad n(E) = 8$

a $\text{P(blue eyes)} = \dfrac{n(E)}{n(U)} = \frac{8}{23}$

b $\text{P(not blue eyes)} = \dfrac{n(E')}{n(U)} = \frac{15}{23}$

or $\text{P(not blue)} = 1 - \text{P(blue eyes)} = 1 - \frac{8}{23} = \frac{15}{23}$

Example 20 **Self Tutor**

In a class of 30 students, 19 study Physics, 17 study Chemistry, and 15 study both of these subjects. Display this information on a Venn diagram and hence determine the probability that a randomly selected class member studies:

a both subjects

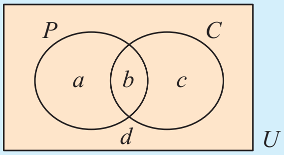
b at least one of the subjects

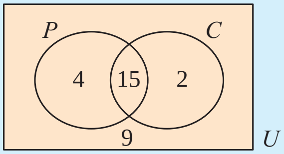
c Physics but not Chemistry

d exactly one of the subjects

e neither subject

f Chemistry if it is known that the student studies Physics.





Let P represent the event of 'studying Physics' and C represent the event of 'studying Chemistry'.

$$\begin{aligned} \text{Now} \quad a+b&=19 && \{\text{as 19 study Physics}\} \\ b+c&=17 && \{\text{as 17 study Chemistry}\} \\ b&=15 && \{\text{as 15 study both}\} \\ a+b+c+d&=30 && \{\text{as there are 30 in the class}\} \end{aligned}$$

$\therefore \quad b=15, \quad a=4, \quad c=2, \quad d=9.$

a P(studies both)

$= \frac{15}{30}$ or $\frac{1}{2}$

b P(studies at least one subject)

$= \frac{4+15+2}{30}$

$= \frac{7}{10}$

c P(P but not C)

$= \frac{4}{30}$

$= \frac{2}{15}$

d P(studies exactly one)

$= \frac{4+2}{30}$

$= \frac{1}{5}$

e P(studies neither)

$= \frac{9}{30}$

$= \frac{3}{10}$

f P(C given P)

$= \frac{15}{15+4}$

$= \frac{15}{19}$

EXERCISE 15I.1

1 Let A be the set of all factors of 6, and B be the set of all positive even integers < 11.

a Describe A and B using set notation.

b Find: **i** $n(A)$ **ii** $A \cup B$ **iii** $A \cap B$.

2 On separate Venn diagrams containing two events A and B that intersect, shade the region representing:

a in A

b in B

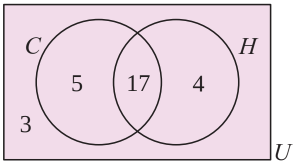
c in both A and B

d in A or B

e in B but not in A

f in exactly one of A or B.

3 The Venn diagram alongside illustrates the number of students in a particular class who study Chemistry (C) and History (H). Determine the number of students:

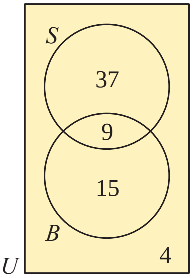


a in the class **b** who study both subjects

c who study at least one of the subjects

d who only study Chemistry.

4 In a survey at an alpine resort, people were asked whether they liked skiing (S) or snowboarding (B). Use the Venn diagram to determine the number of people:



a in the survey

b who liked both activities

c who liked neither activity

d who liked exactly one activity.

5 In a class of 40 students, 19 play tennis, 20 play netball, and 8 play neither of these sports. A student is randomly chosen from the class. Determine the probability that the student:

a plays tennis

b does not play netball

c plays at least one of the sports

d plays one and only one of the sports

e plays netball but not tennis

f plays tennis given he or she plays netball.

6 50 married men were asked whether they gave their wife flowers or chocolates for her last birthday. The results were: 31 gave chocolates, 12 gave flowers, and 5 gave both chocolates and flowers. If one of the married men was chosen at random, determine the probability that he gave his wife:

a chocolates or flowers

b chocolates but not flowers

c neither chocolates nor flowers

d flowers if it is known that he did not give her chocolates.

7 The medical records for a class of 30 children showed that 24 had previously had measles, 12 had previously had measles and mumps, and 26 had previously had at least one of measles or mumps. If one child from the class is selected at random, determine the probability that he or she has had:

a mumps **b** mumps but not measles **c** neither mumps nor measles

d measles if it is known that the child has had mumps.

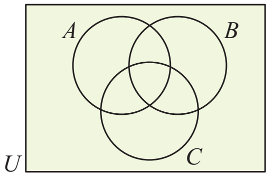
8 If A and B are two non-disjoint sets, shade the region of a Venn diagram representing:

a A' **b** $A' \cap B$ **c** $A \cup B'$ **d** $A' \cap B'$

9 The diagram alongside is the most general case for three events in the same sample space U.
On separate Venn diagram sketches, shade:

a A **b** B' **c** $B \cap C$

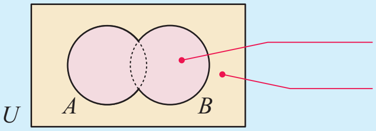
d $A \cup C$ **e** $A \cap B \cap C$ **f** $(A \cup B) \cap C$



USING VENN DIAGRAMS TO VERIFY SET IDENTITIES

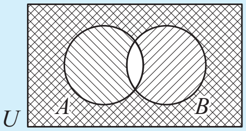
Example 21 **Self Tutor**

Verify that $(A \cup B)' = A' \cap B'$.



this shaded region is $(A \cup B)$

this shaded region is $(A \cup B)'$



▨ represents A'

▧ represents B'

▩ represents $A' \cap B'$

$(A \cup B)'$ and $A' \cap B'$ are represented by the same regions, verifying that $(A \cup B)' = A' \cap B'$.

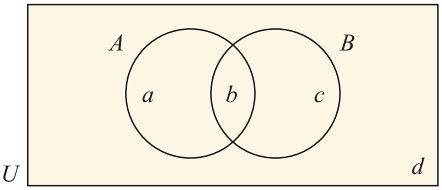
EXERCISE 15I.2

1 Verify that:

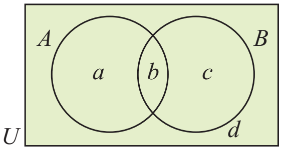
- **a** $(A \cap B)' = A' \cup B'$
- **b** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- **c** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

2 Suppose $S = \{x \mid x \text{ is a positive integer} < 100\}$.
Let $A = \{\text{multiples of } 7 \text{ in } S\}$ and $B = \{\text{multiples of } 5 \text{ in } S\}$.

- **a** How many elements are there in: **i** A **ii** B **iii** $A \cap B$ **iv** $A \cup B$?
- **b** If $n(E)$ represents the number of elements in set E, verify that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.
- **c** Use the figure alongside to establish that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ for all sets A and B in a universal set U.



3



From the Venn diagram, $\text{P}(A) = \dfrac{a+b}{a+b+c+d}$.

- **a** Use the Venn diagram to find:
  i $\text{P}(B)$ **ii** $\text{P}(A \text{ and } B)$ **iii** $\text{P}(A \text{ or } B)$ **iv** $\text{P}(A) + \text{P}(B) - \text{P}(A \text{ and } B)$
- **b** What is the connection between $\text{P}(A \text{ or } B)$ and $\text{P}(A) + \text{P}(B) - \text{P}(A \text{ and } B)$?

J LAWS OF PROBABILITY

THE ADDITION LAW

In the previous exercise we showed that:

For two events A and B, $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$.

This is known as the **addition law of probability**, and can be written as

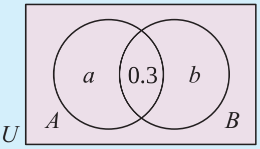
$\text{P}(\textbf{either } A \textbf{ or } B) = \text{P}(A) + \text{P}(B) - \text{P}(\textbf{both } A \textbf{ and } B)$.

Example 22 **Self Tutor**

If $\text{P}(A) = 0.6$, $\text{P}(A \cup B) = 0.7$ and $\text{P}(A \cap B) = 0.3$, find $\text{P}(B)$.

$$\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$$
$$\therefore \quad 0.7 = 0.6 + \text{P}(B) - 0.3$$
$$\therefore \quad \text{P}(B) = 0.4$$

or



Using a Venn diagram with the probabilities on it,

$a + 0.3 = 0.6$ and $a + b + 0.3 = 0.7$

$\therefore \quad a = 0.3$

$\therefore \quad a + b = 0.4$

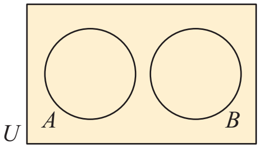
$\therefore \quad 0.3 + b = 0.4$

$\therefore \quad b = 0.1$

$\therefore \quad \text{P}(B) = 0.3 + b = 0.4$

MUTUALLY EXCLUSIVE OR DISJOINT EVENTS

If A and B are **mutually exclusive** events then $\text{P}(A \cap B) = 0$ and so the addition law becomes $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B)$.



Example 23 **Self Tutor**

A box of chocolates contains 6 with hard centres (H) and 12 with soft centres (S).

a Are the events H and S mutually exclusive?

b Find: **i** $\text{P}(H)$ **ii** $\text{P}(S)$ **iii** $\text{P}(H \cap S)$ **iv** $\text{P}(H \cup S)$.

a Chocolates cannot have both a hard and a soft centre.

$\therefore$ H and S are mutually exclusive.

b **i** $\text{P}(H) = \frac{6}{18} = \frac{1}{3}$ **ii** $\text{P}(S) = \frac{12}{18} = \frac{2}{3}$ **iii** $\text{P}(H \cap S) = 0$ **iv** $\text{P}(H \cup S) = \frac{18}{18} = 1$

CONDITIONAL PROBABILITY

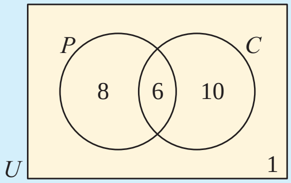
If we have two events A and B, then

> $A \mid B$ is used to represent that 'A occurs knowing that B has occurred'.
> $A \mid B$ is read as "A given B".

Example 24 **Self Tutor**

In a class of 25 students, 14 like pizza and 16 like iced coffee. One student likes neither and 6 students like both. One student is randomly selected from the class. What is the probability that the student:

a likes pizza **b** likes pizza given that he or she likes iced coffee?



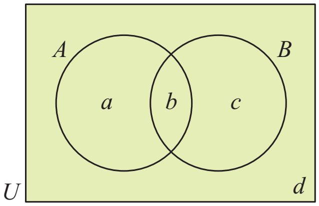
The Venn diagram of the situation is shown.

a P(pizza) $= \frac{14}{25}$ {of the 25 students, 14 like pizza}

b P(pizza | iced coffee) $= \frac{6}{16}$
{of the 16 who like iced coffee, 6 like pizza}

> If A and B are events then $\mathbf{P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}}$.

Proof:



$$\begin{aligned} &\mathrm{P}(A \mid B) \\ &= \frac{b}{b+c} \qquad \{\text{Venn diagram}\} \\ &= \frac{\frac{b}{(a+b+c+d)}}{\frac{b+c}{(a+b+c+d)}} \\ &= \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \end{aligned}$$

It follows that $\mathbf{P(A \cap B) = P(A \mid B)\,P(B)}$ or $\mathbf{P(A \cap B) = P(B \mid A)\,P(A)}$.

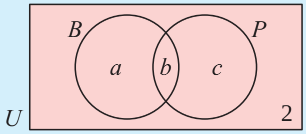
Example 25 **Self Tutor**

In a class of 40 students, 34 like bananas, 22 like pineapple, and 2 dislike both fruits. If a student is randomly selected, find the probability that the student:

a likes both fruits **b** likes at least one fruit

c likes bananas given that he or she likes pineapple

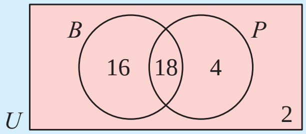
d dislikes pineapple given that he or she likes bananas.



B represents students who like bananas.
P represents students who like pineapple.

We are given that
$$a + b = 34$$
$$b + c = 22$$
$$a + b + c = 38$$

$\therefore \quad c = 38 - 34 = 4$ and so $b = 18$ and $a = 16$



a P(likes both) $= \frac{18}{40}$ $= \frac{9}{20}$

b P(likes at least one) $= \frac{38}{40}$ $= \frac{19}{20}$

c $\text{P}(B \mid P)$ $= \frac{18}{22}$ $= \frac{9}{11}$

d $\text{P}(P' \mid B)$ $= \frac{16}{34}$ $= \frac{8}{17}$

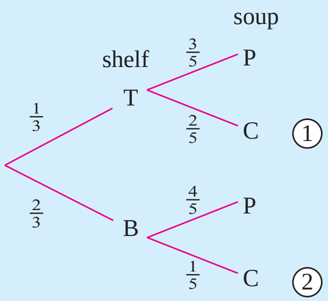
Example 26

Self Tutor

The top shelf in a cupboard contains 3 cans of pumpkin soup and 2 cans of chicken soup. The bottom shelf contains 4 cans of pumpkin soup and 1 can of chicken soup.

Lukas is twice as likely to take a can from the bottom shelf as he is from the top shelf. If he takes one can of soup without looking at the label, determine the probability that it:

a is chicken **b** was taken from top shelf given that it is chicken.



a P(soup is chicken)

$= \frac{1}{3} \times \frac{2}{5} + \frac{2}{3} \times \frac{1}{5}$ {path ① + path ②}

$= \frac{4}{15}$

b P(top shelf | chicken)

$= \dfrac{\text{P(top shelf and chicken)}}{\text{P(chicken)}}$

$= \dfrac{\frac{1}{3} \times \frac{2}{5}}{\frac{4}{15}} \longleftarrow$ path ①

$= \frac{1}{2}$

EXERCISE 15J

1 In a group of 50 students, 40 study Mathematics, 32 study Physics, and each student studies at least one of these subjects.

- **a** Use a Venn diagram to find how many students study both subjects.
- **b** If a student from this group is randomly selected, find the probability that he or she:
  - **i** studies Mathematics but not Physics
  - **ii** studies Physics given that he or she studies Mathematics.

2 In a group of 40 boys, 23 have dark hair, 18 have brown eyes, and 26 have dark hair, brown eyes or both. One of the boys is selected at random. Determine the probability that he has:

a dark hair and brown eyes **b** neither dark hair nor brown eyes

c dark hair but not brown eyes **d** brown eyes given that he has dark hair.

3 50 students went bushwalking. 23 were sunburnt, 22 were bitten by ants, and 5 were both sunburnt and bitten by ants. Determine the probability that a randomly selected student:

a escaped being bitten

b was either bitten or sunburnt

c was neither bitten nor sunburnt

d was bitten, given that he or she was sunburnt

e was sunburnt, given that he or she was not bitten.

4 400 families were surveyed. It was found that 90% had a TV set and 60% had a computer. Every family had at least one of these items. If one of these families is randomly selected, find the probability it has a TV set given that it has a computer.

5 In a certain town three newspapers are published. 20% of the population read A, 16% read B, 14% read C, 8% read A and B, 5% read A and C, 4% read B and C, and 2% read all 3 newspapers. A person is selected at random. Use a Venn diagram to help determine the probability that the person reads:

a none of the papers **b** at least one of the papers

c exactly one of the papers **d** either A or B

e A, given that the person reads at least one paper

f C, given that the person reads either A or B or both.

6 Urn A contains 2 red and 3 blue marbles, and urn B contains 4 red and 1 blue marble. Peter selects an urn by tossing a coin, and takes a marble from that urn.

a Determine the probability that it is red.

b Given that the marble is red, what is the probability that it came from B?

7 The probability that Greta's mother takes her shopping is $\frac{2}{5}$. When Greta goes shopping with her mother she gets an icecream 70% of the time. When Greta does not go shopping with her mother she gets an icecream 30% of the time.
Determine the probability that:

a Greta's mother buys her an icecream when shopping.

b Greta went shopping with her mother, given that her mother buys her an icecream.

8 On a given day, machine A has a 10% chance of malfunctioning and machine B has a 7% chance of the same. Given that at least one of the machines malfunctioned today, what is the chance that machine B malfunctioned?

9 On any day, the probability that a boy eats his prepared lunch is 0.5. The probability that his sister eats her lunch is 0.6. The probability that the girl eats her lunch given that the boy eats his is 0.9. Determine the probability that:

a both eat their lunch **b** the boy eats his lunch given that the girl eats hers

c at least one of them eats their lunch.

10 The probability that a randomly selected person has cancer is 0.02. The probability that he or she reacts positively to a test which detects cancer is 0.95 if he or she has cancer, and 0.03 if he or she does not. Determine the probability that a randomly tested person:

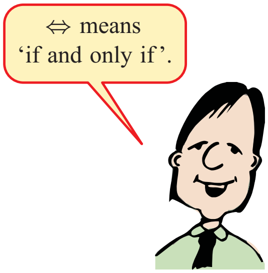
a reacts positively **b** has cancer given that he or she reacts positively.

11 A double-headed, a double-tailed, and an ordinary coin are placed in a tin can. One of the coins is randomly chosen without identifying it. The coin is tossed and falls "heads". Determine the probability that the coin is the "double-header".

K INDEPENDENT EVENTS

A and B are **independent events** if the occurrence of each one of them does not affect the probability that the other occurs.

This means that $\text{P}(A \mid B) = \text{P}(A \mid B') = \text{P}(A)$.



So, as $\text{P}(A \cap B) = \text{P}(A \mid B)\,\text{P}(B)$,

A and B are **independent events** $\Leftrightarrow$ $\text{P}(A \cap B) = \text{P}(A)\,\text{P}(B)$.

Example 27 **Self Tutor**

When two coins are tossed, A is the event of getting 2 heads. When a die is rolled, B is the event of getting a 5 or 6. Show that A and B are independent events.

$\text{P}(A) = \frac{1}{4}$ and $\text{P}(B) = \frac{2}{6}$.

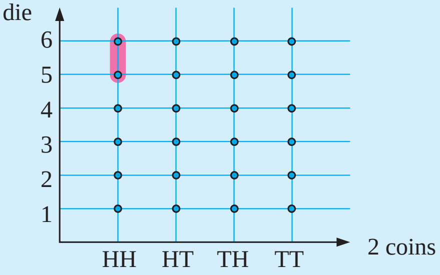
Therefore, $\text{P}(A)\ \text{P}(B) = \frac{1}{4} \times \frac{2}{6} = \frac{1}{12}$

$\text{P}(A \cap B)$

$= \text{P}(2 \text{ heads } \textbf{and} \text{ a 5 or a 6})$

$= \frac{2}{24}$

$= \frac{1}{12}$



Since $\text{P}(A \cap B) = \text{P}(A)\,\text{P}(B)$, the events A and B are independent.

Example 28 **Self Tutor**

$\text{P}(A) = \frac{1}{2}$, $\text{P}(B) = \frac{1}{3}$ and $\text{P}(A \cup B) = p$. Find p if:

a A and B are mutually exclusive **b** A and B are independent.

a If A and B are mutually exclusive, $A \cap B = \varnothing$ and so $\text{P}(A \cap B) = 0$

But $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$

$\therefore\ p = \frac{1}{2} + \frac{1}{3} - 0 = \frac{5}{6}$

b If A and B are independent, $\text{P}(A \cap B) = \text{P}(A)\,\text{P}(B) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

$\therefore\ \text{P}(A \cup B) = \frac{1}{2} + \frac{1}{3} - \frac{1}{6}$ and hence $p = \frac{2}{3}$.

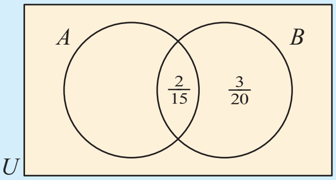
Example 29 **Self Tutor**

Given $\mathrm{P}(A)=\frac{2}{5}$, $\mathrm{P}(B \mid A)=\frac{1}{3}$ and $\mathrm{P}(B \mid A')=\frac{1}{4}$ find: **a** $\mathrm{P}(B)$ **b** $\mathrm{P}(A \cap B')$

$\mathrm{P}(B \mid A)=\dfrac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)}$ so $\mathrm{P}(B \cap A)=\mathrm{P}(B \mid A)\,\mathrm{P}(A)=\frac{1}{3}\times\frac{2}{5}=\frac{2}{15}$

Similarly, $\mathrm{P}(B \cap A')=\mathrm{P}(B \mid A')\,\mathrm{P}(A')=\frac{1}{4}\times\frac{3}{5}=\frac{3}{20}$

$\therefore$ the Venn diagram is:



a $\mathrm{P}(B)=\frac{2}{15}+\frac{3}{20}=\frac{17}{60}$

b $\mathrm{P}(A \cap B')=\mathrm{P}(A)-\mathrm{P}(A \cap B)$

$=\frac{2}{5}-\frac{2}{15}$

$=\frac{4}{15}$

EXERCISE 15K

1 If $\mathrm{P}(R)=0.4$, $\mathrm{P}(S)=0.5$ and $\mathrm{P}(R \cup S)=0.7$, are R and S independent events?

2 If $\mathrm{P}(A)=\frac{2}{5}$, $\mathrm{P}(B)=\frac{1}{3}$ and $\mathrm{P}(A \cup B)=\frac{1}{2}$, find:

a $\mathrm{P}(A \cap B)$ **b** $\mathrm{P}(B \mid A)$ **c** $\mathrm{P}(A \mid B)$

Are A and B independent events?

3 If $\mathrm{P}(X)=0.5$, $\mathrm{P}(Y)=0.7$ and X and Y are independent events, determine the probability of the occurrence of:

a both X and Y **b** X or Y **c** neither X nor Y

d X but not Y **e** X given that Y occurs.

4 The probabilities that A, B and C can solve a particular problem are $\frac{3}{5}$, $\frac{2}{3}$ and $\frac{1}{2}$ respectively. If they all try, determine the probability that at least one of the group solves the problem.

5 **a** Find the probability of getting at least one six when a die is rolled 3 times.

b If a die is rolled n times, find the smallest n such that P(at least one 6 in n throws) $> 99\%$.

6 A and B are independent events. Prove that A' and B' are also independent events.

7 Suppose $\mathrm{P}(A \cap B)=0.1$ and $\mathrm{P}(A \cap B')=0.4$. Find $\mathrm{P}(A \cup B')$ given that A and B are independent.

8 Suppose $\mathrm{P}(C)=\frac{9}{20}$, $\mathrm{P}(C \mid D')=\frac{3}{7}$ and $\mathrm{P}(C \mid D)=\frac{6}{13}$.

a Find: **i** $\mathrm{P}(D)$ **ii** $\mathrm{P}(C' \cup D')$

b Are C and D independent events? Give a reason for your answer.

REVIEW SET 15A NON-CALCULATOR

1 List the different orders in which 4 people A, B, C and D could line up. If they line up at random, determine the probability that:

a A is next to C **b** there is exactly one person between A and C.

2 A coin is tossed and a square spinner labelled A, B, C, D, is twirled. Determine the probability of obtaining:

a a head and consonant **b** a tail and C **c** a tail or a vowel.

3 The probability that a man will be alive in 25 years is $\frac{3}{5}$, and the probability that his wife will be alive is $\frac{2}{3}$. Determine the probability that in 25 years:

a both will be alive **b** at least one will be alive **c** only the wife will be alive.

4 Given $\text{P}(Y) = 0.35$ and $\text{P}(X \cup Y) = 0.8$, and that X and Y are mutually exclusive events, find:

a $\text{P}(X \cap Y)$ **b** $\text{P}(X)$

c the probability that X occurs or Y occurs, but not both X and Y.

5 What is meant by: **a** independent events **b** disjoint events?

6 Graph the sample space of all possible outcomes when a pair of dice is rolled. Hence determine the probability of getting:

a a sum of 7 or 11 **b** a sum of at least 8.

7 In a group of 40 students, 22 study Economics, 25 study Law, and 3 study neither of these subjects. Determine the probability that a randomly chosen student studies:

a both Economics and Law **b** at least one of these subjects

c Economics given that he or she studies Law.

8 The probability that a particular salesman will leave his sunglasses behind in any store is $\frac{1}{5}$. Suppose the salesman visits two stores in succession and leaves his sunglasses behind in one of them. What is the probability that the salesman left his sunglasses in the first store?

REVIEW SET 15B CALCULATOR

1 Niklas and Rolf play tennis with the winner being the first to win two sets. Niklas has a 40% chance of beating Rolf in any set. Draw a tree diagram showing the possible outcomes and hence determine the probability that Niklas will win the match.

2 If I buy 4 tickets in a 500 ticket lottery, and the prizes are drawn without replacement, determine the probability that I will win:

a the first 3 prizes **b** at least one of the first 3 prizes.

3 The students in a school are all vaccinated against measles. 48% of the students are males, of whom 16% have an allergic reaction to the vaccine. 35% of the girls also have an allergic reaction. If a student is randomly chosen from the school, what is the probability that the student:

a has an allergic reaction

b is female given that a reaction occurs?

4 On any one day it could rain with 25% chance and be windy with 36% chance. Draw a tree diagram showing the possibilities with regard to wind and rain on a particular day. Hence determine the probability that on a particular day there will be:

a rain and wind **b** rain or wind.

5 A, B and C have 10%, 20% and 30% chance of independently solving a certain maths problem. If they all try independently of one another, what is the probability that this group will solve the problem?

6 **a** Expand $\left(\frac{3}{5}+\frac{2}{5}\right)^4$.

b A tin contains 20 pens of which 12 have blue ink. Four pens are randomly selected, with replacement, from the tin. What is the probability that:

i two of them have blue ink **ii** at most two have blue ink?

7 A survey of 200 people included 90 females. It found that 60 people smoked, 40 of whom were male.

a Use the given information to complete the table:

	Female	*Male*	*Total*
smoker			
non-smoker			
Total			

b A person is selected at random. Find the probability that this person is:

i a female non-smoker **ii** a male given the person was a non-smoker.

c If two people from the survey are selected at random, calculate the probability that both of them are non-smoking females.

REVIEW SET 15C

1 Systematically list the possible sexes of a 4-child family. Hence determine the probability that a randomly selected 4-child family has two children of each sex.

2 A bag contains 3 red, 4 yellow and 5 blue marbles. Two marbles are randomly selected from the bag without replacement. What is the probability that:

a both are blue **b** they have the same colour

c at least one is red **d** exactly one is yellow?

3 A class contains 25 students. 13 play tennis, 14 play volleyball, and 1 plays neither of these sports. If a student is randomly selected from the class, determine the probability that the student:

a plays both tennis and volleyball **b** plays at least one of these sports

c plays volleyball given that he or she does not play tennis.

4 An urn contains three red balls and six blue balls.

a A ball is drawn at random and found to be blue. What is the probability that a second draw with no replacement will also produce a blue ball?

b Two balls are drawn without replacement and the second is found to be red. What is the probability that the first ball was also red?

5 A school photocopier has a 95% chance of working on any particular day. Find the probability that it will be working on at least one of the next two days.

6 Jon goes cycling on three random mornings of each week. When he goes cycling he has eggs for breakfast 70% of the time. When he does not go cycling he has eggs for breakfast 25% of the time. Determine the probability that he:

a has eggs for breakfast **b** goes cycling given that he has eggs for breakfast.

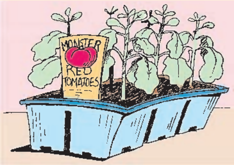
7 **a** Expand $\left(\frac{4}{5}+\frac{1}{5}\right)^5$.

b With every attempt, Jack has an 80% chance of kicking a goal. In one quarter of a match he has 5 kicks for goal. Determine the probability that he scores:

- **i** 3 goals then misses twice
- **ii** 3 goals and misses twice.

INVESTIGATION 7 HOW MANY SHOULD I PLANT?

Suppose you own a seedling business which supplies garden centres with punnets containing six tomato seedlings each.



For each seed you plant in a punnet, there is an 85% chance it will germinate and grow into a seedling of saleable size.

To efficiently run your business, you want to know how many seeds to plant in each punnet to be at least 95% sure of getting at least six healthy seedlings.

SIMULATION

What to do:

1 Click on the simulation icon. Set the number of rows of rods in the sorting chamber to 6, which represents planting 6 seeds. Now set the number of balls to 5000, so we simulate planting 5000 punnets with 6 seeds each. Switch the ball speed **off**. Move the sliding bar to 85%, to simulate the probabilty of each seed germinating and growing to a saleable size.

2 Click START . You should get a result something like:

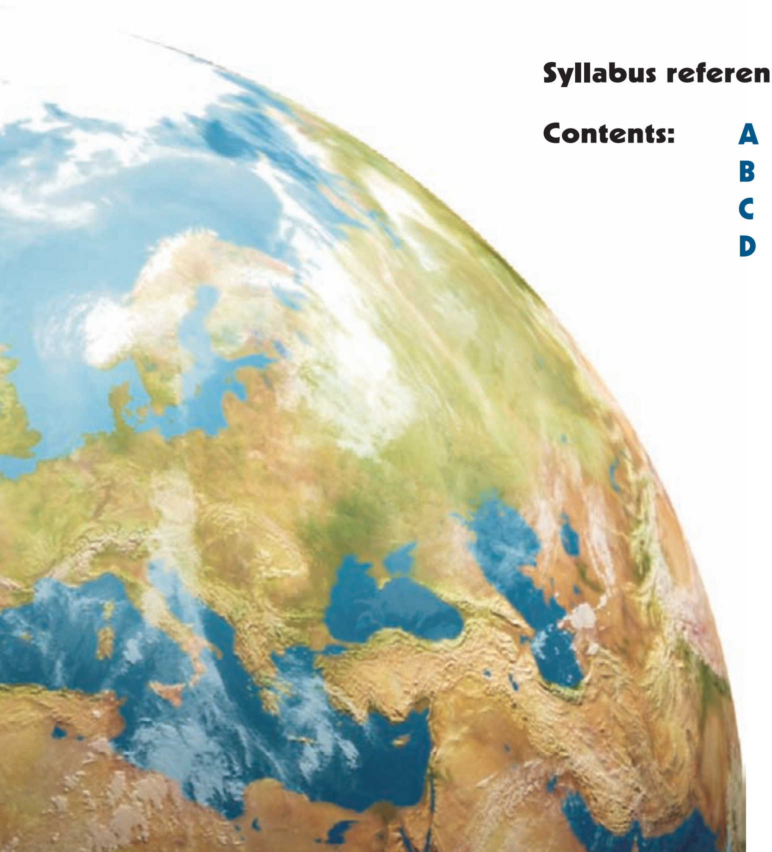
Number of seedlings	0	1	2	3	4	5	6
Number of punnets	0	2	32	211	894	2006	1855
Percentage of punnets	0.0%	0.0%	0.6%	4.2%	17.9%	40.1%	37.1%

In this case we have only planted 6 seeds. We see that the experimental probability of getting 'at least 6 seedlings' is only 37.1%.

3 Increase the number of rows of rods in the chamber to simulate planting more seeds in each punnet. Continue until you find the correct number of seeds to consistently give 95% probability of at least 6 seeds germinating.

Chapter 16

Introduction to calculus



Syllabus reference: 7.1, 7.5

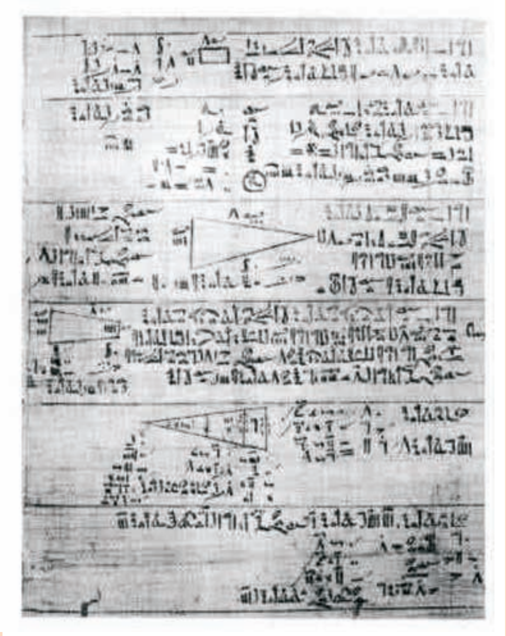
Contents:

Calculus is a major branch of mathematics which builds on algebra, trigonometry, and analytic geometry. It has widespread applications in science, engineering, and financial mathematics.

The study of calculus is divided into two fields, **differential calculus** and **integral calculus**, both of which we will study in this course. These fields are linked by the **fundamental theorem of calculus** which we will study in **Chapter 21**.

HISTORICAL NOTE

Calculus is a Latin word meaning 'pebble'. Ancient Romans used stones for counting.



The history of calculus begins with the **Egyptian Moscow papyrus** from about 1850 BC. Its study continued in Egypt before being taken up by the Greek mathematician **Archimedes of Syracuse**.

It was further developed through the centuries by mathematicians of many nations.

Two of the most important contributors were **Gottfried Wilhelm Leibniz** and **Sir Isaac Newton** who independently developed the **fundamental theorem of calculus**.

A LIMITS

The idea of a **limit** is essential to differential calculus. We will see that it is necessary for finding the gradient of a tangent to a curve at any point on the curve.

Consider the following table of values for $f(x) = x^2$ in the vicinity of $x = 2$.

x	1	1.9	1.99	1.999	1.9999	2.0001	2.001	2.01	2.1	3
$f(x)$	1	3.61	3.9601	3.996 00	3.999 60	4.000 40	4.004 00	4.0401	4.41	9

Notice that as x approaches 2 from the left, then $f(x)$ approaches 4 from below.

Likewise, as x approaches 2 from the right, then $f(x)$ approaches 4 from above.

We say that as x approaches 2 from either direction, $f(x)$ approaches a limit of 4, and write

$$\lim_{x \to 2} x^2 = 4.$$

INFORMAL DEFINITION OF A LIMIT

The following definition of a limit is informal but adequate for the purposes of this course:

> If $f(x)$ can be made as close as we like to some real number A by making x sufficiently close to a, we say that $f(x)$ approaches a **limit** of A as x approaches a, and we write
>
> $$\lim_{x\to a} f(x) = A.$$

We also say that as x approaches a, $f(x)$ **converges** to A.

Notice that we have not used the actual value of $f(x)$ when $x = a$, or in other words $f(a)$. This is very important to the concept of limits.

For example, if $f(x) = \dfrac{5x+x^2}{x}$ and we wish to find the limit as $x \to 0$, it is tempting for us to simply substitute $x = 0$ into $f(x)$.

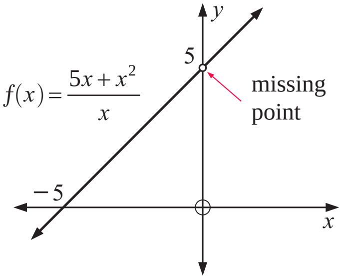
Not only do we get the meaningless value of $\frac{0}{0}$, but also we destroy the basic limit method.

Observe that if $f(x) = \dfrac{5x+x^2}{x}$ then $f(x) \begin{cases} = 5 + x \text{ if } x \neq 0 \\ \text{is undefined if } x = 0. \end{cases}$

$f(x)$ has the graph shown alongside. It is the straight line $y = x + 5$ with the point $(0, 5)$ missing, called a **point of discontinuity** of the function.

However, even though this point is missing, the *limit* of $f(x)$ as x approaches 0 does exist. In particular, as $x \to 0$ from either direction, $f(x) \to 5$.

$$\therefore \lim_{x\to 0} \frac{5x+x^2}{x} = 5$$



In practice we do not need to graph functions each time to determine limits, and most can be found algebraically.

Example 1

Self Tutor

Evaluate: **a** $\lim_{x\to 2} x^2$ **b** $\lim_{x\to 0} \dfrac{5x+x^2}{x}$

a x^2 can be made as close as we like to 4 by making x sufficiently close to 2.

$$\therefore \lim_{x\to 2} x^2 = 4.$$

b $\dfrac{5x+x^2}{x} \begin{cases} = 5 + x \text{ if } x \neq 0 \\ \text{is undefined if } x = 0. \end{cases}$

$$\begin{aligned} \therefore \lim_{x\to 0} \frac{5x+x^2}{x} &= \lim_{x\to 0} \frac{x(5+x)}{x} \\ &= \lim_{x\to 0} 5 + x \quad \text{since } x \neq 0 \\ &= 5 \end{aligned}$$

RULES FOR LIMITS

If $f(x)$ and $g(x)$ are functions and c is a constant:

- $\lim\limits_{x \to a} c = c$
- $\lim\limits_{x \to a} c\,f(x) = c \lim\limits_{x \to a} f(x)$
- $\lim\limits_{x \to a} [f(x) \pm g(x)] = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$
- $\lim\limits_{x \to a} [f(x)g(x)] = \lim\limits_{x \to a} f(x) \times \lim\limits_{x \to a} g(x)$
- $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} f(x) \div \lim\limits_{x \to a} g(x)$ provided $\lim\limits_{x \to a} g(x) \neq 0$.

Example 2 **Self Tutor**

Use the rules for limits to evaluate:

a $\lim\limits_{x \to 3} (x+2)(x-1)$ **b** $\lim\limits_{x \to 1} \dfrac{x^2+2}{x-2}$

Describe the results in terms of convergence.

a As $x \to 3$, $x+2 \to 5$ and $x-1 \to 2$

$\therefore \lim\limits_{x \to 3} (x+2)(x-1) = 5 \times 2 = 10$

As $x \to 3$, $(x+2)(x-1)$ *converges* to 10.

b As $x \to 1$, $x^2+2 \to 3$ and $x-2 \to -1$

$$\therefore \lim_{x \to 1} \frac{x^2+2}{x-2} = \frac{3}{-1} = -3$$

As $x \to 1$, $\dfrac{x^2+2}{x-2}$ *converges* to -3.

LIMITS AT INFINITY

We can use the idea of limits to discuss the behaviour of functions for extreme values of x.

We write $x \to \infty$ to mean when x gets as large as we like and positive,
and $x \to -\infty$ to mean when x gets as large as we like and negative.

We read $x \to \infty$ as "x tends to plus infinity" and $x \to -\infty$ as "x tends to minus infinity".

Notice that as $x \to \infty$, $1 < x < x^2 < x^3 < ......$

$$\therefore \lim_{x \to \infty} \frac{1}{x} = 0, \quad \lim_{x \to \infty} \frac{x}{x^2} = 0, \quad \text{and so on.}$$

Example 3

Self Tutor

Evaluate the following limits:

a $\displaystyle\lim_{x\to\infty}\frac{2x+3}{x-4}$ **b** $\displaystyle\lim_{x\to\infty}\frac{x^2-3x+2}{1-x^2}$

a
$$\begin{aligned}&\lim_{x\to\infty}\frac{2x+3}{x-4}\\&=\lim_{x\to\infty}\frac{2+\frac{3}{x}}{1-\frac{4}{x}} &&\{\text{dividing each term in both numerator and denominator by } x\}\\&=\frac{2}{1} &&\{\text{as } x\to\infty,\ \tfrac{3}{x}\to 0 \text{ and } \tfrac{4}{x}\to 0\}\\&=2\end{aligned}$$

or
$$\begin{aligned}&\lim_{x\to\infty}\frac{2x+3}{x-4}\\&=\lim_{x\to\infty}\frac{2(x-4)+8+3}{x-4}\\&=\lim_{x\to\infty}\left(2+\frac{11}{x-4}\right)\\&=2 \quad \left\{\text{as } \lim_{x\to\infty}\frac{11}{x-4}=0\right\}\end{aligned}$$

b
$$\begin{aligned}&\lim_{x\to\infty}\frac{x^2-3x+2}{1-x^2}\\&=\lim_{x\to\infty}\frac{1-\frac{3}{x}+\frac{2}{x^2}}{\frac{1}{x^2}-1} &&\{\text{dividing each term by } x^2\}\\&=\frac{1}{-1} &&\{\text{as } x\to\infty,\ \tfrac{3}{x}\to 0,\ \tfrac{2}{x^2}\to 0, \text{ and } \tfrac{1}{x^2}\to 0\}\\&=-1\end{aligned}$$

or
$$\begin{aligned}&\lim_{x\to\infty}\frac{x^2-3x+2}{1-x^2}\\&=\lim_{x\to\infty}\frac{-x^2+3x-2}{x^2-1}\\&=\lim_{x\to\infty}\frac{-(x^2-1)+3x-3}{x^2-1}\\&=\lim_{x\to\infty}\left(-1+\frac{3x-3}{x^2-1}\right)\\&=-1 \quad \left\{\text{as } \lim_{x\to\infty}\frac{3x-3}{x^2-1}=0\right\}\end{aligned}$$

EXERCISE 16A

1 Evaluate:

a $\displaystyle\lim_{x\to 3}(x+4)$ **b** $\displaystyle\lim_{x\to -1}(5-2x)$ **c** $\displaystyle\lim_{x\to 4}(3x-1)$

d $\displaystyle\lim_{x\to 2}5x^2-3x+2$ **e** $\displaystyle\lim_{h\to 0}h^2(1-h)$ **f** $\displaystyle\lim_{x\to -1}\frac{1-2x}{x^2+1}$

g $\displaystyle\lim_{x\to 0}(x^2+5)$ **h** $\displaystyle\lim_{x\to -2}\frac{4}{x}$

2 Evaluate the following limits by looking for a common factor in the numerator and denominator:

a $\displaystyle\lim_{x\to 0}\frac{x^2-3x}{x}$ **b** $\displaystyle\lim_{h\to 0}\frac{2h^2+6h}{h}$ **c** $\displaystyle\lim_{h\to 0}\frac{h^3-8h}{h}$

d $\displaystyle\lim_{x\to 1}\frac{x^2-x}{x^2-1}$ **e** $\displaystyle\lim_{x\to 2}\frac{x^2-2x}{x^2-4}$ **f** $\displaystyle\lim_{x\to 3}\frac{x^2-x-6}{x^2-5x+6}$

3 Evaluate:

a $\lim\limits_{x\to\infty} \frac{1}{x}$ **b** $\lim\limits_{x\to\infty} \frac{3x-2}{x+1}$ **c** $\lim\limits_{x\to\infty} \frac{1-2x}{3x+2}$

d $\lim\limits_{x\to\infty} \frac{x}{1-x}$ **e** $\lim\limits_{x\to\infty} \frac{x^2+3}{x^2-1}$ **f** $\lim\limits_{x\to\infty} \frac{x^2-2x+4}{x^2+x-1}$

B FINDING ASYMPTOTES USING LIMITS

Rational functions are functions of the form $\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomials.

Rational functions are characterised by the existence of **asymptotes** which may be vertical, horizontal, or oblique.

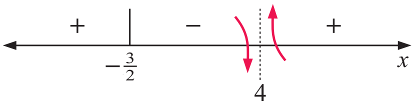
An **oblique asymptote** is neither horizontal nor vertical, but we will not discuss these in this course.

We can investigate the asymptotes of a function by using limits.

Consider the function $f : x \mapsto \frac{2x+3}{x-4}$ which has domain $\{x \in \mathbb{R},\ x \neq 4\}$.

1 There is a vertical asymptote (VA) at $x = 4$.
To discuss the behaviour near the VA, we find what happens to $f(x)$ as $x \to 4$ from the left and right.

- First we draw a sign diagram of $f(x)$.



- Hence as $x \to 4$ (left), $f(x) \to -\infty$
  $x \to 4$ (right), $f(x) \to +\infty$.

2 Is there another type of asymptote?

Now $f(x) = \frac{2x+3}{x-4} = \frac{2(x-4)+11}{x-4} = 2 + \frac{11}{x-4}$.

- As $x \to +\infty$, $f(x) \to 2$ (above) {as $\frac{11}{x-4} \to 0$ from above}
- As $x \to -\infty$, $f(x) \to 2$ (below) {as $\frac{11}{x-4} \to 0$ from below}

Hence, there is a horizontal asymptote (HA) at $y = 2$.
This horizontal asymptote corresponds to the answer in **Example 3** part **a**.

Example 4 **Self Tutor**

Find any asymptotes of the function $f : x \mapsto \frac{x^2-3x+2}{1-x^2}$ and discuss the behaviour of $f(x)$ near these asymptotes.

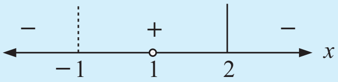
We notice that $f(x) = \frac{(x-2)\cancelto{-1}{(x-1)}}{\cancel{(1-x)}(1+x)} = \frac{-(x-2)}{1+x}$ provided $x \neq 1$.

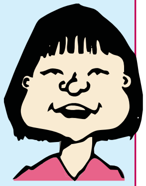
So, there is a point of discontinuity at $x = 1$.

Also, when $x = -1$, $f(x)$ is undefined. {dividing by zero}

This indicates that $x = -1$ is a vertical asymptote.

The sign diagram for $f(x)$ is:





The open circle ∘ indicates a discontinuity.

As $x \to -1$ (left), $f(x) \to -\infty$.
As $x \to -1$ (right), $f(x) \to +\infty$.

$\therefore$ $x = -1$ is a VA.

For $x \neq 1$, $f(x) = \dfrac{-(x-2)}{1+x} = \dfrac{-x+2}{x+1} = \dfrac{-(x+1)+3}{x+1} = -1 + \dfrac{3}{x+1}$

As $x \to \infty$, $\dfrac{3}{x+1} \to 0$ and is > 0 $\therefore$ $f(x) \to -1$ (above)

As $x \to -\infty$, $\dfrac{3}{x+1} \to 0$ and is < 0 $\therefore$ $f(x) \to -1$ (below)

$\therefore$ HA is $y = -1$ (see **Example 3** part **b**)

EXERCISE 16B

1 For each of the following, determine all asymptotes and discuss the behaviour of the graph near its asymptotes:

a $f(x) = \dfrac{3x-2}{x+3}$

b $y = \dfrac{x^2-1}{x^2+1}$

c $f(x) = \dfrac{x}{x^2+1}$

d $f(x) = \dfrac{x-2}{x^2+x-2}$

INVESTIGATION 1 LIMITS IN NUMBER SEQUENCES

The sequence 0.3, 0.33, 0.333, ... can be defined by the general term: $x_n = 0.333...3$ where there are n 3s.

What to do:

n	x_n	$3x_n$	$1 - 3x_n$
1			
2			
3			
4			
5			
10			

1 Copy and complete the table alongside:

2 Consider x_{100} which contains 100 3s. In the number $(1 - 3x_{100})$, how many 0s do we need between the decimal point and the 1?

3 In the limit as n tends to infinity, x_n contains an infinite number of 3s. In the number $(1 - 3x_n)$, how many 0s do we need to write before the 1?

4 Using your answer to **3**, state the limit of $1 - 3x_n$ as $n \to \infty$.

5 Hence state $\lim\limits_{n \to \infty} x_n$, which is the exact value of $0.\overline{3}$.

C RATES OF CHANGE

A **rate** is a comparison between two quantities with different units.

We often judge performances by using rates. For example:

- Sir Donald Bradman's batting rate at Test cricket level was 99.94 *runs per innings.*
- Michael Jordan's basketball scoring rate was 20.0 *points per game.*
- Rangi's typing rate is 63 *words per minute* with an error rate of 2.3 *errors per page.*

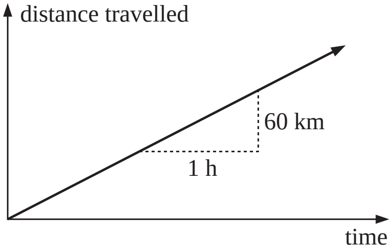
Speed is a commonly used rate. It is the rate of change in distance per unit of time. We are familiar with the formula

$$\textbf{average speed} = \frac{\textbf{distance travelled}}{\textbf{time taken}}.$$

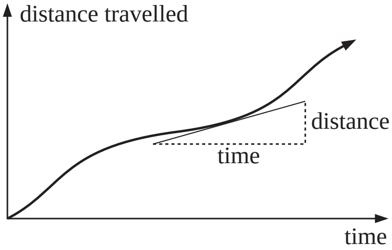
However, if a car has an average speed of 60 km h^{-1} for a journey, it does not mean that the car travels at exactly 60 km h^{-1} for the whole time.

In fact, the speed will probably vary continuously throughout the journey. So, how can we calculate the car's speed at any particular time?

Suppose we are given a graph of the car's distance travelled against time taken. If this graph is a straight line, then we know the speed is constant and is given by the gradient of the line.

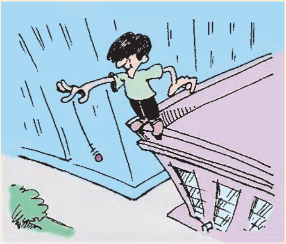


If the graph is a curve, then the car's instantaneous speed is given by the gradient of the tangent to the curve at that time.

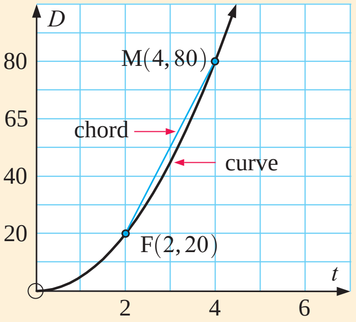


INVESTIGATION 2 INSTANTANEOUS SPEED

When a ball bearing is dropped from the top of a tall building, the distance it has fallen after t seconds is recorded, and the following graph of distance against time obtained.



In this investigation we will try to measure the speed of the ball at the instant when $t = 2$ seconds.



The *average* speed in the time interval $2 \leqslant t \leqslant 4$ is

$$= \frac{\text{distance travelled}}{\text{time taken}}$$

$$= \frac{(80 - 20)\text{ m}}{(4 - 2)\text{ s}}$$

$$= \tfrac{60}{2}\text{ m s}^{-1}$$

$$= 30\text{ m s}^{-1}$$

DEMO

What to do:

1. Click on the icon to start the demonstration. F is the point where $t = 2$ seconds, and M is another point on the curve. To start with M is at $t = 4$ seconds.
   The number in the box marked *gradient* is the gradient of the chord [FM]. This the *average speed* of the ball bearing in the interval from F to M. For M at $t = 4$ seconds, you should see the average speed is 30 m s^{-1}.
2. Click on M and drag it slowly towards F. Copy and complete the table alongside with the gradient of the chord when M is at various times t.

t	gradient of [FM]
3	
2.5	
2.1	
2.01	

3. Observe what happens as M reaches F. Explain why this is so.
4. When $t = 2$ seconds, what do you suspect the instantaneous speed of the ball bearing is?
5. Move M to the origin, and then slide it towards F from the left. Copy and complete the table alongside with the gradient of the chord when M is at various times t.

t	gradient of [FM]
0	
1.5	
1.9	
1.99	

6. Do your results agree with those in **4**?

From the investigation you should have discovered that:

> The **instantaneous rate of change** of a variable at a particular instant is given by the **gradient of the tangent** to the graph at that point.

THE TANGENT TO A CURVE

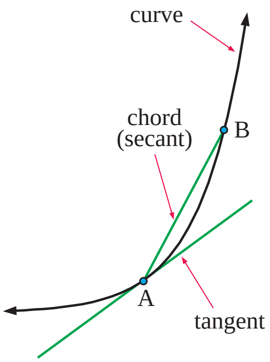
A **chord** or **secant** of a curve is a straight line segment which joins any two points on the curve.

The gradient of the chord [AB] measures the average rate of change of the function for the given change in x-values.

A **tangent** is a straight line which *touches* a curve at a point.

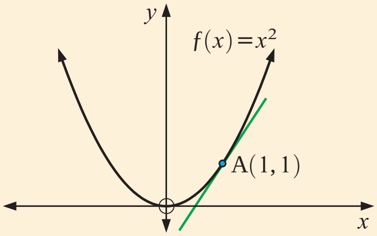
The gradient of the tangent at point A measures the instantaneous rate of change of the function at point A.

In the limit as B approaches A, the gradient of the chord [AB] will be the gradient of the tangent at A.



The **gradient of the tangent** at $x = a$ is defined as the **gradient of the curve** at the point where $x = a$, and is the instantaneous rate of change in $f(x)$ with respect to x at that point.

INVESTIGATION 3 THE GRADIENT OF A TANGENT



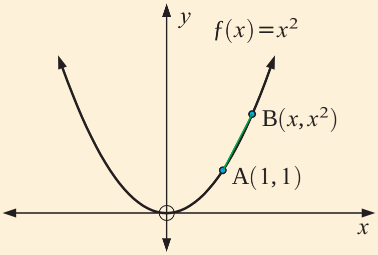
Given a curve $f(x)$, we wish to find the gradient of the tangent at the point $(a, f(a))$.

For example, the point A(1, 1) lies on the curve $f(x) = x^2$. What is the gradient of the tangent at A?

DEMO

What to do:

1 Suppose B lies on $f(x) = x^2$ and B has coordinates (x, x^2).



a Show that the chord [AB] has gradient

$$\frac{f(x)-f(1)}{x-1} \quad \text{or} \quad \frac{x^2-1}{x-1}.$$

b Copy and complete the table alongside:

x	Point B	gradient of [AB]
5	(5, 25)	6
3		
2		
1.5		
1.1		
1.01		
1.001		

2 Comment on the gradient of [AB] as x gets closer to 1.

3 Repeat the process as x gets closer to 1, but from the left of A.

4 Click on the icon to view a demonstration of the process.

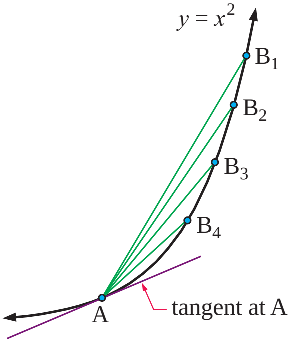
5 What do you suspect is the gradient of the tangent at A?

Fortunately we do not have to use a graph and table of values each time we wish to find the gradient of a tangent. Instead we can use an algebraic and geometric approach which involves **limits**.

LIMIT ARGUMENT

From the investigation, the gradient of [AB] $= \dfrac{x^2-1}{x-1}$

$$\therefore \quad \text{gradient of [AB]} = \frac{(x+1)(x-1)}{x-1} = x+1 \text{ provided that } x \neq 1$$

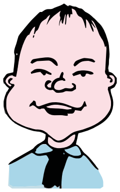


In the limit as B approaches A, $x \to 1$ and the gradient of [AB] $\to$ the gradient of the tangent at A.

So, the gradient of the tangent at the point A is

$$\begin{aligned} m_T &= \lim_{x\to 1} \frac{x^2-1}{x-1} \\ &= \lim_{x\to 1} x+1, \quad x \neq 1 \\ &= 2 \end{aligned}$$

As B approaches A, the gradient of [AB] approaches or **converges** to 2.



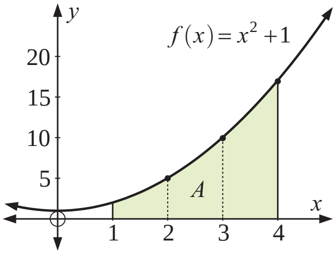
Limit arguments like that above form the foundation of differential calculus.

D CALCULATION OF AREAS UNDER CURVES

Consider the function $f(x) = x^2 + 1$.

We wish to estimate the area A enclosed by $y = f(x)$, the x-axis, and the vertical lines $x = 1$ and $x = 4$.

Suppose we divide the x-interval into three strips of width 1 unit as shown.



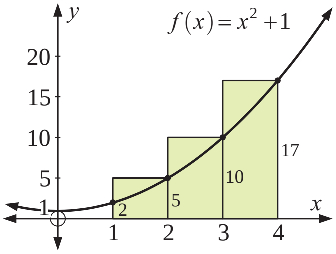
The diagram alongside shows **upper rectangles**, which are rectangles with top edges at the maximum value of the curve on that interval.

The area of the upper rectangles,

$A_U = 1 \times f(2) + 1 \times f(3) + 1 \times f(4)$

$= 5 + 10 + 17$

$= 32$ units^2



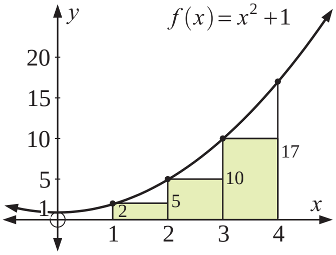
The next diagram shows **lower rectangles**, which are rectangles with top edges at the minimum value of the curve on that interval.

The area of the lower rectangles,

$A_L = 1 \times f(1) + 1 \times f(2) + 1 \times f(3)$

$= 2 + 5 + 10$

$= 17$ units^2



Now clearly $A_L < A < A_U$, so the required area lies between 17 units^2 and 32 units^2.

If the interval $1 \leqslant x \leqslant 4$ was divided into 6 equal intervals, each of length $\frac{1}{2}$, then

$$\begin{aligned} A_U &= \tfrac{1}{2}f(1\tfrac{1}{2}) + \tfrac{1}{2}f(2) + \tfrac{1}{2}f(2\tfrac{1}{2}) + \tfrac{1}{2}f(3) + \tfrac{1}{2}f(3\tfrac{1}{2}) + \tfrac{1}{2}f(4) \\ &= \tfrac{1}{2}(\tfrac{13}{4} + 5 + \tfrac{29}{4} + 10 + \tfrac{53}{4} + 17) \\ &= 27.875 \text{ units}^2 \\ \text{and} \quad A_L &= \tfrac{1}{2}f(1) + \tfrac{1}{2}f(1\tfrac{1}{2}) + \tfrac{1}{2}f(2) + \tfrac{1}{2}f(2\tfrac{1}{2}) + \tfrac{1}{2}f(3) + \tfrac{1}{2}f(3\tfrac{1}{2}) \\ &= \tfrac{1}{2}(2 + \tfrac{13}{4} + 5 + \tfrac{29}{4} + 10 + \tfrac{53}{4}) \\ &= 20.375 \text{ units}^2 \end{aligned}$$

From this refinement we conclude that the required area lies between 20.375 and 27.875 units^2.

As we create more subdivisions, the estimates A_L and A_U will become more and more accurate. In fact, as the subdivision width is reduced further and further, both A_L and A_U will **converge** to A.

EXERCISE 16D.1

1 Consider the area between $y = x$ and the x-axis from $x = 0$ to $x = 1$.

- **a** Divide the interval into 5 strips of equal width, then estimate the area using:
  - **i** upper rectangles
  - **ii** lower rectangles.
- **b** Calculate the actual area and compare it with your answers in **a**.

2 Consider the area between $y = \dfrac{1}{x}$ and the x-axis from $x = 2$ to $x = 4$. Divide the interval into 6 strips of equal width, then estimate the area using:

- **a** upper rectangles
- **b** lower rectangles.

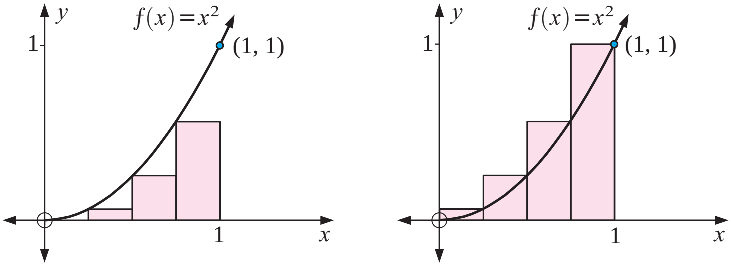
USING TECHNOLOGY

By subdividing the horizontal axis into small enough intervals, we can in theory find estimates for the area under a curve which are as close as we like to the actual value.

We illustrate this process by estimating the area A between the graph of $y = x^2$ and the x-axis for $0 \leqslant x \leqslant 1$.

This example is of historical interest. **Archimedes** (287 - 212 BC) found the exact area. In an article that contains 24 propositions he developed the essential theory of what is now known as integral calculus.

Consider $f(x) = x^2$ and divide the interval $0 \leqslant x \leqslant 1$ into 4 equal subdivisions.



$A_L = \frac{1}{4}(0)^2 + \frac{1}{4}(\frac{1}{4})^2 + \frac{1}{4}(\frac{1}{2})^2 + \frac{1}{4}(\frac{3}{4})^2$ ≈ 0.219 and $A_U = \frac{1}{4}(\frac{1}{4})^2 + \frac{1}{4}(\frac{1}{2})^2 + \frac{1}{4}(\frac{3}{4})^2 + \frac{1}{4}(1)^2$ ≈ 0.469

Now suppose there are n subdivisions, each of width $\frac{1}{n}$.

AREA FINDER

We can use technology to help calculate A_L and A_U for large values of n.

Click on the appropriate icon to access our **area finder** software or instructions for the procedure on a **graphics calculator**.

TI-*n*spire
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The table alongside summarises the results you should obtain for $n = 4$, 10, 25 and 50.

The exact value of A is in fact $\frac{1}{3}$. Notice how both A_L and A_U are converging to this value as n increases.

n	A_L	A_U	*Average*
4	0.218 75	0.468 75	0.343 75
10	0.285 00	0.385 00	0.335 00
25	0.313 60	0.353 60	0.333 60
50	0.323 40	0.343 40	0.333 40

EXERCISE 16D.2

1 Use rectangles to find lower and upper sums for the area between the graph of $y = x^2$ and the x-axis for $1 \leqslant x \leqslant 2$. Use $n = 10$, 25, 50, 100 and 500.
Give your answers to 4 decimal places.
As n gets larger, both A_L and A_U converge to the same number which is a simple fraction. What is it?

2 **a** Use rectangles to find lower and upper sums for the areas between the graphs of each of the following functions and the x-axis for $0 \leqslant x \leqslant 1$.
Use values of $n = 5$, 10, 50, 100, 500, 1000 and 10 000.
Give your answer to 5 decimal places in each case.

i $y = x^3$ **ii** $y = x$ **iii** $y = x^{\frac{1}{2}}$ **iv** $y = x^{\frac{1}{3}}$

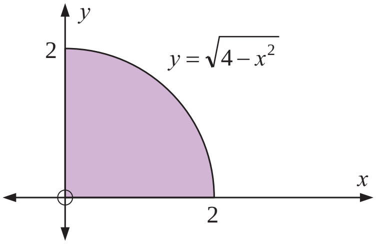
b For each case in **a**, A_L and A_U converge to the same number which is a simple fraction. What fractions are they?

c On the basis of your answer to **b**, conjecture what the area between the graph of $y = x^a$ and the x-axis for $0 \leqslant x \leqslant 1$ might be for any number $a > 0$.

3 Consider the quarter circle of centre $(0, 0)$ and radius 2 units illustrated.

Its area is $\frac{1}{4}$(full circle of radius 2)

$$= \frac{1}{4} \times \pi \times 2^2$$
$$= \pi \text{ units}^2$$



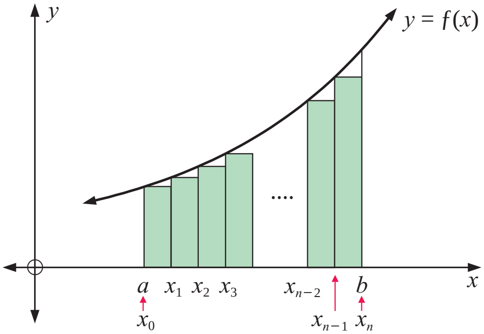
a Estimate the area using lower and upper rectangles for $n = 10$, 50, 100, 200, 1000 and 10 000. Hence, find rational bounds for π.

b Archimedes found the famous approximation $3\frac{10}{71} < \pi < 3\frac{1}{7}$.
For what value of n is your estimate for π better than that of Archimedes?

THE DEFINITE INTEGRAL

Consider the lower and upper rectangle sums for a function which is positive and increasing on the interval $a \leqslant x \leqslant b$.

We divide the interval into n subdivisions of width $w = \dfrac{b-a}{n}$.



Since the function is increasing,

$$A_L = w\,f(x_0) + w\,f(x_1) + w\,f(x_2) + \ldots. + w\,f(x_{n-2}) + w\,f(x_{n-1})$$

$$= w\sum_{i=0}^{n-1} f(x_i)$$

and $A_U = w\,f(x_1) + w\,f(x_2) + w\,f(x_3) + \ldots. + w\,f(x_{n-1}) + w\,f(x_n)$

$$= w\sum_{i=1}^{n} f(x_i)$$

Notice that $A_U - A_L = w\,(f(x_n) - f(x_0))$

$$= \frac{1}{n}(b-a)\,(f(b) - f(a))$$

$$\therefore \quad \lim_{n\to\infty}(A_U - A_L) = 0 \qquad \{\text{since } \lim_{n\to\infty}\frac{1}{n} = 0\}$$

$$\therefore \quad \lim_{n\to\infty} A_L = \lim_{n\to\infty} A_U$$

$\therefore$ since $A_L < A < A_U$, $\quad \lim_{n\to\infty} A_L = A = \lim_{n\to\infty} A_U$

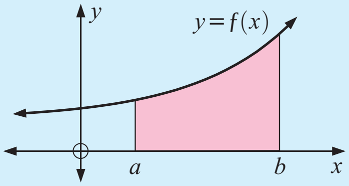
This fact is true for all positive continuous functions on the interval $a \leqslant x \leqslant b$.

The value A is known as the "**definite integral** of $f(x)$ from a to b", written

$$A = \int_a^b f(x)\,dx.$$

If $f(x) \geqslant 0$ for all $a \leqslant x \leqslant b$, then

$\displaystyle\int_a^b f(x)\,dx$ is the shaded area.



HISTORICAL NOTE

The word **integration** means "*to put together into a whole*". An **integral** is the "whole" produced from integration, since the areas $f(x_i) \times w$ of the thin rectangular strips are put together into one whole area.

The symbol $\int$ is called an **integral sign**. In the time of **Newton** and **Leibniz** it was the stretched out letter s, but it is no longer part of the alphabet.

Example 5

Self Tutor

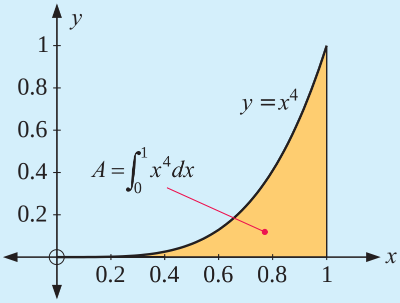
a Sketch the graph of $y = x^4$ for $0 \leqslant x \leqslant 1$. Shade the area described by $\int_0^1 x^4\,dx$.

AREA FINDER

b Use technology to calculate the lower and upper rectangle sums for n equal subdivisions where $n = 5$, 10, 50, 100 and 500.

c Use the information in **b** to find $\int_0^1 x^4\,dx$ to 2 significant figures.

a



b

n	A_L	A_U
5	0.1133	0.3133
10	0.1533	0.2533
50	0.1901	0.2101
100	0.1950	0.2050
500	0.1990	0.2010

c When $n = 500$, $A_L \approx A_U \approx 0.20$, to 2 significant figures.

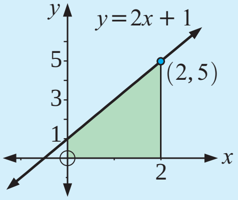
$\therefore$ since $A_L < \int_0^1 x^4\,dx < A_U$, $\int_0^1 x^4\,dx \approx 0.20$

Example 6

Self Tutor

Use graphical evidence and known area facts to find: **a** $\int_0^2 (2x+1)\,dx$ **b** $\int_0^1 \sqrt{1-x^2}\,dx$

a



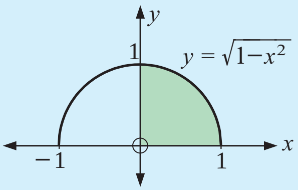
$$\int_0^2 (2x+1)\,dx$$

$= \text{shaded area}$

$= \left(\frac{1+5}{2}\right) \times 2$

$= 6$

b If $y = \sqrt{1-x^2}$ then $y^2 = 1 - x^2$ and so $x^2 + y^2 = 1$ which is the equation of the unit circle. $y = \sqrt{1-x^2}$ is the upper half.



$$\int_0^1 \sqrt{1-x^2}\,dx$$
$$= \text{shaded area}$$
$$= \tfrac{1}{4}(\pi r^2) \quad \text{where} \quad r = 1$$
$$= \tfrac{\pi}{4}$$

EXERCISE 16D.3

1 **a** Sketch the graph of $y = \sqrt{x}$ for $0 \leqslant x \leqslant 1$.

Shade the area described by $\displaystyle\int_0^1 \sqrt{x}\,dx$.

b Find the lower and upper rectangle sums for $n = 5$, 10, 50, 100 and 500.

c Use the information in **b** to find $\displaystyle\int_0^1 \sqrt{x}\,dx$ to 2 significant figures.

2 **a** Sketch the graph of $y = \sqrt{1+x^3}$ and the x-axis for $0 \leqslant x \leqslant 2$.

b Find the lower and upper rectangle sums for $n = 50$, 100, 500.

c What is your best estimate for $\displaystyle\int_0^2 \sqrt{1+x^3}\,dx$?

3 Use graphical evidence and known area facts to find:

a $\displaystyle\int_1^3 (1+4x)\,dx$ **b** $\displaystyle\int_{-1}^2 (2-x)\,dx$ **c** $\displaystyle\int_{-2}^2 \sqrt{4-x^2}\,dx$

INVESTIGATION 4 ESTIMATING $\int_{-3}^{3} e^{-\frac{x^2}{2}}\,dx$

The integral $\displaystyle\int_{-3}^{3} e^{-\frac{x^2}{2}}\,dx$ is of considerable interest to statisticians.

In this investigation we shall estimate the value of this integral using upper and lower rectangular sums for $n = 4500$.

This value of n is too large for most calculators to handle in a single list, so we will perform it in sections.

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TI-84
Casio

What to do:

1 Sketch the graph of $y = e^{-\frac{x^2}{2}}$ for $-3 \leqslant x \leqslant 3$.

2 Calculate the upper and lower rectangular sums for the three intervals $0 \leqslant x \leqslant 1$, $1 \leqslant x \leqslant 2$ and $2 \leqslant x \leqslant 3$ using $n = 750$ for each.

3 Combine the upper rectangular sums and the lower rectangular sums you found in **2** to obtain an upper and lower rectangular sum for $0 \leqslant x \leqslant 3$ for $n = 2250$.

4 Use the fact that the function $y = e^{-\frac{x^2}{2}}$ is symmetric to find upper and lower rectangular sums for $-3 \leqslant x \leqslant 0$ for $n = 2250$.

AREA FINDER

5 Use your results of **3** and **4** to find an estimate for $\int_{-3}^{3} e^{-\frac{x^2}{2}}\,dx$.
How accurate is your estimate?

6 Compare your estimate in **5** with $\sqrt{2\pi}$.

REVIEW SET 16

1 Evaluate the limits:

a $\lim\limits_{x\to 2} \dfrac{x^2-4}{2-x}$

b $\lim\limits_{x\to 4} \dfrac{\sqrt{x}-2}{x-4}$

c $\lim\limits_{x\to 4} \dfrac{x^2-16}{x-4}$

d $\lim\limits_{x\to \infty} \dfrac{1-2x-x^2}{2x^2-4}$

2 Find any asymptotes of the following functions and discuss the behaviour of the graph near them:

a $y = e^{x-2} - 3$

b $y = \ln(x^2+3)$

c $f(x) = e^{-x}\ln x$

d $y = x + \ln(2x-3)$

3 **a** Sketch the region between the curve $y = \dfrac{4}{1+x^2}$ and the x-axis for $0 \leqslant x \leqslant 1$. Divide the interval into 5 equal parts and display the 5 upper and lower rectangles.

b Find the lower and upper rectangle sums for $n = 5$, 50, 100 and 500.

c Give your best estimate for $\int_0^1 \dfrac{4}{1+x^2}\,dx$ and compare this answer with π.

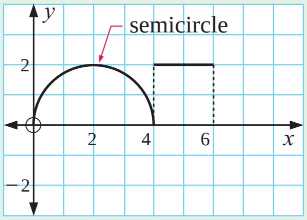
4 Consider the graph of $f(x) = e^{-x}$.

a Sketch the graph of $y = f(x)$ for $0 \leqslant x \leqslant 2$.

b Find upper and lower bounds for $\int_0^2 e^{-x}\,dx$ using upper and lower rectangles with 8 subdivisions.

c Use technology to find, correct to 4 significant figures, upper and lower bounds for $\int_0^2 e^{-x}\,dx$ when $n = 100$.

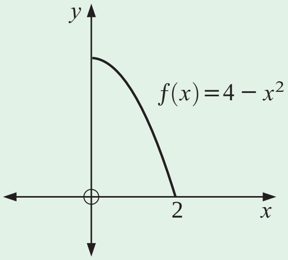
5 The graph of $y = f(x)$ is illustrated:
Evaluate the following using area interpretation:



a $\int_0^4 f(x)\,dx$

b $\int_4^6 f(x)\,dx$

6



a Use *four* upper and lower rectangles to find rationals A and B such that:

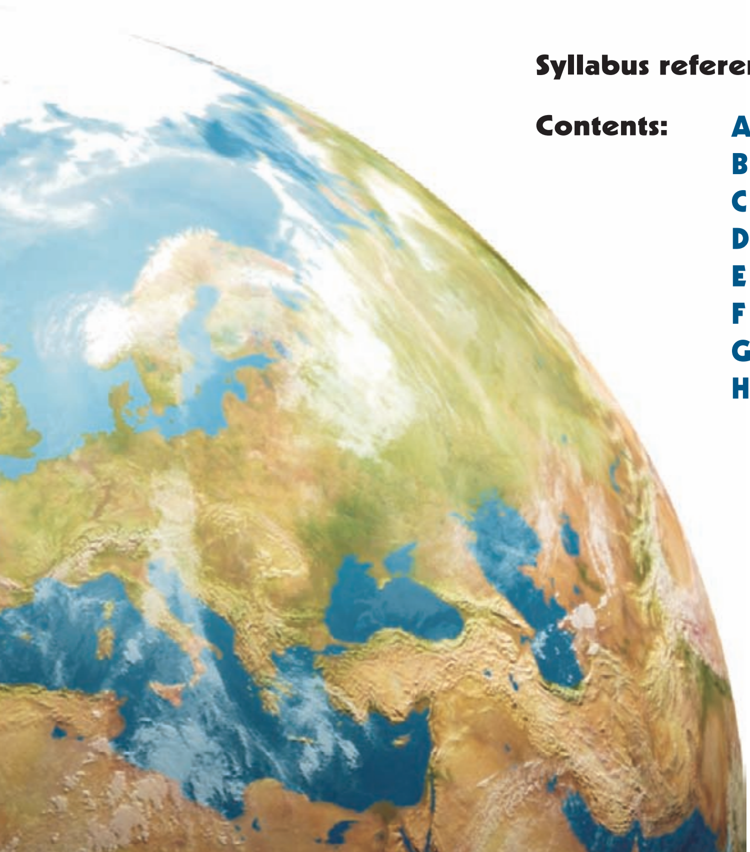
$$A < \int_0^2 (4 - x^2)\,dx < B.$$

b Hence, find a good estimate for

$$\int_0^2 (4 - x^2)\,dx.$$

Chapter 17

Differential calculus



Syllabus reference: 7.1, 7.2, 7.7

Contents:

In the previous chapter we discussed how the topic of calculus is divided into two fields: differential calculus and integral calculus. In this chapter we begin to examine **differential calculus** and how it relates to rate problems and the gradient of curves.

HISTORICAL NOTE

The topic of **differential calculus** originated in the 17th century with the work of **Sir Isaac Newton** and **Gottfried Wilhelm Leibniz**. These mathematicians developed the necessary theory while attempting to find algebraic methods for solving problems dealing with:

- the **gradients of tangents** to curves at any point on the curve, and
- finding the **rate of change** in one variable with respect to another.

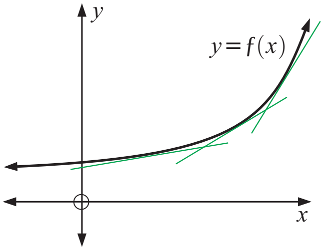
Isaac Newton 1642 – 1727

Gottfried Leibniz 1646 – 1716

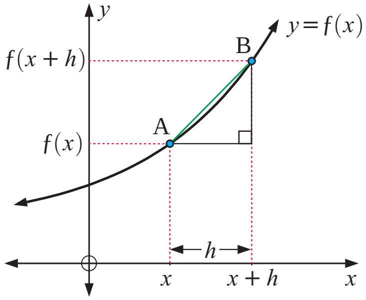
A THE DERIVATIVE FUNCTION

For a non-linear function with equation $y = f(x)$, the gradients of the tangents at various points are different.

Our task is to determine a **gradient function** so that when we replace x by some value a then we will be able to find the gradient of the tangent to $y = f(x)$ at $x = a$.



Consider a general function $y = f(x)$ where A is $(x, f(x))$ and B is $(x + h, f(x + h))$.



The chord [AB] has gradient
$$= \frac{f(x+h) - f(x)}{x + h - x}$$
$$= \frac{f(x+h) - f(x)}{h}.$$

If we now let B approach A, then the gradient of [AB] approaches the gradient of the tangent at A.

So, the gradient of the tangent at the variable point $(x, f(x))$ is the limiting value of

$\dfrac{f(x+h) - f(x)}{h}$ as h approaches 0, or $\displaystyle\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$.

This formula gives the gradient of the tangent for any value of the variable x. Since there is only one value of the gradient for each value of x, the formula is actually a function.

The **gradient function**, also known as the **derived function** or **derivative function** or simply the **derivative** is defined as

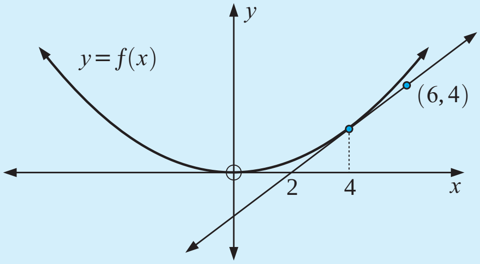
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

We read the derivative function as "eff dashed x".

Example 1 **Self Tutor**

Consider the graph alongside.

Find $f(4)$ and $f'(4)$.



The graph shows the tangent to the curve $y = f(x)$ at the point where $x = 4$.

The gradient of this tangent is $f'(4)$.

The tangent passes through (2, 0) and (6, 4), so $f'(4) = \frac{4-0}{6-2} = 1$.

The equation of the tangent is $\frac{y-0}{x-2} = 1$

$\therefore \quad y = x - 2$

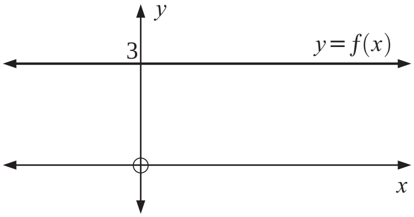
When $x = 4$, $y = 2$, so the point of contact is (4, 2)

$\therefore \quad f(4) = 2$ So, $f(4) = 2$ and $f'(4) = 1$.

EXERCISE 17A.1

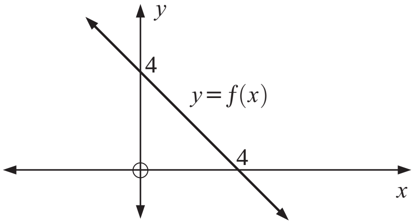
1 Using the graph alongside, find:

a $f(2)$ **b** $f'(2)$

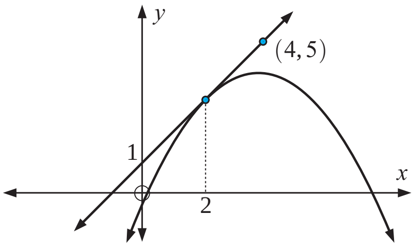


2 Using the graph alongside, find:

a $f(0)$ **b** $f'(0)$



3 Consider the graph alongside.
Find $f(2)$ and $f'(2)$.



INVESTIGATION 1 FINDING GRADIENTS OF FUNCTIONS

The software on the CD can be used to find the gradient of the function $f(x)$ at any point. By sliding the point along the graph we can observe the changing gradient of the tangent. We can hence generate the gradient function $f'(x)$.

GRADIENT FUNCTIONS

What to do:

1 Consider the functions $f(x) = 0$, $f(x) = 2$ and $f(x) = 4$.

  a For each of these functions, what is the gradient?

  b Is the gradient constant for all values of x?

2 Consider the function $f(x) = mx + c$.

  a State the gradient of the function.

  b Is the gradient of the tangent constant for all values of x?

  c Use the CD software to graph the following functions and observe the gradient function $f'(x)$. Hence verify that your answer in **b** is correct.

    i $f(x) = x - 1$ **ii** $f(x) = 3x + 2$ **iii** $f(x) = -2x + 1$

3 **a** Observe the function $f(x) = x^2$ using the CD software.

  b What *type* of function is the gradient function $f'(x)$?

4 **a** Observe the following quadratic functions using the CD software:

    i $f(x) = x^2 + x - 2$ **ii** $f(x) = 2x^2 - 3$
    iii $f(x) = -x^2 + 2x - 1$ **iv** $f(x) = -3x^2 - 3x + 6$

  b What types of functions are the gradient functions $f'(x)$ for the functions in **a**?

5 **a** Observe the function $f(x) = \ln x$ using the CD software.

  b What *type* of function is the gradient function $f'(x)$?

  c What is the *domain* of the gradient function $f'(x)$?

6 **a** Observe the function $f(x) = e^x$ using the CD software.

  b What is the gradient function $f'(x)$?

To find the gradient function $f'(x)$ for a general function $f(x)$, we need to evaluate the limit $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$. We call this the **method of first principles**.

Example 2

Self Tutor

Use the definition of $f'(x)$ to find the gradient function of $f(x) = x^2$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{\cancel{x^2} + 2hx + h^2 - \cancel{x^2}}{h} \\ &= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} \\ &= \lim_{h\to 0} (2x+h) \qquad \{\text{as } h \neq 0\} \\ &= 2x \end{aligned}$$

EXERCISE 17A.2

1 Find, from first principles, the gradient function of $f(x)$ where $f(x)$ is:

a x **b** 5 **c** x^3 **d** x^4

Reminder: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

2 Use your answer to **1** to predict a formula for $f'(x)$ where $f(x) = x^n$, $n \in \mathbb{N}$.

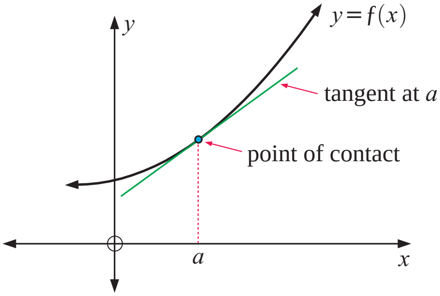
3 Find, from first principles, $f'(x)$ given that $f(x)$ is:

a $2x + 5$ **b** $-x + 4$ **c** $x^2 - 3x$

d $2x^2 + x - 1$ **e** $-x^2 + 5x - 3$ **f** $x^3 - 2x^2 + 3$

B DERIVATIVES AT A GIVEN x-VALUE

Suppose we are given a function $f(x)$ and asked to find its derivative at the point where $x = a$. This is actually the gradient of the tangent to the curve at $x = a$, which we write as $f'(a)$.



There are two methods for finding $f'(a)$ using first principles:

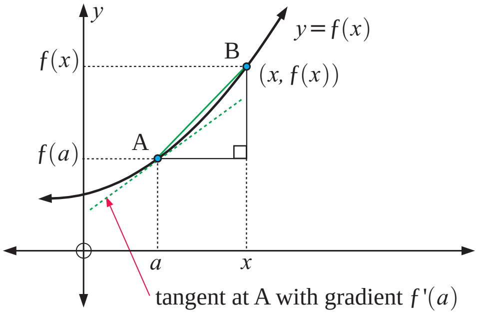
The first method is to start with the definition of the gradient function.

Since $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$,

we can simply substitute $x = a$ to give

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

The second method is to consider two points on the graph of $y = f(x)$: a fixed point $A(a, f(a))$ and a variable point $B(x, f(x))$.



The gradient of chord [AB] $= \dfrac{f(x) - f(a)}{x - a}$.

In the limit as B approaches A, $x \to a$ and the gradient of chord [AB] $\to$ the gradient of the tangent at A.

$$\therefore \quad f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Thus $\boldsymbol{f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}}$ is an alternative definition for the gradient of the tangent at $x = a$.

You can also find the gradient of the tangent at a given point on a function using your graphics calculator. Instructions for doing this can be found in the graphics calculator chapter at the start of the book.

Example 3 **Self Tutor**

Find, from first principles, the gradient of the tangent to $y = 2x^2 + 3$ at the point where $x = 2$.

Let $f(x) = 2x^2 + 3$

$\therefore \quad f(2) = 2(2)^2 + 3 = 11$

and $f'(2) = \lim\limits_{x \to 2} \dfrac{f(x) - f(2)}{x - 2}$

$$\begin{aligned}\therefore \quad f'(2) &= \lim_{x \to 2} \frac{2x^2 + 3 - 11}{x - 2} \\ &= \lim_{x \to 2} \frac{2x^2 - 8}{x - 2} \\ &= \lim_{x \to 2} \frac{2(x+2)\cancel{(x-2)}}{\cancel{x-2}\,{}_1} \quad \{\text{as } x \neq 2\} \\ &= 2 \times 4 \\ &= 8\end{aligned}$$

Example 4 **Self Tutor**

Use the first principles formula $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ to find the instantaneous rate of change in $f(x) = x^2 + 2x$ at the point where $x = 5$.

$f(5) = 5^2 + 2(5) = 35$

$$\begin{aligned}\text{So,} \quad f'(5) &= \lim_{h \to 0} \frac{f(5+h) - f(5)}{h} \\ &= \lim_{h \to 0} \frac{(5+h)^2 + 2(5+h) - 35}{h}\end{aligned}$$

$$= \lim_{h \to 0} \frac{\cancel{25} + 10h + h^2 + \cancel{10} + 2h - \cancel{35}}{h}$$

$$= \lim_{h \to 0} \frac{h^2 + 12h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(h+12)}{\cancel{h}_1} \qquad \{\text{as } h \neq 0\}$$

$$= 12$$

$\therefore$ the instantaneous rate of change in $f(x)$ at $x = 5$ is 12.

EXERCISE 17B

1 Use the formula $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ to find the gradient of the tangent to:

- **a** $f(x) = 3x + 5$ at $x = -2$
- **b** $f(x) = 5 - 2x^2$ at $x = 3$
- **c** $f(x) = x^2 + 3x - 4$ at $x = 3$
- **d** $f(x) = 5 - 2x - 3x^2$ at $x = -2$

2 Use the first principles formula $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ to find:

- **a** $f'(2)$ for $f(x) = x^3$
- **b** $f'(3)$ for $f(x) = x^4$.

C SIMPLE RULES OF DIFFERENTIATION

Differentiation is the process of finding a derivative or gradient function.

There are a number of rules associated with differentiation. These rules can be used to differentiate more complicated functions without having to resort to the tedious method of first principles.

INVESTIGATION 2 SIMPLE RULES OF DIFFERENTIATION

In this investigation we attempt to differentiate functions of the form x^n, cx^n where c is a constant, and functions which are a sum or difference of terms of the form cx^n.

What to do:

1 Differentiate from first principles: **a** x^2 **b** x^3 **c** x^4

2 Consider the binomial expansion:

$$(x+h)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}h + \binom{n}{2} x^{n-2}h^2 + ... + \binom{n}{n} h^n$$

$$= x^n + nx^{n-1}h + \binom{n}{2} x^{n-2}h^2 + ... + h^n$$

Use the first principles formula $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ to find the derivative of $f(x) = x^n$.

3 Find, from first principles, the derivatives of:

a x^2 **b** $4x^2$ **c** x^3 **d** $2x^3$

4 Use **3** to copy and complete: "If $f(x) = cx^n$, then $f'(x) = ...$"

5 Use first principles to find $f'(x)$ for:

a $f(x) = x^2 + 3x$ **b** $f(x) = x^3 - 2x^2$

6 Use **3** and **5** to copy and complete: "If $f(x) = u(x) + v(x)$ then $f'(x) = ...$"

$f(x)$	$f'(x)$	*Name of rule*
c (a constant)	0	**differentiating a constant**
x^n	nx^{n-1}	**differentiating x^n**
$c\,u(x)$	$c\,u'(x)$	**constant times a function**
$u(x)+v(x)$	$u'(x)+v'(x)$	**addition rule**

Each of these rules can be proved using the first limit definition of $f'(x)$.

For example:

- If $f(x) = c\,u(x)$ where c is a constant then $f'(x) = c\,u'(x)$.

Proof:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{c\,u(x+h)-c\,u(x)}{h} \\ &= \lim_{h\to 0} c\left[\frac{u(x+h)-u(x)}{h}\right] \\ &= c \lim_{h\to 0} \frac{u(x+h)-u(x)}{h} \\ &= c\,u'(x) \end{aligned}$$

- If $f(x) = u(x) + v(x)$ then $f'(x) = u'(x) + v'(x)$

Proof:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \left(\frac{u(x+h)+v(x+h)-[u(x)+v(x)]}{h}\right) \\ &= \lim_{h\to 0} \left(\frac{u(x+h)-u(x)+v(x+h)-v(x)}{h}\right) \\ &= \lim_{h\to 0} \frac{u(x+h)-u(x)}{h} + \lim_{h\to 0} \frac{v(x+h)-v(x)}{h} \\ &= u'(x)+v'(x) \end{aligned}$$

Using the rules we have now developed we can differentiate sums of powers of x.

For example, if $f(x) = 3x^4 + 2x^3 - 5x^2 + 7x + 6$ then

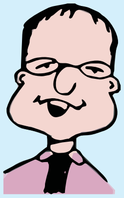
$$\begin{aligned} f'(x) &= 3(4x^3) + 2(3x^2) - 5(2x) + 7(1) + 0 \\ &= 12x^3 + 6x^2 - 10x + 7 \end{aligned}$$

Example 5 **Self Tutor**

Find $f'(x)$ for $f(x)$ equal to: **a** $5x^3 + 6x^2 - 3x + 2$ **b** $7x - \dfrac{4}{x} + \dfrac{3}{x^3}$

a
$$\begin{aligned} f(x) &= 5x^3 + 6x^2 - 3x + 2 \\ \therefore\ f'(x) &= 5(3x^2) + 6(2x) - 3(1) \\ &= 15x^2 + 12x - 3 \end{aligned}$$

b
$$\begin{aligned} f(x) &= 7x - \frac{4}{x} + \frac{3}{x^3} \\ &= 7x - 4x^{-1} + 3x^{-3} \\ \therefore\ f'(x) &= 7(1) - 4(-1x^{-2}) + 3(-3x^{-4}) \\ &= 7 + 4x^{-2} - 9x^{-4} \\ &= 7 + \frac{4}{x^2} - \frac{9}{x^4} \end{aligned}$$



Remember that $\dfrac{1}{x^n} = x^{-n}$.

Example 6 **Self Tutor**

Find the gradient function of $f(x) = x^2 - \dfrac{4}{x}$ and hence find the gradient of the tangent to the function at the point where $x = 2$.

$$\begin{aligned} f(x) &= x^2 - \frac{4}{x} \\ &= x^2 - 4x^{-1} \end{aligned}$$

$$\begin{aligned} \therefore\ f'(x) &= 2x - 4(-1x^{-2}) \\ &= 2x + 4x^{-2} \\ &= 2x + \frac{4}{x^2} \end{aligned}$$

Now $f'(2) = 4 + 1 = 5$.
So, the tangent has gradient $= 5$.

Example 7 **Self Tutor**

Find the gradient function for each of the following:

a $f(x) = 3\sqrt{x} + \dfrac{2}{x}$ **b** $g(x) = x^2 - \dfrac{4}{\sqrt{x}}$

a
$$\begin{aligned} f(x) &= 3\sqrt{x} + \frac{2}{x} = 3x^{\frac{1}{2}} + 2x^{-1} \\ \therefore\ f'(x) &= 3(\tfrac{1}{2}x^{-\frac{1}{2}}) + 2(-1x^{-2}) \\ &= \tfrac{3}{2}x^{-\frac{1}{2}} - 2x^{-2} \\ &= \frac{3}{2\sqrt{x}} - \frac{2}{x^2} \end{aligned}$$

b
$$\begin{aligned} g(x) &= x^2 - \frac{4}{\sqrt{x}} = x^2 - 4x^{-\frac{1}{2}} \\ \therefore\ g'(x) &= 2x - 4(-\tfrac{1}{2}x^{-\frac{3}{2}}) \\ &= 2x + 2x^{-\frac{3}{2}} \\ &= 2x + \frac{2}{x\sqrt{x}} \end{aligned}$$

ALTERNATIVE NOTATION

If we are given a function $f(x)$ then $f'(x)$ represents the derivative function.

If we are given y in terms of x then y' or $\frac{dy}{dx}$ are commonly used to represent the derivative.

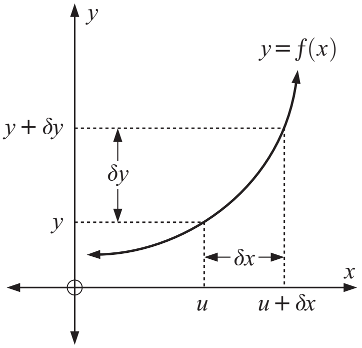
$\frac{dy}{dx}$ reads "dee y by dee x" or "the derivative of y with respect to x".

$\frac{dy}{dx}$ is **not a fraction**. However, the notation $\frac{dy}{dx}$ is a result of taking the limit of a fraction. If we replace h by δx and $f(x+h)-f(x)$ by δy, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \quad \text{becomes}$$

$$f'(x) = \lim_{\delta x\to 0} \frac{\delta y}{\delta x}$$

$$= \frac{dy}{dx}.$$



Example 8 **Self Tutor**

If $y = 3x^2 - 4x$, find $\frac{dy}{dx}$ and interpret its meaning.

As $y = 3x^2 - 4x$, $\frac{dy}{dx} = 6x - 4$.

$\frac{dy}{dx}$ is:
- the gradient function or derivative of $y = 3x^2 - 4x$ from which the gradient at any point can be found
- the instantaneous rate of change in y as x changes.

EXERCISE 17C

1 Find $f'(x)$ given that $f(x)$ is:

a x^3 **b** $2x^3$ **c** $7x^2$ **d** $6\sqrt{x}$

e $3\sqrt[3]{x}$ **f** $x^2 + x$ **g** $4 - 2x^2$ **h** $x^2 + 3x - 5$

i $\frac{1}{2}x^4 - 6x^2$ **j** $\frac{3x-6}{x}$ **k** $\frac{2x-3}{x^2}$ **l** $\frac{x^3+5}{x}$

m $\frac{x^3+x-3}{x}$ **n** $\frac{1}{\sqrt{x}}$ **o** $(2x-1)^2$ **p** $(x+2)^3$

2 Find $\frac{dy}{dx}$ for:

a $y = 2.5x^3 - 1.4x^2 - 1.3$ **b** $y = \pi x^2$ **c** $y = \frac{1}{5x^2}$

d $y = 100x$ **e** $y = 10(x+1)$ **f** $y = 4\pi x^3$

3 Differentiate with respect to x:

a $6x+2$ **b** $x\sqrt{x}$ **c** $(5-x)^2$ **d** $\dfrac{6x^2-9x^4}{3x}$

e $(x+1)(x-2)$ **f** $\dfrac{1}{x^2}+6\sqrt{x}$ **g** $4x-\dfrac{1}{4x}$ **h** $x(x+1)(2x-5)$

4 Find the gradient of the tangent to:

a $y=x^2$ at $x=2$

b $y=\dfrac{8}{x^2}$ at the point $(9,\ \frac{8}{81})$

c $y=2x^2-3x+7$ at $x=-1$

d $y=\dfrac{2x^2-5}{x}$ at the point $(2,\ \frac{3}{2})$

e $y=\dfrac{x^2-4}{x^2}$ at the point $(4,\ \frac{3}{4})$

f $y=\dfrac{x^3-4x-8}{x^2}$ at $x=-1$

Check your answers using technology.

5 Suppose $f(x)=x^2+(b+1)x+2c$, $f(2)=4$, and $f'(-1)=2$.

Find the constants b and c.

6 Find the gradient function of $f(x)$ where $f(x)$ is:

a $4\sqrt{x}+x$ **b** $\sqrt[3]{x}$ **c** $-\dfrac{2}{\sqrt{x}}$ **d** $2x-\sqrt{x}$

e $\dfrac{4}{\sqrt{x}}-5$ **f** $3x^2-x\sqrt{x}$ **g** $\dfrac{5}{x^2\sqrt{x}}$ **h** $2x-\dfrac{3}{x\sqrt{x}}$

7 **a** If $y=4x-\dfrac{3}{x}$, find $\dfrac{dy}{dx}$ and interpret its meaning.

b The position of a car moving along a straight road is given by $S=2t^2+4t$ metres where t is the time in seconds. Find $\dfrac{dS}{dt}$ and interpret its meaning.

c The cost of producing x toasters each week is given by $C=1785+3x+0.002x^2$ dollars. Find $\dfrac{dC}{dx}$ and interpret its meaning.

D THE CHAIN RULE

In **Chapter 1** we defined the **composite** of two functions g and f as $(g\circ f)(x)$ or $g(f(x))$.

We can often write complicated functions as the composite of two or more simpler functions.

For example $y=(x^2+3x)^4$ could be rewritten as $y=u^4$ where $u=x^2+3x$, or as $y=(g\circ f)(x)=g(f(x))$ where $g(x)=x^4$ and $f(x)=x^2+3x$.

Example 9 **Self Tutor**

Find: **a** $g(f(x))$ if $g(x)=\sqrt{x}$ and $f(x)=2-3x$

b $g(x)$ and $f(x)$ such that $g(f(x))=\dfrac{1}{x-x^2}$.

a $g(f(x))$
$= g(2-3x)$
$= \sqrt{2-3x}$

b $g(f(x)) = \dfrac{1}{x-x^2} = \dfrac{1}{f(x)}$

$\therefore \quad g(x) = \dfrac{1}{x}$ and $f(x) = x - x^2$

EXERCISE 17D.1

1 Find $g(f(x))$ if:

a $g(x) = x^2$ and $f(x) = 2x+7$
b $g(x) = 2x+7$ and $f(x) = x^2$
c $g(x) = \sqrt{x}$ and $f(x) = 3-4x$
d $g(x) = 3-4x$ and $f(x) = \sqrt{x}$
e $g(x) = \dfrac{2}{x}$ and $f(x) = x^2+3$
f $g(x) = x^2+3$ and $f(x) = \dfrac{2}{x}$

2 Find $g(x)$ and $f(x)$ such that $g(f(x))$ is:

a $(3x+10)^3$
b $\dfrac{1}{2x+4}$
c $\sqrt{x^2-3x}$
d $\dfrac{10}{(3x-x^2)^3}$

DERIVATIVES OF COMPOSITE FUNCTIONS

The reason we are interested in writing complicated functions as composite functions is to make finding derivatives easier.

INVESTIGATION 3 DIFFERENTIATING COMPOSITES

The purpose of this investigation is to gain insight into how we can differentiate composite functions.

Based on our previous rule "if $y = x^n$ then $\dfrac{dy}{dx} = nx^{n-1}$", we might suspect that if $y = (2x+1)^2$ then $\dfrac{dy}{dx} = 2(2x+1)^1 = 2(2x+1)$. But is this so?

What to do:

1 Consider $y = (2x+1)^2$. Expand the brackets and then find $\dfrac{dy}{dx}$. Is $\dfrac{dy}{dx} = 2(2x+1)$?

2 Consider $y = (3x+1)^2$. Expand the brackets and then find $\dfrac{dy}{dx}$. Is $\dfrac{dy}{dx} = 2(3x+1)^1$?

3 Consider $y = (ax+1)^2$. Expand the brackets and find $\dfrac{dy}{dx}$. Is $\dfrac{dy}{dx} = 2(ax+1)^1$?

4 If $y = u^2$ where u is a function of x, what do you suspect $\dfrac{dy}{dx}$ will be equal to?

5 Consider $y = (x^2+3x)^2$. Expand it and find $\dfrac{dy}{dx}$.
Does your answer agree with the rule you suggested in **4**?

In the previous investigation you probably found that if $y = u^2$ then

$$\frac{dy}{dx} = 2u \times \frac{du}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Now consider $y = (2x+1)^3$ which has the form $y = u^3$ where $u = 2x+1$.

$$\begin{aligned}\text{Expanding we have}\quad y &= (2x+1)^3\\ &= (2x)^3 + 3(2x)^2 1 + 3(2x)1^2 + 1^3 \quad \{\text{binomial expansion}\}\\ &= 8x^3 + 12x^2 + 6x + 1\\ \therefore\quad \frac{dy}{dx} &= 24x^2 + 24x + 6\\ &= 6(4x^2 + 4x + 1)\\ &= 6(2x+1)^2\\ &= 3(2x+1)^2 \times 2\\ &= 3u^2 \times \frac{du}{dx} \quad \text{which is again } \frac{dy}{du}\frac{du}{dx}.\end{aligned}$$

From the above examples we formulate the **chain rule**:

If $y = g(u)$ where $u = f(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$.

This rule is extremely important and enables us to differentiate complicated functions much faster.

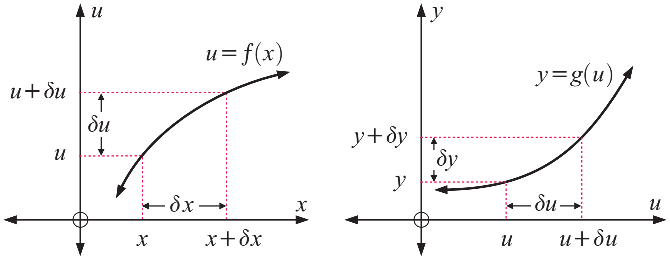
For example, we can readily see that for any function $f(x)$:

If $y = [f(x)]^n$ then $\dfrac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$.

A non-examinable proof of the chain rule is included for completeness.

Proof: Consider $y = g(u)$ where $u = f(x)$.

For a small change of δx in x, there is a small change of $f(x+\delta x) - f(x) = \delta u$ in u and a small change of δy in y.



Now $\dfrac{\delta y}{\delta x} = \dfrac{\delta y}{\delta u} \times \dfrac{\delta u}{\delta x}$ {fraction multiplication}

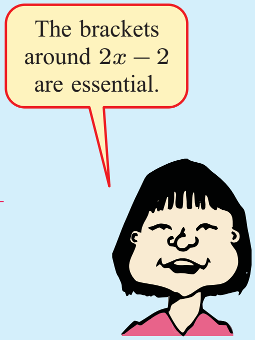
As $\delta x \to 0$, $\delta u \to 0$ also. $\therefore \displaystyle\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta u \to 0} \frac{\delta y}{\delta u} \times \lim_{\delta x \to 0} \frac{\delta u}{\delta x}$ {limit rule}

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Example 10 **Self Tutor**

Find $\dfrac{dy}{dx}$ if: **a** $y=(x^2-2x)^4$ **b** $y=\dfrac{4}{\sqrt{1-2x}}$

a
$$\begin{aligned}
& y=(x^2-2x)^4\\
\therefore\quad & y=u^4 \quad \text{where} \quad u=x^2-2x\\
\text{Now}\quad & \frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\}\\
& =4u^3(2x-2)\\
& =4(x^2-2x)^3(2x-2)
\end{aligned}$$



b
$$\begin{aligned}
& y=\frac{4}{\sqrt{1-2x}}\\
\therefore\quad & y=4u^{-\frac{1}{2}} \quad \text{where} \quad u=1-2x\\
\text{Now}\quad & \frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\}\\
& =4\times(-\tfrac{1}{2}u^{-\frac{3}{2}})\times(-2)\\
& =4u^{-\frac{3}{2}}\\
& =4(1-2x)^{-\frac{3}{2}}
\end{aligned}$$

EXERCISE 17D.2

1 Write in the form au^n, clearly stating what u is:

a $\dfrac{1}{(2x-1)^2}$
b $\sqrt{x^2-3x}$
c $\dfrac{2}{\sqrt{2-x^2}}$
d $\sqrt[3]{x^3-x^2}$
e $\dfrac{4}{(3-x)^3}$
f $\dfrac{10}{x^2-3}$

2 Find the gradient function $\dfrac{dy}{dx}$ for:

a $y=(4x-5)^2$
b $y=\dfrac{1}{5-2x}$
c $y=\sqrt{3x-x^2}$
d $y=(1-3x)^4$
e $y=6(5-x)^3$
f $y=\sqrt[3]{2x^3-x^2}$
g $y=\dfrac{6}{(5x-4)^2}$
h $y=\dfrac{4}{3x-x^2}$
i $y=2\left(x^2-\dfrac{2}{x}\right)^3$

3 Find the gradient of the tangent to:

a $y=\sqrt{1-x^2}$ at $x=\frac{1}{2}$
b $y=(3x+2)^6$ at $x=-1$
c $y=\dfrac{1}{(2x-1)^4}$ at $x=1$
d $y=6\times\sqrt[3]{1-2x}$ at $x=0$
e $y=\dfrac{4}{x+2\sqrt{x}}$ at $x=4$
f $y=\left(x+\dfrac{1}{x}\right)^3$ at $x=1$

Check your answers using technology.

4 If $y = x^3$ then $x = y^{\frac{1}{3}}$.

a Find $\dfrac{dy}{dx}$ and $\dfrac{dx}{dy}$ and hence show that $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 1$.

b Explain why $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 1$ whenever these derivatives exist for any general function $y = f(x)$.

E THE PRODUCT RULE

If $f(x) = u(x) + v(x)$ then $f'(x) = u'(x) + v'(x)$.

So, the derivative of a sum of two functions is the sum of the derivatives.

We now consider the case $f(x) = u(x)v(x)$. Is $f'(x) = u'(x)v'(x)$?

In other words, does the derivative of a product of two functions equal the product of the derivatives of the two functions?

The following example shows that this cannot be true:

If $f(x) = x\sqrt{x}$ we could say $f(x) = u(x)v(x)$ where $u(x) = x$ and $v(x) = \sqrt{x}$.

Now $f(x) = x^{\frac{3}{2}}$ so $f'(x) = \frac{3}{2}x^{\frac{1}{2}}$. But $u'(x)v'(x) = 1 \times \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}} \neq f'(x)$.

THE PRODUCT RULE

If $u(x)$ and $v(x)$ are two functions of x and $y = uv$ then

$$\boldsymbol{\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}} \quad \text{or} \quad \boldsymbol{y' = u'(x)v(x) + u(x)v'(x)}.$$

Consider again the example $f(x) = x\sqrt{x}$.

This is a product $u(x)v(x)$ where $u(x) = x$ and $v(x) = x^{\frac{1}{2}}$

$\therefore$ $u'(x) = 1$ and $v'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

According to the product rule

$$\begin{aligned} f'(x) = u'v + uv' &= 1 \times x^{\frac{1}{2}} + x \times \tfrac{1}{2}x^{-\frac{1}{2}} \\ &= x^{\frac{1}{2}} + \tfrac{1}{2}x^{\frac{1}{2}} \\ &= \tfrac{3}{2}x^{\frac{1}{2}} \quad \text{which is correct } \checkmark \end{aligned}$$

For completeness we prove the product rule:

Proof: Let $y = u(x)v(x)$. Suppose there is a small change of δx in x which causes corresponding changes of δu in u, δv in v, and δy in y.

$$\begin{aligned} \text{As } y = uv, \quad y + \delta y &= (u + \delta u)(v + \delta v) \\ \therefore \quad y + \delta y &= uv + (\delta u)v + u(\delta v) + \delta u \delta v \\ \therefore \quad \delta y &= (\delta u)v + u(\delta v) + \delta u \delta v \end{aligned}$$

$$\therefore \quad \frac{\delta y}{\delta x} = \left(\frac{\delta u}{\delta x}\right) v + u\left(\frac{\delta v}{\delta x}\right) + \left(\frac{\delta u}{\delta x}\right)\delta v \qquad \{\text{dividing each term by } \delta x\}$$

$$\therefore \quad \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \left(\lim_{\delta x \to 0}\frac{\delta u}{\delta x}\right) v + u\left(\lim_{\delta x \to 0}\frac{\delta v}{\delta x}\right) + 0 \qquad \{\text{as } \delta x \to 0, \ \delta v \to 0 \ \text{ also}\}$$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$$

Example 11 **Self Tutor**

Find $\frac{dy}{dx}$ if: **a** $y = \sqrt{x}(2x+1)^3$ **b** $y = x^2(x^2-2x)^4$

a $y = \sqrt{x}(2x+1)^3$ is the product of $u = x^{\frac{1}{2}}$ and $v = (2x+1)^3$

$\therefore \quad u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 3(2x+1)^2 \times 2$ {chain rule}

$= 6(2x+1)^2$

Now $\frac{dy}{dx} = u'v + uv'$ {product rule}

$= \frac{1}{2}x^{-\frac{1}{2}}(2x+1)^3 + x^{\frac{1}{2}} \times 6(2x+1)^2$

$= \frac{1}{2}x^{-\frac{1}{2}}(2x+1)^3 + 6x^{\frac{1}{2}}(2x+1)^2$

b $y = x^2(x^2-2x)^4$ is the product of $u = x^2$ and $v = (x^2-2x)^4$

$\therefore \quad u' = 2x$ and $v' = 4(x^2-2x)^3(2x-2)$ {chain rule}

Now $\frac{dy}{dx} = u'v + uv'$ {product rule}

$= 2x(x^2-2x)^4 + x^2 \times 4(x^2-2x)^3(2x-2)$

$= 2x(x^2-2x)^4 + 4x^2(x^2-2x)^3(2x-2)$

EXERCISE 17E

1 Find $\frac{dy}{dx}$ using the product rule:

a $y = x^2(2x-1)$
b $y = 4x(2x+1)^3$
c $y = x^2\sqrt{3-x}$
d $y = \sqrt{x}(x-3)^2$
e $y = 5x^2(3x^2-1)^2$
f $y = \sqrt{x}(x-x^2)^3$

2 Find the gradient of the tangent to:

a $y = x^4(1-2x)^2$ at $x = -1$
b $y = \sqrt{x}(x^2-x+1)^2$ at $x = 4$
c $y = x\sqrt{1-2x}$ at $x = -4$
d $y = x^3\sqrt{5-x^2}$ at $x = 1$

3 If $y = \sqrt{x}(3-x)^2$ show that $\frac{dy}{dx} = \frac{(3-x)(3-5x)}{2\sqrt{x}}$.

Find the x-coordinates of all points on $y = \sqrt{x}(3-x)^2$ where the tangent is horizontal.

F THE QUOTIENT RULE

Expressions like $\dfrac{x^2+1}{2x-5}$, $\dfrac{\sqrt{x}}{1-3x}$ and $\dfrac{x^3}{(x-x^2)^4}$ are called **quotients**.

Quotient functions have the form $Q(x) = \dfrac{u(x)}{v(x)}$.

$$\text{Notice that} \quad u(x) = Q(x)\,v(x)$$

$$\text{and by the product rule} \quad u'(x) = Q'(x)\,v(x) + Q(x)\,v'(x)$$

$$\therefore \quad u'(x) - Q(x)\,v'(x) = Q'(x)\,v(x)$$

$$\therefore \quad Q'(x)\,v(x) = u'(x) - \frac{u(x)}{v(x)}v'(x)$$

$$\therefore \quad Q'(x)\,v(x) = \frac{u'(x)\,v(x) - u(x)\,v'(x)}{v(x)}$$

$$\therefore \quad Q'(x) = \frac{u'(x)\,v(x) - u(x)\,v'(x)}{[v(x)]^2}$$

and this formula is called the **quotient rule**.

So, if $\boldsymbol{Q(x) = \dfrac{u(x)}{v(x)}}$ then $\boldsymbol{Q'(x) = \dfrac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}}$

or if $\boldsymbol{y = \dfrac{u}{v}}$ where u and v are functions of x then $\boldsymbol{\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}}$.

Example 12 Self Tutor

Use the quotient rule to find $\dfrac{dy}{dx}$ if: **a** $y = \dfrac{1+3x}{x^2+1}$ **b** $y = \dfrac{\sqrt{x}}{(1-2x)^2}$

a $y = \dfrac{1+3x}{x^2+1}$ is a quotient with $u = 1+3x$ and $v = x^2+1$

$\therefore \quad u' = 3$ and $v' = 2x$

$$\text{Now} \quad \frac{dy}{dx} = \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\}$$

$$= \frac{3(x^2+1) - (1+3x)2x}{(x^2+1)^2}$$

$$= \frac{3x^2 + 3 - 2x - 6x^2}{(x^2+1)^2}$$

$$= \frac{3 - 2x - 3x^2}{(x^2+1)^2}$$

b $y = \dfrac{\sqrt{x}}{(1-2x)^2}$ is a quotient where $u = x^{\frac{1}{2}}$ and $v = (1-2x)^2$

$\therefore \quad u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(1-2x)^1 \times (-2)$ {chain rule}

$= -4(1-2x)$

$$
\begin{aligned}
\text{Now}\quad \frac{dy}{dx} &= \frac{u'v-uv'}{v^2} \qquad \{\text{quotient rule}\}\\
&= \frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x)^2 - x^{\frac{1}{2}}\times(-4(1-2x))}{(1-2x)^4}\\
&= \frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x)^2 + 4x^{\frac{1}{2}}(1-2x)}{(1-2x)^4}\\
&= \frac{\cancel{(1-2x)}\left[\frac{1-2x}{2\sqrt{x}} + 4\sqrt{x}\left(\frac{2\sqrt{x}}{2\sqrt{x}}\right)\right]}{(1-2x)^{\cancel{4}\,3}} \qquad \{\text{look for common factors}\}\\
&= \frac{1-2x+8x}{2\sqrt{x}(1-2x)^3}\\
&= \frac{6x+1}{2\sqrt{x}(1-2x)^3}
\end{aligned}
$$

Note: Simplification of $\frac{dy}{dx}$ as in the above example is often unnecessary, especially if you want to find the gradient of a tangent at a given point. In such cases you can substitute a value for x without simplifying.

EXERCISE 17F

1 Use the quotient rule to find $\frac{dy}{dx}$ if:

a $y=\frac{1+3x}{2-x}$ **b** $y=\frac{x^2}{2x+1}$ **c** $y=\frac{x}{x^2-3}$

d $y=\frac{\sqrt{x}}{1-2x}$ **e** $y=\frac{x^2-3}{3x-x^2}$ **f** $y=\frac{x}{\sqrt{1-3x}}$

2 Find the gradient of the tangent to:

a $y=\frac{x}{1-2x}$ at $x=1$ **b** $y=\frac{x^3}{x^2+1}$ at $x=-1$

c $y=\frac{\sqrt{x}}{2x+1}$ at $x=4$ **d** $y=\frac{x^2}{\sqrt{x^2+5}}$ at $x=-2$

3 **a** If $y=\frac{2\sqrt{x}}{1-x}$, show that $\frac{dy}{dx}=\frac{x+1}{\sqrt{x}(1-x)^2}$.

b For what values of x is $\frac{dy}{dx}$ **i** zero **ii** undefined?

4 **a** If $y=\frac{x^2-3x+1}{x+2}$, show that $\frac{dy}{dx}=\frac{x^2+4x-7}{(x+2)^2}$.

b For what values of x is $\frac{dy}{dx}$ **i** zero **ii** undefined?

c What is the graphical significance of your answers in **b**?

G TANGENTS AND NORMALS

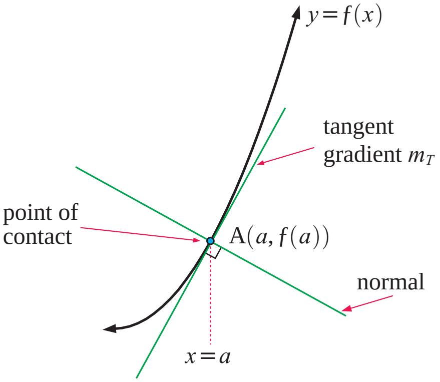
Consider a curve $y = f(x)$.

If A is the point with x-coordinate a, then the gradient of the tangent at this point is $f'(a) = m_T$.

The equation of the tangent is

$$\frac{y - f(a)}{x - a} = f'(a) \quad \{\text{equating gradients}\}$$

or $y - f(a) = f'(a)(x - a)$.



Alternatively, the equation of the tangent at the point A$(a,\ b)$ is

$$\boldsymbol{\frac{y - b}{x - a} = f'(a) \quad \textit{or} \quad y - b = f'(a)(x - a).}$$

A **normal** to a curve is a line which is perpendicular to the tangent at the point of contact.

The gradients of perpendicular lines are negative reciprocals of each other.

Thus, the gradient of the **normal** at $x = a$ is $m_N = -\dfrac{1}{f'(a)}$.

For example, if $f(x) = x^2$ then $f'(x) = 2x$.

At $x = 2$, $m_T = f'(2) = 4$ and $m_N - \dfrac{1}{f'(2)} = -\frac{1}{4}$.

So, at $x = 2$ the tangent has gradient 4 and the normal has gradient $-\frac{1}{4}$.

Since $f(2) = 4$, the tangent has equation $y - 4 = 4(x - 2)$ or $y = 4x - 4$
and the normal has equation $y - 4 = -\frac{1}{4}(x - 2)$ or $y = -\frac{1}{4}x + \frac{9}{2}$.

Reminder: If a line has gradient $\frac{4}{5}$ say, and passes through $(2,\ -3)$ say, another quick way to write down its equation is $4x - 5y = 4(2) - 5(-3)$, or $4x - 5y = 23$.

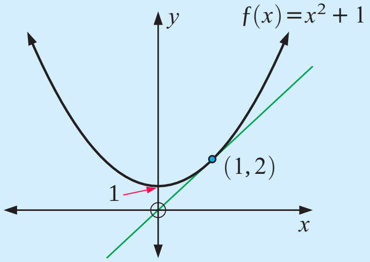
If the gradient was $-\frac{4}{5}$, we would have:
$4x + 5y = 4(2) + 5(-3)$, or $4x + 5y = -7$.

You can also find the equations of tangents at a given point using your graphics calculator. Instructions for doing this can be found at the start of the book.

Example 13

Self Tutor

Find the equation of the tangent to $f(x) = x^2 + 1$ at the point where $x = 1$.



Since $f(1) = 1 + 1 = 2$, the point of contact is $(1, 2)$.

Now $f'(x) = 2x$, so $m_T = f'(1) = 2$

$\therefore$ the tangent has equation $\dfrac{y-2}{x-1} = 2$

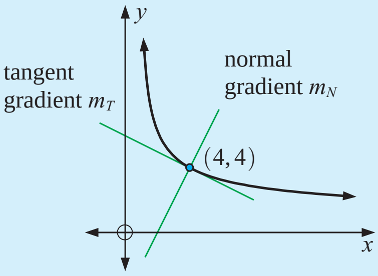
which is $y - 2 = 2x - 2$ or $y = 2x$.

Example 14

Self Tutor

Find the equation of the normal to $y = \dfrac{8}{\sqrt{x}}$ at the point where $x = 4$.

When $x = 4$, $y = \frac{8}{\sqrt{4}} = \frac{8}{2} = 4$. So, the point of contact is $(4, 4)$.



Now as $y = 8x^{-\frac{1}{2}}$, $\dfrac{dy}{dx} = -4x^{-\frac{3}{2}}$

$\therefore$ when $x = 4$, $m_T = -4 \times 4^{-\frac{3}{2}} = -\frac{1}{2}$

$\therefore$ the normal at $(4, 4)$ has gradient $m_N = \frac{2}{1}$.

$\therefore$ the equation of the normal is

$2x - 1y = 2(4) - 1(4)$ or $2x - y = 4$.

Example 15

Self Tutor

Find the equations of any horizontal tangents to $y = x^3 - 12x + 2$.

Since $y = x^3 - 12x + 2$, $\dfrac{dy}{dx} = 3x^2 - 12$

Horizontal tangents have gradient 0, so $3x^2 - 12 = 0$

$\therefore$ $3(x^2 - 4) = 0$

$\therefore$ $3(x + 2)(x - 2) = 0$

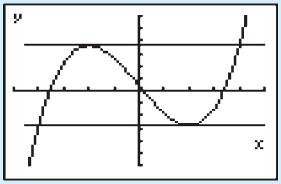
$\therefore$ $x = -2$ or 2

When $x = 2$, $y = 8 - 24 + 2 = -14$

When $x = -2$, $y = -8 + 24 + 2 = 18$

$\therefore$ the points of contact are $(2, -14)$ and $(-2, 18)$

$\therefore$ the tangents are $y = -14$ and $y = 18$.



Example 16 **Self Tutor**

Find the equation of the tangent to $y = \sqrt{10 - 3x}$ at the point where $x = 3$.

Let $f(x) = (10 - 3x)^{\frac{1}{2}}$

$\therefore\quad f'(x) = \frac{1}{2}(10 - 3x)^{-\frac{1}{2}} \times (-3)$

$\therefore\quad m_T = f'(3) = \frac{1}{2}(1)^{-\frac{1}{2}} \times (-3) = -\frac{3}{2}$

When $x = 3$, $y = \sqrt{10 - 9} = 1$

$\therefore$ the point of contact is (3, 1).

So, the tangent has equation $\dfrac{y - 1}{x - 3} = -\dfrac{3}{2}$ which is $2y - 2 = -3x + 9$

or $3x + 2y = 11$

EXERCISE 17G

1 Find the equation of the tangent to:

a $y = x - 2x^2 + 3$ at $x = 2$

b $y = \sqrt{x} + 1$ at $x = 4$

c $y = x^3 - 5x$ at $x = 1$

d $y = \dfrac{4}{\sqrt{x}}$ at (1, 4)

e $y = \dfrac{3}{x} - \dfrac{1}{x^2}$ at $(-1, -4)$

f $y = 3x^2 - \dfrac{1}{x}$ at $x = -1$.

Check your answers using technology.

2 Find the equation of the normal to:

a $y = x^2$ at the point (3, 9)

b $y = x^3 - 5x + 2$ at $x = -2$

c $y = \dfrac{5}{\sqrt{x}} - \sqrt{x}$ at the point (1, 4)

d $y = 8\sqrt{x} - \dfrac{1}{x^2}$ at $x = 1$.

3 **a** Find the equations of the horizontal tangents to $y = 2x^3 + 3x^2 - 12x + 1$.

b Find the points of contact where horizontal tangents meet the curve $y = 2\sqrt{x} + \dfrac{1}{\sqrt{x}}$.

c Find k if the tangent to $y = 2x^3 + kx^2 - 3$ at the point where $x = 2$ has gradient 4.

d Find the equation of the tangent to $y = 1 - 3x + 12x^2 - 8x^3$ which is parallel to the tangent at (1, 2).

4 **a** The tangent to the curve $y = x^2 + ax + b$ where a and b are constants, is $2x + y = 6$ at the point where $x = 1$. Find the values of a and b.

b The normal to the curve $y = a\sqrt{x} + \dfrac{b}{\sqrt{x}}$ where a and b are constants, has equation $4x + y = 22$ at the point where $x = 4$. Find the values of a and b.

c Show that the equation of the tangent to $y = 2x^2 - 1$ at the point where $x = a$, is $4ax - y = 2a^2 + 1$.

5 Find the equation of the tangent to:

a $y = \sqrt{2x + 1}$ at $x = 4$

b $y = \dfrac{1}{2 - x}$ at $x = -1$

c $f(x) = \dfrac{x}{1 - 3x}$ at $(-1, -\frac{1}{4})$

d $f(x) = \dfrac{x^2}{1 - x}$ at $(2, -4)$.

6 Find the equation of the normal to:

a $y = \dfrac{1}{(x^2+1)^2}$ at $(1, \frac{1}{4})$

b $y = \dfrac{1}{\sqrt{3-2x}}$ at $x = -3$

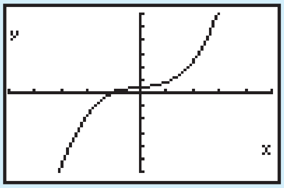
c $f(x) = \sqrt{x}(1-x)^2$ at $x = 4$

d $f(x) = \dfrac{x^2-1}{2x+3}$ at $x = -1$.

7 $y = a\sqrt{1-bx}$ where a and b are constants, has a tangent with equation $3x + y = 5$ at the point where $x = -1$. Find a and b.

Example 17 **Self Tutor**

Find the coordinates of the point(s) where the tangent to $y = x^3 + x + 2$ at (1, 4) meets the curve again.



Let $f(x) = x^3 + x + 2$

$\therefore \quad f'(x) = 3x^2 + 1$

$\therefore \quad m_T = f'(1) = 3 + 1 = 4$

$\therefore$ the tangent at (1, 4) has gradient 4

and its equation is $4x - y = 4(1) - 4$ or $y = 4x$.

Now $y = 4x$ meets $y = x^3 + x + 2$ where $x^3 + x + 2 = 4x$

$\therefore \quad x^3 - 3x + 2 = 0$

Since the tangent *touches* the curve when $x = 1$, $x = 1$ must be a repeated zero of $x^3 - 3x + 2$.

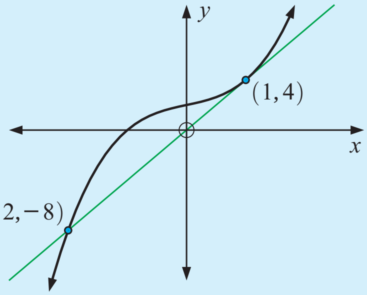
$\therefore \quad (x-1)^2(x+2) = 0$ {by inspection}

$\uparrow$ $x^2 \times x = x^3$ $\qquad$ $\uparrow$ $(-1)^2 \times 2 = 2$

$\therefore \quad x = 1$ or -2

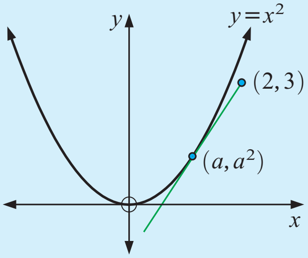
When $x = -2$, $y = (-2)^3 + (-2) + 2 = -8$

$\therefore$ the tangent meets the curve again at $(-2, -8)$.



Example 18 **Self Tutor**

Find the equations of the tangents to $y = x^2$ from the external point (2, 3).



Let (a, a^2) lie on $f(x) = x^2$.

Now $f'(x) = 2x$, so $f'(a) = 2a$

$\therefore$ at (a, a^2) the gradient of the tangent is $\dfrac{2a}{1}$

$\therefore$ its equation is $\dfrac{y - a^2}{x - a} = 2a$

or $y - a^2 = 2ax - 2a^2$

or $y = 2ax - a^2$

But this tangent passes through (2, 3).

$$\therefore \quad 3 = 2a(2) - a^2$$
$$\therefore \quad a^2 - 4a + 3 = 0$$
$$\therefore \quad (a-1)(a-3) = 0$$
$$\therefore \quad a = 1 \text{ or } 3$$

If $a = 1$, the tangent has equation $y = 2x - 1$ with point of contact (1, 1).

If $a = 3$, the tangent has equation $y = 6x - 9$ with point of contact (3, 9).

8 **a** Find where the tangent to the curve $y = x^3$ at the point where $x = 2$, meets the curve again.

b Find where the tangent to the curve $y = -x^3 + 2x^2 + 1$ at the point where $x = -1$, meets the curve again.

c Find where the tangent to the curve $y = x^3 + \frac{4}{x}$ at the point where $x = 1$, meets the curve again.

9 **a** Find the equation of the tangent to $y = x^2 - x + 9$ at the point where $x = a$. Hence, find the equations of the two tangents from (0, 0) to the curve. State the coordinates of the points of contact.

b Find the equations of the tangents to $y = x^3$ from the external point (−2, 0).

c Find the equation(s) of the normal(s) to $y = \sqrt{x}$ from the external point (4, 0).

10 Consider $f(x) = \dfrac{8}{x^2}$.

a Sketch the graph of the function.

b Find the equation of the tangent at the point where $x = a$.

c If the tangent in **b** cuts the x-axis at A and the y-axis at B, find the coordinates of A and B.

d Find the area of triangle OAB and discuss the area of the triangle as $a \to \infty$.

H THE SECOND DERIVATIVE

Given a function $f(x)$, the derivative $f'(x)$ is known as the **first derivative**.

The **second derivative** of $f(x)$ is the derivative of $f'(x)$, or **the derivative of the first derivative**.

We use $f''(x)$ or y'' or $\dfrac{d^2y}{dx^2}$ to represent the second derivative.

$f''(x)$ reads "*f double dashed of x*".

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ reads "*dee two y by dee x squared*".

THE SECOND DERIVATIVE IN CONTEXT

Michael rides up a hill and down the other side to his friend's house. The dots on the graph show Michael's position at various times t.

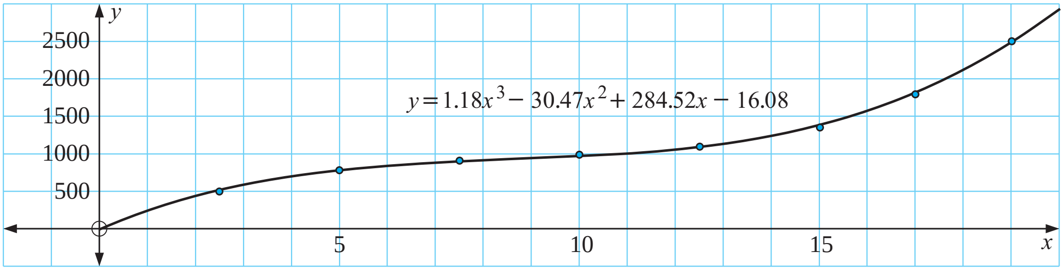
DEMO

The distance Michael has travelled at various times is given in the following table:

Time (t min)	0	2.5	5	7.5	10	12.5	15	17	19
Distance travelled (s m)	0	498	782	908	989	1096	1350	1792	2500

The model $s \approx 1.18t^3 - 30.47t^2 + 284.52t - 16.08$ metres fits this data well, although the model gives $s(0) \approx -16.08$ m whereas the actual data gives $s(0) = 0$. This sort of problem often occurs when modelling from data.

A graph of the data points and the model is given below:



Now $\dfrac{ds}{dt} \approx 3.54t^2 - 60.94t + 284.52$ metres per minute is the instantaneous rate of change in displacement per unit of time, or instantaneous velocity.

The instantaneous rate of change in velocity at any point in time is Michael's **acceleration**,

$$\text{so} \quad \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2} \quad \text{is the instantaneous acceleration.}$$

$$\text{In this case} \quad \frac{d^2s}{dt^2} \approx 7.08t - 60.94 \quad \text{metres per minute per minute.}$$

We see that when $t = 12$, $s \approx 1050$ m,

$$\frac{ds}{dt} \approx 63 \quad \text{metres per minute,}$$

$$\text{and} \quad \frac{d^2s}{dt^2} \approx 24 \quad \text{metres per minute per minute.}$$

We will examine displacement, velocity and acceleration in greater detail in the next chapter.

Example 19

Self Tutor

Find $f''(x)$ given that $f(x) = x^3 - \dfrac{3}{x}$.

$$\begin{aligned} \text{Now} \quad f(x) &= x^3 - 3x^{-1} \\ \therefore \quad f'(x) &= 3x^2 + 3x^{-2} \\ \therefore \quad f''(x) &= 6x - 6x^{-3} \\ &= 6x - \frac{6}{x^3} \end{aligned}$$

EXERCISE 17H

1 Find $f''(x)$ given that:

a $f(x) = 3x^2 - 6x + 2$

b $f(x) = 2x^3 - 3x^2 - x + 5$

c $f(x) = \dfrac{2}{\sqrt{x}} - 1$

d $f(x) = \dfrac{2-3x}{x^2}$

e $f(x) = (1-2x)^3$

f $f(x) = \dfrac{x+2}{2x-1}$

2 Find $\dfrac{d^2y}{dx^2}$ given that:

a $y = x - x^3$

b $y = x^2 - \dfrac{5}{x^2}$

c $y = 2 - \dfrac{3}{\sqrt{x}}$

d $y = \dfrac{4-x}{x}$

e $y = (x^2 - 3x)^3$

f $y = x^2 - x + \dfrac{1}{1-x}$

3 Find x when $f''(x) = 0$ for:

a $f(x) = 2x^3 - 6x^2 + 5x + 1$

b $f(x) = \dfrac{x}{x^2+2}$

4 Consider the function $f(x) = 2x^3 - x$.
Complete the following table by indicating whether $f(x)$, $f'(x)$ and $f''(x)$ are positive (+), negative (–), or zero (0) at the given values of x.

x	-1	0	1
$f(x)$	$-$		
$f'(x)$			
$f''(x)$			

REVIEW SET 17A NON-CALCULATOR

1 If $f(x) = 7 + x - 3x^2$, find: **a** $f(3)$ **b** $f'(3)$ **c** $f''(3)$.

2 Find the equation of the tangent to $y = -2x^2$ at the point where $x = -1$.

3 Find $\dfrac{dy}{dx}$ for: **a** $y = 3x^2 - x^4$ **b** $y = \dfrac{x^3 - x}{x^2}$

4 Find, from first principles, the derivative of $f(x) = x^2 + 2x$.

5 Find the equation of the normal to $y = \dfrac{1-2x}{x^2}$ at the point where $x = 1$.

6 The tangent to $y = \dfrac{ax+b}{\sqrt{x}}$ at $x = 1$ is $2x - y = 1$. Find a and b.

7 Determine the derivative with respect to x of:

a $f(x) = (x^2+3)^4$ **b** $g(x) = \dfrac{\sqrt{x+5}}{x^2}$

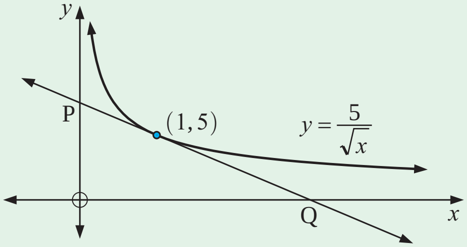
8 Find $f''(2)$ for: **a** $f(x) = 3x^2 - \dfrac{1}{x}$ **b** $f(x) = \sqrt{x}$

9 The curve $f(x) = 2x^3 + ax + b$ has a tangent with gradient 10 at the point $(-2, 33)$. Find the values of a and b.

10 Find x if $f''(x) = 0$ and $f(x) = 2x^4 - 4x^3 - 9x^2 + 4x + 7$.

11 The line through A(2, 4) and B(0, 8) is a tangent to $y = \dfrac{a}{(x+2)^2}$. Find a.

12 Find the coordinates of P and Q.



REVIEW SET 17B CALCULATOR

1 Differentiate with respect to x:

a $5x - 3x^{-1}$ **b** $(3x^2+x)^4$ **c** $(x^2+1)(1-x^2)^3$

2 Determine the equation of any horizontal tangents to the curve with equation $y = x^3 - 3x^2 - 9x + 2$.

3 The tangent to $y = x^2\sqrt{1-x}$ at $x = -3$ cuts the axes at A and B. Determine the area of triangle OAB.

4 Find where the tangent to $y = 2x^3 + 4x - 1$ at $(1, 5)$ cuts the curve again.

5 Find a given that the tangent to $y = \dfrac{4}{(ax+1)^2}$ at $x = 0$ passes through $(1, 0)$.

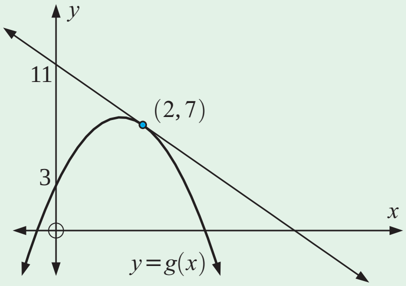
6 Find all points on the curve $y = 2x^3 + 3x^2 - 10x + 3$ where the gradient of the tangent is 2.

7 If $y = \sqrt{5-4x}$, find: **a** $\dfrac{dy}{dx}$ **b** $\dfrac{d^2y}{dx^2}$

8 Find the equation of the normal to $y = \dfrac{x+1}{x^2-2}$ at the point where $x = 1$.

9 Show that $y = 2 - \dfrac{7}{1+2x}$ has no horizontal tangents.

10 Find the equation of the quadratic $g(x)$ in the form $g(x) = ax^2 + bx + c$.



11 **a** Sketch the graph of $x \mapsto \frac{4}{x}$ for $x > 0$.

b Find the equation of the tangent to the function at the point where $x = k$, $k > 0$.

c If the tangent in **b** cuts the x-axis at A and the y-axis at B, find the coordinates of A and B.

d What can be deduced about the area of triangle OAB?

e Find k if the normal to the curve at $x = k$ passes through the point $(1, 1)$.

REVIEW SET 17C

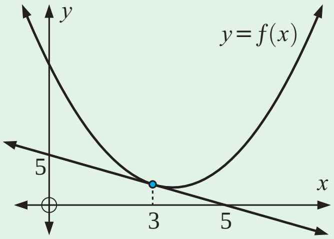
1 Differentiate with respect to x: **a** $y = x^3\sqrt{1-x^2}$ **b** $y = \dfrac{x^2-3x}{\sqrt{x+1}}$

2 Find the equation of the normal to $y = \dfrac{1}{\sqrt{x}}$ at the point where $x = 4$.

3 Use the rules of differentiation to find $\dfrac{dy}{dx}$ for:

a $y = \dfrac{4}{\sqrt{x}} - 3x$ **b** $y = \left(x - \dfrac{1}{x}\right)^4$ **c** $y = \sqrt{x^2-3x}$

4 Use the graph alongside to find $f(3)$ and $f'(3)$.



5 Differentiate with respect to x: **a** $f(x) = \dfrac{(x+3)^3}{\sqrt{x}}$ **b** $f(x) = x^4\sqrt{x^2+3}$

6 $y = 2x$ is a tangent to the curve $y = x^3 + ax + b$ at $x = 1$. Find a and b.

7 The tangent to $y = x^3 + ax^2 - 4x + 3$ at $x = 1$ is parallel to the line $y = 3x$. Find the value of a and the equation of the tangent at $x = 1$. Where does the tangent cut the curve again?

8 If the normal to $f(x) = \dfrac{3x}{1+x}$ at $(2,\ 2)$ cuts the axes at B and C, determine the length of [BC].

9 Find $\dfrac{d^2y}{dx^2}$ for: **a** $y = 3x^4 - \dfrac{2}{x}$ **b** $y = x^3 - x + \dfrac{1}{\sqrt{x}}$

10 The curve $f(x) = 3x^3 + ax^2 + b$ has tangent with gradient 0 at the point $(-2,\ 14)$. Find a and b and hence $f''(-2)$.

11 Show that the curves whose equations are $y = \sqrt{3x+1}$ and $y = \sqrt{5x - x^2}$ have the same gradient at their point of intersection. Find the equation of the common tangent at this point.

Chapter 18

Applications of differential calculus

Syllabus reference: 7.3, 7.6, 7.7

Contents:

We saw in the previous chapter that one application of differential calculus is in finding the equations of tangents and normals to curves. There are many other uses, however, including the following which we will consider in this course:

- functions of time
- rates of change
- motion in a straight line (displacement, velocity and acceleration)
- curve properties (monotonicity and concavity)
- optimisation (maxima and minima)
- applications in economics

A TIME RATE OF CHANGE

There are countless quantities in the real world that vary with time.

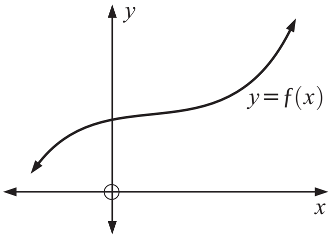
For example:
- temperature varies continuously
- the height of a tree varies as it grows
- the prices of stocks and shares vary with each day's trading.

Varying quantities can be modelled using functions of time.

For example, we could use:

- $s(t)$ to model the distance travelled by a runner
- $H(t)$ to model the height of a person riding in a Ferris wheel
- $C(t)$ to model the capacity of a person's lungs, which changes when the person breathes.

We saw in the previous chapter that if $y = f(x)$ then $f'(x)$ or $\dfrac{dy}{dx}$ is the gradient of the tangent at any value of x, and also the rate of change in y with respect to x.



We can likewise find the derivative of a function of time to tell us the **rate** at which something is happening.

For the examples above:

- $\dfrac{ds}{dt}$ or $s'(t)$ is the instantaneous *speed* of the runner.
  It might have units metres per second or m s^{-1}.
- $\dfrac{dH}{dt}$ or $H'(t)$ is the instantaneous rate of ascent of the person in the Ferris wheel.
  It might also have units metres per second or m s^{-1}.
- $\dfrac{dC}{dt}$ or $C'(t)$ is the person's instantaneous rate of change in lung capacity.
  It might have units litres per second or L s^{-1}.

EXERCISE 18A

1 The estimated future profits of a small business are given by $P(t) = 2t^2 - 12t + 118$ thousand dollars, where t is the time in years from now.

- **a** What is the current annual profit?
- **b** Find $\dfrac{dP}{dt}$ and state its units.
- **c** What is the significance of $\dfrac{dP}{dt}$?
- **d** For what values of t will the profit:
  - **i** decrease **ii** increase on the previous year?
- **e** What is the minimum profit and when does it occur?
- **f** Find $\dfrac{dP}{dt}$ when $t = 4$, 10 and 25. What do these figures represent?

2 Water is draining from a swimming pool such that the remaining volume of water after t minutes is $V = 200(50 - t)^2$ m^3. Find:

- **a** the average rate at which the water leaves the pool in the first 5 minutes
- **b** the instantaneous rate at which the water is leaving at $t = 5$ minutes.

3 When a ball is thrown, its height above the ground is given by $s(t) = 1.2 + 28.1t - 4.9t^2$ metres where t is the time in seconds.

- **a** From what distance above the ground was the ball released?
- **b** Find $s'(t)$ and state what it represents.
- **c** Find t when $s'(t) = 0$. What is the significance of this result?
- **d** What is the maximum height reached by the ball?
- **e** Find the ball's speed: **i** when released **ii** at $t = 2$ s **iii** at $t = 5$ s.
  State the significance of the sign of the derivative.
- **f** How long will it take for the ball to hit the ground?
- **g** What is the significance of $\dfrac{d^2s}{dt^2}$?

4 A shell is accidentally fired vertically from a mortar at ground level and reaches the ground again after 14.2 seconds. Its height above the ground at any time t seconds is given by $s(t) = bt - 4.9t^2$ metres where b is constant.

- **a** Show that the initial velocity of the shell is b m s^{-1} upwards.
- **b** Find the initial velocity of the shell.

B GENERAL RATES OF CHANGE

We have seen previously that if $s(t)$ is a displacement function then $s'(t)$ or $\dfrac{ds}{dt}$ is the instantaneous rate of change in displacement with respect to time, which is called velocity.

In general, $\dfrac{dy}{dx}$ gives the **rate of change in y with respect to x**.

We can see that: If y increases as x increases, then $\dfrac{dy}{dx}$ will be positive.

If y decreases as x increases, then $\dfrac{dy}{dx}$ will be negative.

Example 1 **Self Tutor**

According to a psychologist, the ability of a person to understand spatial concepts is given by $A = \frac{1}{3}\sqrt{t}$ where t is the age in years, $5 \leqslant t \leqslant 18$.

a Find the rate of improvement in ability to understand spatial concepts when a person is: **i** 9 years old **ii** 16 years old.

b Explain why $\dfrac{dA}{dt} > 0$, $5 \leqslant t \leqslant 18$. Comment on the significance of this result.

c Explain why $\dfrac{d^2A}{dt^2} < 0$, $5 \leqslant t \leqslant 18$. Comment on the significance of this result.

a $A = \frac{1}{3}\sqrt{t} = \frac{1}{3}t^{\frac{1}{2}}$ $\quad\therefore\quad \dfrac{dA}{dt} = \frac{1}{6}t^{-\frac{1}{2}} = \dfrac{1}{6\sqrt{t}}$

i When $t = 9$, $\dfrac{dA}{dt} = \frac{1}{18}$ $\quad\therefore$ the rate of improvement is $\frac{1}{18}$ units per year for a 9 year old.

ii When $t = 16$, $\dfrac{dA}{dt} = \frac{1}{24}$ $\quad\therefore$ the rate of improvement is $\frac{1}{24}$ units per year for a 16 year old.

b As $\sqrt{t}$ is never negative, $\dfrac{1}{6\sqrt{t}}$ is never negative

$\therefore\ \dfrac{dA}{dt} > 0$ for all $5 \leqslant t \leqslant 18$.

This means that the ability to understand spatial concepts increases with age.

c $\dfrac{dA}{dt} = \frac{1}{6}t^{-\frac{1}{2}}$ so $\dfrac{d^2A}{dt^2} = -\frac{1}{12}t^{-\frac{3}{2}} = -\dfrac{1}{12t\sqrt{t}}$

$\therefore\ \dfrac{d^2A}{dt^2} < 0$ for all $5 \leqslant t \leqslant 18$.

This means that while the ability to understand spatial concepts increases with time, the rate of increase slows down with age.

Example 2 **Self Tutor**

The cost of producing x items in a factory each day is given by

$$C(x) = \underbrace{0.000\,13x^3 + 0.002x^2}_{\text{cost of labour}} + \underset{\substack{\uparrow \\ \text{raw material} \\ \text{costs}}}{5x} + \underset{\substack{\uparrow \\ \text{fixed or overhead costs such as} \\ \text{heating, cooling, maintenance, rent}}}{2200}$$

a Find $C'(x)$, which is called the marginal cost function.

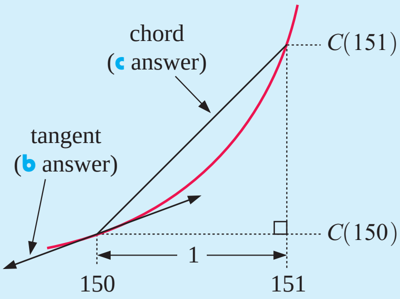
b Find the marginal cost when 150 items are produced. Interpret this result.

c Find $C(151) - C(150)$. Compare this with the answer in **b**.

a The marginal cost function is
$C'(x) = 0.000\,39x^2 + 0.004x + 5$

b $C'(150) = \$14.38$

This is the rate at which the costs are increasing with respect to the production level x when 150 items are made per day. It gives an estimate of the cost for making the 151st item.



c $C(151) - C(150) \approx \$3448.19 - \$3433.75$
$\approx \$14.44$

This is the actual cost of making the 151st item each week, so the answer in **b** gives a good estimate.

EXERCISE 18B

You are encouraged to use technology to graph the function for each question. This is often useful in interpreting results.

1 The quantity of a chemical in human skin which is responsible for its 'elasticity' is given by $Q = 100 - 10\sqrt{t}$ where t is the age of a person in years.

a Find Q at: **i** $t = 0$ **ii** $t = 25$ **iii** $t = 100$ years.

b At what rate is the quantity of the chemical changing at the age of:

i 25 years **ii** 50 years?

c Show that the rate at which the skin loses the chemical is decreasing for all $t > 0$.

2 The height of *pinus radiata*, grown in ideal conditions, is given by $H = 20 - \dfrac{97.5}{t+5}$ metres, where t is the number of years after the tree was planted from an established seedling.


©iStockphoto/Nicola Stratford

a How high was the tree at the time of its planting?

b Find the height of the tree at $t = 4$, $t = 8$ and $t = 12$ years.

c Find the rate at which the tree is growing at $t = 0$, 5 and 10 years.

d Show that $\dfrac{dH}{dt} > 0$ for all $t \geqslant 0$. What is the significance of this result?

3 The total cost of running a train from Paris to Marseille is given by

$C(v) = \frac{1}{5}v^2 + \dfrac{200\,000}{v}$ euros where v is the average speed of the train in $\text{km}\,\text{h}^{-1}$.

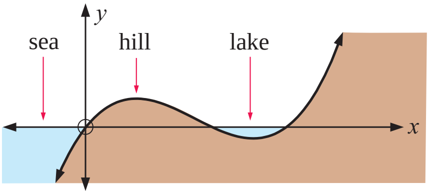
a Find the total cost of the journey if the average speed is:

i $50\ \text{km}\,\text{h}^{-1}$ **ii** $100\ \text{km}\,\text{h}^{-1}$.

b Find the rate of change in the cost of running the train at speeds of:

i $30\ \text{km}\,\text{h}^{-1}$ **ii** $90\ \text{km}\,\text{h}^{-1}$.

c At what speed will the cost be a minimum?

4 

Alongside is a land and sea profile where the x-axis is sea level. The function $y = \frac{1}{10}x(x-2)(x-3)$ km gives the height of the land or sea bed relative to sea level.

a Find where the lake is located relative to the shore line of the sea.

b Find $\frac{dy}{dx}$ and interpret its value when $x = \frac{1}{2}$ and when $x = 1\frac{1}{2}$ km.

c Find the deepest point of the lake and the depth at this point.

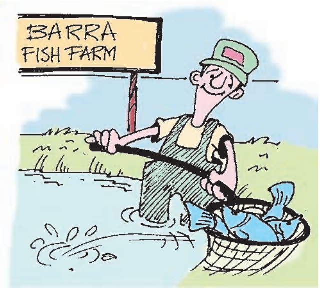
5 A tank contains 50 000 litres of water. The tap is left fully on and all the water drains from the tank in 80 minutes. The volume of water remaining in the tank after t minutes is given by $V = 50\,000\left(1 - \frac{t}{80}\right)^2$ where $0 \leqslant t \leqslant 80$.

a Find $\frac{dV}{dt}$ and draw the graph of $\frac{dV}{dt}$ against t.

b At what time was the outflow fastest?

c Show that $\frac{d^2V}{dt^2}$ is always constant and positive. Interpret this result.

6 A fish farm grows and harvests barramundi in a large dam. The population of fish after t years is given by the function $P(t)$. The rate of change in the population $\frac{dP}{dt}$ is modelled by $\frac{dP}{dt} = aP\left(1 - \frac{P}{b}\right) - cP$ where a, b and c are known constants. a is the birth rate of the barramundi, b is the maximum carrying capacity of the dam and c is the harvest rate each year.



a Explain why the fish population is stable when $\frac{dP}{dt} = 0$.

b If the birth rate is 6%, the maximum carrying capacity is 24 000, and 5% is harvested each year, find the stable population.

c If the harvest rate changes to 4%, what will the stable population increase to?

7 Seablue make denim jeans. The cost model for making x pairs per day is

$$C(x) = 0.0003x^3 + 0.02x^2 + 4x + 2250 \text{ dollars.}$$

a Find the marginal cost function $C'(x)$.

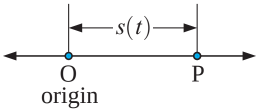
b Find $C'(220)$. What does it estimate?

c Find $C(221) - C(220)$. What does this represent?

d Find $C''(x)$ and the value of x when $C''(x) = 0$. What is the significance of this point?

C MOTION IN A STRAIGHT LINE

DISPLACEMENT



Suppose an object P moves along a straight line so that its position s from an origin O is given as some function of time t.

We write $s = s(t)$ where $t \geqslant 0$.

$s(t)$ is a **displacement function** and for any value of t it gives the displacement from O.

$s(t)$ is a vector quantity. Its magnitude is the distance from O, and its sign indicates the direction from O.

It is clear that:

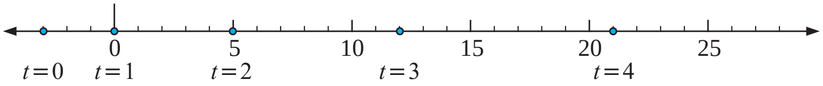
if $s(t) > 0$, P is located to the **right of O**
if $s(t) = 0$, P is located **at O**
if $s(t) < 0$, P is located to the **left of O**.

MOTION GRAPHS

Consider $s(t) = t^2 + 2t - 3$ cm.

$s(0) = -3$ cm, $s(1) = 0$ cm, $s(2) = 5$ cm, $s(3) = 12$ cm, $s(4) = 21$ cm.

To appreciate the motion of P we draw a **motion graph**.



DEMO

Click on the demo icon to get a better idea of the motion.

Fully animated, we not only get a good idea of the position of P, but also of what is happening to its velocity and acceleration.

VELOCITY AND ACCELERATION

AVERAGE VELOCITY

The **average velocity** of an object moving in a straight line in the time interval from $t = t_1$ to $t = t_2$ is the ratio of the change in displacement to the time taken.

If $s(t)$ is the displacement function then **average velocity** $= \dfrac{\boldsymbol{s(t_2) - s(t_1)}}{\boldsymbol{t_2 - t_1}}$.

On a graph of $s(t)$, the average velocity is the gradient of a chord.

INSTANTANEOUS VELOCITY

In **Chapter 17** we established that $\dfrac{ds}{dt} = s'(t)$ is the instantaneous rate of change in displacement per unit of time, or instantaneous velocity.

If $s(t)$ is the displacement function of an object moving in a straight line, then

$$\boldsymbol{v(t) = s'(t) = \lim_{h\to 0} \frac{s(t+h) - s(t)}{h}}$$

is the **instantaneous velocity** or **velocity function** of the object at time t.

On a graph of $s(t)$, the instantaneous velocity is the gradient of a tangent.

AVERAGE ACCELERATION

If an object moves in a straight line with velocity function $v(t)$ then its **average acceleration** on the time interval from $t = t_1$ to $t = t_2$ is the ratio of the change in velocity to the time taken.

$$\textbf{average acceleration} = \boldsymbol{\frac{v(t_2) - v(t_1)}{t_2 - t_1}}.$$

INSTANTANEOUS ACCELERATION

If a particle moves in a straight line with velocity function $v(t)$, then the **instantaneous acceleration** at time t is $\boldsymbol{a(t) = v'(t) = \lim_{h\to 0} \frac{v(t+h) - v(t)}{h}}$.

Example 3 **Self Tutor**

A particle moves in a straight line with displacement from O given by $s(t) = 3t - t^2$ metres at time t seconds. Find:

a the average velocity in the time interval from $t = 2$ to $t = 5$ seconds

b the average velocity in the time interval from $t = 2$ to $t = 2 + h$ seconds

c $\lim_{h\to 0} \frac{s(2+h) - s(2)}{h}$ and comment on its significance.

a average velocity

$$= \frac{s(5) - s(2)}{5 - 2}$$
$$= \frac{(15 - 25) - (6 - 4)}{3}$$
$$= \frac{-10 - 2}{3}$$
$$= -4 \text{ m s}^{-1}$$

b average velocity

$$= \frac{s(2+h) - s(2)}{2 + h - 2}$$
$$= \frac{3(2+h) - (2+h)^2 - 2}{h}$$
$$= \frac{6 + 3h - 4 - 4h - h^2 - 2}{h}$$
$$= \frac{-h - h^2}{h}$$
$$= -1 - h \text{ m s}^{-1} \text{ provided } h \neq 0$$

c
$$\lim_{h\to 0} \frac{s(2+h) - s(2)}{h} = \lim_{h\to 0} (-1 - h)$$
$$= -1 \text{ m s}^{-1}$$

This is the instantaneous velocity of the particle at time $t = 2$ seconds.

EXERCISE 18C.1

1 A particle P moves in a straight line with a displacement function of $s(t) = t^2 + 3t - 2$ metres, where $t \geqslant 0$, t in seconds.

a Find the average velocity from $t = 1$ to $t = 3$ seconds.

b Find the average velocity from $t = 1$ to $t = 1 + h$ seconds.

c Find the value of $\lim\limits_{h \to 0} \dfrac{s(1+h) - s(1)}{h}$ and comment on its significance.

d Find the average velocity from time t to time $t + h$ seconds and interpret $\lim\limits_{h \to 0} \dfrac{s(t+h) - s(t)}{h}$.

2 A particle P moves in a straight line with a displacement function of $s(t) = 5 - 2t^2$ cm, where $t \geqslant 0$, t in seconds.

a Find the average velocity from $t = 2$ to $t = 5$ seconds.

b Find the average velocity from $t = 2$ to $t = 2 + h$ seconds.

c Find the value of $\lim\limits_{h \to 0} \dfrac{s(2+h) - s(2)}{h}$ and state the meaning of this value.

d Interpret $\lim\limits_{h \to 0} \dfrac{s(t+h) - s(t)}{h}$.

3 A particle moves in a straight line with velocity function $v(t) = 2\sqrt{t} + 3$ cm s^{-1}, $t \geqslant 0$.

a Find the average acceleration from $t = 1$ to $t = 4$ seconds.

b Find the average acceleration from $t = 1$ to $t = 1 + h$ seconds.

c Find the value of $\lim\limits_{h \to 0} \dfrac{v(1+h) - v(1)}{h}$. Interpret this value.

d Interpret $\lim\limits_{h \to 0} \dfrac{v(t+h) - v(t)}{h}$.

4 An object moves in a straight line with displacement function $s(t)$ and velocity function $v(t)$, $t \geqslant 0$. State the meaning of:

a $\lim\limits_{h \to 0} \dfrac{s(4+h) - s(4)}{h}$ **b** $\lim\limits_{h \to 0} \dfrac{v(4+h) - v(4)}{h}$

VELOCITY AND ACCELERATION FUNCTIONS

If a particle P moves in a straight line and its position is given by the displacement function $s(t)$, $t \geqslant 0$, then:

- the **velocity** of P at time t is given by
  $\boldsymbol{v(t) = s'(t)}$ {the derivative of the displacement function}
- the **acceleration** of P at time t is given by
  $\boldsymbol{a(t) = v'(t) = s''(t)}$ {the derivative of the velocity function}
- $s(0)$, $v(0)$ and $a(0)$ give us the position, velocity and acceleration of the particle at time $t = 0$, and these are called the **initial conditions**.

SIGN INTERPRETATION

Suppose a particle P moves in a straight line with displacement function $s(t)$ relative to an origin O. Its velocity function is $v(t)$ and its acceleration function is $a(t)$.

We can use **sign diagrams** to interpret:

- where the particle is located relative to O
- the direction of motion and where a change of direction occurs
- when the particle's velocity is increasing or decreasing.

SIGNS OF $s(t)$:

$s(t)$	*Interpretation*
$= 0$	P is at O
> 0	P is located to the right of O
< 0	P is located to the left of O

SIGNS OF $v(t)$:

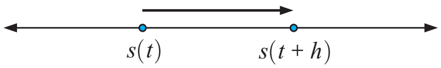
$v(t)$	*Interpretation*
$= 0$	P is instantaneously at rest
> 0	P is moving to the right
< 0	P is moving to the left

$$v(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

If $v(t) > 0$ then $s(t+h) - s(t) > 0$

$\therefore \quad s(t+h) > s(t)$

For $h > 0$ the particle is moving from $s(t)$ to $s(t+h)$.



$\therefore$ P is moving to the right.

SIGNS OF $a(t)$:

$a(t)$	*Interpretation*
> 0	velocity is increasing
< 0	velocity is decreasing
$= 0$	velocity may be a maximum or minimum

A useful table:

Phrase used in a question	t	s	v	a
initial conditions	0			
at the origin		0		
stationary			0	
reverses			0	
maximum height			0	
constant velocity				0
max. or min. velocity				0

When a particle reverses direction, its velocity must change sign.

We need a sign diagram of a to determine if the point is a maximum or minimum.

SPEED

As we have seen, velocities have size (magnitude) and sign (direction). In contrast, the speed of an object is a measure of *how fast* it is travelling, regardless of the direction of travel. Speed is therefore a *scalar* quantity which has size but no sign. Speed cannot be negative.

The **speed** at any instant is the magnitude of the object's velocity.

If $S(t)$ represents speed then we write $S = |v|$.

To determine when the speed of an object is increasing or decreasing, we need to employ a **sign test**.

- If the signs of $v(t)$ and $a(t)$ are the same (both positive or both negative), then the **speed** of P is **increasing**.
- If the signs of $v(t)$ and $a(t)$ are opposite, then the **speed** of P is **decreasing**.

We prove *the first* of these as follows:

Proof: Let $S = |v|$ be the speed of P at any instant, so $S = \begin{cases} v & \text{if } v \geqslant 0 \\ -v & \text{if } v < 0. \end{cases}$

Case 1: If $v > 0$, $S = v$ and $\therefore \dfrac{dS}{dt} = \dfrac{dv}{dt} = a(t)$

If $a(t) > 0$ then $\dfrac{dS}{dt} > 0$ which implies that S is increasing.

Case 2: If $v < 0$, $S = -v$ and $\therefore \dfrac{dS}{dt} = -\dfrac{dv}{dt} = -a(t)$

If $a(t) < 0$ then $\dfrac{dS}{dt} > 0$ which also implies that S is increasing.

Thus if $v(t)$ and $a(t)$ have the same sign then the speed of P is increasing.

INVESTIGATION

DISPLACEMENT, VELOCITY AND ACCELERATION GRAPHS

In this investigation we examine the motion of a projectile which is fired in a vertical direction. The projectile is affected by gravity, which is responsible for the projectile's constant acceleration.

MOTION DEMO

We then extend the investigation to consider other cases of motion in a straight line.

What to do:

1. Click on the icon to examine vertical projectile motion in a straight line. Observe first the displacement along the line, then look at the velocity or rate of change in displacement.

2 Examine the three graphs:
- *displacement* v *time*
- *velocity* v *time*
- *acceleration* v *time*

Comment on the shapes of these graphs.

3 Pick from the menu or construct functions of your own choosing to investigate the relationship between displacement, velocity and acceleration.

You are encouraged to use the motion demo above to answer questions in the following exercise.

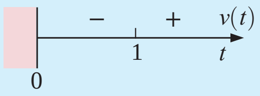
Example 4 **Self Tutor**

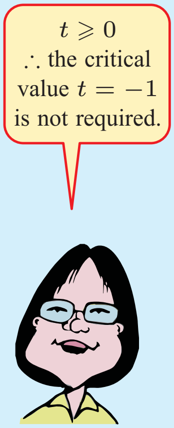
A particle moves in a straight line with position relative to some origin O given by $s(t) = t^3 - 3t + 1$ cm, where t is the time in seconds, $t \geqslant 0$.

a Find expressions for the particle's velocity and acceleration, and draw sign diagrams for each of them.
b Find the initial conditions and hence describe the motion at this instant.
c Describe the motion of the particle at $t = 2$ seconds.
d Find the position of the particle when changes in direction occur.
e Draw a motion diagram for the particle.
f For what time interval(s) is the particle's speed increasing?
g What is the total distance travelled in the time from $t = 0$ to $t = 2$ seconds?

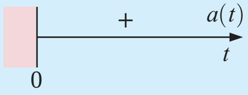
a $s(t) = t^3 - 3t + 1$ cm

$\therefore$ $v(t) = 3t^2 - 3$ $\{$as $v(t) = s'(t)\}$
$= 3(t^2 - 1)$
$= 3(t+1)(t-1)$ cm s^{-1}
which has sign diagram




$t \geqslant 0$
$\therefore$ the critical value $t = -1$ is not required.

and $a(t) = 6t$ cm s^{-2} $\{$as $a(t) = v'(t)\}$
which has sign diagram



b When $t = 0$, $s(0) = 1$ cm
$v(0) = -3$ cm s^{-1}
$a(0) = 0$ cm s^{-2}

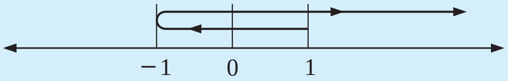
$\therefore$ the particle is 1 cm to the right of O, moving to the left at a speed of 3 cm s^{-1}.

c When $t = 2$, $s(2) = 8 - 6 + 1 = 3$ cm
$v(2) = 12 - 3 = 9$ cm s^{-1}
$a(2) = 12$ cm s^{-2}

$\therefore$ the particle is 3 cm to the right of O, moving to the right at a speed of 9 cm s^{-1}. Since a and v have the same sign, the speed is increasing.

d Since $v(t)$ changes sign when $t = 1$, a change of direction occurs at this instant.
$s(1) = 1 - 3 + 1 = -1$, so the particle changes direction when it is 1 cm to the left of O.

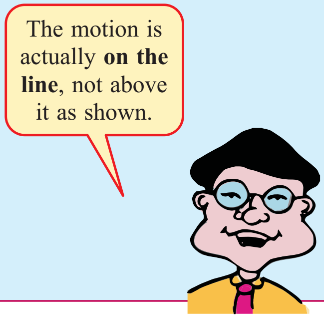
e



As $t \to \infty$, $s(t) \to \infty$ and $v(t) \to \infty$.

f Speed is increasing when $v(t)$ and $a(t)$ have the same sign. This is for $t \geqslant 1$.

g Total distance travelled $= 2 + 4 = 6$ cm.



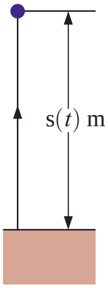
Note: In later chapters on integral calculus we will see another technique for finding the distances travelled and displacement over time.

EXERCISE 18C.2

1 An object moves in a straight line with position given by $s(t) = t^2 - 4t + 3$ cm from an origin O, where t is in seconds, $t \geqslant 0$.

- **a** Find expressions for the object's velocity and acceleration at any instant and draw sign diagrams for each function.
- **b** Find the initial conditions and explain what is happening to the object at that instant.
- **c** Describe the motion of the object at time $t = 2$ seconds.
- **d** At what time(s) does the object reverse direction? Find the position of the object at these instants.
- **e** Draw a motion diagram for the object.
- **f** For what time intervals is the speed of the object decreasing?

2 A stone is projected vertically so that its position above ground level after t seconds is given by $s(t) = 98t - 4.9t^2$ metres, $t \geqslant 0$.



- **a** Find the velocity and acceleration functions for the stone and draw sign diagrams for each function.
- **b** Find the initial position and velocity of the stone.
- **c** Describe the stone's motion at times $t = 5$ and $t = 12$ seconds.
- **d** Find the maximum height reached by the stone.
- **e** Find the time taken for the stone to hit the ground.

3 A particle moves in a straight line with displacement function $s(t) = 12t - 2t^3 - 1$ centimetres where t is in seconds, $t \geqslant 0$.

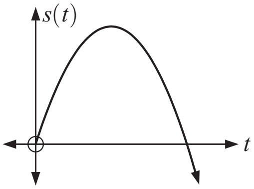
- **a** Find velocity and acceleration functions for the particle's motion.
- **b** Find the initial conditions and interpret their meaning.
- **c** Find the times and positions when the particle reverses direction.
- **d** At what times is the particle's: **i** speed increasing **ii** velocity increasing?

4 The position of a particle moving along the x-axis is given by $x(t) = t^3 - 9t^2 + 24t$ metres where t is in seconds, $t \geqslant 0$.

- **a** Draw sign diagrams for the particle's velocity and acceleration functions.

- **b** Find the position of the particle at the times when it reverses direction, and hence draw a motion diagram for the particle.
- **c** At what times is the particle's: **i** speed decreasing **ii** velocity decreasing?
- **d** Find the total distance travelled by the particle in the first 5 seconds of motion.

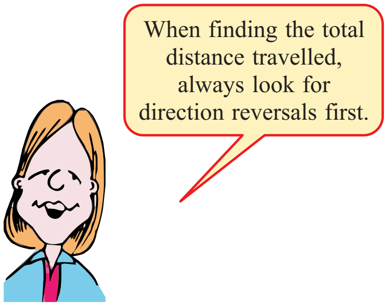
5 In an experiment, an object was fired vertically from the earth's surface. From the results, a two-dimensional graph of the position $s(t)$ metres above the earth's surface was plotted, where t was the time in seconds. It was noted that the graph was *parabolic*.



Assuming a constant gravitational acceleration g and an initial velocity of $v(0)$, show that:

- **a** $v(t) = v(0) + gt$
- **b** $s(t) = v(0) \times t + \frac{1}{2}gt^2$.

Hint: Assume that $s(t) = at^2 + bt + c$.



When finding the total distance travelled, always look for direction reversals first.

D SOME CURVE PROPERTIES

In this section we consider some properties of curves which can be established using derivatives. These include intervals in which curves are increasing and decreasing, and the stationary points of functions.

INCREASING AND DECREASING INTERVALS

The concepts of increasing and decreasing are closely linked to **intervals** of a function's domain.

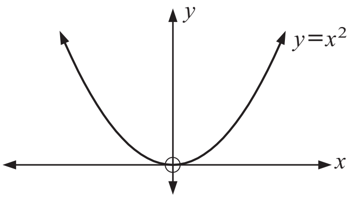
Some examples of intervals and their graphical representations are:

Algebraic form	*Geometric form*
$x \geqslant 2$	 2
$x > 2$	 2
$x \leqslant 4$	 4
$x < 4$	 4
$2 \leqslant x \leqslant 4$	 2 4
$2 \leqslant x < 4$	 2 4

Suppose S is an interval in the domain of $f(x)$, so $f(x)$ is defined for all x in S.

- $f(x)$ is **increasing** on S $\Leftrightarrow$ $f(a) < f(b)$ for all $a, b \in S$ such that $a < b$.
- $f(x)$ is **decreasing** on S $\Leftrightarrow$ $f(a) > f(b)$ for all $a, b \in S$ such that $a < b$.

For example:



$y = x^2$ is decreasing for $x \leqslant 0$ and increasing for $x \geqslant 0$.

Important:

People often get confused about the point $x = 0$. They wonder how the curve can be both increasing and decreasing at the same point when it is clear that the tangent is horizontal. The answer is that increasing and decreasing are associated with *intervals*, not particular values for x. We must clearly state that $y = x^2$ is decreasing *on the interval* $x \leqslant 0$ and increasing *on the interval* $x \geqslant 0$.

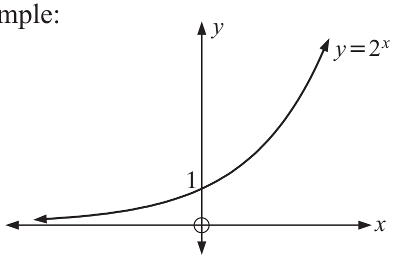
We can deduce when a curve is increasing or decreasing by considering $f'(x)$ on the interval in question. For most functions that we deal with in this course:

- $f(x)$ is **increasing** on S $\Leftrightarrow$ $f'(x) \geqslant 0$ for all x in S
- $f(x)$ is **decreasing** on S $\Leftrightarrow$ $f'(x) \leqslant 0$ for all x in S.

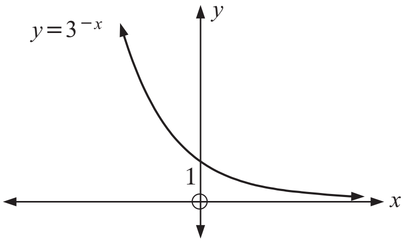
MONOTONICITY

Many functions are either increasing or decreasing for all $x \in \mathbb{R}$. We say these functions are **monotone increasing** or **monotone decreasing**.

For example:

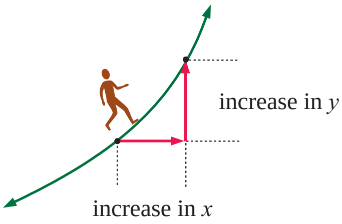


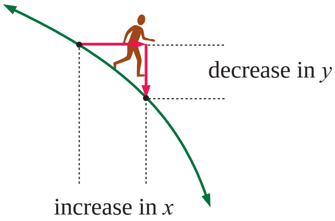
$y = 2^x$ is increasing for all x.



$y = 3^{-x}$ is decreasing for all x.

Notice that:
- for an **increasing** function, an increase in x produces an increase in y
- for a **decreasing** function, an increase in x produces a decrease in y.





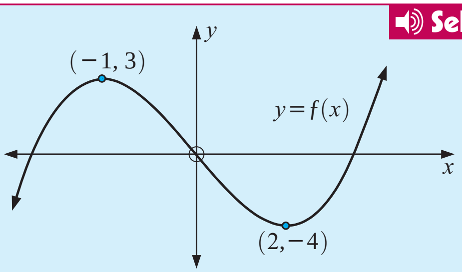
Example 5

Self Tutor

Find intervals where $f(x)$ is:

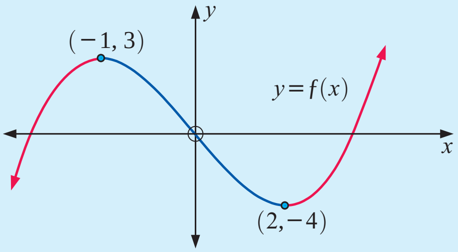
a increasing

b decreasing.



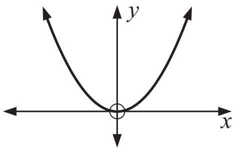
a $f(x)$ is increasing for $x \leqslant -1$ and for $x \geqslant 2$ since $f'(x) \geqslant 0$ on these intervals.

b $f(x)$ is decreasing for $-1 \leqslant x \leqslant 2$.

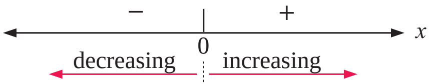


Sign diagrams for the derivative are extremely useful for determining intervals where a function is increasing or decreasing. Consider the following examples:

- $f(x) = x^2$

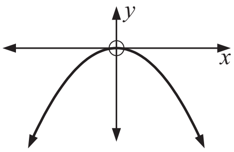


$f'(x) = 2x$ which has sign diagram

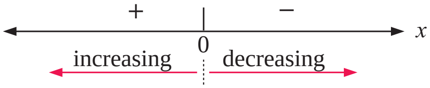


$\therefore$ $f(x) = x^2$ is decreasing for $x \leqslant 0$
and increasing for $x \geqslant 0$.

- $f(x) = -x^2$

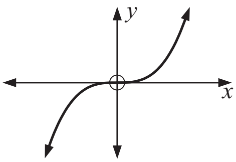


$f'(x) = -2x$ which has sign diagram

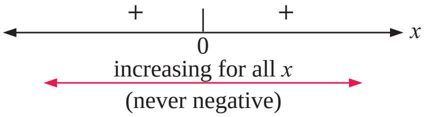


$\therefore$ $f(x) = -x^2$ is increasing for $x \leqslant 0$
and decreasing for $x \geqslant 0$.

- $f(x) = x^3$

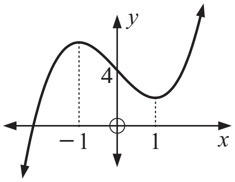


$f'(x) = 3x^2$ which has sign diagram



$\therefore$ $f(x)$ is monotone increasing.

- $f(x) = x^3 - 3x + 4$



$$\begin{aligned} f'(x) &= 3x^2 - 3 \\ &= 3(x^2 - 1) \\ &= 3(x + 1)(x - 1) \end{aligned}$$

which has sign diagram

$\therefore$ $f(x)$ is increasing for $x \leqslant -1$ and for $x \geqslant 1$, and decreasing for $-1 \leqslant x \leqslant 1$.

Example 6

Self Tutor

Find the intervals where the following functions are increasing or decreasing:

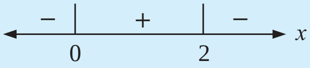
a $f(x) = -x^3 + 3x^2 + 5$ **b** $f(x) = 3x^4 - 8x^3 + 2$

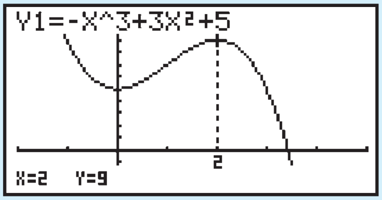
a $f(x) = -x^3 + 3x^2 + 5$

$\therefore \quad f'(x) = -3x^2 + 6x$

$\therefore \quad f'(x) = -3x(x-2)$

which has sign diagram




Y1=-X^3+3X²+5

X=2 Y=9

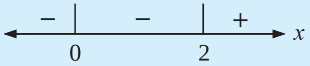
So, $f(x)$ is decreasing for $x \leqslant 0$ and for $x \geqslant 2$, and increasing for $0 \leqslant x \leqslant 2$.

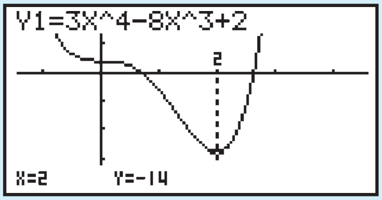
b $f(x) = 3x^4 - 8x^3 + 2$

$\therefore \quad f'(x) = 12x^3 - 24x^2$

$= 12x^2(x-2)$

which has sign diagram




Y1=3X^4-8X^3+2

X=2 Y=-14

So, $f(x)$ is decreasing for $x \leqslant 2$, and increasing for $x \geqslant 2$.

Remember that $f(x)$ must be defined for all x on an interval before we can classify the interval as increasing or decreasing. We must exclude points where a function is undefined, and need to take care with vertical asymptotes.

Example 7

Self Tutor

Consider $f(x) = \dfrac{2x-3}{x^2+2x-3}$.

a Show that $f'(x) = \dfrac{-2x(x-3)}{(x+3)^2(x-1)^2}$ and draw its sign diagram.

b Hence, find intervals where $y = f(x)$ is increasing or decreasing.

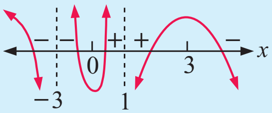
a $f(x) = \dfrac{2x-3}{x^2+2x-3}$

$$f'(x) = \frac{2(x^2+2x-3) - (2x-3)(2x+2)}{(x^2+2x-3)^2} \qquad \{\text{quotient rule}\}$$

$$= \frac{2x^2+4x-6-[4x^2-2x-6]}{((x-1)(x+3))^2}$$

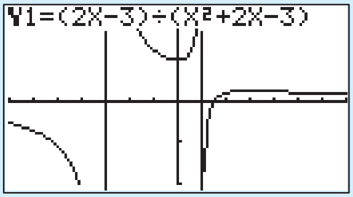
$$= \frac{-2x^2+6x}{(x-1)^2(x+3)^2}$$

$$= \frac{-2x(x-3)}{(x-1)^2(x+3)^2}$$ which has sign diagram



b $f(x)$ is increasing for $0 \leqslant x < 1$
and for $1 < x \leqslant 3$.

$f(x)$ is decreasing for $x < -3$
and for $-3 < x \leqslant 0$
and for $x \geqslant 3$.

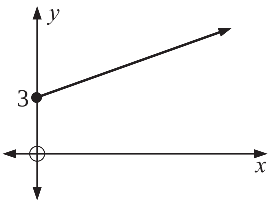


```
Y1=(2X-3)÷(X²+2X-3)
```

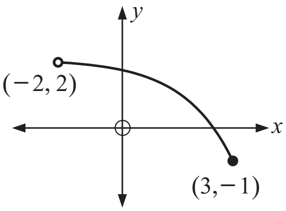
EXERCISE 18D.1

1 Write down the intervals where the graphs are: **i** increasing **ii** decreasing.

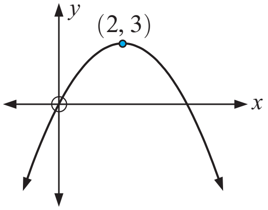
a



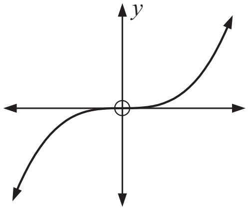
b



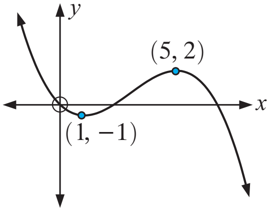
c



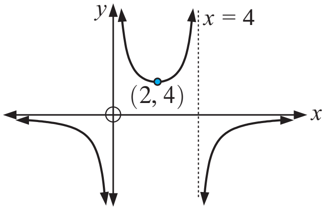
d



e



f



2 Find intervals where $f(x)$ is increasing or decreasing:

a $f(x) = x^2$

b $f(x) = -x^3$

c $f(x) = 2x^2 + 3x - 4$

d $f(x) = \sqrt{x}$

e $f(x) = \dfrac{2}{\sqrt{x}}$

f $f(x) = x^3 - 6x^2$

g $f(x) = -2x^3 + 4x$

h $f(x) = -4x^3 + 15x^2 + 18x + 3$

i $f(x) = 3x^4 - 16x^3 + 24x^2 - 2$

j $f(x) = 2x^3 + 9x^2 + 6x - 7$

k $f(x) = x^3 - 6x^2 + 3x - 1$

l $f(x) = x - 2\sqrt{x}$

m $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + 11$

n $f(x) = x^4 - 4x^3 + 2x^2 + 4x + 1$

3 Consider $f(x) = \dfrac{4x}{x^2+1}$.

a Show that $f'(x) = \dfrac{-4(x+1)(x-1)}{(x^2+1)^2}$ and draw its sign diagram.

b Hence, find intervals where $y = f(x)$ is increasing or decreasing.

4 Consider $f(x)=\dfrac{4x}{(x-1)^2}$.

- **a** Show that $f'(x)=\dfrac{-4(x+1)}{(x-1)^3}$ and draw its sign diagram.
- **b** Hence, find intervals where $y=f(x)$ is increasing or decreasing.

5 Consider $f(x)=\dfrac{-x^2+4x-7}{x-1}$.

- **a** Show that $f'(x)=\dfrac{-(x+1)(x-3)}{(x-1)^2}$ and draw its sign diagram.
- **b** Hence, find intervals where $y=f(x)$ is increasing or decreasing.

6 Find intervals where $f(x)$ is increasing or decreasing if:

- **a** $f(x)=\dfrac{x^3}{x^2-1}$
- **b** $f(x)=x^2+\dfrac{4}{x-1}$

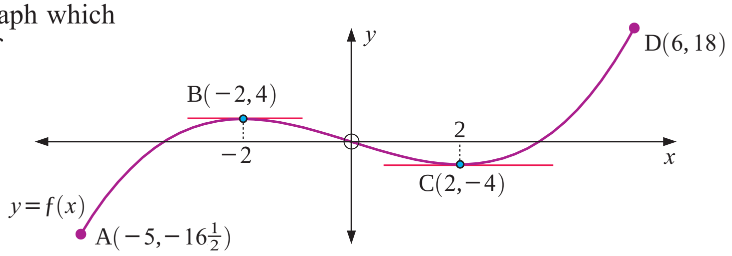
STATIONARY POINTS

A **stationary point** of a function is a point such that $f'(x)=0$.

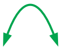
It could be a local maximum, local minimum, or horizontal inflection.

TURNING POINTS (MAXIMA AND MINIMA)

Consider the following graph which has a restricted domain of $-5\leqslant x\leqslant 6$.



A is a **global minimum** as it is the minimum value of y on the entire domain.

B is a **local maximum** as it is a turning point where the curve has shape  and $f'(x)=0$ at that point.

C is a **local minimum** as it is a turning point where the curve has shape  and $f'(x)=0$ at that point.

D is a **global maximum** as it is the maximum value of y on the entire domain.

For many functions, a local maximum or minimum is also the global maximum or minimum.

For example, for $y=x^2$ the point (0, 0) is a local minimum and is also the global minimum.

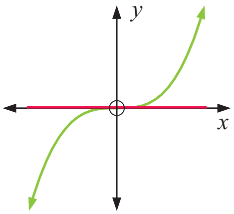
HORIZONTAL OR STATIONARY POINTS OF INFLECTION

It is not always true that whenever we find a value of x where $f'(x)=0$ we have a local maximum or minimum.

For example, $f(x) = x^3$ has $f'(x) = 3x^2$
and $f'(x) = 0$ when $x = 0$.

The x-axis is a tangent to the curve which actually crosses over the curve at $O(0,\ 0)$. This tangent is horizontal but $O(0,\ 0)$ is neither a local maximum nor a local minimum.

It is called a **horizontal inflection** (or **inflexion**) as the curve changes its curvature or shape.

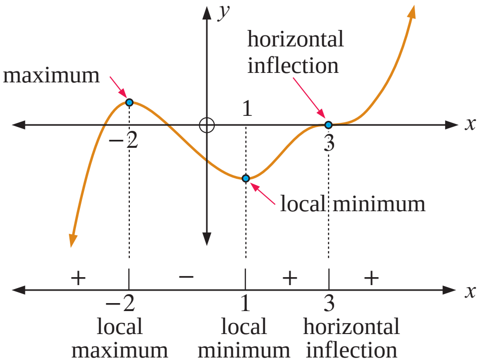


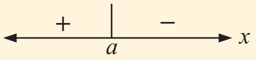
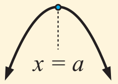
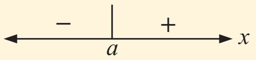
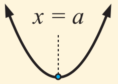
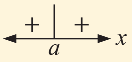
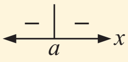
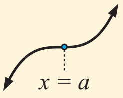
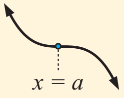
SIGN DIAGRAMS

Consider the graph alongside.

The sign diagram of its gradient function is shown directly beneath it.

We can use the sign diagram to describe the stationary points of the function.



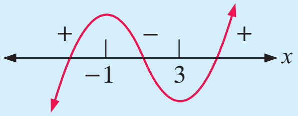
Stationary point	*Sign diagram of $f'(x)$ near $x = a$*	*Shape of curve near $x = a$*
local maximum		
local minimum		
horizontal inflection or stationary inflection	 or 	 or 

Example 8 **Self Tutor**

Find and classify all stationary points of $f(x) = x^3 - 3x^2 - 9x + 5$.

$$\begin{aligned} f(x) &= x^3 - 3x^2 - 9x + 5 \\ \therefore\quad f'(x) &= 3x^2 - 6x - 9 \\ &= 3(x^2 - 2x - 3) \\ &= 3(x-3)(x+1) \end{aligned}$$

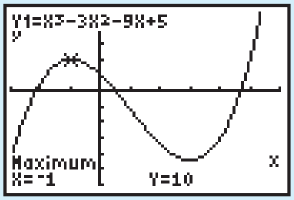
which has sign diagram:



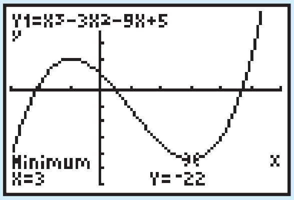
So, we have a local maximum at $x = -1$ and a local minimum at $x = 3$.

$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 5$
$= 10$

$f(3) = 3^3 - 3 \times 3^2 - 9 \times 3 + 5$
$= -22$



There is a local maximum at $(-1, 10)$.



There is a local minimum at $(3, -22)$.

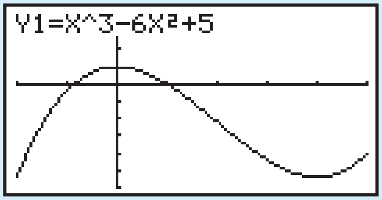
If we are asked to find the greatest or least value on an interval, then we should always check the endpoints. We seek the *global* maximum or minimum on the given domain.

Example 9 **Self Tutor**

Find the greatest and least value of $x^3 - 6x^2 + 5$ on the interval $-2 \leqslant x \leqslant 5$.

First we graph $y = x^3 - 6x^2 + 5$ on $-2 \leqslant x \leqslant 5$.

In this case the greatest value is clearly at the local maximum when $\frac{dy}{dx} = 0$.



Now $\frac{dy}{dx} = 3x^2 - 12x$
$= 3x(x-4)$

and $\frac{dy}{dx} = 0$ when $x = 0$ or 4.

So, the greatest value is $f(0) = 5$ when $x = 0$.

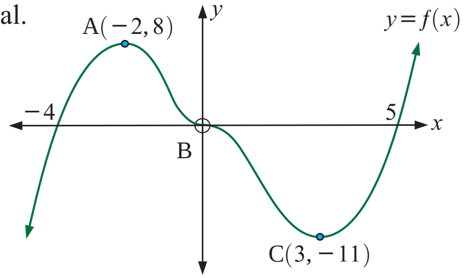
The least value is either $f(-2)$ or $f(4)$, whichever is smaller.

Now $f(-2) = -27$ and $f(4) = -27$

$\therefore$ least value is -27 when $x = -2$ *and* when $x = 4$.

EXERCISE 18D.2

1 The tangents at points A, B and C are horizontal.



- **a** Classify points A, B and C.
- **b** Draw a sign diagram for the gradient function $f'(x)$ for all x.
- **c** State intervals where $y = f(x)$ is:
  - **i** increasing **ii** decreasing.
- **d** Draw a sign diagram for $f(x)$ for all x.
- **e** Comment on the differences between the sign diagrams found above.

2 For each of the following functions, find and classify the stationary points, and hence sketch the function showing all important features.

a $f(x) = x^2 - 2$
b $f(x) = x^3 + 1$
c $f(x) = x^3 - 3x + 2$
d $f(x) = x^4 - 2x^2$
e $f(x) = x^3 - 6x^2 + 12x + 1$
f $f(x) = \sqrt{x} + 2$
g $f(x) = x - \sqrt{x}$
h $f(x) = x^4 - 6x^2 + 8x - 3$
i $f(x) = 1 - x\sqrt{x}$
j $f(x) = x^4 - 2x^2 - 8$

3 At what value of x does the quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$, have a stationary point? Under what conditions is the stationary point a local maximum or a local minimum?

4 $f(x) = 2x^3 + ax^2 - 24x + 1$ has a local maximum at $x = -4$. Find a.

5 $f(x) = x^3 + ax + b$ has a stationary point at $(-2, 3)$.

a Find the values of a and b.
b Find the position and nature of all stationary points.

6 The cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ touches the line with equation $y = 9x + 2$ at the point $(0, 2)$, and has a stationary point at $(-1, -7)$. Find $P(x)$.

7 Find the greatest and least value of:

a $x^3 - 12x - 2$ for $-3 \leqslant x \leqslant 5$
b $4 - 3x^2 + x^3$ for $-2 \leqslant x \leqslant 3$

8 A manufacturing company makes door hinges. They have a standing order filled by producing 50 each hour, but production of more than 150 per hour is useless as they will not sell. The cost function for making x hinges per hour is:

$$C(x) = 0.0007x^3 - 0.1796x^2 + 14.663x + 160 \text{ dollars}$$

where $50 \leqslant x \leqslant 150$. Find the minimum and maximum hourly costs, and the production levels when each occurs.

E RATIONAL FUNCTIONS

Rational functions have the form $f(x) = \dfrac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ are polynomials.

For example, $f(x) = \dfrac{2x - 1}{x^2 + 2}$ and $f(x) = \dfrac{x^2 - 4}{x^2 - 3x + 2}$ are rational functions.

In this course we consider rational functions for which $g(x)$ and $h(x)$ are either linear or quadratic. We have already considered some rational functions of the form $y = \dfrac{\text{linear}}{\text{linear}}$ earlier in the text. We saw that one feature of these functions is the presence of **asymptotes**. These are lines (or curves) that the graph of the function approaches when x or y takes large values.

Vertical asymptotes can be found by solving $h(x) = 0$.

Horizontal asymptotes can be found by finding what value $f(x)$ approaches as $x \to \pm\infty$.

Oblique asymptotes are neither horizontal nor vertical. They are not covered in this course.

FUNCTIONS OF THE FORM $y = \dfrac{\textbf{linear}}{\textbf{linear}}$

Example 10 **Self Tutor**

Consider $f(x) = \dfrac{2x-3}{x+2}$.

a Write $f(x)$ in the form $a + \dfrac{b}{x+2}$ and hence find the equations of its asymptotes.

b Find $f'(x)$ and draw its sign diagram.

c Find the axes intercepts.

d Sketch its graph.

a
$$f(x) = \frac{2x-3}{x+2} = \frac{2(x+2)-7}{x+2} = 2 - \frac{7}{x+2}$$

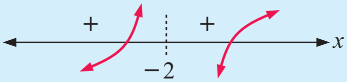
Vertical asymptote is $x + 2 = 0$ or $x = -2$.

As $x \to \pm\infty$, $f(x) \to 2$.

$\therefore$ horizontal asymptote is $y = 2$.

b
$$f'(x) = \frac{2(x+2)-(2x-3)1}{(x+2)^2} = \frac{2x+4-2x+3}{(x+2)^2} = \frac{7}{(x+2)^2}$$

which has sign diagram:



As $f'(x)$ is never zero, $y = f(x)$ has no turning points.

c It cuts the x-axis when $y = 0$

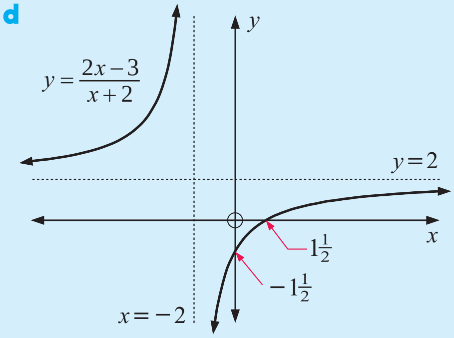
$\therefore$ $2x - 3 = 0$

$\therefore$ $x = \frac{3}{2} = 1\frac{1}{2}$

It cuts the y-axis when $x = 0$

$\therefore$ $y = -\frac{3}{2} = -1\frac{1}{2}$

d



EXERCISE 18E.1

1 Write the following functions in the form $f(x) = a + \dfrac{b}{cx+d}$ and hence determine the equations of their asymptotes:

a $f(x) = \dfrac{3x-2}{x+1}$ **b** $f(x) = \dfrac{x-4}{2x-1}$ **c** $f(x) = \dfrac{4-2x}{x-1}$

2 For each of the following functions:

- **i** state the equations of the asymptotes, giving reasons for your answers
- **ii** find $f'(x)$ and draw its sign diagram
- **iii** find the axes intercepts
- **iv** sketch the graph of $y = f(x)$.

a $f(x) = -3 + \dfrac{1}{4-x}$

b $f(x) = \dfrac{x}{x+2}$

c $f(x) = \dfrac{4x+3}{x-2}$

d $f(x) = \dfrac{1-x}{x+2}$

FUNCTIONS OF THE FORM $y = \dfrac{\text{linear}}{\text{quadratic}}$

For functions of this form, we notice that as $x \to \pm\infty$, $f(x) \to 0$. They will therefore all have the horizontal asymptote $y = 0$.

Example 11 **Self Tutor**

Consider $f(x) = \dfrac{3x-9}{x^2-x-2}$.

a Determine the equations of any asymptotes.

b Find $f'(x)$ and determine the position and nature of any stationary points.

c Find the axes intercepts.

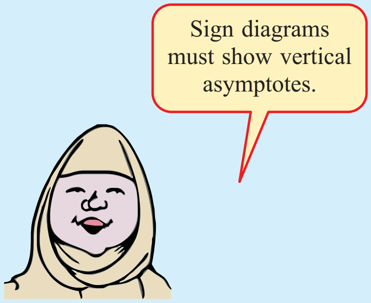
d Sketch the graph of the function.

a $f(x) = \dfrac{3x-9}{x^2-x-2} = \dfrac{3x-9}{(x-2)(x+1)}$

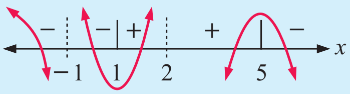
Vertical asymptotes are $x = 2$ and $x = -1$ {when the denominator is 0}

Horizontal asymptote is $y = 0$ {as $x \to \pm\infty$, $f(x) \to 0$}

b
$$f'(x) = \frac{3(x^2-x-2)-(3x-9)(2x-1)}{(x-2)^2(x+1)^2} \quad \text{\{quotient rule\}}$$
$$= \frac{3x^2-3x-6-[6x^2-21x+9]}{(x-2)^2(x+1)^2}$$
$$= \frac{-3x^2+18x-15}{(x-2)^2(x+1)^2}$$
$$= \frac{-3(x^2-6x+5)}{(x-2)^2(x+1)^2}$$
$$= \frac{-3(x-5)(x-1)}{(x-2)^2(x+1)^2}$$



$f'(x)$ has sign diagram:



There is a local maximum when $x = 5$
and a local minimum when $x = 1$

The local maximum is $(5, \frac{1}{3})$.

The local minimum is $(1, 3)$.

c Cuts the x-axis when $y = 0$

$\therefore \quad 3x - 9 = 0$ or $x = 3$

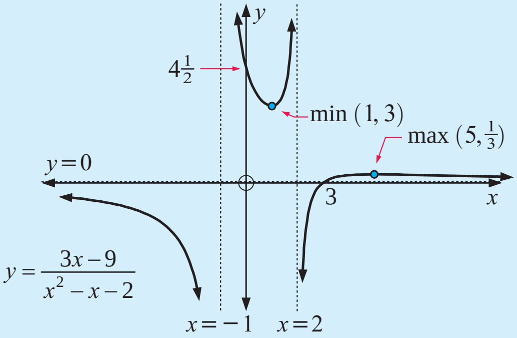
So, the x-intercept is 3.

Cuts the y-axis when $x = 0$

$\therefore \quad y = \dfrac{-9}{-2} = 4\frac{1}{2}$

So, the y-intercept is $4\frac{1}{2}$.

d



EXERCISE 18E.2

1 Determine the equations of the asymptotes of:

a $y = \dfrac{2x}{x^2 - 4}$ **b** $y = \dfrac{1-x}{(x+2)^2}$ **c** $y = \dfrac{3x+2}{x^2+1}$

2 For each of the following functions:

- **i** determine the equation(s) of the asymptotes
- **ii** find $f'(x)$ and hence determine the position and nature of any stationary points
- **iii** find the axes intercepts
- **iv** sketch the graph of the function, showing all information in **i**, **ii** and **iii**.

a $f(x) = \dfrac{4x}{x^2+1}$ **b** $f(x) = \dfrac{4x}{x^2 - 4x - 5}$

c $f(x) = \dfrac{4x}{(x-1)^2}$ **d** $f(x) = \dfrac{3x-3}{(x+2)^2}$

FUNCTIONS OF THE FORM $y = \dfrac{\textbf{quadratic}}{\textbf{quadratic}}$

Functions such as $y = \dfrac{2x^2 - x + 3}{x^2 + x - 2}$ have a **horizontal asymptote** which can be found by dividing every term by x^2.

Notice that $y = \dfrac{2 - \frac{1}{x} + \frac{3}{x^2}}{1 + \frac{1}{x} - \frac{2}{x^2}}$ so as $x \to \pm\infty$, $y \to \dfrac{2}{1} = 2$.

Example 12 **Self Tutor**

Consider $f(x) = \dfrac{x^2 - 3x + 2}{x^2 + 3x + 2}$.

a Determine the equations of its asymptotes.

b Find $f'(x)$ and determine the position and nature of any turning points.

c Find the axes intercepts.

d Sketch the graph of the function.

a $f(x) = \dfrac{x^2-3x+2}{x^2+3x+2} = \dfrac{1-\frac{3}{x}+\frac{2}{x^2}}{1+\frac{3}{x}+\frac{2}{x^2}}$ so as $x \to \pm\infty$, $y \to 1$

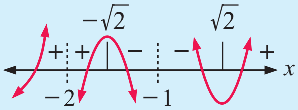
$\therefore$ horizontal asymptote is $y = 1$.

$f(x) = \dfrac{x^2-3x+2}{(x+1)(x+2)}$ $\therefore$ vertical asymptotes are $x = -1$ and $x = -2$.

b $$f'(x) = \frac{(2x-3)(x^2+3x+2)-(x^2-3x+2)(2x+3)}{(x+1)^2(x+2)^2}$$

$$= \frac{6x^2-12}{(x+1)^2(x+2)^2} \qquad \{\text{on simplifying}\}$$

$$= \frac{6(x+\sqrt{2})(x-\sqrt{2})}{(x+1)^2(x+2)^2}$$ and has sign diagram



So, we have a local maximum at $x = -\sqrt{2}$ and a local minimum at $x = \sqrt{2}$.

The local maximum is $(-\sqrt{2}, -34.0)$. The local minimum is $(\sqrt{2}, -0.0294)$.

c Cuts the x-axis when $y = 0$

$\therefore \quad x^2-3x+2 = 0$

$\therefore \quad (x-1)(x-2) = 0$

$\therefore \quad x = 1$ or 2

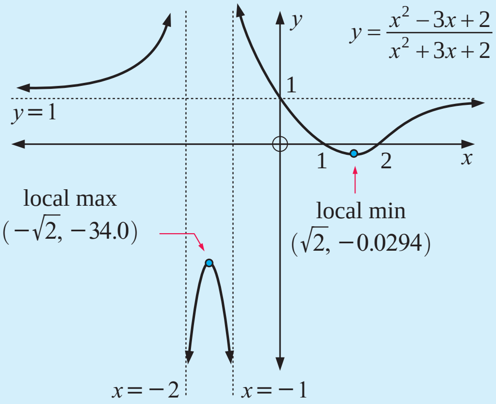
So, the x-intercepts are 1 and 2.

Cuts the y-axis when $x = 0$

$\therefore \quad y = \frac{2}{2} = 1$

So, the y-intercept is 1.

d



EXERCISE 18E.3

1 Determine the equations of the asymptotes of:

a $y = \dfrac{2x^2-x+2}{x^2-1}$ **b** $y = \dfrac{-x^2+2x-1}{x^2+x+1}$ **c** $y = \dfrac{3x^2-x+2}{(x+2)^2}$

2 For each of the following functions:

i determine the equation(s) of the asymptotes

ii determine the position and nature of any stationary points

iii find the axes intercepts

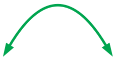
iv sketch the function, showing all information obtained in **i**, **ii** and **iii**.

a $y = \dfrac{x^2-x}{x^2-x-6}$ **b** $y = \dfrac{x^2-1}{x^2+1}$

c $y = \dfrac{x^2-5x+4}{x^2+5x+4}$ **d** $y = \dfrac{x^2-6x+5}{(x+1)^2}$

F INFLECTIONS AND SHAPE

When a curve, or part of a curve, has shape:

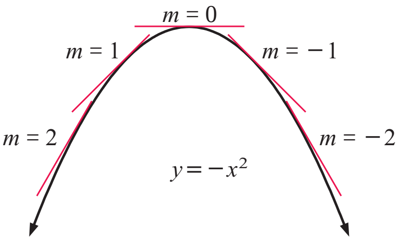


we say that the shape is **concave downwards**

we say that the shape is **concave upwards**.

TEST FOR SHAPE

Consider the **concave downwards** curve:



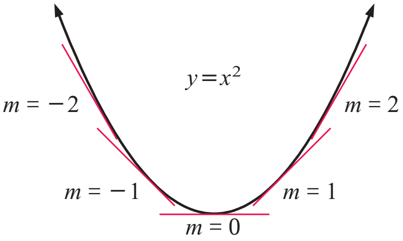
Wherever we are on the curve, as x is increased, the gradient of the tangent decreases.

$\therefore$ $f'(x)$ is decreasing

$\therefore$ its derivative is negative,

which means $f''(x) < 0$.

Likewise, if the curve is **concave upwards**:



Wherever we are on the curve, as x is increased, the gradient of the tangent increases.

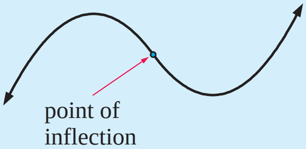
$\therefore$ $f'(x)$ is increasing

$\therefore$ its derivative is positive,

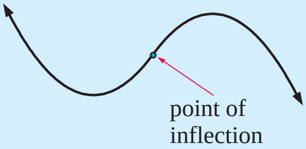
which means $f''(x) > 0$.

POINTS OF INFLECTION (INFLEXION)

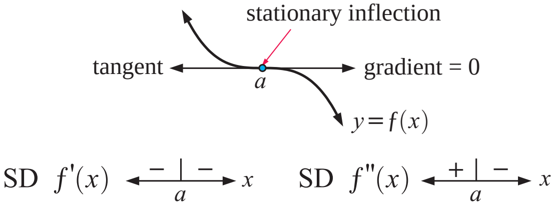
A **point of inflection** is a point on a curve at which there is a change of **curvature** or shape.



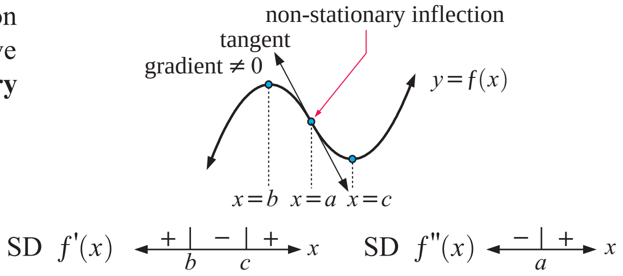
or



If the tangent at a point of inflection is horizontal then this point is also a stationary point. We say that we have a **horizontal** or **stationary inflection**.



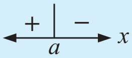
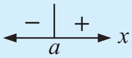
If the tangent at a point of inflection is not horizontal we say that we have a **non-horizontal** or **non-stationary inflection**.



The tangent at the point of inflection, also called the **inflecting tangent**, crosses the curve at that point.

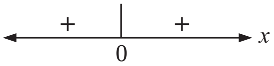
There is a **point of inflection** at $x = a$ if $f''(a) = 0$ **and** the sign of $f''(x)$ changes on either side of $x = a$.

The point of inflection corresponds to a change in curvature.

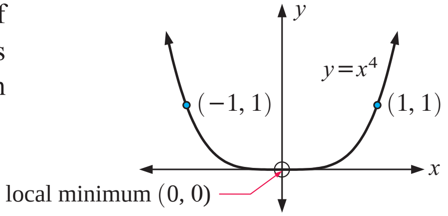
In the vicinity of a, $f''(x)$ has sign diagram either  or 

Observe that if $f(x) = x^4$ then

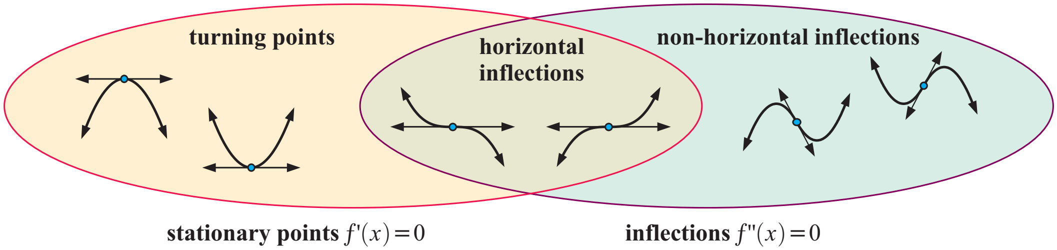
$f'(x) = 4x^3$ and $f''(x) = 12x^2$ and $f''(x)$ has sign diagram

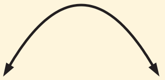


Although $f''(0) = 0$ we do not have a point of inflection at (0, 0) because the sign of $f''(x)$ does not change on either side of $x = 0$. In fact the graph of $f(x) = x^4$ is:



SUMMARY



	A curve is **concave downwards** on an interval S if $f''(x) \leqslant 0$ for all x in S.
	A curve is **concave upwards** on an interval S if $f''(x) \geqslant 0$ for all x in S.
If $f''(x)$ changes sign at $x = a$, and $f''(a) = 0$, then we have a • **horizontal inflection** if $f'(a) = 0$ • **non-horizontal inflection** if $f'(a) \neq 0$.	

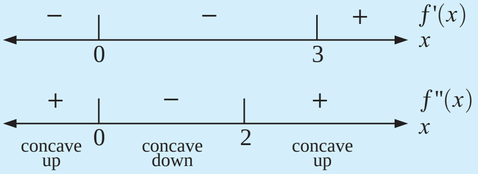
Click on the demo icon to examine some standard functions for turning points, points of inflection, and intervals where the function is increasing, decreasing, and concave up or down.

DEMO

Example 13 **Self Tutor**

Find and classify all points of inflection of $f(x) = x^4 - 4x^3 + 5$.

$$\begin{aligned} f(x) &= x^4 - 4x^3 + 5 \\ \therefore\ f'(x) &= 4x^3 - 12x^2 = 4x^2(x-3) \\ \therefore\ f''(x) &= 12x^2 - 24x \\ &= 12x(x-2) \\ \therefore\ f''(x) &= 0 \text{ when } x = 0 \text{ or } 2 \end{aligned}$$



Since the signs of $f''(x)$ change about $x = 0$ and $x = 2$, these two points are points of inflection.

$$\begin{aligned} \text{Also}\quad f'(0) &= 0, & f'(2) &= 32 - 48 \neq 0 \\ \text{and}\quad f(0) &= 5, & f(2) &= 16 - 32 + 5 = -11 \end{aligned}$$

Thus $(0,\ 5)$ is a horizontal inflection, and $(2,\ -11)$ is a non-horizontal inflection.

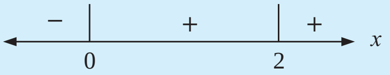
Example 14 **Self Tutor**

Consider $f(x) = 3x^4 - 16x^3 + 24x^2 - 9$.

- **a** Find and classify all points where $f'(x) = 0$.
- **b** Find and classify all points of inflection.
- **c** Find intervals where the function is increasing or decreasing.
- **d** Find intervals where the function is concave up or down.
- **e** Sketch the function showing *all* important features.

a

$$\begin{aligned} f(x) &= 3x^4 - 16x^3 + 24x^2 - 9 \\ \therefore\ f'(x) &= 12x^3 - 48x^2 + 48x \\ &= 12x(x^2 - 4x + 4) \\ &= 12x(x-2)^2 \end{aligned}$$

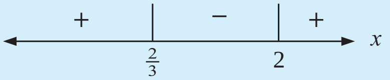


Now $f(0) = -9$ and $f(2) = 7$

$\therefore$ $(0,\ -9)$ is a local minimum and $(2,\ 7)$ is a horizontal inflection.

b

$$\begin{aligned} f''(x) &= 36x^2 - 96x + 48 \\ &= 12(3x^2 - 8x + 4) \\ &= 12(x-2)(3x-2) \end{aligned}$$

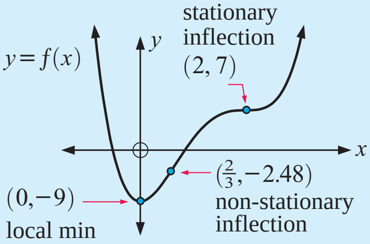


$\therefore$ $(2,\ 7)$ is a horizontal inflection and $\left(\frac{2}{3},\ f(\frac{2}{3})\right)$ or $\left(\frac{2}{3},\ -2.48\right)$ is a non-horizontal inflection.

c $f(x)$ is decreasing for $x \leqslant 0$
$f(x)$ is increasing for $x \geqslant 0$.

d $f(x)$ is concave up for $x \leqslant \frac{2}{3}$ and $x \geqslant 2$
$f(x)$ is concave down for $\frac{2}{3} \leqslant x \leqslant 2$.

e



EXERCISE 18F.1

1 Find and classify all points of inflection of:

a $f(x) = x^2 + 3$

b $f(x) = 2 - x^3$

c $f(x) = x^3 - 6x^2 + 9x + 1$

d $f(x) = x^3 + 6x^2 + 12x + 5$

e $f(x) = -3x^4 - 8x^3 + 2$

f $f(x) = 3 - \dfrac{1}{\sqrt{x}}$

2 For each of the following functions:

- **i** find and classify all points where $f'(x) = 0$
- **ii** find and classify all points of inflection
- **iii** find intervals where the function is increasing or decreasing
- **iv** find intervals where the function is concave up or down
- **v** sketch the function showing *all* important features.

a $f(x) = x^2$

b $f(x) = x^3$

c $f(x) = \sqrt{x}$

d $f(x) = x^3 - 3x^2 - 24x + 1$

e $f(x) = 3x^4 + 4x^3 - 2$

f $f(x) = (x-1)^4$

g $f(x) = x^4 - 4x^2 + 3$

h $f(x) = 3 - \dfrac{4}{\sqrt{x}}$

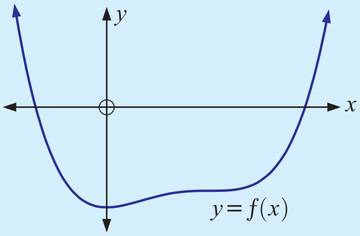
ZEROS OF $f'(x)$ AND $f''(x)$

$f''(x) = 0$ corresponds to the stationary points of $y = f'(x)$, so $f''(x)$ will change sign at a local maximum or minimum of $y = f'(x)$. Such points correspond to the points of inflection of $y = f(x)$.

If a local maximum or minimum of $y = f'(x)$ *touches* the x-axis, then it corresponds to a stationary point of inflection of $y = f(x)$. Otherwise it corresponds to a non-stationary point of inflection.

Example 15 **Self Tutor**

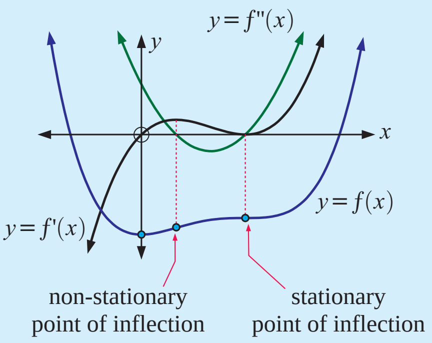
Using the graph of $y = f(x)$ alongside, sketch the graphs of $y = f'(x)$ and $y = f''(x)$.



The local minimum corresponds to $f'(x) = 0$ and $f''(x) \neq 0$.

The non-stationary point of inflection corresponds to $f'(x) \neq 0$ and $f''(x) = 0$.

The stationary point of inflection corresponds to $f'(x) = 0$ and $f''(x) = 0$.

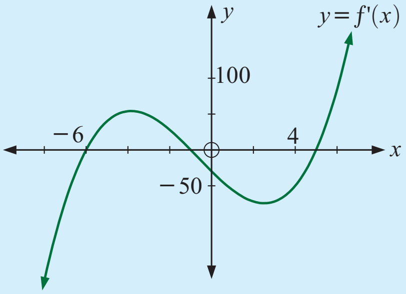


Example 16

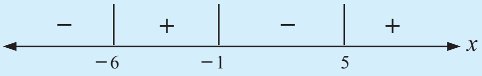
Self Tutor

The graph alongside shows a gradient function $y = f'(x)$.

Sketch a graph which could be $y = f(x)$, showing clearly the x-values corresponding to all stationary points and points of inflection.

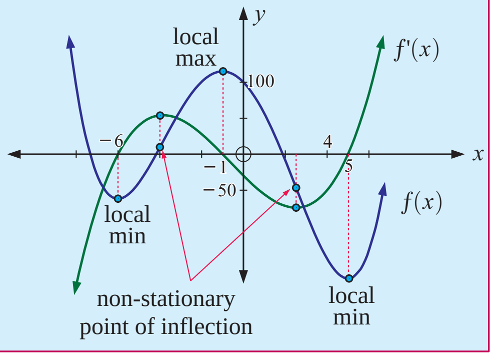


Sign diagram of $f'(x)$ is:



$f'(x)$ is a maximum when $x = -4$ and a minimum when $x \approx 2\frac{1}{2}$.

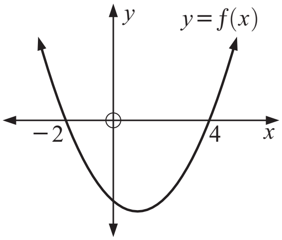
At these points $f''(x) = 0$ but $f'(x) \neq 0$, so they correspond to non-stationary points of inflection.



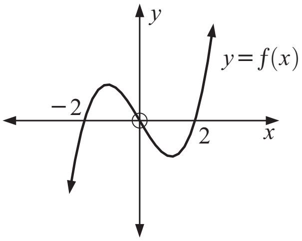
EXERCISE 18F.2

1 Using the graphs of $y = f(x)$ below, sketch the graphs of $y = f'(x)$ and $y = f''(x)$. Show clearly the axes intercepts and turning points.

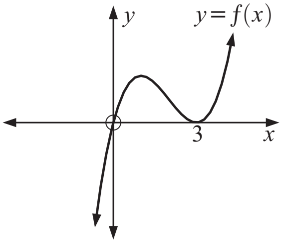
a



b

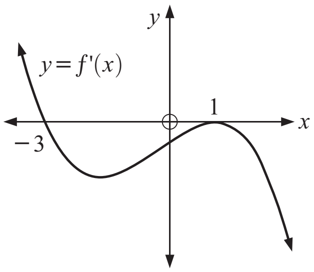


c

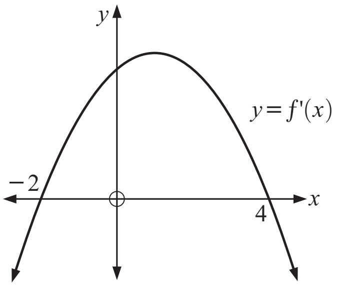


2 For the graphs of $y = f'(x)$ below, sketch a graph which could be $y = f(x)$. Show clearly the location of any stationary points and points of inflection.

a



b



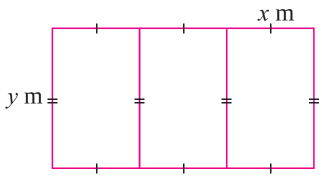
G OPTIMISATION

There are many problems for which we need to find the **maximum** or **minimum** value of a function. We can solve such problems using differential calculus techniques. The solution is often referred to as the **optimum** solution and the process is called **optimisation**.

Consider the following problem:

An industrial shed is to have a total floor space of 600 m^2 and is to be divided into 3 rectangular rooms of equal size. The walls, internal and external, will cost $60 per metre to build. What dimensions should the shed have to minimise the cost of the walls?

We let each room be x m by y m as shown.



Clearly $x > 0$ and $y > 0$.

The total length of wall material is $L = 6x + 4y$ m.

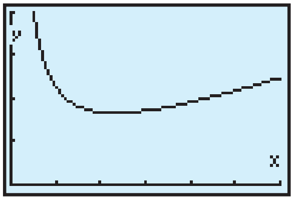
We know that the total area is 600 m^2,

so $3x \times y = 600$ and hence $y = \dfrac{200}{x}$.

Knowing this relationship enables us to write L in terms of one variable, in this case x.

$$L = 6x + 4\left(\frac{200}{x}\right) = \left(6x + \frac{800}{x}\right) \text{ m}$$

The cost is $60 per metre, so the total cost is $C(x) = 60\left(6x + \dfrac{800}{x}\right)$ dollars.



Now $C(x) = 360x + 48\,000x^{-1}$

$\therefore \quad C'(x) = 360 - 48\,000x^{-2}$

$\therefore \quad C'(x) = 0$ when $360 = \dfrac{48\,000}{x^2}$

$\therefore \quad x^2 = \dfrac{48\,000}{360} \approx 133.333$

$\therefore \quad x \approx 11.547$

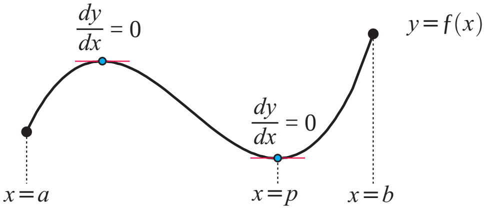
Now when $x \approx 11.547$, $y \approx \dfrac{200}{11.547} \approx 17.321$ and $C(11.547) \approx 8313.84$ dollars.

So, the minimum cost is about \$8310 when the shed is 34.6 m by 17.3 m.

WARNING

> The maximum or minimum value does not always occur when the first derivative is zero. It is essential to also examine the values of the function at the endpoint(s) of the domain for global maxima and minima.

For example:



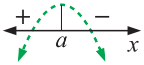
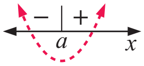
The maximum value of y occurs at the endpoint $x = b$. The minimum value of y occurs at the local minimum $x = p$.

TESTING OPTIMAL SOLUTIONS

If one is trying to optimise a function $f(x)$ and we find values of x such that $f'(x) = 0$, there are several tests we can use to see whether we have a maximum or a minimum solution:

SIGN DIAGRAM TEST

If near to $x = a$ where $f'(a) = 0$ the sign diagram is:

-  we have a **local maximum**
-  we have a **local minimum**.

SECOND DERIVATIVE TEST

At $x = a$ where $f'(a) = 0$:

- If $\dfrac{d^2y}{dx^2} < 0$ we have  shape, which is a **local maximum**
- If $\dfrac{d^2y}{dx^2} > 0$ we have  shape, which is a **local minimum**.

GRAPHICAL TEST

If the graph of $y = f(x)$ shows:

-  we have a **local maximum**
-  we have a **local minimum**.

OPTIMISATION PROBLEM SOLVING METHOD

Step 1: Draw a large, clear diagram of the situation.

Step 2: Construct a formula with the variable to be **optimised** (maximised or minimised) as the subject. It should be written in terms of **one** convenient **variable**, x say. You should write down what restrictions there are on x.

Step 3: Find the **first derivative** and find the values of x when it is **zero**.

Step 4: If there is a restricted domain such as $a \leqslant x \leqslant b$, the maximum or minimum may occur either when the derivative is zero or else at an endpoint.

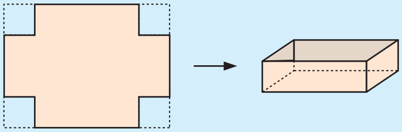
Show using the **sign diagram test**, the **second derivative test** or the **graphical test**, that you have a maximum or a minimum situation.

Example 17

Self Tutor

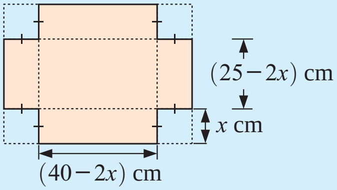
A rectangular cake dish is made by cutting out squares from the corners of a 25 cm by 40 cm rectangle of tin-plate, and then folding the metal to form the container.

What size squares must be cut out to produce the cake dish of maximum volume?



DEMO

Step 1: Let x cm be the side lengths of the squares that are cut out.



Step 2:

$$\begin{aligned}\text{Volume} &= \text{length} \times \text{width} \times \text{depth}\\ &= (40-2x)(25-2x)x\\ &= (1000-80x-50x+4x^2)x\\ &= 1000x-130x^2+4x^3 \text{ cm}^3\end{aligned}$$

Notice that $x > 0$ and $25 - 2x > 0$

$\therefore\ 0 < x < 12.5$

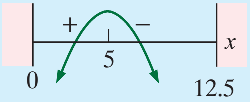
Step 3:

$$\begin{aligned}\frac{dV}{dx} &= 12x^2-260x+1000\\ &= 4(3x^2-65x+250)\\ &= 4(3x-50)(x-5)\end{aligned}$$

$\therefore\ \dfrac{dV}{dx} = 0$ when $x = \frac{50}{3} = 16\frac{2}{3}$ or $x = 5$

Step 4: **Sign diagram test**

$\dfrac{dV}{dx}$ has sign diagram:



or **Second derivative test**

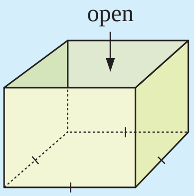
$\dfrac{d^2V}{dx^2} = 24x - 260$ so when $x = 5$, $\dfrac{d^2V}{dx^2} = -140$ which is < 0

$\therefore$ the shape is  and we have a local maximum.

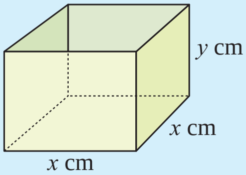
So, the maximum volume is obtained when $x = 5$, which is when 5 cm squares are cut from the corners.

Example 18 **Self Tutor**

A 4 litre container must have a square base, vertical sides, and an open top. Find the most economical shape which minimises the surface area of material needed.



Step 1:

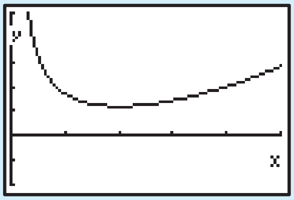


Let the base lengths be x cm and the depth be y cm. The volume

$$V = \text{length} \times \text{width} \times \text{depth}$$
$$\therefore \quad V = x^2y$$
$$\therefore \quad 4000 = x^2y \quad .... \ (1) \quad \{\text{as } 1 \text{ litre} \equiv 1000 \text{ cm}^3\}$$

Step 2: The total surface area

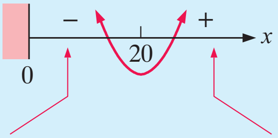
$$A = \text{area of base} + 4(\text{area of one side})$$
$$= x^2 + 4xy$$
$$= x^2 + 4x\left(\frac{4000}{x^2}\right) \qquad \{\text{using } (1)\}$$
$$\therefore \quad A(x) = x^2 + 16\,000x^{-1} \quad \text{where} \quad x > 0$$



Step 3:

$$A'(x) = 2x - 16\,000x^{-2}$$
$$\therefore \quad A'(x) = 0 \quad \text{when} \quad 2x = \frac{16\,000}{x^2}$$
$$\therefore \quad 2x^3 = 16\,000$$
$$\therefore \quad x = \sqrt[3]{8000} = 20$$

Step 4: **Sign diagram test** *or* **Second derivative test**



If $x = 10$,

$$A'(10) = 20 - \frac{16\,000}{100}$$
$$= 20 - 160$$
$$= -140$$

If $x = 30$,

$$A'(30) = 60 - \frac{16\,000}{900}$$
$$\approx 60 - 17.8$$
$$\approx 42.2$$

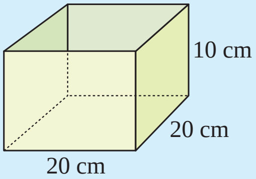
Second derivative test

$$A''(x) = 2 + 32\,000x^{-3}$$
$$= 2 + \frac{32\,000}{x^3}$$

which is always positive as $x^3 > 0$ for all $x > 0$.

Both tests establish that the minimum material is used to make the container when $x = 20$ and $y = \dfrac{4000}{20^2} = 10$.

So,



is the most economical shape.

Sometimes the variable to be optimised is in the form of a single square root function. In these situations it is convenient to square the function and use the fact that if $A > 0$, the optimum value of $A(x)$ occurs at the same value of x as the optimum value of $[A(x)]^2$.

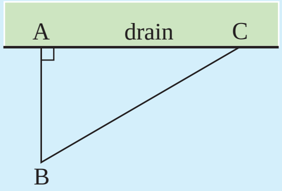
Example 19

Self Tutor

An animal enclosure is a right angled triangle with one leg being a drain. The farmer has 300 m of fencing available for the other two sides, [AB] and [BC].

a If $AB = x$ m, show that $AC = \sqrt{90\,000 - 600x}$.

b Find the maximum area of the triangular enclosure.

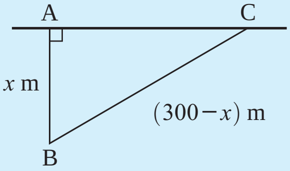


a $(AC)^2 + x^2 = (300 - x)^2$ {Pythagoras}

$\therefore \quad (AC)^2 = 90\,000 - 600x + x^2 - x^2$

$= 90\,000 - 600x$

$\therefore \quad AC = \sqrt{90\,000 - 600x}$



b The area of triangle ABC is

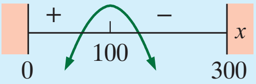
$$A(x) = \tfrac{1}{2}(\text{base} \times \text{altitude})$$
$$= \tfrac{1}{2}(AC \times x)$$
$$= \tfrac{1}{2}x\sqrt{90\,000 - 600x}$$

$0 < x < 300$

$$\therefore \quad [A(x)]^2 = \frac{x^2}{4}(90\,000 - 600x)$$
$$= 22\,500x^2 - 150x^3$$

$$\therefore \quad \frac{d}{dx}[A(x)]^2 = 45\,000x - 450x^2$$
$$= 450x(100 - x)$$

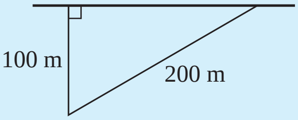
with sign diagram:



$A(x)$ is maximised when $x = 100$

so $A_{\max} = \tfrac{1}{2}(100)\sqrt{90\,000 - 60\,000}$

$\approx 8660 \text{ m}^2$

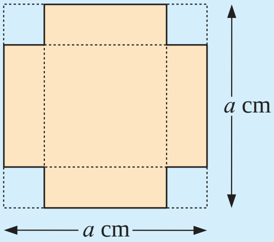


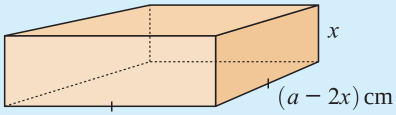
Example 20

Self Tutor

A square sheet of metal has smaller squares cut from its corners as shown.

What sized square should be cut out so that when the sheet is bent into an open box it will hold the maximum amount of liquid?





Let x cm by x cm squares be cut out.

$$\begin{aligned}\text{Volume} &= \text{length} \times \text{width} \times \text{depth}\\ &= (a-2x)\times(a-2x)\times x\end{aligned}$$

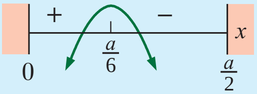
$\therefore \quad V(x) = x(a-2x)^2$

$$\begin{aligned}\text{Now}\quad V'(x) &= 1(a-2x)^2 + x \times 2(a-2x)^1 \times (-2) \quad \{\text{product rule}\}\\ &= (a-2x)[a-2x-4x]\\ &= (a-2x)(a-6x)\end{aligned}$$

$\therefore \quad V'(x) = 0$ when $x = \dfrac{a}{2}$ or $\dfrac{a}{6}$

However, $a - 2x$ must be > 0 and so $x < \dfrac{a}{2}$

$\therefore \quad x = \dfrac{a}{6}$ is the only value in $0 < x < \dfrac{a}{2}$ with $V'(x) = 0$.

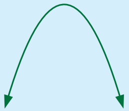


Second derivative test:

$$\begin{aligned}\text{Now}\quad V''(x) &= -2(a-6x) + (a-2x)(-6) \quad \{\text{product rule}\}\\ &= -2a + 12x - 6a + 12x\\ &= 24x - 8a\end{aligned}$$

$\therefore \quad V''\left(\dfrac{a}{6}\right) = 4a - 8a = -4a$ which is < 0

$\therefore$ the volume is maximised when $x = \dfrac{a}{6}$.



convex
downwards

Conclusion: The resulting container has maximum capacity when $x = \dfrac{a}{6}$.

Use **calculus techniques** to answer the following problems.

In cases where finding the zeros of the derivatives is difficult you may use a **graphics calculator** or **graphing package** to help you.

GRAPHING PACKAGE

EXERCISE 18G

1 A manufacturer can produce x fittings per day where $0 \leqslant x \leqslant 10\,000$. The production costs are:

- €1000 per day for the workers
- €2 per day per fitting
- €$\dfrac{5000}{x}$ per day for running costs and maintenance.

How many fittings should be produced daily to minimise costs?

2 For the cost function $C(x) = 720 + 4x + 0.02x^2$ dollars and price function $p(x) = 15 - 0.002x$ dollars, find the production level that will maximise profits.

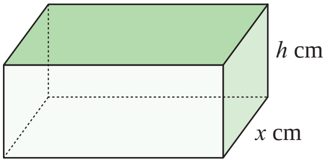
3 The total cost of producing x blankets per day is $\frac{1}{4}x^2 + 8x + 20$ dollars, and for this production level each blanket may be sold for $(23 - \frac{1}{2}x)$ dollars.
How many blankets should be produced per day to maximise the total profit?

4 The cost of running a boat is $£\frac{v^2}{10}$ per hour where v km h^{-1} is the speed of the boat. All other costs amount to £62.50 per hour. Find the speed which will minimise the total cost per kilometre.

5 A duck farmer wishes to build a rectangular enclosure of area 100 m^2. The farmer must purchase wire netting for three of the sides as the fourth side is an existing fence. Naturally, the farmer wishes to minimise the length (and therefore cost) of fencing required to complete the job.

- **a** If the shorter sides have length x m, show that the required length of wire netting to be purchased is $L = 2x + \frac{100}{x}$.
- **b** Use *technology* to help you sketch the graph of $y = 2x + \frac{100}{x}$.
- **c** Find the minimum value of L and the corresponding value of x when this occurs.
- **d** Sketch the optimum situation showing all dimensions.

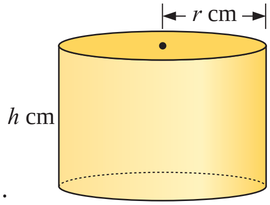
6 Radioactive waste is to be disposed of in fully enclosed lead boxes of inner volume 200 cm^3. The base of the box has dimensions in the ratio 2 : 1.



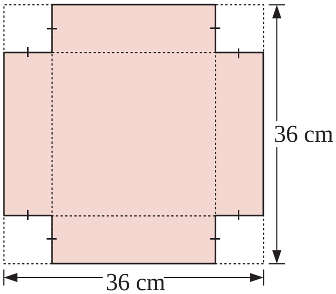
- **a** What is the inner length of the box?
- **b** Explain why $x^2h = 100$.
- **c** Explain why the inner surface area of the box is given by $A(x) = 4x^2 + \frac{600}{x}$ cm^2.
- **d** Use *technology* to help sketch the graph of $y = 4x^2 + \frac{600}{x}$.
- **e** Find the minimum inner surface area of the box and the corresponding value of x.
- **f** Sketch the optimum box shape showing all dimensions.

7 Consider the manufacture of cylindrical tin cans of 1 L capacity where the cost of the metal used is to be minimised. This means that the surface area must be as small as possible.

- **a** Explain why the height h is given by $h = \frac{1000}{\pi r^2}$ cm.
- **b** Show that the total surface area A is given by $A = 2\pi r^2 + \frac{2000}{r}$ cm^2.
- **c** Use *technology* to help you sketch the graph of A against r.
- **d** Find the value of r which makes A as small as possible.
- **e** Sketch the can of smallest surface area.

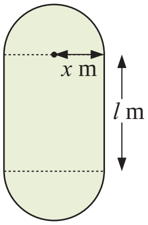


8 Sam has sheets of metal which are 36 cm by 36 cm square. He wants to cut out identical squares which are x cm by x cm from the corners of each sheet. He will then bend the sheets along the dashed lines to form an open container.



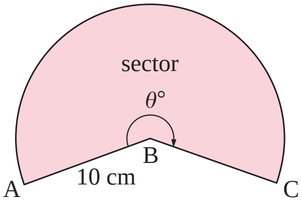
- **a** Show that the capacity of the container is given by $V(x) = x(36 - 2x)^2$ cm^3.
- **b** What sized squares should be cut out to produce the container of greatest capacity?

9 An athletics track has two 'straights' of length l m and two semicircular ends of radius x m. The perimeter of the track is 400 m.

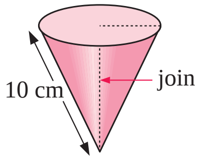


- **a** Show that $l = 200 - \pi x$ and hence write down the possible values that x may have.
- **b** Show that the area inside the track is $A = 400x - \pi x^2$.
- **c** What values of l and x produce the largest area inside the track?

10 A sector of radius 10 cm and angle θ^o is bent to form a conical cup as shown.



becomes

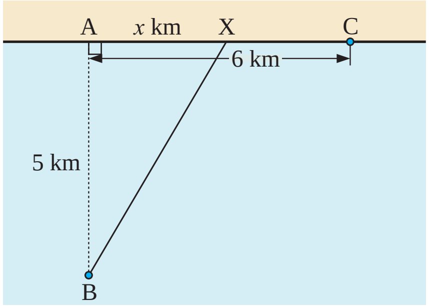


when edges [AB] and [CB] are joined with tape

Suppose the resulting cone has base radius r cm and height h cm.

- **a** Show using the sector that arc AC $= \dfrac{\theta\pi}{18}$.
- **b** Explain why $r = \dfrac{\theta}{36}$.
- **c** Show that $h = \sqrt{100 - \left(\frac{\theta}{36}\right)^2}$.
- **d** Find the cone's capacity V in terms of θ only.
- **e** Use technology to sketch the graph of $V(\theta)$.
- **f** Find θ when $V(\theta)$ is a maximum.

11 B is a row boat 5 km out at sea from A. [AC] is a straight sandy beach 6 km long. Peter can row the boat at 8 km h^{-1} and run along the beach at 17 km h^{-1}. Suppose Peter rows directly from B to point X on [AC] such that AX $= x$ km.

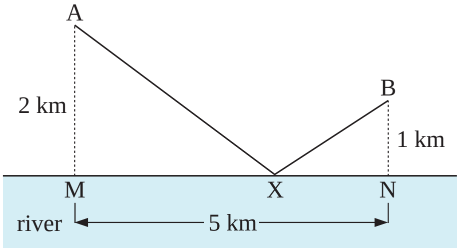


- **a** Explain why $0 \leqslant x \leqslant 6$.
- **b** If $T(x)$ is the *total time* Peter takes to row to X and then run along the beach to C, show that $T(x) = \dfrac{\sqrt{x^2 + 25}}{8} + \dfrac{6 - x}{17}$ hours.
- **c** Find x such that $\dfrac{dT}{dx} = 0$. What is the significance of this value of x? Prove your statement.

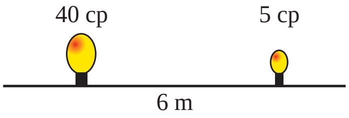
12 A pumphouse is to be placed at some point X along a river.

Two pipelines will then connect the pumphouse to homesteads A and B.

How far from M should point X be so that the total length of pipeline is minimised?

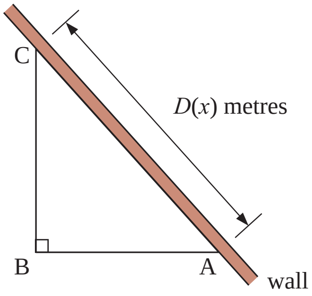


13 Open cylindrical bins are to contain 100 litres. Find the radius and height of the bin shape which minimises the surface area and thus the material needed.

14 

Two lamps with intensities 40 and 5 candle-power are placed 6 m apart. If the intensity of illumination I at any point is directly proportional to the power of the source, and inversely proportional to the square of the distance from the source, find the darkest point on the line joining the two lamps.

15 A right angled triangular pen is made from 24 m of fencing, all used for sides [AB] and [BC]. Side [AC] is an existing brick wall.

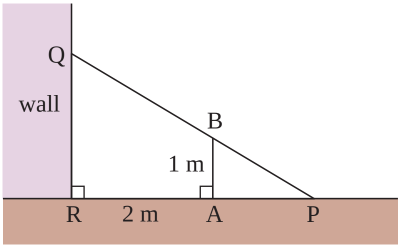


- **a** If $AB = x$ m, find $D(x)$ in terms of x.
- **b** Find $\dfrac{d[D(x)]^2}{dx}$ and hence draw a sign diagram for it.
- **c** Find the smallest and the greatest possible value of $D(x)$ and the design of the pen in each case.

16 At 1.00 pm a ship A leaves port P. It sails in the direction 030^o at 12 km h^{-1}. At the same time, ship B is 100 km due east of P, and is sailing at 8 km h^{-1} towards P.

- **a** Show that the distance $D(t)$ between the two ships is given by $D(t) = \sqrt{304t^2 - 2800t + 10\,000}$ km, where t is the number of hours after 1.00 pm.
- **b** Find the minimum value of $[D(t)]^2$ for all $t \geqslant 0$.
- **c** At what time, to the nearest minute, are the ships closest?

17 [AB] is a 1 m high fence which is 2 m from a vertical wall [RQ]. An extension ladder [PQ] is placed on the fence so that it touches the ground at P and the wall at Q.



- **a** If $AP = x$ m, find QR in terms of x.
- **b** If the ladder has length L m, show that $[L(x)]^2 = (x+2)^2\left(1 + \dfrac{1}{x^2}\right)$.
- **c** Show that $\dfrac{d[L(x)]^2}{dx} = 0$ only when $x = \sqrt[3]{2}$.
- **d** Find, correct to the nearest centimetre, the shortest length of the extension ladder. You must prove that this length is the shortest.

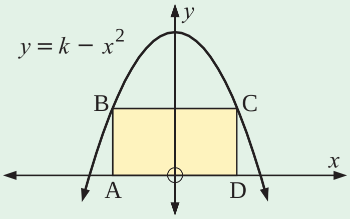
REVIEW SET 18A NON-CALCULATOR

1 A particle P moves in a straight line with position relative to the origin O given by $s(t) = 2t^3 - 9t^2 + 12t - 5$ cm, where t is the time in seconds, $t \geqslant 0$.

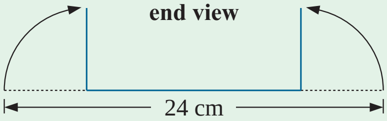
- **a** Find expressions for the particle's velocity and acceleration and draw sign diagrams for each of them.
- **b** Find the initial conditions.
- **c** Describe the motion of the particle at time $t = 2$ seconds.
- **d** Find the times and positions where the particle changes direction.
- **e** Draw a diagram to illustrate the motion of P.
- **f** Determine the time intervals when the particle's speed is increasing.

2 Rectangle ABCD is inscribed within the parabola $y = k - x^2$ and the x-axis, as shown.

- **a** If $\text{OD} = x$, show that the rectangle ABCD has area function $A(x) = 2kx - 2x^3$.
- **b** If the area of ABCD is a maximum when $\text{AD} = 2\sqrt{3}$, find k.



3 A rectangular gutter is formed by bending a 24 cm wide sheet of metal as shown in the illustration.

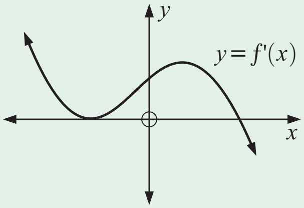


Where must the bends be made in order to maximise the capacity of the gutter?

4 Consider the function $f(x) = \dfrac{3x-2}{x+3}$.

- **a** State the equation of the vertical asymptote.
- **b** Find the axes intercepts.
- **c** Find $f'(x)$ and draw its sign diagram.
- **d** Find the position and nature of any stationary points.

5 Given the graph of $y = f'(x)$ drawn alongside, sketch a possible curve for $y = f(x)$. Show clearly any turning points and points of inflection.



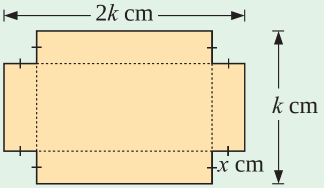
6 Consider $f(x) = \dfrac{x-2}{x^2+x-2}$.

- **a** Determine the equations of any asymptotes.
- **b** Find the position and nature of its turning points.
- **c** Find its axes intercepts.
- **d** Sketch the graph of the function showing the important features of **a**, **b** and **c**.
- **e** For what values of p does $\dfrac{x-2}{x^2+x-2} = p$ have two real distinct solutions?

7 A particle moves in a straight line with position relative to O given by $s(t) = 2t - \frac{4}{t}$ cm, where $t > 0$ is the time in seconds.

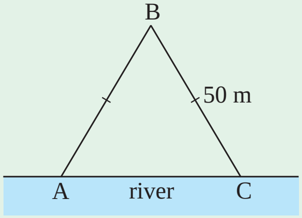
- **a** Find velocity and acceleration functions for the particle's motion and draw sign diagrams for each of them.
- **b** Describe the motion of the particle at $t = 1$ second.
- **c** Does the particle ever change direction? If so, where and when does it do this?
- **d** Draw a diagram to illustrate the motion of the particle.
- **e** Find the time intervals when the:
  - **i** velocity is increasing
  - **ii** speed is increasing.

8 A rectangular sheet of tin-plate is $2k$ cm by k cm. Four squares, each with sides x cm, are cut from its corners. The remainder is bent into the shape of an open rectangular container. Find the value of x which will maximise the capacity of the container.



REVIEW SET 18B **CALCULATOR**

1 A triangular pen is enclosed by two fences [AB] and [BC], each of length 50 m, with the river being the third side.



- **a** If AC $= 2x$ m, show that the area of triangle ABC is $A(x) = x\sqrt{2500 - x^2}$ m^2.
- **b** Find $\frac{d[A(x)]^2}{dx}$ and hence find x such that the area is a maximum.

2 Suppose $f(x) = x^3 + ax$, $a < 0$ has a turning point when $x = \sqrt{2}$.

- **a** Find a.
- **b** Find the position and nature of all stationary points of $y = f(x)$.
- **c** Sketch the graph of $y = f(x)$.

3 A particle moves along the x-axis with position relative to origin O given by $x(t) = 3t - \sqrt{t}$ cm, where t is the time in seconds, $t \geqslant 0$.

- **a** Find expressions for the particle's velocity and acceleration at any time t, and draw sign diagrams for each function.
- **b** Find the initial conditions and hence describe the motion at that instant.
- **c** Describe the motion of the particle at $t = 9$ seconds.
- **d** Find the time and position when the particle reverses direction.
- **e** Determine the time interval when the particle's speed is decreasing.

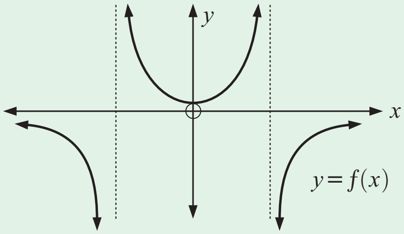
4 The cost per hour of running a freight train is given by $C(v) = \frac{v^2}{30} + \frac{9000}{v}$ dollars where v is the average speed of the train in km h^{-1}.

- **a** Find the cost of running the train for:
  - **i** two hours at 45 km h^{-1}
  - **ii** 5 hours at 64 km h^{-1}.
- **b** Find the rate of change in the hourly cost of running the train at speeds of:
  - **i** 50 km h^{-1}
  - **ii** 66 km h^{-1}.
- **c** At what speed will the cost per hour be a minimum?

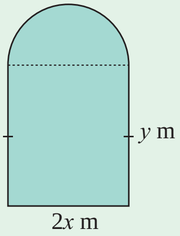
5 Consider the function $f(x) = \frac{x^2 - 1}{x^2 + 1}$.

- **a** Find the axes intercepts.
- **b** Explain why $f(x)$ has no vertical asymptotes.
- **c** Find the position and nature of any stationary points.
- **d** Show that $y = f(x)$ has non-stationary inflection points at $x = \pm\sqrt{\frac{1}{3}}$.
- **e** Sketch the graph of $y = f(x)$ showing all features found in **a**, **b**, **c** and **d** above.

6 The graph of $y = f(x)$ is given. On the same axes sketch the graph of $y = f'(x)$.



7 A 200 m fence is placed around a lawn which has the shape of a rectangle with a semi-circle on one of its sides.



- **a** Using the dimensions shown on the figure, show that $y = 100 - x - \frac{\pi}{2}x$.
- **b** Find the area of the lawn A in terms of x only.
- **c** Find the dimensions of the lawn if it has the maximum possible area.

REVIEW SET 18C

1 Consider the function $f(x) = 2x^3 - 3x^2 - 36x + 7$.

- **a** Find and classify all stationary points and points of inflection.
- **b** Find intervals where the function is increasing and decreasing.
- **c** Find intervals where the function is concave up or down.
- **d** Sketch the graph of $y = f(x)$ showing all important features.

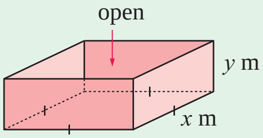
2 Consider the function $f(x) = x^3 - 4x^2 + 4x$.

- **a** Find all axes intercepts.

b Find and classify all stationary points and points of inflection.

c Sketch the graph of $y = f(x)$ showing features from **a** and **b**.

3 A manufacturer of open steel boxes has to make one with a square base and a capacity of 1 m^3. The steel costs \$2 per square metre.



a If the base measures x m by x m and the height is y m, find y in terms of x.

b Hence, show that the total cost of the steel is $C(x) = 2x^2 + \dfrac{8}{x}$ dollars.

c Find the dimensions of the box which would cost the least in steel to make.

4 A particle P moves in a straight line with position from O given by $s(t) = 15t - \dfrac{60}{(t-1)^2}$ cm, where t is the time in seconds, $t \geqslant 0$.

a Find velocity and acceleration functions for P's motion.

b Describe the motion of P at $t = 3$ seconds.

c For what values of t is the particle's speed increasing?

5 The height of a tree t years after it is planted is given by $H(t) = 6\left(1 - \dfrac{2}{t+3}\right)$ metres, $t \geqslant 0$.

a How high was the tree when it was planted?

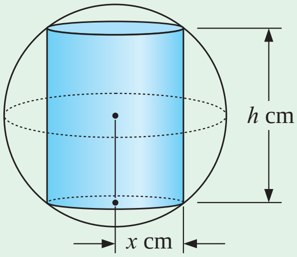
b Determine the height of the tree after $t = 3$, 6 and 9 years.

c Find the rate at which the tree is growing at $t = 0$, 3, 6 and 9 years.

d Show that $H'(t) > 0$ and explain the significance of this result.

e Sketch the graph of $H(t)$ against t.

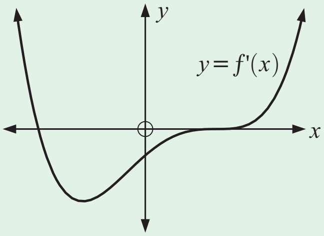
6 A machinist has a spherical ball of brass with diameter 10 cm. The ball is placed in a lathe and machined into a cylinder.



a If the cylinder has radius x cm, show that the cylinder's volume is given by $V(x) = \pi x^2 \sqrt{100 - 4x^2}$ cm^3.

b Hence, find the dimensions of the cylinder of largest volume which can be made.

7



The graph of $y = f'(x)$ is drawn. On the same axes clearly draw a possible graph of $y = f(x)$. Show all turning points and points of inflection.

Chapter 19

Derivatives of exponential and logarithmic functions

Syllabus reference: 7.1, 7.2, 7.3

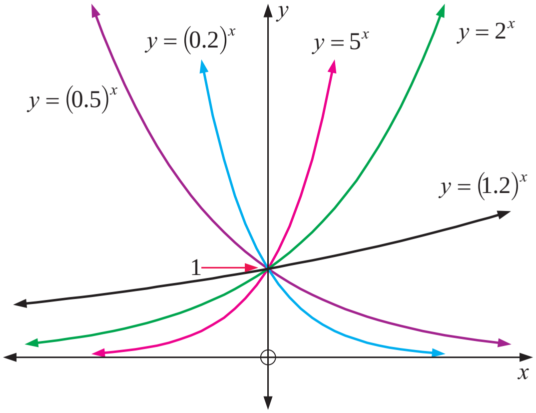
Contents:

In **Chapter 3** we saw that the simplest **exponential functions** have the form $f(x) = a^x$ where a is any positive constant, $a \neq 1$.

The graphs of all members of the exponential family $f(x) = a^x$ have the following properties:

- pass through the point (0, 1)
- asymptotic to the x-axis at one end
- lie above the x-axis for all x
- concave up for all x
- monotone increasing for $a > 1$
- monotone decreasing for $0 < a < 1$.

For example,



A EXPONENTIAL e

INVESTIGATION 1 THE DERIVATIVE OF $y = a^x$

This investigation could be done by using a **graphics calculator** or by clicking on the icon.

The purpose of this investigation is to observe the nature of the derivatives of $f(x) = a^x$ for various values of a.

What to do:

1 For $y = 2^x$ find the gradient of the tangent at $x = 0$, 0.5, 1, 1.5, 2 and 2.5. Use modelling techniques from your graphics calculator or the software provided to show that $\dfrac{dy}{dx} \approx 0.693 \times 2^x$.

CALCULUS DEMO

2 Repeat **1** for the functions

a $y = 3^x$ **b** $y = 5^x$ **c** $y = (0.5)^x$.

3 Use your observations from **1** and **2** to write a statement about the derivative of the general exponential $y = a^x$ for $a > 0$, $a \neq 1$.

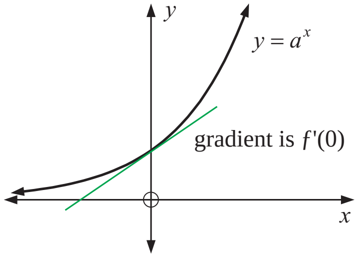
From the investigation you should have discovered that:

If $f(x) = a^x$ then $f'(x) = ka^x$ where k is a constant equal to $f'(0)$.

Proof:

If $f(x) = a^x$,

$$\begin{aligned}
\text{then}\quad f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} && \{\text{first principles definition of derivative}\}\\
&= \lim_{h\to 0} \frac{a^{x+h}-a^x}{h}\\
&= \lim_{h\to 0} \frac{a^x(a^h-1)}{h}\\
&= a^x \times \left(\lim_{h\to 0} \frac{a^h-1}{h}\right) && \{\text{as } a^x \text{ is independent of } h\}\\
\text{But}\quad f'(0) &= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}\\
&= \lim_{h\to 0} \frac{a^h-1}{h}\\
\therefore\quad f'(x) &= a^x f'(0)
\end{aligned}$$



Given this result, if we can find a value of a such that $f'(0) = 1$, then we have found *a function which is its own derivative.*

INVESTIGATION 2 FINDING a WHEN $y = a^x$ AND $\frac{dy}{dx} = a^x$

Click on the icon to graph $f(x) = a^x$ and its derivative function $y = f'(x)$.

DEMO

Experiment with different values of a until the graphs of $f(x) = a^x$ and $y = f'(x)$ appear the same.

Estimate the corresponding value of a to 2 decimal places.

From **Investigation 2** you should have discovered that $f(x) = f'(x) = a^x$ when $a \approx 2.72$.

To find this value of a more accurately we return to the algebraic approach:

We showed that if $f(x) = a^x$ then $f'(x) = a^x \left(\lim_{h\to 0} \frac{a^h-1}{h}\right)$.

So if $f'(x) = a^x$ we require $\lim_{h\to 0} \frac{a^h-1}{h} = 1$.

$\therefore \quad \frac{a^h-1}{h} \approx 1$ for values of h which are close to 0

$\therefore \quad a^h \approx 1 + h$ for h close to 0.

Letting $h = \frac{1}{n}$, $a^{\frac{1}{n}} \approx 1 + \frac{1}{n}$ for large values of n $\{h = \frac{1}{n} \to 0$ as $n \to \infty\}$

$$\therefore \quad a \approx \left(1 + \frac{1}{n}\right)^n \quad \text{for large values of } n.$$

We now examine $\left(1 + \frac{1}{n}\right)^n$ as $n \to \infty$.

n	$\left(1 + \frac{1}{n}\right)^n$	n	$\left(1 + \frac{1}{n}\right)^n$
10	2.593 742 460	10^7	2.718 281 693
10^2	2.704 813 829	10^8	2.718 281 815
10^3	2.716 923 932	10^9	2.718 281 827
10^4	2.718 145 927	10^{10}	2.718 281 828
10^5	2.718 268 237	10^{11}	2.718 281 828
10^6	2.718 280 469	10^{12}	2.718 281 828

In fact as $n \to \infty$, $\left(1 + \frac{1}{n}\right)^n \to 2.718\,281\,828\,459\,045\,235 \ldots.$

and this irrational number is denoted by the symbol e.

$e = 2.718\,281\,828\,459\,045\,235 \ldots.$ and is called **exponential e.**

If $f(x) = e^x$ then $f'(x) = e^x$.

Alternative notation: e^x is sometimes written as $\exp(x)$.

For example, $\exp(1 - x) = e^{1-x}$.

e is an important number with similarities to the number π. Both numbers are irrational (not surds) with non-recurring, non-terminating decimal expansions, and both are discovered naturally.

We also saw e in an earlier chapter when looking at continuous compound interest.

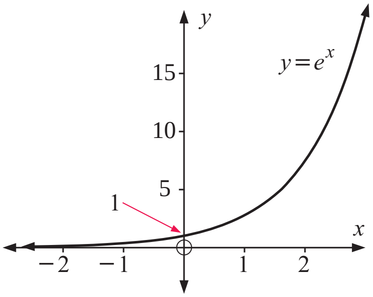
PROPERTIES OF $y = e^x$

Notice that $\frac{dy}{dx} = e^x = y$.

As $x \to \infty$, $y \to \infty$ very rapidly,

and so $\frac{dy}{dx} \to \infty$.

This means that the gradient of the curve is very large for large values of x. The curve increases in steepness as x gets larger.



As $x \to -\infty$, $y \to 0$ and so $\dfrac{dy}{dx} \to 0$.

This means for large negative x, the graph becomes flatter and approaches the asymptote $y = 0$.

$e^x > 0$ for all x, so the range of $f : x \mapsto e^x$ is $\mathbb{R}^+$.

THE DERIVATIVE OF $e^{f(x)}$

The functions e^{-x}, e^{2x+3} and e^{-x^2} are all of the form $e^{f(x)}$. Such functions are often used in problem solving.

In general, $e^{f(x)} > 0$ for all x, no matter what the function $f(x)$.

$$\begin{aligned} \text{Consider} \quad & y = e^{f(x)}. \\ \text{If we let} \quad & u = f(x) \\ \text{then} \quad & y = e^u. \\ \text{But} \quad & \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\} \\ \therefore \quad & \frac{dy}{dx} = e^u \frac{du}{dx} = e^{f(x)} \times f'(x) \end{aligned}$$

Summary:

Function	*Derivative*
e^x	e^x
$e^{f(x)}$	$e^{f(x)} \times f'(x)$

Example 1 **Self Tutor**

Find the gradient function for y equal to:

a $2e^x + e^{-3x}$ **b** x^2e^{-x} **c** $\dfrac{e^{2x}}{x}$

a If $y = 2e^x + e^{-3x}$ then
$$\begin{aligned} \frac{dy}{dx} &= 2e^x + e^{-3x}(-3) \\ &= 2e^x - 3e^{-3x} \end{aligned}$$

b If $y = x^2e^{-x}$ then
$$\begin{aligned} \frac{dy}{dx} &= 2xe^{-x} + x^2e^{-x}(-1) \qquad \{\text{product rule}\} \\ &= 2xe^{-x} - x^2e^{-x} \end{aligned}$$

c If $y = \dfrac{e^{2x}}{x}$ then
$$\begin{aligned} \frac{dy}{dx} &= \frac{e^{2x}(2)x - e^{2x}(1)}{x^2} \qquad \{\text{quotient rule}\} \\ &= \frac{e^{2x}(2x-1)}{x^2} \end{aligned}$$

Example 2 **Self Tutor**

Find the gradient function for y equal to: **a** $(e^x - 1)^3$ **b** $\dfrac{1}{\sqrt{2e^{-x}+1}}$

a
$$\begin{aligned} y &= (e^x - 1)^3 \\ &= u^3 \text{ where } u = e^x - 1 \\ \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \quad \{\text{chain rule}\} \\ &= 3u^2\frac{du}{dx} \\ &= 3(e^x-1)^2 \times e^x \\ &= 3e^x(e^x-1)^2 \end{aligned}$$

b
$$\begin{aligned} y &= (2e^{-x}+1)^{-\frac{1}{2}} \\ &= u^{-\frac{1}{2}} \text{ where } u = 2e^{-x}+1 \\ \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \quad \{\text{chain rule}\} \\ &= -\tfrac{1}{2}u^{-\frac{3}{2}}\frac{du}{dx} \\ &= -\tfrac{1}{2}(2e^{-x}+1)^{-\frac{3}{2}} \times 2e^{-x}(-1) \\ &= e^{-x}(2e^{-x}+1)^{-\frac{3}{2}} \end{aligned}$$

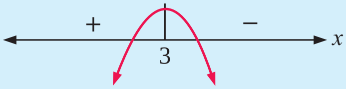
Example 3 **Self Tutor**

Find the exact position and nature of the stationary points of $y = (x-2)e^{-x}$.

$$\begin{aligned} \frac{dy}{dx} &= (1)e^{-x} + (x-2)e^{-x}(-1) \qquad \{\text{product rule}\} \\ &= e^{-x}(1-(x-2)) \\ &= \frac{3-x}{e^x} \quad \text{where } e^x \text{ is positive for all } x. \end{aligned}$$

So, $\dfrac{dy}{dx} = 0$ when $x = 3$.

The sign diagram of $\dfrac{dy}{dx}$ is:



$\therefore$ at $x = 3$ we have a local maximum.

But when $x = 3$, $y = (1)e^{-3} = \dfrac{1}{e^3}$

$\therefore$ the local maximum is $(3, \dfrac{1}{e^3})$.

EXERCISE 19A

1 Find the gradient function for $f(x)$ equal to:

a e^{4x} **b** $e^x + 3$ **c** $\exp(-2x)$ **d** $e^{\frac{x}{2}}$

e $2e^{-\frac{x}{2}}$ **f** $1 - 2e^{-x}$ **g** $4e^{\frac{x}{2}} - 3e^{-x}$ **h** $\dfrac{e^x + e^{-x}}{2}$

i e^{-x^2} **j** $e^{\frac{1}{x}}$ **k** $10(1+e^{2x})$ **l** $20(1-e^{-2x})$

m e^{2x+1} **n** $e^{\frac{x}{4}}$ **o** e^{1-2x^2} **p** $e^{-0.02x}$

2 Find the derivative of:

a xe^x **b** x^3e^{-x} **c** $\dfrac{e^x}{x}$ **d** $\dfrac{x}{e^x}$

e x^2e^{3x} **f** $\dfrac{e^x}{\sqrt{x}}$ **g** $\sqrt{x}e^{-x}$ **h** $\dfrac{e^x+2}{e^{-x}+1}$

3 Find the gradient function for $f(x)$ equal to:

a $(e^x+2)^4$ **b** $\dfrac{1}{1-e^{-x}}$ **c** $\sqrt{e^{2x}+10}$

d $\dfrac{1}{(1-e^{3x})^2}$ **e** $\dfrac{1}{\sqrt{1-e^{-x}}}$ **f** $x\sqrt{1-2e^{-x}}$

4 Suppose $y = Ae^{kx}$ where A and k are constants.

Show that: **a** $\dfrac{dy}{dx} = ky$ **b** $\dfrac{d^2y}{dx^2} = k^2y$.

5 If $y = 2e^{3x} + 5e^{4x}$, show that $\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 12y = 0$.

6 Given $f(x) = e^{kx} + x$ and $f'(0) = -8$, find k.

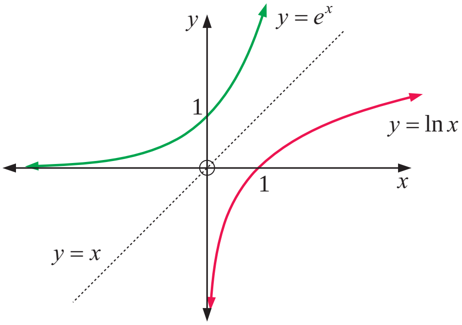
7 Find the position and nature of the turning point(s) of:

a $y = xe^{-x}$ **b** $y = x^2e^x$ **c** $y = \dfrac{e^x}{x}$ **d** $y = e^{-x}(x+2)$

B NATURAL LOGARITHMS

In **Chapter 4** we found that:

- If $e^x = a$ then $x = \ln a$ and vice versa. So, $e^x = a \Leftrightarrow x = \ln a$.
- The graph of $y = \ln x$ is the reflection of the graph of $y = e^x$ in the mirror line $y = x$.



- $y = e^x$ and $y = \ln x$ are inverse functions.
- From the definition $e^x = a \Leftrightarrow x = \ln a$ we observe that $e^{\ln a} = a$.

  This means that any positive real number a can be written as a power of e, or alternatively,

  the **natural logarithm** of any positive number is its power of e, i.e., $\ln e^n = n$.

You should remember that the **laws of logarithms** in base e are identical to those for base 10 and indeed for any base.

For $a > 0$, $b > 0$:
- $\ln(ab) = \ln a + \ln b$
- $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
- $\ln(a^n) = n \ln a$

Notice also that:
- $\ln 1 = 0$ and $\ln e = 1$
- $\ln\left(\frac{1}{a}\right) = -\ln a$
- $\log_b a = \frac{\ln a}{\ln b}$, $b \neq 1$

Example 4 **Self Tutor**

Find algebraically the exact points of intersection of $y = e^x - 3$ and $y = 1 - 3e^{-x}$.

Check your solution using technology.

The functions meet where

$$e^x - 3 = 1 - 3e^{-x}$$

$$\therefore \quad e^x - 4 + 3e^{-x} = 0$$

$$\therefore \quad e^{2x} - 4e^x + 3 = = 0 \quad \{\text{multiplying each term by } e^x\}$$

$$\therefore \quad (e^x - 1)(e^x - 3) = 0$$

$$\therefore \quad e^x = 1 \text{ or } 3$$

$$\therefore \quad x = \ln 1 \text{ or } \ln 3$$

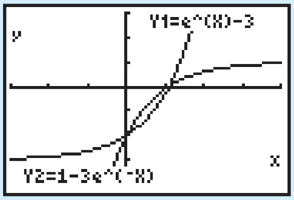
$$\therefore \quad x = 0 \text{ or } \ln 3$$

GRAPHING PACKAGE

When $x = 0$, $y = e^0 - 3 = -2$

When $x = \ln 3$, $e^x = 3$ $\quad \therefore \; y = 3 - 3 = 0$

$\therefore$ the functions meet at $(0, -2)$ and at $(\ln 3, 0)$.



Example 5 **Self Tutor**

Consider the function $y = 2 - e^{-x}$.

- **a** Find the x-intercept.
- **b** Find the y-intercept.
- **c** Show algebraically that the function is increasing for all x.
- **d** Show algebraically that the function is concave down for all x.
- **e** Use technology to help graph $y = 2 - e^{-x}$.
- **f** Explain why $y = 2$ is a horizontal asymptote.

a The x-intercept occurs when $y = 0$, $\therefore \quad e^{-x} = 2$

$\therefore \quad -x = \ln 2$

$\therefore \quad x = -\ln 2$

$\therefore$ the x-intercept is $-\ln 2 \approx -0.693$

b The y-intercept occurs when $x = 0$

$\therefore \quad y = 2 - e^0 = 2 - 1 = 1$

c $\dfrac{dy}{dx} = 0 - e^{-x}(-1) = e^{-x} = \dfrac{1}{e^x}$

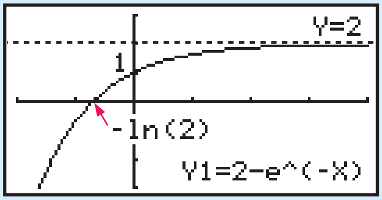
Now $e^x > 0$ for all x, so $\dfrac{dy}{dx} > 0$ for all x

$\therefore$ the function is increasing for all x.

d $\dfrac{d^2y}{dx^2} = e^{-x}(-1) = \dfrac{-1}{e^x}$ which is < 0 for all x

$\therefore$ the function is concave down for all x.

e



f As $x \to \infty$, $e^x \to \infty$
and $e^{-x} \to 0$

$\therefore \quad y \to 2$ (below)

Hence, the horizontal asymptote is $y = 2$.

The first three questions of the following exercise revise logarithmic and exponential functions. If you need help with these you should consult **Chapters 3** and **4**.

EXERCISE 19B

1 Without using a calculator, evaluate:

a $\ln e^2$ **b** $\ln \sqrt{e}$ **c** $\ln\left(\dfrac{1}{e}\right)$ **d** $\ln\left(\dfrac{1}{\sqrt{e}}\right)$

e $e^{\ln 3}$ **f** $e^{2\ln 3}$ **g** $e^{-\ln 5}$ **h** $e^{-2\ln 2}$

2 Write as a power of e: **a** 2 **b** 10 **c** a **d** a^x

3 Solve for x:

a $e^x = 2$ **b** $e^x = -2$ **c** $e^x = 0$

d $e^{2x} = 2e^x$ **e** $e^x = e^{-x}$ **f** $e^{2x} - 5e^x + 6 = 0$

g $e^x + 2 = 3e^{-x}$ **h** $1 + 12e^{-x} = e^x$ **i** $e^x + e^{-x} = 3$

4 Find algebraically the point(s) of intersection of:

a $y = e^x$ and $y = e^{2x} - 6$ **b** $y = 2e^x + 1$ and $y = 7 - e^x$

c $y = 3 - e^x$ and $y = 5e^{-x} - 3$ Check your answers using technology.

5 The function $f(x) = e^{2x} - 3$ cuts the x-axis at A and the y-axis at B.

a Find the coordinates of A and B.

b Show algebraically that the function is increasing for all x.

- **c** Find $f''(x)$ and hence explain why $f(x)$ is concave up for all x.
- **d** Use technology to help graph $y = e^{2x} - 3$.
- **e** Explain why $y = -3$ is a horizontal asymptote.

6 Suppose $f(x) = e^x - 3$ and $g(x) = 3 - 5e^{-x}$.

- **a** Find the x and y-intercepts of both functions.
- **b** Discuss $f(x)$ and $g(x)$ as $x \to \infty$ and as $x \to -\infty$.
- **c** Find algebraically the point(s) of intersection of the functions.
- **d** Sketch the graph of both functions on the same set of axes for $-3 \leqslant x \leqslant 4$. Show all important features on your graph.

7 The function $y = e^x - 3e^{-x}$ cuts the x-axis at P and the y-axis at Q.

- **a** Determine the coordinates of P and Q.
- **b** Prove that the function is increasing for all x.
- **c** Show that $\dfrac{d^2y}{dx^2} = y$. What can be deduced about the concavity of the function above and below the x-axis?
- **d** Use technology to help graph $y = e^x - 3e^{-x}$. Show the features of **a**, **b** and **c** on the graph.

C DERIVATIVES OF LOGARITHMIC FUNCTIONS

INVESTIGATION 3 THE DERIVATIVE OF $\ln x$

If $y = \ln x$, what is the gradient function?

CALCULUS DEMO

What to do:

1. Click on the icon to see the graph of $y = \ln x$. A tangent is drawn to a point on the graph and the gradient of this tangent is given. Watch as the point moves from left to right and a graph of the gradient of the tangent at each point is displayed.
2. What do you think the equation of the gradient function is?
3. Find the gradient at $x = 0.25$, $x = 0.5$, $x = 1$, $x = 2$, $x = 3$, $x = 4$, $x = 5$. Do your results confirm your suggestion in **2**?

From the investigation you should have observed that if $y = \ln x$ then $\dfrac{dy}{dx} = \dfrac{1}{x}$.

The proof of this result is beyond the scope of this course.

By use of the chain rule, we can show that if $y = \ln f(x)$ then $\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$.

Proof: Suppose $y = \ln f(x)$

If we let $u = f(x)$, then $y = \ln u$.

$$\text{Now } \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \{\text{chain rule}\}$$
$$= \frac{1}{u} f'(x)$$
$$= \frac{f'(x)}{f(x)}$$

Summary:

Function	*Derivative*
$\ln x$	$\frac{1}{x}$
$\ln f(x)$	$\frac{f'(x)}{f(x)}$

Example 6 **Self Tutor**

Find the gradient function of:

a $y = \ln(kx)$ where k is a constant

b $y = \ln(1 - 3x)$

c $y = x^3 \ln x$

a If $y = \ln(kx)$ then $\frac{dy}{dx} = \frac{k}{kx} = \frac{1}{x}$

b If $y = \ln(1 - 3x)$

then $\frac{dy}{dx} = \frac{-3}{1 - 3x} = \frac{3}{3x - 1}$

c If $y = x^3 \ln x$

$$\text{then } \frac{dy}{dx} = 3x^2 \ln x + x^3\left(\frac{1}{x}\right) \quad \{\text{product rule}\}$$
$$= 3x^2 \ln x + x^2$$
$$= x^2(3\ln x + 1)$$

$\ln(kx) = \ln k + \ln x$

$= \ln x + \text{constant}$

so $\ln(kx)$ and $\ln x$

both have derivative $\frac{1}{x}$.

The laws of logarithms can help us to differentiate some logarithmic functions more easily.

Example 7 **Self Tutor**

Differentiate with respect to x:

a $y = \ln(xe^{-x})$

b $y = \ln\left[\frac{x^2}{(x+2)(x-3)}\right]$

a If $y = \ln(xe^{-x})$ then $y = \ln x + \ln e^{-x}$ {log of a product law}

$\therefore\ y = \ln x - x$ $\{\ln e^a = a\}$

Differentiating with respect to x, we get $\frac{dy}{dx} = \frac{1}{x} - 1$

b If $y = \ln\left[\frac{x^2}{(x+2)(x-3)}\right]$ then $y = \ln x^2 - \ln[(x+2)(x-3)]$

$$= 2\ln x - [\ln(x+2) + \ln(x-3)]$$
$$= 2\ln x - \ln(x+2) - \ln(x-3)$$
$$\therefore\ \frac{dy}{dx} = \frac{2}{x} - \frac{1}{x+2} - \frac{1}{x-3}$$

EXERCISE 19C

1 Find the gradient function of:

a $y = \ln(7x)$ **b** $y = \ln(2x+1)$ **c** $y = \ln(x - x^2)$

d $y = 3 - 2\ln x$ **e** $y = x^2 \ln x$ **f** $y = \dfrac{\ln x}{2x}$

g $y = e^x \ln x$ **h** $y = (\ln x)^2$ **i** $y = \sqrt{\ln x}$

j $y = e^{-x} \ln x$ **k** $y = \sqrt{x}\ln(2x)$ **l** $y = \dfrac{2\sqrt{x}}{\ln x}$

m $y = 3 - 4\ln(1-x)$ **n** $y = x\ln(x^2+1)$

2 Find $\dfrac{dy}{dx}$ for:

a $y = x\ln 5$ **b** $y = \ln(x^3)$ **c** $y = \ln(x^4 + x)$

d $y = \ln(10 - 5x)$ **e** $y = [\ln(2x+1)]^3$ **f** $y = \dfrac{\ln(4x)}{x}$

g $y = \ln\left(\dfrac{1}{x}\right)$ **h** $y = \ln(\ln x)$ **i** $y = \dfrac{1}{\ln x}$

3 Use the logarithm laws to help differentiate with respect to x:

a $y = \ln\sqrt{1-2x}$ **b** $y = \ln\left(\dfrac{1}{2x+3}\right)$ **c** $y = \ln\left(e^x\sqrt{x}\right)$

d $y = \ln\left(x\sqrt{2-x}\right)$ **e** $y = \ln\left(\dfrac{x+3}{x-1}\right)$ **f** $y = \ln\left(\dfrac{x^2}{3-x}\right)$

g $f(x) = \ln\left((3x-4)^3\right)$ **h** $f(x) = \ln\left(x(x^2+1)\right)$ **i** $f(x) = \ln\left(\dfrac{x^2+2x}{x-5}\right)$

4 **a** By substituting $e^{\ln 2}$ for 2 in $y = 2^x$ find $\dfrac{dy}{dx}$.

b Show that if $y = a^x$ then $\dfrac{dy}{dx} = a^x \times \ln a$.

5 Consider $f(x) = \ln(2x-1) - 3$.

a Find the x-intercept.

b Can $f(0)$ be found? What is the significance of this result?

c Find the gradient of the tangent to the curve at $x = 1$.

d For what values of x does $f(x)$ have meaning?

e Find $f''(x)$ and hence explain why $f(x)$ is concave down whenever $f(x)$ has meaning.

f Graph the function.

6 Consider $f(x) = x\ln x$.

a For what values of x is $f(x)$ defined?

b Show that the minimum value of $f(x)$ is $-\dfrac{1}{e}$.

7 Prove that $\dfrac{\ln x}{x} \leqslant \dfrac{1}{e}$ for all $x > 0$.

Hint: Let $f(x) = \dfrac{\ln x}{x}$ and find its greatest value.

8 Consider the function $f(x) = x - \ln x$.

- **a** Show that the graph of $y = f(x)$ has a local minimum and that this is the only turning point.
- **b** Hence prove that $\ln x \leqslant x - 1$ for all $x > 0$.

D APPLICATIONS

The applications we consider here are:

- tangents and normals
- rates of change
- curve properties
- displacement, velocity and acceleration
- optimisation (maxima and minima)

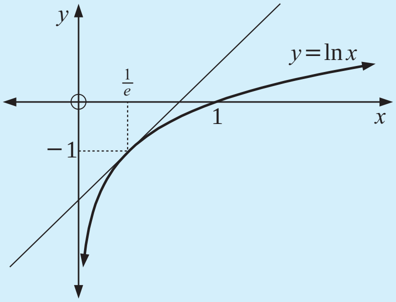
Example 8 **Self Tutor**

Show that the equation of the tangent to $y = \ln x$ at the point where $y = -1$ is $y = ex - 2$.

When $y = -1$, $\ln x = -1$

$\therefore \quad x = -e^{-1} = \dfrac{1}{e}$

$\therefore$ the point of contact is $\left(\dfrac{1}{e}, -1\right)$.



Now $f(x) = \ln x$ has derivative $f'(x) = \dfrac{1}{x}$

$\therefore$ the tangent at $\left(\dfrac{1}{e}, -1\right)$ has gradient $\dfrac{1}{\frac{1}{e}} = e$

$\therefore$ the tangent has equation $\dfrac{y - -1}{x - \frac{1}{e}} = e$

$\therefore \quad y + 1 = e(x - \dfrac{1}{e})$

$\therefore \quad y = ex - 2$

EXERCISE 19D

1 Find the equation of the tangent to the function $f : x \mapsto e^{-x}$ at the point where $x = 1$.

2 Find the equation of the tangent to $y = \ln(2 - x)$ at the point where $x = -1$.

3 The tangent to $y = x^2e^x$ at $x = 1$ cuts the x and y-axes at A and B respectively. Find the coordinates of A and B.

4 Find the equation of the normal to $y = \ln\sqrt{x}$ at the point where $y = -1$.

5 Find the equation of the tangent to $y = e^x$ at the point where $x = a$.
Hence, find the equation of the tangent to $y = e^x$ which passes through the origin.

6 Consider $f(x) = \ln x$.

- **a** For what values of x is $f(x)$ defined?
- **b** Find the signs of $f'(x)$ and $f''(x)$ and comment on the geometrical significance of each.
- **c** Sketch the graph of $f(x) = \ln x$ and find the equation of the normal at the point where $y = 1$.

7 Find, correct to 2 decimal places, the angle between the tangents to $y = 3e^{-x}$ and $y = 2 + e^x$ at their point of intersection.

8 A radioactive substance decays according to the formula $W = 20e^{-kt}$ grams where t is the time in hours.

- **a** Find k given that after 50 hours the weight is 10 grams.
- **b** Find the weight of radioactive substance present at:
  - **i** $t = 0$ hours **ii** $t = 24$ hours **iii** $t = 1$ week.
- **c** How long will it take for the weight to reach 1 gram?
- **d** Find the rate of radioactive decay at: **i** $t = 100$ hours **ii** $t = 1000$ hours.
- **e** Show that $\dfrac{dW}{dt}$ is proportional to the weight of substance remaining.

9 The temperature of a liquid after being placed in a refrigerator is given by
$T = 5 + 95e^{-kt}$ ^{o}C where k is a positive constant and t is the time in minutes.

- **a** Find k if the temperature of the liquid is 20^{o}C after 15 minutes.
- **b** What was the temperature of the liquid when it was first placed in the refrigerator?
- **c** Show that $\dfrac{dT}{dt} = c(T - 5)$ for some constant c.
- **d** At what rate is the temperature changing at:
  - **i** $t = 0$ mins **ii** $t = 10$ mins **iii** $t = 20$ mins?

10 The height of a certain species of shrub t years after it is planted is given by
$H(t) = 20\ln(3t + 2) + 30$ cm, $t \geqslant 0$.

- **a** How high was the shrub when it was planted?
- **b** How long will it take for the shrub to reach a height of 1 m?
- **c** At what rate is the shrub's height changing:
  - **i** 3 years after being planted **ii** 10 years after being planted?

11 In the conversion of sugar solution to alcohol, the chemical reaction obeys the law $A = s(1 - e^{-kt})$, $t \geqslant 0$ where t is the number of hours after the reaction commenced, s is the original sugar concentration (%), and A is the alcohol produced, in litres.

- **a** Find A when $t = 0$.
- **b** If $s = 10$ and $A = 5$ after 3 hours, find k.
- **c** If $s = 10$, find the speed of the reaction at time 5 hours.
- **d** Show that the speed of the reaction is proportional to $A - s$.

12 Consider the function $f(x) = \dfrac{e^x}{x}$.

- **a** Does the graph of $y = f(x)$ have any x or y-intercepts?
- **b** Discuss $f(x)$ as $x \to \infty$ and as $x \to -\infty$.
- **c** Find and classify any stationary points of $y = f(x)$.
- **d** Sketch the graph of $y = f(x)$ showing all important features.
- **e** Find the equation of the tangent to $f(x) = \dfrac{e^x}{x}$ at the point where $x = -1$.

13 A particle P moves in a straight line. Its displacement from the origin O is given by $s(t) = 100t + 200e^{-\frac{t}{5}}$ cm where t is the time in seconds, $t \geqslant 0$.

- **a** Find the velocity and acceleration functions.
- **b** Find the initial position, velocity and acceleration of P.
- **c** Discuss the velocity of P as $t \to \infty$.
- **d** Sketch the graph of the velocity function.
- **e** Find when the velocity of P is 80 cm per second.

14 A psychologist claims that the ability A to memorise simple facts during infancy years can be calculated using the formula $A(t) = t\ln t + 1$ where $0 < t \leqslant 5$, t being the age of the child in years.

- **a** At what age is the child's memorising ability a minimum?
- **b** Sketch the graph of $A(t)$.

15 One of the most common functions used in statistics is the *normal distribution function* $f(x) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$.

- **a** Find the stationary points of the function and find intervals where the function is increasing and decreasing.
- **b** Find all points of inflection.
- **c** Discuss $f(x)$ as $x \to \infty$ and as $x \to -\infty$.
- **d** Sketch the graph of $y = f(x)$ showing all important features.

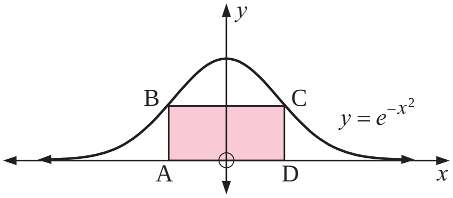
16 A manufacturer of electric kettles performs a cost control study. They discover that to produce x kettles per day, the cost per kettle $C(x)$ is given by

$$C(x) = 4\ln x + \left(\frac{30 - x}{10}\right)^2 \text{ hundred dollars}$$

with a minimum production capacity of 10 kettles per day.
How many kettles should be manufactured to keep the cost per kettle to a minimum?

17 Infinitely many rectangles which sit on the x-axis can be inscribed under the curve $y = e^{-x^2}$.

Determine the coordinates of C such that rectangle ABCD has maximum area.



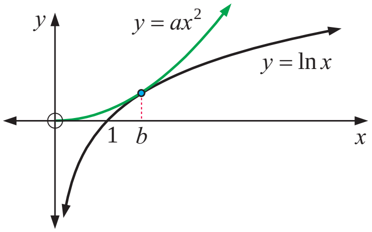
18 The revenue generated when a manufacturer sells x torches per day is given by

$R(x) \approx 1000 \ln\left(1 + \dfrac{x}{400}\right) + 600$ dollars.

Each torch costs the manufacturer $1.50 to produce plus fixed costs of $300 per day.

How many torches should be produced daily to maximise the profits made?

19 A quadratic of the form $y = ax^2$, $a > 0$, touches the logarithmic function $y = \ln x$.



- **a** If the x-coordinate of the point of contact is b, explain why $ab^2 = \ln b$ and $2ab = \dfrac{1}{b}$.
- **b** Deduce that the point of contact is $(\sqrt{e}, \frac{1}{2})$.
- **c** What is the value of a?
- **d** What is the equation of the common tangent?

20 A small population of wasps is observed. After t weeks the population is modelled by

$P(t) = \dfrac{50\,000}{1 + 1000e^{-0.5t}}$ wasps, where $0 \leqslant t \leqslant 25$.

Find when the wasp population is growing fastest.

21 Consider the function $y = Ate^{-bt}$, $t \geqslant 0$ where A and b are positive constants.

- **a** Find the t and y-intercepts of the function.
- **b** Find and describe the stationary point in terms of A and b.
- **c** Find the point of inflection in terms of A and b.
- **d** Graph the function showing the features found above.
- **e** When a new pain killing injection is administered, the effect is modelled by $E(t) = 750te^{-1.5t}$ units, where $t \geqslant 0$ is the time in hours after the injection. At what time is the drug most effective?

REVIEW SET 19A NON-CALCULATOR

1 Find $\dfrac{dy}{dx}$ if: **a** $y = e^{x^3+2}$ **b** $y = \ln\left(\dfrac{x+3}{x^2}\right)$

2 Find the equation of the normal to $y = e^{-x^2}$ at the point where $x = 1$.

3
- **a** Find the exact coordinates of the point of intersection P of the graphs of $f(x) = e^{2x}$ and $g(x) = -e^x + 6$.
- **b** Show that the equation of the tangent to $f(x)$ at P is given by $y = 8x + 4 - 8\ln 2$.

4 Consider the function $f(x) = \dfrac{e^x}{x-1}$.

- **a** Find the x and y-intercepts.
- **b** For what values of x is $f(x)$ defined?
- **c** Find the signs of $f'(x)$ and $f''(x)$ and comment on the geometrical significance of each.

d Sketch the graph of $y = f(x)$.

e Find the equation of the tangent at the point where $x = 2$.

5 Find where the tangent to $y = \ln(x^2 + 3)$ at $x = 0$ cuts the x-axis.

6 At the point where $x = 0$, the tangent to $f(x) = e^{4x} + px + q$ has equation $y = 5x - 7$. Find p and q.

7 Find the exact roots of the following equations:

a $3e^x - 5 = -2e^{-x}$

b $2\ln x - 3\ln\left(\frac{1}{x}\right) = 10$

REVIEW SET 19B CALCULATOR

1 The height of a tree t years after it was planted is given by $H(t) = 60 + 40\ln(2t + 1)$ cm, $t \geqslant 0$.

a How high was the tree when it was planted?

b How long does it take for the tree to reach: **i** 150 cm **ii** 300 cm?

c At what rate is the tree's height increasing after: **i** 2 years **ii** 20 years?

2 A particle P moves in a straight line with position given by $s(t) = 80e^{-\frac{t}{10}} - 40t$ m where t is the time in seconds, $t \geqslant 0$.

a Find the velocity and acceleration functions.

b Find the initial position, velocity, and acceleration of P.

c Discuss the velocity of P as $t \to \infty$.

d Sketch the graph of the velocity function.

e Find when the velocity is -44 metres per second.

3 A shirt maker sells x shirts per day with revenue function

$$R(x) = 200\ln\left(1 + \frac{x}{100}\right) + 1000 \text{ dollars.}$$

The manufacturing costs are determined by the cost function

$$C(x) = (x - 100)^2 + 200 \text{ dollars.}$$

How many shirts should be sold daily to maximise profits? What is the maximum daily profit?

4 The value of a car t years after its purchase is given by $V = 20\,000e^{-0.4t}$ dollars. Calculate:

a the purchase price of the car

b the rate of decrease of the value of the car 10 years after it was purchased.

5 A manufacturer determines that the total weekly cost C of producing x clocks per day is given by $C(x) = 10\ln x + \left(20 - \frac{x}{10}\right)^2$ dollars.

How many clocks per day should be produced to minimise the costs given that at least 50 clocks per day must be made to fill fixed daily orders?

6 A particle P moves in a straight line with position given by $s(t) = 25t - 10\ln t$ cm, $t \geqslant 1$, where t is the time in minutes.

- **a** Find the velocity and acceleration functions.
- **b** Find the position, velocity, and acceleration when $t = e$ minutes.
- **c** Discuss the velocity as $t \to \infty$.
- **d** Sketch the graph of the velocity function.
- **e** Find when the velocity of P is 20 cm per minute.

REVIEW SET 19C

1 Find $\dfrac{dy}{dx}$ if: **a** $y = \ln(x^3 - 3x)$ **b** $y = \dfrac{e^x}{x^2}$

2 Find where the tangent to $y = \ln(x^4 + 3)$ at $x = 1$ cuts the y-axis.

3 Solve exactly for x: **a** $e^{2x} = 3e^x$ **b** $e^{2x} - 7e^x + 12 = 0$

4 Consider the function $f(x) = e^x - x$.

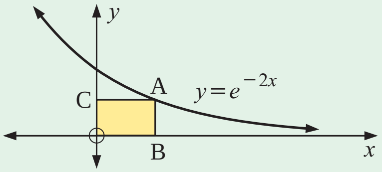
- **a** Find and classify any stationary points of $y = f(x)$.
- **b** Discuss $f(x)$ as $x \to \infty$.
- **c** Find $f''(x)$ and draw its sign diagram. Give a geometrical interpretation for the sign of $f''(x)$.
- **d** Sketch the graph of $y = f(x)$.
- **e** Deduce that $e^x \geqslant x + 1$ for all x.

5 Differentiate with respect to x: **a** $f(x) = \ln(e^x + 3)$ **b** $f(x) = \ln\left[\dfrac{(x+2)^3}{x}\right]$

6 Consider the function $f(x) = x + \ln x$.

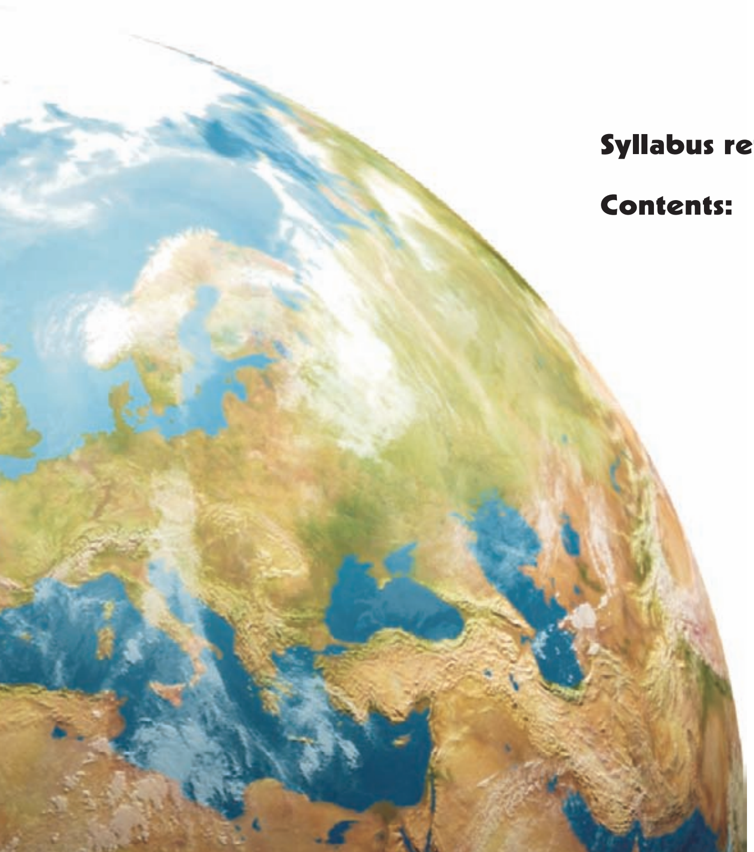
- **a** Find the values of x for which $f(x)$ is defined.
- **b** Find the signs of $f'(x)$ and $f''(x)$ and comment on the geometrical significance of each.
- **c** Sketch the graph of $y = f(x)$.
- **d** Find the equation of the normal at the point where $x = 1$.

7 Infinitely many rectangles can be inscribed under the curve $y = e^{-2x}$ as shown. Determine the coordinates of A such that the rectangle OBAC has maximum area.



Chapter 20

Derivatives of trigonometric functions



Syllabus reference: 7.1, 7.2, 7.3

Contents:

INTRODUCTION

In **Chapter 10** we saw that sine and cosine curves arise naturally from motion in a circle.

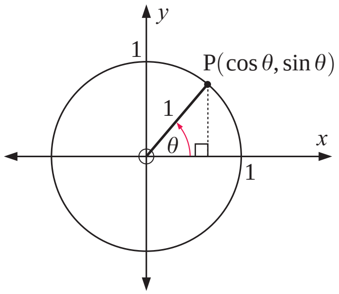
Click on the icon to observe the motion of point P around the unit circle. Observe the graphs of P's height relative to the x-axis, and then P's horizontal displacement from the y-axis. The resulting graphs are those of $y = \sin t$ and $y = \cos t$.

DEMO

Suppose P moves anticlockwise around the unit circle with constant linear speed of 1 unit per second.

After 2π seconds, P will travel 2π units which is one full revolution.

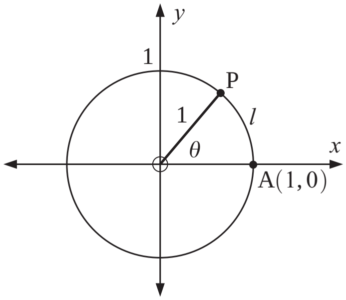
So, after t seconds P will travel through t radians, and at time t, P is at $(\cos t, \sin t)$.



The **angular velocity** of P is the time rate of change in $\mathrm{A\hat{O}P}$.

Angular velocity is only meaningful in motion along a circular or elliptical arc.

For the example above, the angular velocity of P is $\dfrac{d\theta}{dt}$ and $\dfrac{d\theta}{dt} = 1$ radian per second.



If we let l be the arc length AP, the **linear speed** of P is the time rate of change in l, or $\dfrac{dl}{dt}$.

For the example above, $l = \theta r = \theta \times 1 = \theta$ and $\dfrac{dl}{dt} = \dfrac{d\theta}{dt} = 1$ unit per second.

A DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

INVESTIGATION DERIVATIVES OF $\sin t$ AND $\cos t$

Our aim is to use a computer demonstration to investigate the derivatives of $\sin t$ and $\cos t$.

What to do:

1 Click on the icon to observe the graph of $y = \sin t$. A tangent with t-step of length 1 unit moves across the curve, and its y-step is translated onto the gradient graph. Suggest the derivative of the function $y = \sin t$.

DERIVATIVES DEMO

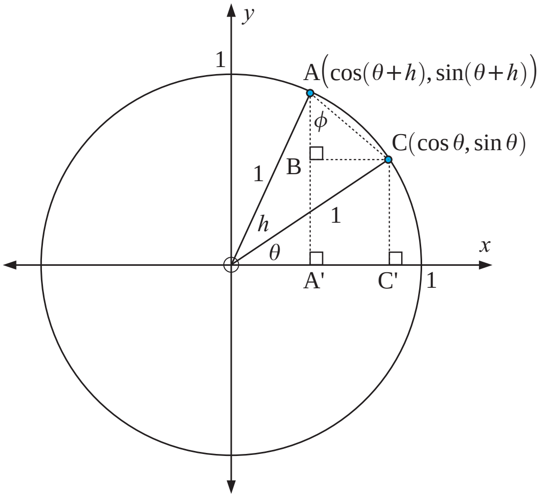
2 Repeat the process in **1** for the graph of $y = \cos t$. Hence suggest the derivative of the function $y = \cos t$.

From the investigation you may have deduced that

$$\frac{d}{dt}(\sin t) = \cos t \quad \text{and} \quad \frac{d}{dt}(\cos t) = -\sin t.$$

We will now demonstrate these derivatives using algebra.

THE DERIVATIVE OF $\sin x$



Consider $f(\theta) = \sin\theta$.

Now $$\frac{f(\theta+h)-f(\theta)}{h}$$
$$= \frac{\sin(\theta+h)-\sin\theta}{h}$$
$$= \frac{\text{AA}'-\text{CC}'}{h}$$
$$= \frac{\text{AA}'-\text{BA}'}{h}$$
$$= \frac{\text{AB}}{h}$$
$$= \frac{\text{AB}}{\text{arc AC}} \quad \ldots\ldots (1)$$

Now angle $\phi = \text{O}\widehat{\text{A}}\text{C} - \text{O}\widehat{\text{A}}\text{A}'$

But $\text{O}\widehat{\text{A}}\text{C} = \dfrac{\pi - h}{2}$ {base angle of isosceles triangle OAC}

$$\therefore \quad \phi = \frac{\pi-h}{2} - \left(\frac{\pi}{2} - h - \theta\right)$$
$$= \theta + \frac{h}{2} \quad \text{on simplifying}$$

As $h \to 0$, $\phi \to \theta$ and $\cos\phi \to \cos\theta$ ...... (2)

Also, as $h \to 0$, $\dfrac{\sin(\theta+h)-\sin\theta}{h} \to \dfrac{\text{AB}}{\text{AC}}$ {arc AC → AC}

$$\therefore \quad \frac{\sin(\theta+h)-\sin\theta}{h} \to \cos\phi \quad \{\text{using } \Delta\text{ABC}\}$$

$$\therefore \quad \frac{\sin(\theta+h)-\sin\theta}{h} \to \cos\theta \quad \{\text{using (2)}\}$$

$$\therefore \quad f'(\theta) = \cos\theta$$

For x in radians, if $f(x) = \sin x$ then $f'(x) = \cos x$.

THE DERIVATIVE OF $\cos x$

$$\begin{aligned}
\text{Consider}\quad & y=\cos x\\
\therefore\quad & y=\sin(\tfrac{\pi}{2}-x)\quad \{\sin(\tfrac{\pi}{2}-x)=\cos x\}\\
\therefore\quad & y=\sin u\quad \text{where}\quad u=\tfrac{\pi}{2}-x\\
\therefore\quad & \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}\qquad \{\text{chain rule}\}\\
& \quad =\cos u\times(-1)\\
& \quad =-\cos u\\
& \quad =-\cos(\tfrac{\pi}{2}-x)\\
& \quad =-\sin x
\end{aligned}$$

For x in radians, if $f(x)=\cos x$ then $f'(x)=-\sin x$.

THE DERIVATIVE OF $\tan x$

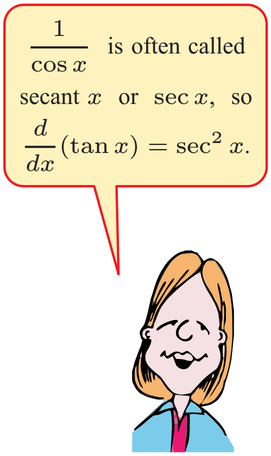
Consider $y=\tan x=\dfrac{\sin x}{\cos x}$

We let $u=\sin x,\quad v=\cos x$

so $\dfrac{du}{dx}=\cos x,\quad \dfrac{dv}{dx}=-\sin x$

$$\begin{aligned}
\therefore\quad \frac{dy}{dx}&=\frac{u'v-uv'}{v^2}\qquad\{\text{quotient rule}\}\\
&=\frac{\cos x\cos x-\sin x(-\sin x)}{[\cos x]^2}\\
&=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}\\
&=\frac{1}{\cos^2 x}\qquad\{\sin^2 x+\cos^2 x=1\}
\end{aligned}$$

DERIVATIVE DEMO



$\dfrac{1}{\cos x}$ is often called secant x or $\sec x$, so $\dfrac{d}{dx}(\tan x)=\sec^2 x$.

Summary:

Function	*Derivative*
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\dfrac{1}{\cos^2 x}$

THE DERIVATIVES OF $\sin[f(x)]$, $\cos[f(x)]$ AND $\tan[f(x)]$

Given $y=\sin[f(x)]$ we can write $y=\sin u$ where $u=f(x)$.

$$\begin{aligned}
\text{Now}\quad \frac{dy}{dx}&=\frac{dy}{du}\times\frac{du}{dx}\qquad\{\text{chain rule}\}\\
&=\cos u\times f'(x)\\
&=\cos[f(x)]\times f'(x)
\end{aligned}$$

We can perform the same procedure for $\cos[f(x)]$ and $\tan[f(x)]$, giving the results in the following table.

Function	*Derivative*
$\sin[f(x)]$	$\cos[f(x)]\, f'(x)$
$\cos[f(x)]$	$-\sin[f(x)]\, f'(x)$
$\tan[f(x)]$	$\dfrac{f'(x)}{\cos^2[f(x)]}$

Example 1 **Self Tutor**

Differentiate with respect to x: **a** $x \sin x$ **b** $4\tan^2(3x)$

a If $y = x\sin x$
then by the product rule

$$\frac{dy}{dx} = (1)\sin x + (x)\cos x$$
$$= \sin x + x\cos x$$

b If $y = 4\tan^2(3x) = 4[\tan(3x)]^2$
then by the chain rule

$$\frac{dy}{dx} = 8[\tan(3x)]^1 \times \frac{d}{dx}[\tan(3x)]$$
$$= 8\tan(3x) \times \frac{3}{\cos^2(3x)}$$
$$= \frac{24\sin(3x)}{\cos^3(3x)}$$

Example 2 **Self Tutor**

Find the equation of the tangent to $y = \tan x$ at the point where $x = \frac{\pi}{4}$.

Let $f(x) = \tan x$ so $f(\frac{\pi}{4}) = \tan\frac{\pi}{4} = 1$

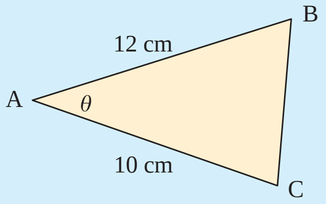
$$f'(x) = \frac{1}{\cos^2 x} \quad \text{so} \quad f'(\tfrac{\pi}{4}) = \frac{1}{\cos^2(\frac{\pi}{4})} = \frac{1}{(\frac{1}{\sqrt{2}})^2} = 2$$

At $(\frac{\pi}{4}, 1)$, the tangent has gradient 2

$\therefore$ the equation is $\dfrac{y-1}{x-\frac{\pi}{4}} = 2$ which is $y - 1 = 2x - \frac{\pi}{2}$
or $y = 2x + (1 - \frac{\pi}{2})$

Example 3 **Self Tutor**

Find the rate of change in the area of triangle ABC as θ changes, at the time when $\theta = 60^o$.



Area $A = \frac{1}{2} \times 10 \times 12 \times \sin\theta$ $\{\text{Area} = \frac{1}{2}ab\sin C$ or $\frac{1}{2}bc\sin A\}$

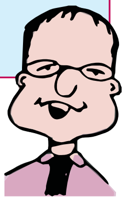
$\therefore$ $A = 60\sin\theta$

$$\therefore \quad \frac{dA}{d\theta} = 60\cos\theta$$

When $\theta = \frac{\pi}{3}$, $\cos\theta = \frac{1}{2}$

$$\therefore \quad \frac{dA}{d\theta} = 30 \text{ cm}^2 \text{ per radian}$$

θ must be in **radians** so the dimensions are correct.



EXERCISE 20A

1 Find $\dfrac{dy}{dx}$ for:

a $y = \sin(2x)$ **b** $y = \sin x + \cos x$ **c** $y = \cos(3x) - \sin x$

d $y = \sin(x+1)$ **e** $y = \cos(3-2x)$ **f** $y = \tan(5x)$

g $y = \sin\left(\frac{x}{2}\right) - 3\cos x$ **h** $y = 3\tan(\pi x)$ **i** $y = 4\sin x - \cos(2x)$

2 Differentiate with respect to x:

a $x^2 + \cos x$ **b** $\tan x - 3\sin x$ **c** $e^x \cos x$ **d** $e^{-x}\sin x$

e $\ln(\sin x)$ **f** $e^{2x}\tan x$ **g** $\sin(3x)$ **h** $\cos\left(\dfrac{x}{2}\right)$

i $3\tan(2x)$ **j** $x\cos x$ **k** $\dfrac{\sin x}{x}$ **l** $x\tan x$

3 Differentiate with respect to x:

a $\sin(x^2)$ **b** $\cos(\sqrt{x})$ **c** $\sqrt{\cos x}$ **d** $\sin^2 x$

e $\cos^3 x$ **f** $\cos x \sin(2x)$ **g** $\cos(\cos x)$ **h** $\cos^3(4x)$

i $\dfrac{1}{\sin x}$ **j** $\dfrac{1}{\cos(2x)}$ **k** $\dfrac{2}{\sin^2(2x)}$ **l** $\dfrac{8}{\tan^3(\frac{x}{2})}$

4 Suppose $f(x) = 2\sin^3 x - 3\sin x$.

a Show that $f'(x) = -3\cos x\cos 2x$. **b** Find $f''(x)$.

5 **a** If $y = \sin(2x+3)$, show that $\dfrac{d^2y}{dx^2} + 4y = 0$.

b If $y = 2\sin x + 3\cos x$, show that $y'' + y = 0$ where y'' represents $\dfrac{d^2y}{dx^2}$.

c Show that the curve with equation $y = \dfrac{\cos x}{1+\sin x}$ cannot have horizontal tangents.

6 Find the equation of:

a the tangent to $y = \sin x$ at the origin

b the tangent to $y = \tan x$ at the origin

c the normal to $y = \cos x$ at the point where $x = \frac{\pi}{6}$

d the normal to $y = \dfrac{1}{\sin(2x)}$ at the point where $x = \frac{\pi}{4}$.

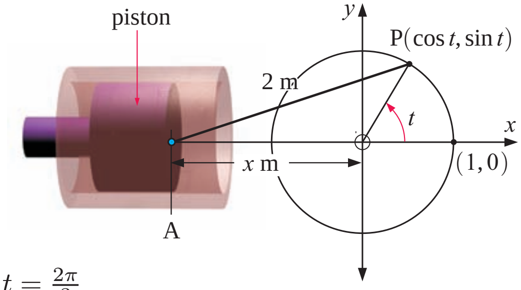
7 On the Indonesian coast, the depth of water at time t hours after midnight is given by $d = 9.3 + 6.8\cos(0.507t)$ metres.

- **a** Is the tide rising or falling at 8.00 am?
- **b** What is the rate of change in the depth of water at 8.00 am?

8 The voltage in a circuit is given by $V(t) = 340\sin(100\pi t)$ where t is the time in seconds. At what rate is the voltage changing:

- **a** when $t = 0.01$ **b** when $V(t)$ is a maximum?

9 A piston is operated by rod [AP] attached to a flywheel of radius 1 m. AP = 2 m. P has coordinates $(\cos t,\ \sin t)$ and point A is $(-x,\ 0)$.



- **a** Show that $x = \sqrt{4 - \sin^2 t} - \cos t$.
- **b** Find the rate at which x is changing at the instant when:
  - **i** $t = 0$ **ii** $t = \frac{\pi}{2}$ **iii** $t = \frac{2\pi}{3}$

10 For each of the following functions, determine the position and nature of the stationary points on the interval $0 \leqslant x \leqslant 2\pi$, then show them on a graph of the function.

- **a** $f(x) = \sin x$ **b** $f(x) = \cos(2x)$ **c** $f(x) = \sin^2 x$
- **d** $f(x) = e^{\sin x}$ **e** $f(x) = \sin(2x) + 2\cos x$

11 A particle P moves along the x-axis with position given by $x(t) = 1 - 2\cos t$ cm where t is the time in seconds.

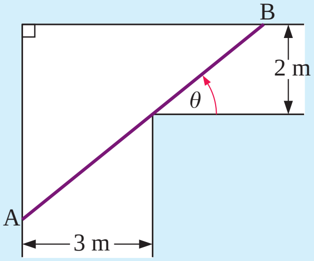
- **a** State the initial position, velocity and acceleration of P.
- **b** Describe the motion when $t = \frac{\pi}{4}$ seconds.
- **c** Find the times when the particle reverses direction on $0 < t < 2\pi$ and find the position of the particle at these instants.
- **d** When is the particle's speed increasing on $0 \leqslant t \leqslant 2\pi$?

12 Show that $y = 4e^{-x}\sin x$ has a maximum when $x = \frac{\pi}{4}$.

B OPTIMISATION WITH TRIGONOMETRY

Example 4 **Self Tutor**

Two corridors meet at right angles and are 2 m and 3 m wide respectively. θ is the angle marked on the given figure. [AB] is a thin metal tube which must be kept horizontal and cannot be bent as it moves around the corner from one corridor to the other.



a Show that the length AB is given by

$$L = \frac{3}{\cos\theta} + \frac{2}{\sin\theta}.$$

b Show that $\dfrac{dL}{d\theta} = 0$ when $\theta = \tan^{-1}\left(\sqrt[3]{\frac{2}{3}}\right) \approx 41.1^o$.

c Find L when $\theta = \tan^{-1}\left(\sqrt[3]{\frac{2}{3}}\right)$ and comment on the significance of this value.

DEMO

a $\cos\theta = \dfrac{3}{a}$ and $\sin\theta = \dfrac{2}{b}$ so $a = \dfrac{3}{\cos\theta}$ and $b = \dfrac{2}{\sin\theta}$

$$\therefore \quad L = a + b = \frac{3}{\cos\theta} + \frac{2}{\sin\theta}$$

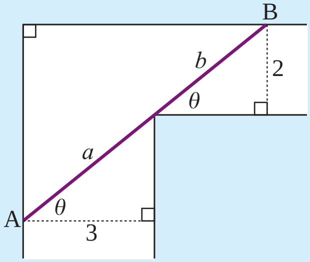
b

$$L = 3[\cos\theta]^{-1} + 2[\sin\theta]^{-1}$$

$$\therefore \quad \frac{dL}{d\theta} = -3[\cos\theta]^{-2} \times (-\sin\theta) - 2[\sin\theta]^{-2} \times \cos\theta$$

$$= \frac{3\sin\theta}{\cos^2\theta} - \frac{2\cos\theta}{\sin^2\theta}$$

$$= \frac{3\sin^3\theta - 2\cos^3\theta}{\cos^2\theta\sin^2\theta}$$



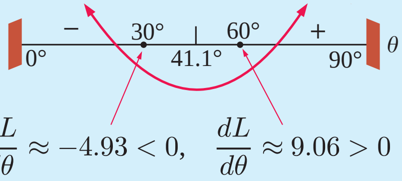
Thus $\dfrac{dL}{d\theta} = 0 \Leftrightarrow 3\sin^3\theta = 2\cos^3\theta$

$$\therefore \quad \tan^3\theta = \tfrac{2}{3}$$

$$\therefore \quad \tan\theta = \sqrt[3]{\tfrac{2}{3}}$$

and so $\theta = \tan^{-1}\left(\sqrt[3]{\frac{2}{3}}\right) \approx 41.1^o$

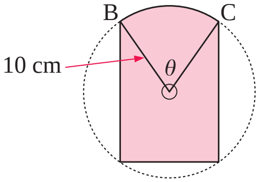
c *Sign diagram* of $\dfrac{dL}{d\theta}$:



Thus, AB is minimised when $\theta \approx 41.1^o$. At this time $L \approx 7.02$ metres, so if we ignore the width of the rod then the greatest length of rod able to be horizontally carried around the corner is 7.02 m.

EXERCISE 20B

1 A circular piece of tinplate of radius 10 cm has 3 segments removed as illustrated.
If θ is the measure of angle COB, show that the remaining area is given by $A = 50(\theta + 3\sin\theta)$.
Hence, find θ to the nearest $\frac{1}{10}$ of a degree when the area A is a maximum.

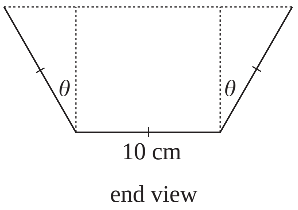


2 A symmetrical gutter is made from a sheet of metal 30 cm wide by bending it twice as shown.

a Deduce that the cross-sectional area is given by $A = 100\cos\theta(1+\sin\theta)$.

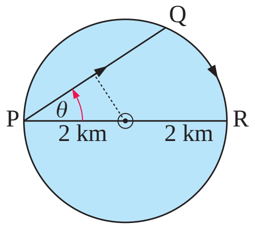
b Show that $\dfrac{dA}{d\theta} = 0$ when $\sin\theta = \frac{1}{2}$ or -1.

c For what value of θ does the gutter have maximum carrying capacity?



3 Hieu can row a boat across a circular lake of radius 2 km at 3 km h^{-1}. He can walk around the edge of the lake at 5 km h^{-1}.

What is the longest possible time Hieu could take to get from P to R by rowing from P to Q and then walking from Q to R?



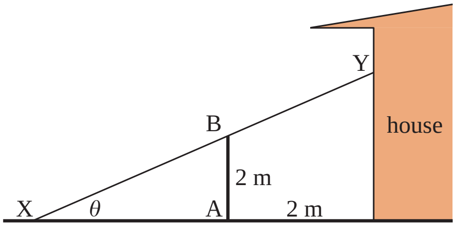
4 Fence [AB] is 2 m high and is 2 m from a house. [XY] is a ladder which touches the ground at X, the house at Y, and the fence at B.

a If L is the length of [XY], show that

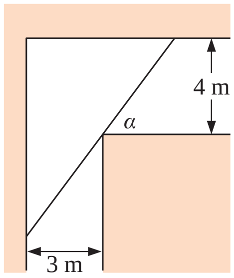
$$L = \frac{2}{\cos\theta} + \frac{2}{\sin\theta}$$

b Show that $\dfrac{dL}{d\theta} = \dfrac{2\sin^3\theta - 2\cos^3\theta}{\sin^2\theta\cos^2\theta}$.

c What is the length of the shortest ladder [XY] which touches at X, B and Y?



5 In **Example 4**, suppose the corridors are those in a hospital and are 4 m wide and 3 m wide respectively. What is the maximum length of thin metal tube that can be moved around the corner? Remember it must be kept horizontal and must not be bent.



REVIEW SET 20

1 Differentiate with respect to x:

a $\sin(5x)\ln(x)$

b $\sin(x)\cos(2x)$

c $e^{-2x}\tan x$

d $10x - \sin(10x)$

e $\ln\left(\dfrac{1}{\cos x}\right)$

f $\sin(5x)\ln(2x)$

2 Show that the equation of the tangent to $y = x\tan x$ at $x = \frac{\pi}{4}$ is $(2+\pi)x - 2y = \dfrac{\pi^2}{4}$.

3 Find $f'(x)$ and $f''(x)$ for:

a $f(x) = 3\sin x - 4\cos(2x)$

b $f(x) = \sqrt{x}\cos(4x)$

4 A particle moves in a straight line along the x-axis with position given by $x(t) = 3 + \sin(2t)$ cm after t seconds.

a Find the initial position, velocity and acceleration of the particle.

b Find the times when the particle changes direction during $0 \leqslant t \leqslant \pi$ seconds.

c Find the total distance travelled by the particle in the first π seconds.

5 Consider $f(x) = \sqrt{\cos x}$ for $0 \leqslant x \leqslant 2\pi$.

a For what values of x is $f(x)$ meaningful?

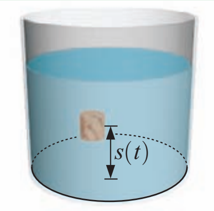
b Find $f'(x)$ and hence find intervals where $f(x)$ is increasing and decreasing.

c Sketch the graph of $y = f(x)$ on $0 \leqslant x \leqslant 2\pi$.

6 A cork bobs up and down in a bucket of water such that the distance from the centre of the cork to the bottom of the bucket is given by $s(t) = 30 + \cos(\pi t)$ cm, $t \geqslant 0$ seconds.

a Find the cork's velocity at times $t = 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2$ s.

b Find the time intervals when the cork is falling.

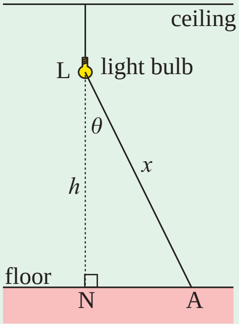


7 A light bulb hangs from the ceiling at height h metres above the floor, directly above point N. At any point A on the floor which is x metres from the light bulb, the illumination I is given by

$$I = \frac{\sqrt{8}\cos\theta}{x^2} \text{ units.}$$

a If NA = 1 metre, show that at A, $I = \sqrt{8}\cos\theta\sin^2\theta$.

b The light bulb may be lifted or lowered to change the intensity at A. Assuming NA = 1 metre, find the height the bulb should be above the floor for greatest illumination at A.



8 Find the equation of:

a the tangent to $y = \dfrac{1}{\sin x}$ at the point where $x = \frac{\pi}{3}$

b the normal to $y = \cos(\frac{x}{2})$ at the point where $x = \frac{\pi}{2}$.

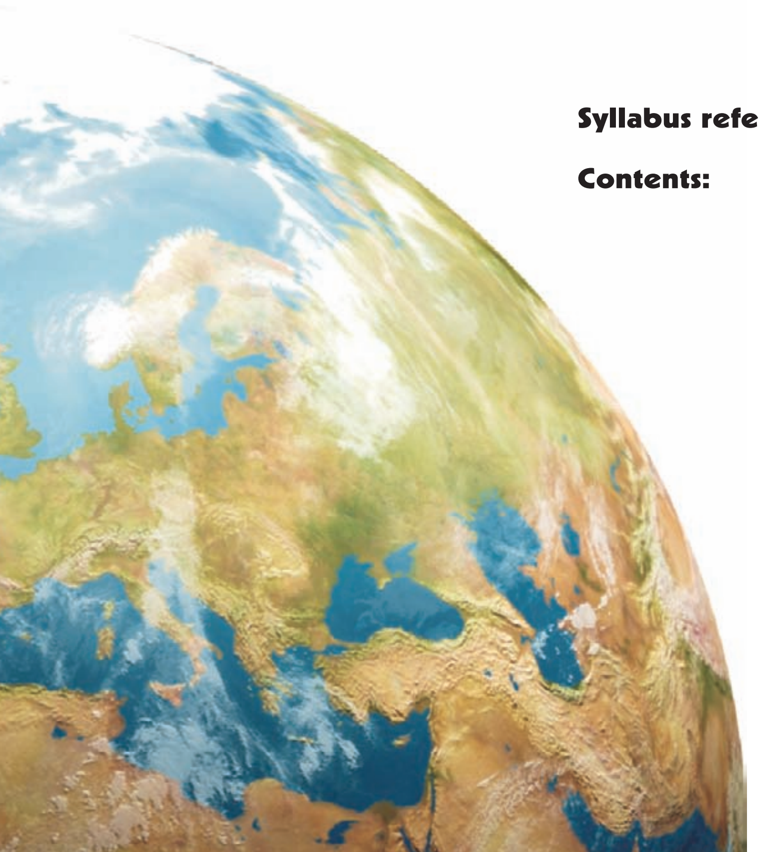
9 The function f is defined by $f : x \mapsto -10\sin 2x\cos 2x$, $0 \leqslant x \leqslant \pi$.

a Write down an expression for $f(x)$ in the form $k\sin 4x$.

b Solve $f'(x) = 0$, giving exact answers.

Chapter 21

Integration



Syllabus reference: 7.4, 7.5

Contents:

In the previous chapters we used differential calculus to find the derivatives of many types of functions. We also used it in problem solving, in particular to find the gradients of graphs and rates of changes, and to solve optimisation problems.

In this chapter we consider **integral calculus**. This involves **antidifferentiation** which is the reverse process of differentiation. Integral calculus also has many useful applications, including:

- finding areas where curved boundaries are involved
- finding volumes of revolution
- finding distances travelled from velocity functions
- finding hydrostatic pressure
- finding work done by a force
- finding centres of mass and moments of inertia
- solving problems in economics and biology
- solving problems in statistics
- solving differential equations.

A ANTIDIFFERENTIATION

In many problems in calculus we know the rate of change of one variable with respect to another, but we do not have a formula which relates the variables. In other words, we know $\dfrac{dy}{dx}$, but we need to know y in terms of x.

Examples of problems we need to solve include:

- The gradient function $f'(x)$ of a curve is $2x+3$ and the curve passes through the origin. What is the function $y=f(x)$?
- The rate of change in temperature is $\dfrac{dT}{dt}=10e^{-t}$ ^{o}C per minute where $t \geqslant 0$. What is the temperature function given that initially the temperature was 11^{o}C?

The process of finding y from $\dfrac{dy}{dx}$ or $f(x)$ from $f'(x)$ is the reverse process of differentiation. We call it **antidifferentiation**.

$$\begin{tikzcd} y \text{ or } f(x) \arrow[r, bend left, "\text{differentiation}"] & \dfrac{dy}{dx} \text{ or } f'(x) \arrow[l, bend left, "\substack{\text{antidifferentiation}\\ \text{or integration}}"] \end{tikzcd}$$

Consider the following problem: If $\dfrac{dy}{dx} = x^2$, what is y in terms of x?

From our work on differentiation, we know that when we differentiate power functions the index reduces by 1. We hence know that y must involve x^3.

Now if $y = x^3$ then $\dfrac{dy}{dx} = 3x^2$, so if we start with $y = \frac{1}{3}x^3$ then $\dfrac{dy}{dx} = x^2$.

However, in all of the cases $y = \frac{1}{3}x^3 + 2$, $y = \frac{1}{3}x^3 + 100$ and $y = \frac{1}{3}x^3 - 7$ we find $\dfrac{dy}{dx} = x^2$.

In fact, there are infinitely many functions of the form $y = \frac{1}{3}x^3 + c$ where c is an arbitrary constant which will give $\dfrac{dy}{dx} = x^2$. Ignoring the arbitrary constant, we say that $\frac{1}{3}x^3$ is the **antiderivative** of x^2. It is the simplest function which when differentiated gives x^2.

If $F(x)$ is a function where $F'(x) = f(x)$ we say that:

- the **derivative** of $F(x)$ is $f(x)$ and
- the **antiderivative** of $f(x)$ is $F(x)$.

Example 1 **Self Tutor**

Find the antiderivative of: **a** x^3 **b** e^{2x} **c** $\dfrac{1}{\sqrt{x}}$

a We know that the derivative of x^4 involves x^3.

Since $\dfrac{d}{dx}\left(x^4\right) = 4x^3$, $\dfrac{d}{dx}\left(\frac{1}{4}x^4\right) = x^3$

$\therefore$ the antiderivative of x^3 is $\frac{1}{4}x^4$.

b Since $\dfrac{d}{dx}\left(e^{2x}\right) = e^{2x} \times 2$, $\dfrac{d}{dx}\left(\frac{1}{2}e^{2x}\right) = \frac{1}{2} \times e^{2x} \times 2 = e^{2x}$

$\therefore$ the antiderivative of e^{2x} is $\frac{1}{2}e^{2x}$.

c $\dfrac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$ Now $\dfrac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}}$ $\therefore$ $\dfrac{d}{dx}(2x^{\frac{1}{2}}) = 2(\frac{1}{2})x^{-\frac{1}{2}} = x^{-\frac{1}{2}}$

$\therefore$ the antiderivative of $\dfrac{1}{\sqrt{x}}$ is $2\sqrt{x}$.

EXERCISE 21A

1 **a** Find the antiderivative of:

i x **ii** x^2 **iii** x^5 **iv** x^{-2} **v** x^{-4} **vi** $x^{\frac{1}{3}}$ **vii** $x^{-\frac{1}{2}}$

b From your answers in **a**, predict a general rule for the antiderivative of x^n.

2 **a** Find the antiderivative of:

i e^{2x} **ii** e^{5x} **iii** $e^{\frac{1}{2}x}$ **iv** $e^{0.01x}$ **v** $e^{\pi x}$ **vi** $e^{\frac{x}{3}}$

b From your answers in **a**, predict a general rule for the antiderivative of e^{kx} where k is a constant.

3 Find the antiderivative of:

a $6x^2 + 4x$ by differentiating $x^3 + x^2$ **b** e^{3x+1} by differentiating e^{3x+1}

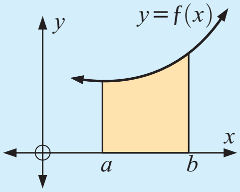
c $\sqrt{x}$ by differentiating $x\sqrt{x}$ **d** $(2x+1)^3$ by differentiating $(2x+1)^4$.

B THE FUNDAMENTAL THEOREM OF CALCULUS

Sir Isaac Newton and **Gottfried Wilhelm Leibniz** showed the link between differential calculus and the definite integral or limit of an area sum we saw in **Chapter 16**. This link is called the **fundamental theorem of calculus**. The beauty of this theorem is that it enables us to evaluate complicated summations.

We have already observed in **Chapter 16** that:

If $f(x)$ is a continuous positive function on an interval $a \leqslant x \leqslant b$ then the area under the curve between $x = a$ and $x = b$ is $\displaystyle\int_a^b \boldsymbol{f(x)\,dx}$.

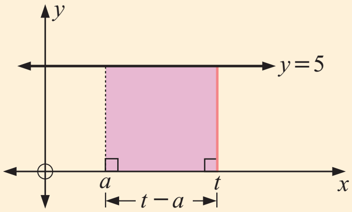


INVESTIGATION THE AREA FUNCTION

Consider the constant function $f(x) = 5$.

The corresponding area function is

$$\begin{aligned} A(t) &= \int_a^t 5\,dx \\ &= \text{shaded area in graph} \\ &= (t-a)5 \\ &= 5t - 5a \end{aligned}$$



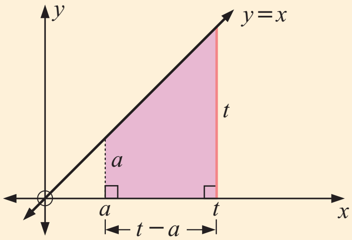
$\therefore$ we can write $A(t)$ in the form $F(t) - F(a)$ where $F(t) = 5t$ or equivalently, $F(x) = 5x$.

What to do:

1 What is the derivative $F'(x)$ of the function $F(x) = 5x$? How does this relate to the function $f(x)$?

2 Consider the simplest linear function $f(x) = x$.
The corresponding area function is

$$A(t) = \int_a^t x\,dx$$
$$= \text{shaded area in graph}$$
$$= \left(\frac{t+a}{2}\right)(t-a)$$

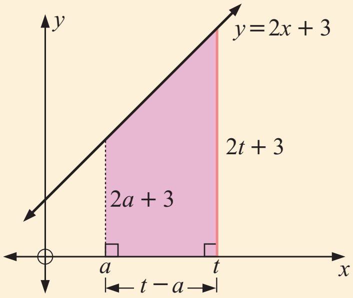


a Can you write $A(t)$ in the form $F(t) - F(a)$?

b If so, what is the derivative $F'(x)$? How does it relate to the function $f(x)$?

3 Consider $f(x) = 2x + 3$. The corresponding area function is

$$A(t) = \int_a^t (2x+3)\,dx$$
$$= \text{shaded area in graph}$$
$$= \left(\frac{2t+3+2a+3}{2}\right)(t-a)$$



a Can you write $A(t)$ in the form $F(t) - F(a)$?

b If so, what is the derivative $F'(x)$?
How does it relate to the function $f(x)$?

4 Repeat the procedure in **2** and **3** for finding the area functions of

a $f(x) = \frac{1}{2}x + 3$ **b** $f(x) = 5 - 2x$

Do your results fit with your earlier observations?

5 If $f(x) = 3x^2 + 4x + 5$, predict what $F(x)$ would be without performing the algebraic procedure.

From the investigation you should have discovered that, for $f(x) \geqslant 0$,

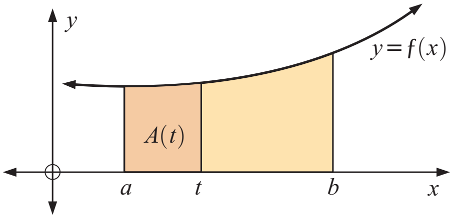
$$\int_a^t f(x)\,dx = F(t) - F(a) \quad \text{where} \quad F'(x) = f(x). \quad F(x) \text{ is the } \textbf{antiderivative} \text{ of } f(x).$$

The following argument shows why this is true for all functions $f(x) \geqslant 0$.

Consider a function $y = f(x)$ which has antiderivative $F(x)$ and an area function $A(t)$ which is the area from $x = a$ to $x = t$,

So, $A(t) = \displaystyle\int_a^t f(x)\,dx$.

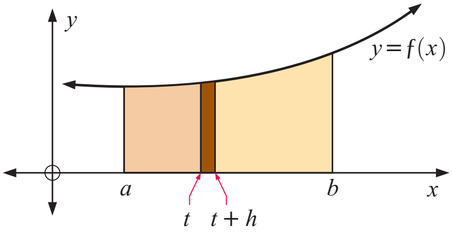
$A(t)$ is clearly an increasing function and
$A(a) = 0$ ..... (1)



Now consider a narrow strip of the region between $x = t$ and $x = t + h$.

The area of this strip is $A(t+h) - A(t)$.

Since the narrow strip is contained within two rectangles then



$$\text{area of smaller rectangle} \leqslant A(t+h) - A(t) \leqslant \text{area of larger rectangle}$$

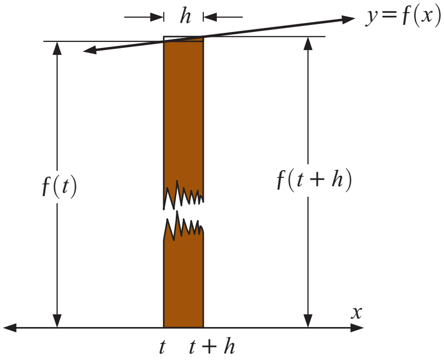
$$\therefore \quad hf(t) \leqslant A(t+h) - A(t) \leqslant hf(t+h)$$

$$\therefore \quad f(t) \leqslant \frac{A(t+h) - A(t)}{h} \leqslant f(t+h)$$

Taking the limit as $h \to 0$ gives

$$f(t) \leqslant A'(t) \leqslant f(t)$$

$$\therefore \quad A'(t) = f(t)$$



enlarged strip

The area function $A(t)$ is an antiderivative of $f(t)$, so $A(t)$ and $F(t)$ differ by a constant.

$$\therefore \quad A(t) = F(t) + c$$

Letting $t = a$, $\quad A(a) = F(a) + c$

But $\quad A(a) = 0 \quad$ {from (1)}

so $\quad c = -F(a)$

$$\therefore \quad A(t) = F(t) - F(a)$$

$$\therefore \quad \text{letting } t = b, \qquad \int_a^b f(x)\,dx = F(b) - F(a)$$

This result is in fact true for all continuous functions $f(x)$, and can be stated as the **fundamental theorem of calculus**:

For a continuous function $f(x)$ with antiderivative $F(x)$, $\displaystyle\boldsymbol{\int_a^b f(x)\,dx = F(b) - F(a)}$.

The fundamental theorem of calculus has many applications beyond the calculation of areas.

For example, given a velocity function $v(t)$ we know that $\dfrac{ds}{dt} = v$.

So, $s(t)$ is the antiderivative of $v(t)$ and by the fundamental theorem of calculus,

$\displaystyle\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$ gives the **displacement** over the time interval $t_1 \leqslant t \leqslant t_2$.

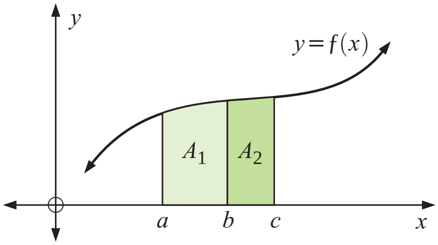
PROPERTIES OF DEFINITE INTEGRALS

The following properties of definite integrals can all be deduced from the fundamental theorem of calculus:

- $\int_a^a f(x)\,dx = 0$
- $\int_a^b c\,dx = c(b-a)$ $\{c$ is a constant$\}$
- $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
- $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$
- $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
- $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Example proof:

$$\begin{aligned}&\int_a^b f(x)\,dx + \int_b^c f(x)\,dx\\ &= F(b) - F(a) + F(c) - F(b)\\ &= F(c) - F(a)\\ &= \int_a^c f(x)\,dx\end{aligned}$$



$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = A_1 + A_2 = \int_a^c f(x)\,dx$$

Example 2

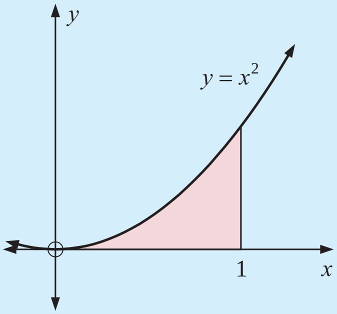
Self Tutor

Use the fundamental theorem of calculus to find the area:

a between the x-axis and $y = x^2$ from $x = 0$ to $x = 1$

b between the x-axis and $y = \sqrt{x}$ from $x = 1$ to $x = 9$.

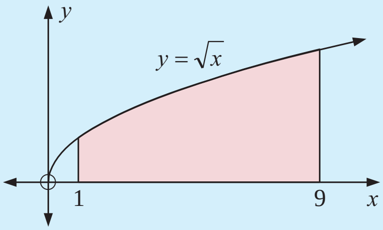
a

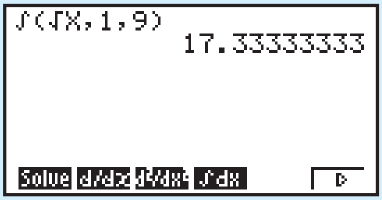


$f(x) = x^2$ has antiderivative $F(x) = \frac{x^3}{3}$

$$\begin{aligned}\therefore \text{ the area } &= \int_0^1 x^2\,dx\\ &= F(1) - F(0)\\ &= \tfrac{1}{3} - 0\\ &= \tfrac{1}{3} \text{ units}^2\end{aligned}$$

b





```
∫(√X,1,9)
                17.33333333
```

$f(x)=\sqrt{x}=x^{\frac{1}{2}}$ has antiderivative

$$F(x)=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}=\tfrac{2}{3}x\sqrt{x}$$

$$\begin{aligned}\therefore\quad \text{the area } &= \int_1^9 x^{\frac{1}{2}}\,dx\\ &= F(9)-F(1)\\ &= \tfrac{2}{3}\times 27-\tfrac{2}{3}\times 1\\ &= 17\tfrac{1}{3}\text{ units}^2\end{aligned}$$

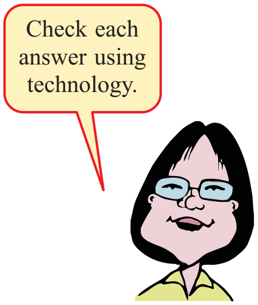
EXERCISE 21B

1 Use the fundamental theorem of calculus to show that:

- **a** $\displaystyle\int_a^a f(x)\,dx=0$ and explain the result graphically
- **b** $\displaystyle\int_a^b c\,dx=c(b-a)$ where c is a constant
- **c** $\displaystyle\int_b^a f(x)\,dx=-\int_a^b f(x)\,dx$
- **d** $\displaystyle\int_a^b c\,f(x)\,dx=c\int_a^b f(x)\,dx$ where c is a constant
- **e** $\displaystyle\int_a^b [f(x)+g(x)]\,dx=\int_a^b f(x)\,dx+\int_a^b g(x)\,dx$

2 Use the fundamental theorem of calculus to find the area between the x-axis and:

- **a** $y=x^3$ from $x=0$ to $x=1$
- **b** $y=x^3$ from $x=1$ to $x=2$
- **c** $y=x^2+3x+2$ from $x=1$ to $x=3$
- **d** $y=\sqrt{x}$ from $x=0$ to $x=2$
- **e** $y=e^x$ from $x=0$ to $x=1.5$
- **f** $y=\dfrac{1}{\sqrt{x}}$ from $x=1$ to $x=4$
- **g** $y=x^3+2x^2+7x+4$ from $x=1$ to $x=1.25$.



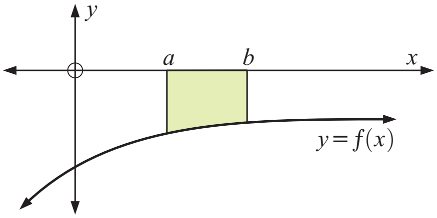
3 Using technology, find correct to 3 significant figures, the area between the x-axis and:

- **a** $y=e^{x^2}$ from $x=0$ to $x=1.5$
- **b** $y=(\ln x)^2$ from $x=2$ to $x=4$
- **c** $y=\sqrt{9-x^2}$ from $x=1$ to $x=2$.

4 **a** Use the fundamental theorem of calculus to show that

$$\int_a^b (-f(x))\,dx = -\int_a^b f(x)\,dx$$

b Use the result in **a** to show that if $f(x) \leqslant 0$ for all x on $a \leqslant x \leqslant b$ then the shaded area $= -\int_a^b f(x)\,dx$.



c Calculate the following integrals, and give graphical interpretations of your answers:

i $\int_0^1 (-x^2)\,dx$ **ii** $\int_0^1 (x^2 - x)\,dx$ **iii** $\int_{-2}^0 3x\,dx$

d Use graphical evidence and known area facts to find $\int_0^2 \left(-\sqrt{4-x^2}\right) dx$.

C INTEGRATION

Earlier we showed that the **antiderivative** of x^2 was $\frac{1}{3}x^3$.

We showed that any function of the form $\frac{1}{3}x^3 + c$ where c is any constant, has derivative x^2.

We say that the **integral** of x^2 is $\frac{1}{3}x^3 + c$ and write $\int x^2\,dx = \frac{1}{3}x^3 + c$.

We read this as "the integral of x^2 with respect to x".

In general, if $F'(x) = f(x)$ then $\int f(x)\,dx = F(x) + c$.

This process is known as **indefinite integration.** It is indefinite because it is not being applied over a particular interval.

DISCOVERING INTEGRALS

Since integration is the reverse process of differentiation we can sometimes discover integrals by differentiation. For example:

- if $F(x) = x^4$ then $F'(x) = 4x^3$

$$\therefore \int 4x^3\,dx = x^4 + c$$

- if $F(x) = \sqrt{x} = x^{\frac{1}{2}}$ then $F'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$$\therefore \int \frac{1}{2\sqrt{x}}dx = \sqrt{x} + c$$

The following rules may prove useful:

- Any constant may be written in front of the integral sign.

$$\int k\,f(x)\,dx = k\int f(x)\,dx, \quad k \text{ is a constant}$$

Proof: Consider differentiating $kF(x)$ where $F'(x) = f(x)$.

$$\frac{d}{dx}(k\,F(x)) = k\,F'(x) = k\,f(x)$$

$$\therefore \int k\,f(x)\,dx = k\,F(x)$$

$$= k\int f(x)\,dx$$

- The integral of a sum is the sum of the separate integrals. This rule enables us to integrate term by term.

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Example 3 **Self Tutor**

If $y = x^4 + 2x^3$, find $\frac{dy}{dx}$.

Hence find $\int (2x^3 + 3x^2)\,dx$.

If $y = x^4 + 2x^3$,

then $\frac{dy}{dx} = 4x^3 + 6x^2$

$$\therefore \int 4x^3 + 6x^2\,dx = x^4 + 2x^3 + c$$

$$\therefore \int 2(2x^3 + 3x^2)\,dx = x^4 + 2x^3 + c$$

$$\therefore 2\int (2x^3 + 3x^2)\,dx = x^4 + 2x^3 + c$$

$$\therefore \int (2x^3 + 3x^2)\,dx = \tfrac{1}{2}x^4 + x^3 + c$$

EXERCISE 21C.1

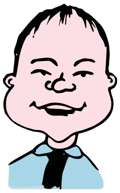
1 If $y = x^7$, find $\frac{dy}{dx}$. Hence find $\int x^6\,dx$.

2 If $y = x^3 + x^2$, find $\frac{dy}{dx}$. Hence find $\int (3x^2 + 2x)\,dx$.

3 If $y = e^{2x+1}$, find $\frac{dy}{dx}$. Hence find $\int e^{2x+1}\,dx$.

4 If $y = (2x+1)^4$ find $\frac{dy}{dx}$. Hence find $\int (2x+1)^3\,dx$.

5 If $y = x\sqrt{x}$, find $\frac{dy}{dx}$. Hence find $\int \sqrt{x}\,dx$.



We can always check that an integral is correct by differentiating the answer. It should give us the **integrand**, the function we originally integrated.

6 If $y = \dfrac{1}{\sqrt{x}}$, find $\dfrac{dy}{dx}$. Hence find $\displaystyle\int \frac{1}{x\sqrt{x}}\,dx$.

7 If $y = \cos 2x$, find $\dfrac{dy}{dx}$. Hence find $\displaystyle\int \sin 2x\,dx$.

8 If $y = \sin(1-5x)$, find $\dfrac{dy}{dx}$. Hence find $\displaystyle\int \cos(1-5x)\,dx$.

9 Prove the rule $\displaystyle\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$.

10 Find $\dfrac{dy}{dx}$ if $y = \sqrt{1-4x}$. Hence find $\displaystyle\int \frac{1}{\sqrt{1-4x}}\,dx$.

11 By considering $\dfrac{d}{dx}\ln(5-3x+x^2)$, find $\displaystyle\int \frac{4x-6}{5-3x+x^2}\,dx$.

RULES FOR DIFFERENTIATION

In earlier chapters we developed rules to help us differentiate functions more efficiently. Following is a summary of these rules:

Function	*Derivative*	*Name*
c, a constant	0	
$mx+c$, m and c are constants	m	
x^n	nx^{n-1}	**power rule**
$cu(x)$	$cu'(x)$	
$u(x)+v(x)$	$u'(x)+v'(x)$	**addition rule**
$u(x)v(x)$	$u'(x)v(x)+u(x)v'(x)$	**product rule**
$\dfrac{u(x)}{v(x)}$	$\dfrac{u'(x)v(x)-u(x)v'(x)}{[v(x)]^2}$	**quotient rule**
$y=f(u)$ where $u=u(x)$	$\dfrac{dy}{dx}=\dfrac{dy}{du}\dfrac{du}{dx}$	**chain rule**
e^x	e^x	
$e^{f(x)}$	$e^{f(x)}f'(x)$	
$\ln x$	$\dfrac{1}{x}$	
$\ln f(x)$	$\dfrac{f'(x)}{f(x)}$	
$[f(x)]^n$	$n[f(x)]^{n-1}f'(x)$	
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
$\tan x$	$\dfrac{1}{\cos^2 x}$	

These rules or combinations of them can be used to differentiate almost all functions.

However, the task of finding **antiderivatives** is not so easy and cannot be written as a simple list of rules as we did above. In fact, huge books of different types of functions and their integrals have been written. Fortunately our course is restricted to a few special cases.

RULES FOR INTEGRATION

For k a constant, $\frac{d}{dx}(kx+c)=k$ $\quad\therefore\quad$ $\boldsymbol{\int k\,dx = kx + c}$

If $n \neq -1$, $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}+c\right)=\frac{(n+1)x^n}{n+1}=x^n$

$\therefore\quad$ $\boldsymbol{\int x^n\,dx = \frac{x^{n+1}}{n+1}+c,\ \ n\neq -1}$

$\frac{d}{dx}(e^x+c)=e^x$ $\quad\therefore\quad$ $\boldsymbol{\int e^x\,dx = e^x + c}$

If $x>0$, $\frac{d}{dx}(\ln x+c)=\frac{1}{x}$ $\quad\therefore\quad$ $\boldsymbol{\int \frac{1}{x}\,dx = \ln x + c,\ \ x>0}$

$\frac{d}{dx}(\sin x+c)=\cos x$ $\quad\therefore\quad$ $\boldsymbol{\int \cos x\,dx = \sin x + c}$

$\frac{d}{dx}(-\cos x+c)=\sin x+c$ $\quad\therefore\quad$ $\boldsymbol{\int \sin x\,dx = -\cos x + c}$

Function	*Integral*
k, a constant	$kx+c$
x^n	$\frac{x^{n+1}}{n+1}+c,\ \ n\neq -1$
e^x	e^x+c
$\frac{1}{x}$	$\ln x+c,\ \ x>0$
$\cos x$	$\sin x+c$
$\sin x$	$-\cos x+c$

In each case, c is an arbitrary constant called the **constant of integration** or **integrating constant**.

You should remember that you can always check your integration by differentiating the resulting function.

Example 4 **Self Tutor**

Find: **a** $\int (x^3 - 2x^2 + 5)\, dx$ **b** $\int \left(\frac{1}{x^3} - \sqrt{x}\right) dx$

a
$$\int (x^3 - 2x^2 + 5)\, dx$$
$$= \frac{x^4}{4} - \frac{2x^3}{3} + 5x + c$$

b
$$\int \left(\frac{1}{x^3} - \sqrt{x}\right) dx$$
$$= \int (x^{-3} - x^{\frac{1}{2}})\, dx$$
$$= \frac{x^{-2}}{-2} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$= -\frac{1}{2x^2} - \tfrac{2}{3}x^{\frac{3}{2}} + c$$

Example 5 **Self Tutor**

Integrate with respect to x:

a $2\sin x - \cos x$ **b** $-\frac{2}{x} + 3e^x$

a
$$\int [2\sin x - \cos x]dx$$
$$= 2(-\cos x) - \sin x + c$$
$$= -2\cos x - \sin x + c$$

b
$$\int \left[-\frac{2}{x} + 3e^x\right] dx$$
$$= -2\ln x + 3e^x + c \quad \text{provided} \quad x > 0$$

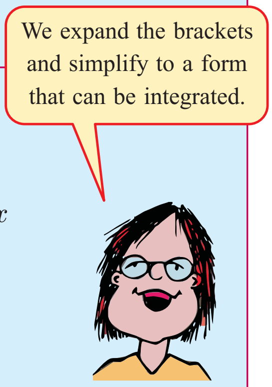
There is no product or quotient rule for integration. Consequently we often have to carry out multiplication or division before we integrate.

Example 6 **Self Tutor**

Find: **a** $\int \left(3x + \frac{2}{x}\right)^2 dx$ **b** $\int \left(\frac{x^2 - 2}{\sqrt{x}}\right) dx$

a
$$\int \left(3x + \frac{2}{x}\right)^2 dx$$
$$= \int \left(9x^2 + 12 + \frac{4}{x^2}\right) dx$$
$$= \int (9x^2 + 12 + 4x^{-2})\, dx$$
$$= \frac{9x^3}{3} + 12x + \frac{4x^{-1}}{-1} + c$$
$$= 3x^3 + 12x - \frac{4}{x} + c$$

b
$$\int \left(\frac{x^2 - 2}{\sqrt{x}}\right) dx$$
$$= \int \left(\frac{x^2}{\sqrt{x}} - \frac{2}{\sqrt{x}}\right) dx$$
$$= \int (x^{\frac{3}{2}} - 2x^{-\frac{1}{2}})\, dx$$
$$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= \tfrac{2}{5}x^2\sqrt{x} - 4\sqrt{x} + c$$



We can find the constant of integration c if we are given a point on the curve.

Example 7 **Self Tutor**

Find $f(x)$ given that $f'(x) = x^3 - 2x^2 + 3$ and $f(0) = 2$.

Since $f'(x) = x^3 - 2x^2 + 3,$

$$f(x) = \int (x^3 - 2x^2 + 3)\, dx$$

$$\therefore \quad f(x) = \frac{x^4}{4} - \frac{2x^3}{3} + 3x + c$$

But $f(0) = 2$, so $0 - 0 + 0 + c = 2$ and so $c = 2$

Thus $f(x) = \dfrac{x^4}{4} - \dfrac{2x^3}{3} + 3x + 2$

Example 8 **Self Tutor**

Find $f(x)$ given that $f'(x) = 2\sin x - \sqrt{x}$ and $f(0) = 4$.

$$f(x) = \int \left[2\sin x - x^{\frac{1}{2}}\right] dx$$

$$\therefore \quad f(x) = 2 \times (-\cos x) - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$\therefore \quad f(x) = -2\cos x - \tfrac{2}{3}x^{\frac{3}{2}} + c$$

But $f(0) = -2\cos 0 - 0 + c$

$\therefore \quad 4 = -2 + c$ and so $c = 6$

Thus $f(x) = -2\cos x - \frac{2}{3}x^{\frac{3}{2}} + 6.$

If we are given the second derivative we need to integrate twice to find the function. This creates two integrating constants, so we need two other facts about the curve in order to determine these constants.

Example 9 **Self Tutor**

Find $f(x)$ given that $f''(x) = 12x^2 - 4$, $f'(0) = -1$ and $f(1) = 4$.

If $f''(x) = 12x^2 - 4$

then $f'(x) = \dfrac{12x^3}{3} - 4x + c$ {integrating with respect to x}

$\therefore \quad f'(x) = 4x^3 - 4x + c$

But $f'(0) = -1$ so $0 - 0 + c = -1$ and so $c = -1$

Thus $f'(x) = 4x^3 - 4x - 1$

$\therefore \quad f(x) = \dfrac{4x^4}{4} - \dfrac{4x^2}{2} - x + d$ {integrating again}

$\therefore \quad f(x) = x^4 - 2x^2 - x + d$

But $f(1) = 4$ so $1 - 2 - 1 + d = 4$ and so $d = 6$

Thus $f(x) = x^4 - 2x^2 - x + 6$

EXERCISE 21C.2

1 Find:

a $\displaystyle\int (x^4 - x^2 - x + 2)\,dx$ **b** $\displaystyle\int (\sqrt{x} + e^x)\,dx$ **c** $\displaystyle\int \left(3e^x - \frac{1}{x}\right) dx$

d $\displaystyle\int \left(x\sqrt{x} - \frac{2}{x}\right) dx$ **e** $\displaystyle\int \left(\frac{1}{x\sqrt{x}} + \frac{4}{x}\right) dx$ **f** $\displaystyle\int \left(\frac{1}{2}x^3 - x^4 + x^{\frac{1}{3}}\right) dx$

g $\displaystyle\int \left(x^2 + \frac{3}{x}\right) dx$ **h** $\displaystyle\int \left(\frac{1}{2x} + x^2 - e^x\right) dx$ **i** $\displaystyle\int \left(5e^x + \frac{1}{3}x^3 - \frac{4}{x}\right) dx$

2 Integrate with respect to x:

a $3\sin x - 2$ **b** $4x - 2\cos x$ **c** $\sin x - 2\cos x + e^x$

d $x^2\sqrt{x} - 10\sin x$ **e** $\dfrac{x(x-1)}{3} + \cos x$ **f** $-\sin x + 2\sqrt{x}$

3 Find:

a $\displaystyle\int (x^2 + 3x - 2)\,dx$ **b** $\displaystyle\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$ **c** $\displaystyle\int \left(2e^x - \frac{1}{x^2}\right) dx$

d $\displaystyle\int \frac{1 - 4x}{x\sqrt{x}}\,dx$ **e** $\displaystyle\int (2x + 1)^2\,dx$ **f** $\displaystyle\int \frac{x^2 + x - 3}{x}\,dx$

g $\displaystyle\int \frac{2x - 1}{\sqrt{x}}\,dx$ **h** $\displaystyle\int \frac{x^2 - 4x + 10}{x^2\sqrt{x}}\,dx$ **i** $\displaystyle\int (x + 1)^3\,dx$

4 Find:

a $\displaystyle\int \left(\sqrt{x} + \frac{1}{2}\cos x\right) dx$ **b** $\displaystyle\int \left(2e^t - 4\sin t\right) dt$ **c** $\displaystyle\int \left(3\cos t - \frac{1}{t}\right) dt$

5 Find y if:

a $\dfrac{dy}{dx} = 6$ **b** $\dfrac{dy}{dx} = 4x^2$ **c** $\dfrac{dy}{dx} = 5\sqrt{x} - x^2$

d $\dfrac{dy}{dx} = \dfrac{1}{x^2}$ **e** $\dfrac{dy}{dx} = 2e^x - 5$ **f** $\dfrac{dy}{dx} = 4x^3 + 3x^2$

6 Find y if:

a $\dfrac{dy}{dx} = (1 - 2x)^2$ **b** $\dfrac{dy}{dx} = \sqrt{x} - \dfrac{2}{\sqrt{x}}$ **c** $\dfrac{dy}{dx} = \dfrac{x^2 + 2x - 5}{x^2}$

7 Find $f(x)$ if:

a $f'(x) = x^3 - 5\sqrt{x} + 3$ **b** $f'(x) = 2\sqrt{x}(1 - 3x)$ **c** $f'(x) = 3e^x - \dfrac{4}{x}$

8 Find $f(x)$ given that:

a $f'(x) = 2x - 1$ and $f(0) = 3$

b $f'(x) = 3x^2 + 2x$ and $f(2) = 5$

c $f'(x) = e^x + \dfrac{1}{\sqrt{x}}$ and $f(1) = 1$

d $f'(x) = x - \dfrac{2}{\sqrt{x}}$ and $f(1) = 2$.

9 Find $f(x)$ given that:

a $f'(x) = x^2 - 4\cos x$ and $f(0) = 3$

b $f'(x) = 2\cos x - 3\sin x$ and $f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

10 Find $f(x)$ given that:

a $f''(x) = 2x + 1$, $f'(1) = 3$ and $f(2) = 7$

b $f''(x) = 15\sqrt{x} + \dfrac{3}{\sqrt{x}}$, $f'(1) = 12$ and $f(0) = 5$

c $f''(x) = \cos x$, $f'(\frac{\pi}{2}) = 0$ and $f(0) = 3$

d $f''(x) = 2x$ and the points $(1,\ 0)$ and $(0,\ 5)$ lie on the curve.

D INTEGRATING $f(ax+b)$

In this section we deal with integrals of functions which are composite with the linear function $ax + b$.

Notice that $\dfrac{d}{dx}\left(\dfrac{1}{a}e^{ax+b}\right) = \dfrac{1}{a}e^{ax+b} \times a = e^{ax+b}$

$$\therefore \quad \boxed{\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c}$$

Likewise if $n \neq -1$,

$$\frac{d}{dx}\left(\frac{1}{a(n+1)}(ax+b)^{n+1}\right) = \frac{1}{a(n+1)}(n+1)(ax+b)^n \times a,$$
$$= (ax+b)^n$$

$$\therefore \quad \boxed{\int (ax+b)^n\,dx = \frac{1}{a}\,\frac{(ax+b)^{n+1}}{n+1} + c, \quad n \neq -1}$$

Also, $\dfrac{d}{dx}\left(\dfrac{1}{a}\ln(ax+b)\right) = \dfrac{1}{a}\left(\dfrac{a}{ax+b}\right) = \dfrac{1}{ax+b}$ for $ax + b > 0$

$$\therefore \quad \boxed{\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(ax+b) + c, \quad ax+b>0}$$

We can perform the same process for the circular functions:

$$\frac{d}{dx}(\sin(ax+b)) = a\cos(ax+b)$$

$$\therefore \quad \int a\cos(ax+b)\,dx = \sin(ax+b)+c$$

$$\therefore \quad a\int \cos(ax+b)\,dx = \sin(ax+b)+c$$

So, $$\boldsymbol{\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)+c}$$

Likewise we can show $$\boldsymbol{\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)+c}$$

SUMMARY OF INTEGRALS FOR FUNCTIONS OF THE FORM $f(ax+b)$

Function	*Integral*
e^{ax+b}	$\frac{1}{a}e^{ax+b}+c$
$(ax+b)^n$	$\frac{1}{a}\frac{(ax+b)^{n+1}}{n+1}+c, \quad n \neq -1$
$\frac{1}{ax+b}$	$\frac{1}{a}\ln(ax+b)+c, \quad ax+b>0$
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b)+c$
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b)+c$

Example 10 **Self Tutor**

Find: **a** $\int (2x+3)^4\,dx$ **b** $\int \frac{1}{\sqrt{1-2x}}\,dx$

a $\int (2x+3)^4\,dx$

$= \frac{1}{2} \times \frac{(2x+3)^5}{5} + c$

$= \frac{1}{10}(2x+3)^5 + c$

b $\int \frac{1}{\sqrt{1-2x}}\,dx$

$= \int (1-2x)^{-\frac{1}{2}}\,dx$

$= \frac{1}{-2} \times \frac{(1-2x)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= -\sqrt{1-2x} + c$

Example 11 Self Tutor

Find: **a** $\int (2e^{2x} - e^{-3x})\,dx$ **b** $\int \frac{4}{1-2x}\,dx$

a $\int (2e^{2x} - e^{-3x})\,dx$

$= 2(\frac{1}{2})e^{2x} - (\frac{1}{-3})e^{-3x} + c$

$= e^{2x} + \frac{1}{3}e^{-3x} + c$

b $\int \frac{4}{1-2x}\,dx$

$= 4\int \frac{1}{1-2x}\,dx$

$= 4\left(\frac{1}{-2}\right)\ln(1-2x) + c, \quad 1 - 2x > 0$

$= -2\ln(1-2x) + c, \quad x < \frac{1}{2}$

Example 12 Self Tutor

Integrate with respect to x:

$2\sin(3x) + \cos(4x + \pi)$

$\int (2\sin(3x) + \cos(4x+\pi))\,dx$

$= 2 \times \frac{1}{3}(-\cos(3x)) + \frac{1}{4}\sin(4x+\pi) + c$

$= -\frac{2}{3}\cos(3x) + \frac{1}{4}\sin(4x+\pi) + c$

INTEGRALS OF POWERS OF CIRCULAR FUNCTIONS

Integrals involving $\sin^2(ax+b)$ and $\cos^2(ax+b)$ can be found by first using

$$\boldsymbol{\sin^2\theta = \tfrac{1}{2} - \tfrac{1}{2}\cos(2\theta)} \text{ or } \boldsymbol{\cos^2\theta = \tfrac{1}{2} + \tfrac{1}{2}\cos(2\theta)}.$$

These formulae are simply rearrangements of $\cos(2\theta)$ formulae.

For example,

- $\sin^2(3x - \frac{\pi}{2})$ becomes $\frac{1}{2} - \frac{1}{2}\cos(6x - \pi)$
- $\cos^2\left(\frac{x}{2}\right)$ becomes $\frac{1}{2} + \frac{1}{2}\cos 2\left(\frac{x}{2}\right) = \frac{1}{2} + \frac{1}{2}\cos x$.

Example 13 Self Tutor

Integrate $(2 - \sin x)^2$.

$\int (2 - \sin x)^2 dx$

$= \int (4 - 4\sin x + \sin^2 x)\,dx$

$= \int \left(4 - 4\sin x + \frac{1}{2} - \frac{1}{2}\cos(2x)\right)dx$

$= \int \left(\frac{9}{2} - 4\sin x - \frac{1}{2}\cos(2x)\right)dx$

$= \frac{9}{2}x + 4\cos x - \frac{1}{2} \times \frac{1}{2}\sin(2x) + c$

$= \frac{9}{2}x + 4\cos x - \frac{1}{4}\sin(2x) + c$

EXERCISE 21D

1 Find:

a $\int (2x+5)^3\,dx$ **b** $\int \frac{1}{(3-2x)^2}\,dx$ **c** $\int \frac{4}{(2x-1)^4}\,dx$

d $\int (4x-3)^7\,dx$ **e** $\int \sqrt{3x-4}\,dx$ **f** $\int \frac{10}{\sqrt{1-5x}}\,dx$

g $\int 3(1-x)^4\,dx$ **h** $\int \frac{4}{\sqrt{3-4x}}\,dx$

2 Integrate with respect to x:

a $\sin(3x)$ **b** $2\cos(-4x)+1$ **c** $3\cos\left(\frac{x}{2}\right)$

d $3\sin(2x)-e^{-x}$ **e** $2\sin\left(2x+\frac{\pi}{6}\right)$ **f** $-3\cos\left(\frac{\pi}{4}-x\right)$

g $\cos(2x)+\sin(2x)$ **h** $2\sin(3x)+5\cos(4x)$ **i** $\frac{1}{2}\cos(8x)-3\sin x$

3 **a** Find $y=f(x)$ given $\frac{dy}{dx}=\sqrt{2x-7}$ and that $y=11$ when $x=8$.

b Function $f(x)$ has gradient function $\frac{4}{\sqrt{1-x}}$ and passes through the point $(-3, -11)$.
Find the point on the graph of $y=f(x)$ with x-coordinate -8.

4 Integrate with respect to x:

a $\cos^2 x$ **b** $\sin^2 x$ **c** $1+\cos^2(2x)$

d $3-\sin^2(3x)$ **e** $\frac{1}{2}\cos^2(4x)$ **f** $(1+\cos x)^2$

5 Find:

a $\int 3(2x-1)^2\,dx$ **b** $\int (x^2-x)^2\,dx$ **c** $\int (1-3x)^3\,dx$

d $\int (1-x^2)^2\,dx$ **e** $\int 4\sqrt{5-x}\,dx$ **f** $\int (x^2+1)^3\,dx$

6 Find:

a $\int \left(2e^x+5e^{2x}\right)\,dx$ **b** $\int \left(3e^{5x-2}\right)\,dx$ **c** $\int \left(e^{7-3x}\right)\,dx$

d $\int \frac{1}{2x-1}\,dx$ **e** $\int \frac{5}{1-3x}\,dx$ **f** $\int \left(e^{-x}-\frac{4}{2x+1}\right)\,dx$

g $\int (e^x+e^{-x})^2\,dx$ **h** $\int (e^{-x}+2)^2\,dx$ **i** $\int \left(x-\frac{5}{1-x}\right)\,dx$

7 Find y given that:

a $\frac{dy}{dx}=(1-e^x)^2$ **b** $\frac{dy}{dx}=1-2x+\frac{3}{x+2}$ **c** $\frac{dy}{dx}=e^{-2x}+\frac{4}{2x-1}$

8 To find $\int \frac{1}{4x}\,dx$, Tracy's answer was $\int \frac{1}{4x}\,dx = \frac{1}{4}\ln(4x) + c,\ x > 0$

and Nadine's answer was $\int \frac{1}{4x}\,dx = \frac{1}{4}\int \frac{1}{x}\,dx = \frac{1}{4}\ln(x) + c,\ x > 0.$

Which of them has found the correct answer? Prove your statement.

9 Suppose $f'(x) = p\sin\left(\frac{1}{2}x\right)$, $f(0) = 1$ and $f(2\pi) = 0$. Find p and $f(x)$.

10 Consider a function g such that $g''(x) = -\sin 2x$.
Show that the gradients of the tangents to $y = g(x)$ when $x = \pi$ and $x = -\pi$ are equal.

11 **a** If $f'(x) = 2e^{-2x}$ and $f(0) = 3$, find $f(x)$.

b If $f'(x) = 2x - \dfrac{2}{1-x}$ and $f(-1) = 3$, find $f(x)$.

c If a curve has gradient function $\sqrt{x} + \frac{1}{2}e^{-4x}$ and passes through (1, 0), find the equation of the function.

12 Show that $(\sin x + \cos x)^2 = 1 + \sin 2x$, and hence determine $\int (\sin x + \cos x)^2\,dx$.

13 Show that $(\cos x + 1)^2 = \frac{1}{2}\cos 2x + 2\cos x + \frac{3}{2}$, and hence determine $\int (\cos x + 1)^2\,dx$.

E DEFINITE INTEGRALS

We have seen from the **fundamental theorem of calculus**:

If $F(x)$ is the antiderivative of $f(x)$ where $f(x)$ is continuous on the interval $a \leqslant x \leqslant b$

then the **definite integral** of $f(x)$ on this interval is $\boldsymbol{\int_a^b f(x)\,dx = F(b) - F(a)}$.

$\int_a^b f(x)\,dx$ reads "the integral from $x = a$ to $x = b$ of $f(x)$ with respect to x".

It is called a **definite** integral because there are lower and upper limits for the integration and it therefore results in a numerical answer.

When calculating definite integrals we can omit the constant of integration c as this will always cancel out in the subtraction process.

It is common to write $F(b) - F(a)$ as $[F(x)]_a^b$, and so $\boldsymbol{\int_a^b f(x)dx = [F(x)]_a^b}$.

Since definite integrals result in numerical answers, we are able to evaluate them using technology. Instructions for this can be found in the graphics calculator instructions at the start of the book.

Example 14 **Self Tutor**

Find $\displaystyle\int_1^3 (x^2+2)\,dx$.

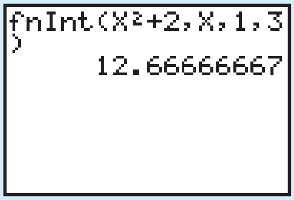
$$\int_1^3 (x^2+2)\,dx$$

$$= \left[\frac{x^3}{3} + 2x\right]_1^3$$

$$= \left(\tfrac{3^3}{3} + 2(3)\right) - \left(\tfrac{1^3}{3} + 2(1)\right)$$

$$= (9+6) - (\tfrac{1}{3} + 2)$$

$$= 12\tfrac{2}{3}$$



```
fnInt(X²+2,X,1,3
)
      12.66666667
```

Example 15 **Self Tutor**

Evaluate: **a** $\displaystyle\int_0^{\frac{\pi}{3}} \sin x\,dx$ **b** $\displaystyle\int_1^4 \left(2x + \frac{3}{x}\right) dx$

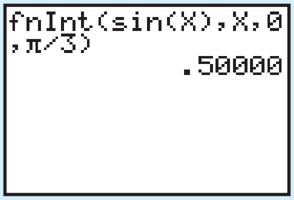
a $\displaystyle\int_0^{\frac{\pi}{3}} \sin x\,dx$

$$= [-\cos x]_0^{\frac{\pi}{3}}$$

$$= (-\cos\tfrac{\pi}{3}) - (-\cos 0)$$

$$= -\tfrac{1}{2} + 1$$

$$= \tfrac{1}{2}$$



```
fnInt(sin(X),X,0
,π/3)
           .50000
```

b $\displaystyle\int_1^4 \left(2x + \frac{3}{x}\right) dx$

$$= \left[x^2 + 3\ln x\right]_1^4 \quad \{\text{since } x > 0\}$$

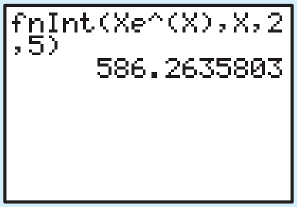
$$= (16 + 3\ln 4) - (1 + 3\ln 1)$$

$$= 15 + 6\ln 2$$

Some integrals are difficult or even impossible to evaluate. In these cases you are expected to use a graphics calculator to evaluate the integral.

Example 16 **Self Tutor**

Evaluate $\displaystyle\int_2^5 xe^x\,dx$ to an accuracy of 4 significant figures.



```
fnInt(Xe^(X),X,2
,5)
      586.2635803
```

$$\int_2^5 xe^x\,dx \approx 586.3$$

EXERCISE 21E.1

1 Evaluate the following and check with your graphics calculator:

a $\displaystyle\int_0^1 x^3\,dx$ **b** $\displaystyle\int_0^2 (x^2 - x)\,dx$ **c** $\displaystyle\int_0^1 e^x\,dx$

d $\displaystyle\int_0^{\frac{\pi}{6}} \cos x\,dx$ **e** $\displaystyle\int_1^4 \left(x - \frac{3}{\sqrt{x}}\right) dx$ **f** $\displaystyle\int_4^9 \frac{x-3}{\sqrt{x}}\,dx$

g $\displaystyle\int_1^3 \frac{1}{x}\,dx$ **h** $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x\,dx$ **i** $\displaystyle\int_1^2 (e^{-x}+1)^2\,dx$

j $\displaystyle\int_2^6 \frac{1}{\sqrt{2x-3}}\,dx$ **k** $\displaystyle\int_0^1 e^{1-x}\,dx$ **l** $\displaystyle\int_0^{\frac{\pi}{6}} \sin(3x)\,dx$

m $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2 x\,dx$ **n** $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$

2 Evaluate using technology:

a $\displaystyle\int_1^3 \ln x\,dx$ **b** $\displaystyle\int_{-1}^1 e^{-x^2}\,dx$ **c** $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \sin(\sqrt{x})\,dx$

PROPERTIES OF DEFINITE INTEGRALS

Earlier in the chapter we proved the following properties of definite integrals using the fundamental theorem of calculus:

- $\displaystyle\int_a^b [-f(x)]\,dx = -\int_a^b f(x)\,dx$
- $\displaystyle\int_a^b cf(x)dx = c\int_a^b f(x)dx,$ c is any constant
- $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)dx$
- $\displaystyle\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

EXERCISE 21E.2

Use questions **1** to **4** to check the properties of definite integrals.

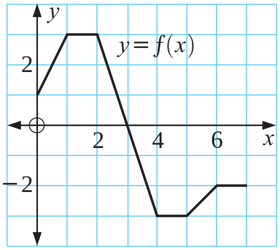
1 Find: **a** $\displaystyle\int_1^4 \sqrt{x}\,dx$ and $\displaystyle\int_1^4 (-\sqrt{x})\,dx$ **b** $\displaystyle\int_0^1 x^7\,dx$ and $\displaystyle\int_0^1 (-x^7)\,dx$

2 Find: **a** $\displaystyle\int_0^1 x^2\,dx$ **b** $\displaystyle\int_1^2 x^2\,dx$ **c** $\displaystyle\int_0^2 x^2\,dx$ **d** $\displaystyle\int_0^1 3x^2\,dx$

3 Find: **a** $\displaystyle\int_0^2 (x^3-4x)\,dx$ **b** $\displaystyle\int_2^3 (x^3-4x)\,dx$ **c** $\displaystyle\int_0^3 (x^3-4x)\,dx$

4 Find: **a** $\int_0^1 x^2\,dx$ **b** $\int_0^1 \sqrt{x}\,dx$ **c** $\int_0^1 (x^2+\sqrt{x})\,dx$

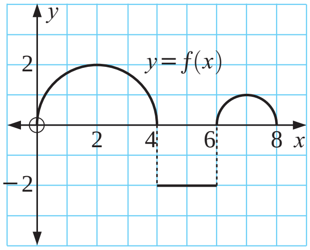
5 Evaluate the following integrals using area interpretation:



a $\int_0^3 f(x)\,dx$ **b** $\int_3^7 f(x)\,dx$

c $\int_2^4 f(x)\,dx$ **d** $\int_0^7 f(x)\,dx$

6 Evaluate the following integrals using area interpretation:



a $\int_0^4 f(x)\,dx$ **b** $\int_4^6 f(x)\,dx$

c $\int_6^8 f(x)\,dx$ **d** $\int_0^8 f(x)\,dx$

7 Write as a single integral:

a $\int_2^4 f(x)\,dx + \int_4^7 f(x)\,dx$

b $\int_1^3 g(x)\,dx + \int_3^8 g(x)\,dx + \int_8^9 g(x)\,dx$

8 **a** If $\int_1^3 f(x)\,dx = 2$ and $\int_1^6 f(x)\,dx = -3$, find $\int_3^6 f(x)\,dx$.

b If $\int_0^2 f(x)\,dx = 5$, $\int_4^6 f(x)\,dx = -2$ and $\int_0^6 f(x)\,dx = 7$, find $\int_2^4 f(x)\,dx$.

9 Given that $\int_{-1}^1 f(x)\,dx = -4$, determine the value of:

a $\int_1^{-1} f(x)\,dx$ **b** $\int_{-1}^1 (2+f(x))\,dx$ **c** $\int_{-1}^1 2f(x)\,dx$

d k such that $\int_{-1}^1 kf(x)\,dx = 7$

10 If $g(2)=4$ and $g(3)=5$, calculate $\int_2^3 \left(g'(x)-1\right)\,dx$.

REVIEW SET 21A NON-CALCULATOR

1 Integrate with respect to x:

a $\dfrac{4}{\sqrt{x}}$ **b** $\dfrac{3}{1-2x}$ **c** $\sin(4x-5)$ **d** e^{4-3x}

2 Find the exact value of: **a** $\displaystyle\int_{-5}^{-1} \sqrt{1-3x}\,dx$ **b** $\displaystyle\int_{0}^{\frac{\pi}{2}} \cos\left(\frac{x}{2}\right)\,dx$

3 By differentiating $y=\sqrt{x^2-4}$, find $\displaystyle\int \frac{x}{\sqrt{x^2-4}}\,dx$.

4 Find the values of b such that $\displaystyle\int_0^b \cos x\,dx = \frac{1}{\sqrt{2}}$, $0<b<\pi$.

5 Integrate with respect to x: **a** $4\sin^2\left(\dfrac{x}{2}\right)$ **b** $(2-\cos x)^2$

6 By differentiating $(3x^2+x)^3$, find $\displaystyle\int (3x^2+x)^2(6x+1)\,dx$.

7 If $\displaystyle\int_1^4 f(x)\,dx = 3$, determine:

a $\displaystyle\int_1^4 (f(x)+1)\,dx$ **b** $\displaystyle\int_1^2 f(x)\,dx - \int_4^2 f(x)\,dx$

8 If $\displaystyle\int_0^a e^{1-2x}\,dx = \frac{e}{4}$, find a in the form $\ln k$.

9 Given that $f''(x)=2\sin(2x)$, $f'(\frac{\pi}{2})=0$ and $f(0)=3$, find the exact value of $f(\frac{\pi}{2})$.

10 Evaluate: $\displaystyle\int_0^{\frac{\pi}{6}} \sin^2\left(\frac{x}{2}\right)\,dx$

REVIEW SET 21B CALCULATOR

1 Find y if: **a** $\dfrac{dy}{dx}=(x^2-1)^2$ **b** $\dfrac{dy}{dx}=400-20e^{-\frac{x}{2}}$

2 Evaluate correct to 4 significant figures:

a $\displaystyle\int_{-2}^{0} 4e^{-x^2}\,dx$ **b** $\displaystyle\int_0^1 \frac{10x}{\sqrt{3x+1}}\,dx$

3 Find $\dfrac{d}{dx}(\ln x)^2$ and hence find $\displaystyle\int \frac{\ln x}{x}\,dx$.

4 A curve $y=f(x)$ has $f''(x)=18x+10$. Find $f(x)$ if $f(0)=-1$ and $f(1)=13$.

5 Evaluate the following correct to 6 significant figures:

a $\displaystyle\int_3^4 \frac{x}{\sqrt{2x+1}}\,dx$ **b** $\displaystyle\int_0^1 x^2 e^{x+1}\,dx$

6 Suppose $f''(x) = 3x^2 + 2x$ and $f(0) = f(2) = 3$. Find:

a $f(x)$ **b** the equation of the normal to $y = f(x)$ at $x = 2$.

7 **a** Find $(e^x + 2)^3$ using the binomial expansion.

b Hence find the exact value of $\displaystyle\int_0^1 (e^x + 2)^3\,dx$.

c Check your answer to **b** using technology.

REVIEW SET 21C

1 Find:

a $\displaystyle\int \left(2e^{-x} - \frac{1}{x} + 3\right) dx$ **b** $\displaystyle\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$ **c** $\displaystyle\int (3 + e^{2x-1})^2\,dx$

2 Given that $f'(x) = x^2 - 3x + 2$ and $f(1) = 3$, find $f(x)$.

3 Find the exact value of $\displaystyle\int_2^3 \frac{1}{\sqrt{3x-4}}\,dx$.

4 Evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \cos^2\left(\frac{x}{2}\right) dx$

5 Find $\displaystyle\frac{d}{dx}(e^{-2x}\sin x)$ and hence find $\displaystyle\int_0^{\frac{\pi}{2}} \left[e^{-2x}(\cos x - 2\sin x)\right] dx$

6 Find $\displaystyle\int (2x+3)^n\,dx$ for all integers n.

7 A function has gradient function $2\sqrt{x} + \dfrac{a}{\sqrt{x}}$ and passes through the points (0, 2) and (1, 4). Find a and hence explain why the function $y = f(x)$ has no stationary points.

8 $\displaystyle\int_a^{2a} (x^2 + ax + 2)\,dx = \frac{73a}{2}$. Find a.

Chapter 22

Applications of integration

Syllabus reference: 7.5, 7.6

Contents:

We have already seen how definite integrals can be related to the areas between functions and the x-axis. In this chapter we explore this relationship further, and consider other applications of integral calculus such as motion problems.

INVESTIGATION $\int_a^b f(x)\,dx$ AND AREAS

Does $\displaystyle\int_a^b f(x)\,dx$ always give us an area?

What to do:

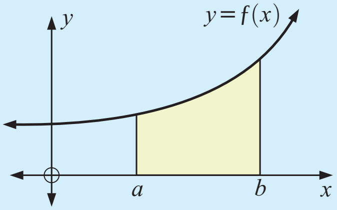
1. Find $\displaystyle\int_0^1 x^3\,dx$ and $\displaystyle\int_{-1}^1 x^3\,dx$.
2. Explain why the first integral in **1** gives an area, whereas the second integral does not. Graphical evidence is essential.
3. Find $\displaystyle\int_{-1}^0 x^3\,dx$ and explain why the answer is negative.
4. Check that $\displaystyle\int_{-1}^0 x^3\,dx + \int_0^1 x^3\,dx = \int_{-1}^1 x^3\,dx$.

A FINDING AREAS BETWEEN CURVES

We have already established in **Chapter 21** that:

If $f(x)$ is positive and continuous on the interval $a \leqslant x \leqslant b$, then the area bounded by $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$

is given by $\quad A = \displaystyle\int_a^b f(x)\,dx$ or $\displaystyle\int_a^b y\,dx$.



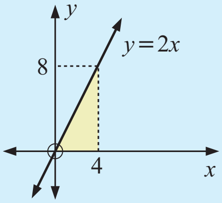
AREAS BETWEEN CURVES AND THE x-AXIS

Example 1 Self Tutor

Find the area of the region enclosed by $y = 2x$, the x-axis, $x = 0$ and $x = 4$ by using:

a a geometric argument **b** integration.

a



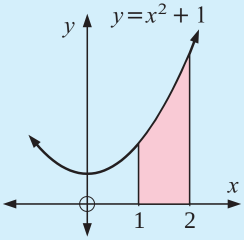
$$\text{Area} = \tfrac{1}{2} \times 4 \times 8$$
$$= 16 \text{ units}^2$$

b $$\text{Area} = \int_0^4 2x\,dx$$
$$= \left[x^2\right]_0^4$$
$$= 4^2 - 0^2$$
$$= 16 \text{ units}^2$$

Example 2

Self Tutor

Find the area of the region enclosed by $y = x^2 + 1$, the x-axis, $x = 1$ and $x = 2$.

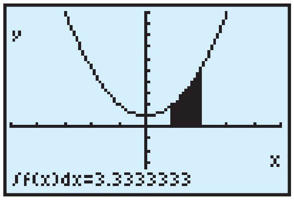


$$\text{Area} = \int_1^2 (x^2 + 1)\,dx$$
$$= \left[\frac{x^3}{3} + x\right]_1^2$$
$$= \left(\tfrac{8}{3} + 2\right) - \left(\tfrac{1}{3} + 1\right)$$
$$= 3\tfrac{1}{3} \text{ units}^2$$


It is helpful to sketch the region.

We can check this result using a graphics calculator or graphing package.

GRAPHING PACKAGE

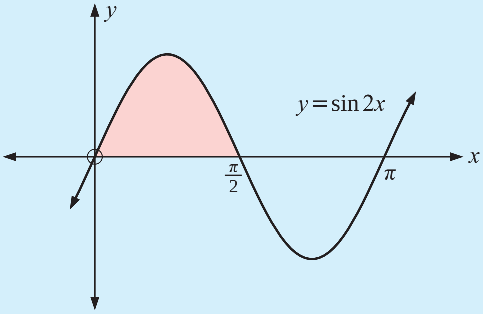


∫f(x)dx=3.3333333

Example 3

Self Tutor

Find the area enclosed by one arch of the curve $y = \sin 2x$.



The period of $y = \sin 2x$ is $\frac{2\pi}{2} = \pi$.

$\therefore$ the first positive x-intercept is $\frac{\pi}{2}$.

$$\text{The required area} = \int_0^{\frac{\pi}{2}} \sin 2x\,dx$$
$$= \left[\tfrac{1}{2}(-\cos 2x)\right]_0^{\frac{\pi}{2}}$$
$$= -\tfrac{1}{2}\left[\cos 2x\right]_0^{\frac{\pi}{2}}$$
$$= -\tfrac{1}{2}(\cos \pi - \cos 0)$$
$$= 1 \text{ unit}^2$$

EXERCISE 22A.1

1 Find the area of each of the regions described below by using:

i a geometric argument **ii** integration

a $y = 5$, the x-axis, $x = -6$ and $x = 0$

b $y = x$, the x-axis, $x = 4$ and $x = 5$

c $y = -3x$, the x-axis, $x = -3$ and $x = 0$

d $y = -x$, the x-axis, $x = 0$ and $x = 2$

2 Find the exact value of the area of the region bounded by:

a $y = x^2$, the x-axis and $x = 1$

b $y = \sin x$, the x-axis, $x = 0$ and $x = \pi$

c $y = x^3$, the x-axis, $x = 1$ and $x = 4$

d $y = e^x$, the x-axis, the y-axis and $x = 1$

e the x-axis and the part of $y = 6 + x - x^2$ above the x-axis

f the axes and $y = \sqrt{9 - x}$

g $y = \dfrac{1}{x}$, the x-axis, $x = 1$ and $x = 4$

h $y = \dfrac{1}{x}$, the x-axis, $x = -1$ and $x = -3$

i $y = 2 - \dfrac{1}{\sqrt{x}}$, the x-axis and $x = 4$

j $y = e^x + e^{-x}$, the x-axis, $x = -1$ and $x = 1$

Use technology to check your answers.

3 Find the area enclosed by one arch of the curve $y = \cos 3x$.

4 Use technology to find the area of the region bounded by:

a $y = \ln x$, the x-axis, $x = 1$ and $x = 4$

b $y = x \sin x$, the x-axis, $x = 1$ and $x = \frac{\pi}{2}$

c $y = x^2 e^{-x}$, the x-axis, $x = 0$ and $x = 2.8$.

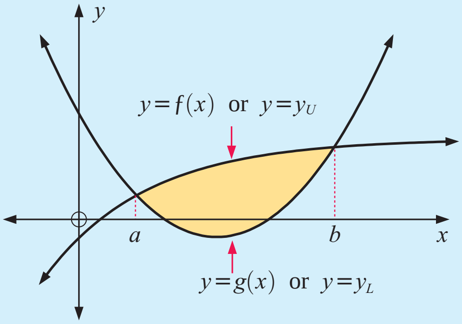
AREA BETWEEN TWO FUNCTIONS

If two functions $f(x)$ and $g(x)$ intersect at $x = a$ and $x = b$, and $f(x) \geqslant g(x)$ for all $a \leqslant x \leqslant b$, then the area of the shaded region between their points of intersection is given by

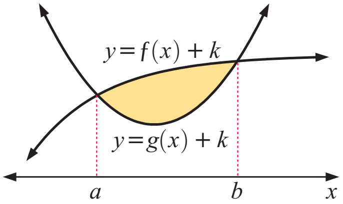
$$\int_a^b [f(x) - g(x)]\,dx.$$

Alternatively, if we describe the upper and lower functions as $y = y_U$ and $y = y_L$ respectively, then the area is

$$\int_a^b [y_U - y_L]\,dx.$$



Proof: If we translate each curve vertically through $\binom{0}{k}$ until it is completely above the x-axis, the area does not change.



Area of shaded region

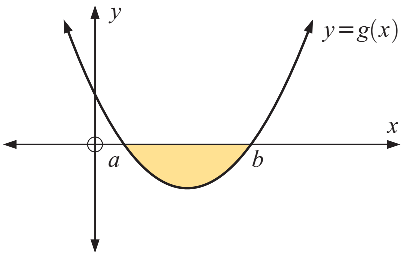
$$= \int_a^b [f(x) + k]\,dx - \int_a^b [g(x) + k]\,dx$$

$$= \int_a^b [f(x) - g(x)]\,dx$$

We can see immediately that if $y = f(x) = 0$

then the enclosed area is $\displaystyle\int_a^b [-g(x)]\,dx$

or $\displaystyle -\int_a^b g(x)dx.$



Example 4

Self Tutor

Use $\displaystyle\int_a^b [y_U - y_L]\,dx$ to find the area bounded by the x-axis and $y = x^2 - 2x$.

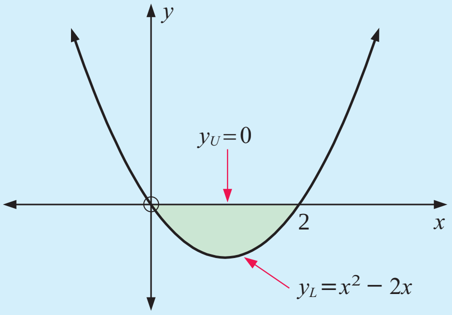
The curve cuts the x-axis when $y = 0$

$\therefore\ x^2 - 2x = 0$

$\therefore\ x(x-2) = 0$

$\therefore\ x = 0$ or 2

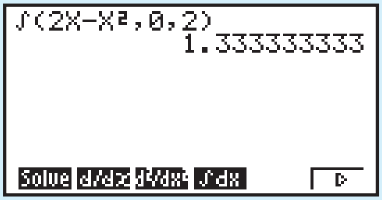
$\therefore$ the x-intercepts are 0 and 2.



$$\text{Area} = \int_0^2 [y_U - y_L]\,dx$$

$$= \int_0^2 [0 - (x^2 - 2x)]\,dx$$

$$= \int_0^2 (2x - x^2)\,dx$$



```
∫(2X−X²,0,2)
               1.333333333
```

$$= \left[x^2 - \frac{x^3}{3}\right]_0^2$$

$$= (4 - \tfrac{8}{3}) - (0)$$

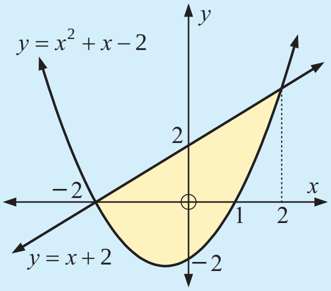
$\therefore$ the area is $\frac{4}{3}$ units^2.

Example 5 **Self Tutor**

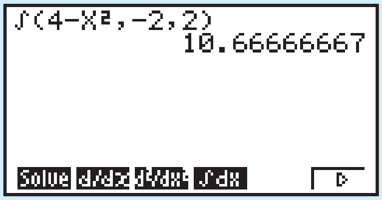
Find the area of the region enclosed by $y = x + 2$ and $y = x^2 + x - 2$.

$y = x + 2$ meets $y = x^2 + x - 2$ where

$$x^2 + x - 2 = x + 2$$
$$\therefore \quad x^2 - 4 = 0$$
$$\therefore \quad (x+2)(x-2) = 0$$
$$\therefore \quad x = \pm 2$$



$$\text{Area} = \int_{-2}^{2} [y_U - y_L]\, dx$$
$$= \int_{-2}^{2} [(x+2) - (x^2 + x - 2)]\, dx$$
$$= \int_{-2}^{2} (4 - x^2)\, dx$$
$$= \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$
$$= (8 - \tfrac{8}{3}) - (-8 + \tfrac{8}{3})$$
$$= 10\tfrac{2}{3} \text{ units}^2$$



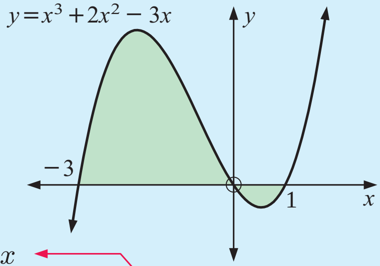
```
∫(4-X²,-2,2)
              10.66666667
```

Example 6 **Self Tutor**

Find the total area of the regions contained by $y = f(x)$ and the x-axis for $f(x) = x^3 + 2x^2 - 3x$.

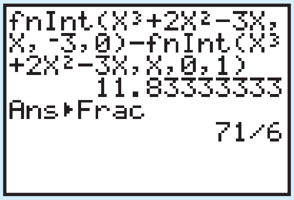
$$f(x) = x^3 + 2x^2 - 3x$$
$$= x(x^2 + 2x - 3)$$
$$= x(x-1)(x+3)$$

$\therefore$ $y = f(x)$ cuts the x-axis at 0, 1, and -3.



Total area

$$= \int_{-3}^{0} (x^3 + 2x^2 - 3x)\, dx - \int_{0}^{1} (x^3 + 2x^2 - 3x)\, dx$$
$$= \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]_{-3}^{0} - \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]_{0}^{1}$$
$$= (0 - -11\tfrac{1}{4}) - (-\tfrac{7}{12} - 0)$$
$$= 11\tfrac{5}{6} \text{ units}^2$$



```
fnInt(X³+2X²-3X,
X,-3,0)-fnInt(X³
+2X²-3X,X,0,1)
      11.83333333
Ans▸Frac
             71/6
```

EXERCISE 22A.2

1 Find the exact value of the area bounded by:

- **a** the x-axis and $y = x^2 + x - 2$
- **b** the x-axis, $y = e^{-x} - 1$ and $x = 2$
- **c** the x-axis and the part of $y = 3x^2 - 8x + 4$ below the x-axis
- **d** $y = \cos x$, the x-axis, $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$
- **e** $y = x^3 - 4x$, the x-axis, $x = 1$ and $x = 2$
- **f** $y = \sin x - 1$, the x-axis, $x = 0$ and $x = \frac{\pi}{2}$
- **g** one arch of $y = \sin^2 x$ and the x-axis.

2 Find the area of the region enclosed by $y = x^2 - 2x$ and $y = 3$.

3 Consider the graphs of $y = x - 3$ and $y = x^2 - 3x$.

- **a** Sketch the graphs on the same set of axes.
- **b** Find the coordinates of the points where the graphs meet.
- **c** Find the area of the region enclosed by the two graphs.

4 Determine the area of the region enclosed by $y = \sqrt{x}$ and $y = x^2$.

5
- **a** On the same set of axes, graph $y = e^x - 1$ and $y = 2 - 2e^{-x}$, showing axes intercepts and asymptotes.
- **b** Find algebraically the points of intersection of $y = e^x - 1$ and $y = 2 - 2e^{-x}$.
- **c** Find the area of the region enclosed by the two curves.

6 Determine exactly the area of the region bounded by $y = 2e^x$, $y = e^{2x}$ and $x = 0$.

7 On the same set of axes, draw the graphs of the functions $y = 2x$ and $y = 4x^2$. Determine exactly the area of the region enclosed by these functions.

8 Sketch the circle with equation $x^2 + y^2 = 9$.

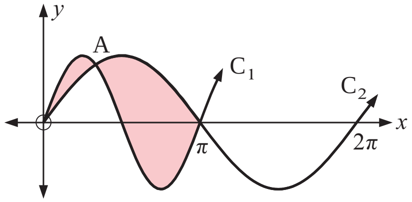
- **a** Explain why the upper half of the circle has equation $y = \sqrt{9 - x^2}$.
- **b** Hence, determine $\int_0^3 \sqrt{9 - x^2}\, dx$ without actually integrating the function.
- **c** Check your answer using technology.

9 Find the area enclosed by the function $y = f(x)$ and the x-axis for:

- **a** $f(x) = x^3 - 9x$
- **b** $f(x) = -x(x-2)(x-4)$
- **c** $f(x) = x^4 - 5x^2 + 4$.

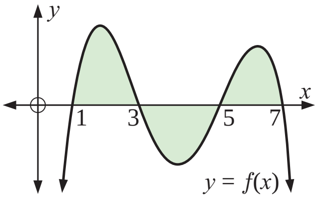
10 The illustrated curves are those of $y = \sin x$ and $y = \sin(2x)$.

- **a** Identify each curve.
- **b** Find algebraically the coordinates of A.
- **c** Find the total area enclosed by C_1 and C_2 for $0 \leqslant x \leqslant \pi$.



11 **a** Explain why the total area shaded is *not* equal to $\int_1^7 f(x)\,dx$.

b What is the total shaded area equal to in terms of integrals?



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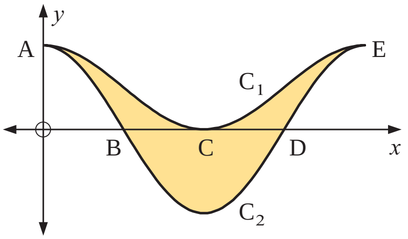


The illustrated curves are those of $y = \cos(2x)$ and $y = \cos^2 x$.

a Identify each curve.

b Determine the coordinates of A, B, C, D and E.

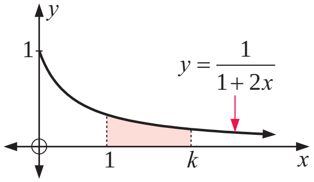
c Show that the area of the shaded region is $\frac{\pi}{2}$ units^2.

13 Find, correct to 3 significant figures, the areas of the regions enclosed by the curves:

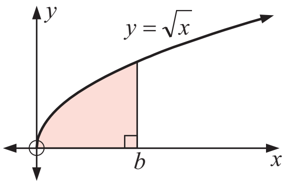
a $y = e^{-x^2}$ and $y = x^2 - 1$

b $y = x^x$ and $y = 4x - \frac{1}{10}x^4$

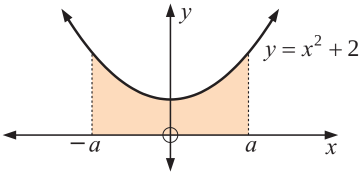
14 The shaded area is 0.2 units^2.
Find k, correct to 4 decimal places.



15 The shaded area is 1 unit^2.
Find b, correct to 4 decimal places.



16 The shaded area is $6a$ units^2.
Find the exact value of a.



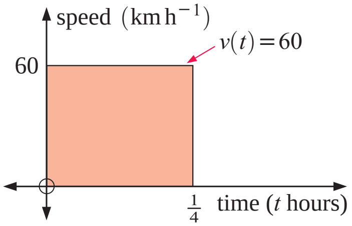
B MOTION PROBLEMS

DISTANCES FROM VELOCITY GRAPHS

Suppose a car travels at a constant positive velocity of 60 km h^{-1} for 15 minutes.

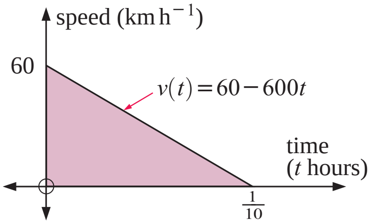
We know the distance travelled $= \text{speed} \times \text{time}$
$= 60 \text{ km h}^{-1} \times \frac{1}{4} \text{ h}$
$= 15 \text{ km}.$

When we graph *speed* against *time*, the graph is a horizontal line and it is clear that the distance travelled is the area shaded.



So, the distance travelled can also be found by the definite integral $\int_0^{\frac{1}{4}} 60\,dt = 15$.

Now suppose the speed decreases at a constant rate so that the car, initially travelling at 60 km h^{-1}, stops in 6 minutes.



In this case the average speed must be 30 km h^{-1},

so the distance travelled $= 30\ \text{km h}^{-1} \times \frac{1}{10}$ h

$= 3$ km

But the triangle has area $= \frac{1}{2} \times \text{base} \times \text{altitude}$

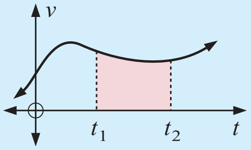
$= \frac{1}{2} \times \frac{1}{10} \times 60 = 3$

So, once again the shaded area gives us the distance travelled, and we can find it using the definite integral $\int_0^{\frac{1}{10}} (60 - 600t)\,dt = 3$.

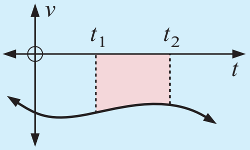
These results suggest that: distance travelled $= \int_{t_1}^{t_2} v(t)\,dt$ provided we do not change direction.

If we have a change of direction within the time interval then the velocity will change sign. In such cases we need to add the components of area above and below the t-axis.

For a velocity-time function $v(t)$ where $v(t) \geqslant 0$ on the interval $t_1 \leqslant t \leqslant t_2$, **distance travelled** $= \int_{t_1}^{t_2} v(t)\,dt$.

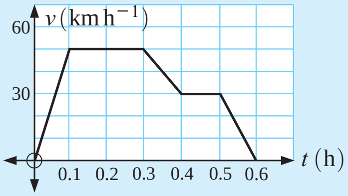


For a velocity-time function $v(t)$ where $v(t) \leqslant 0$ on the interval $t_1 \leqslant t \leqslant t_2$, **distance travelled** $= -\int_{t_1}^{t_2} v(t)\,dt$.



Example 7 **Self Tutor**

The velocity-time graph for a train journey is illustrated in the graph alongside. Find the total distance travelled by the train.



Total distance travelled

$= $ total area under the graph

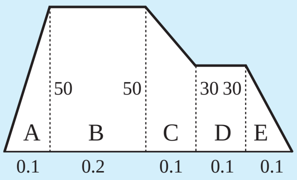
$= $ area A + area B + area C + area D + area E

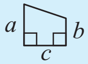
$= \frac{1}{2}(0.1)50 + (0.2)50 + \left(\frac{50+30}{2}\right)(0.1) + (0.1)30$

$\qquad + \frac{1}{2}(0.1)30$

$= 2.5 + 10 + 4 + 3 + 1.5$

$= 21$ km

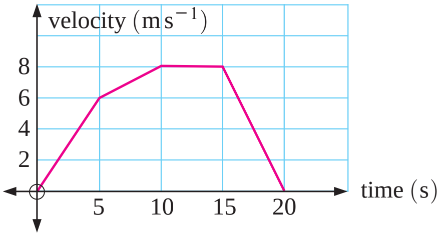




area $= \dfrac{a+b}{2} \times c$

EXERCISE 22B.1

1 A runner has the velocity-time graph shown. Find the total distance travelled by the runner.



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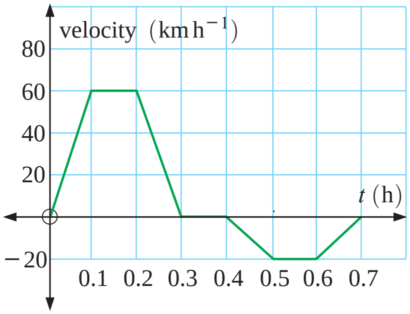


A car travels along a straight road with the velocity-time function illustrated.

a What is the significance of the graph:

  i above the t-axis

  ii below the t-axis?

b Find the total distance travelled by the car.

c Find the final displacement of the car.

3 A cyclist rides off from rest, accelerating at a constant rate for 3 minutes until she reaches 40 km h^{-1}. She then maintains a constant speed for 4 minutes until reaching a hill. She slows down at a constant rate to 30 km h^{-1} in one minute, then continues at this rate for 10 minutes. At the top of the hill she reduces her speed uniformly and is stationary 2 minutes later.

a Draw a graph to show the cyclist's motion.

b How far has the cyclist travelled?

DISPLACEMENT AND VELOCITY FUNCTIONS

In this section we are concerned with **motion in a straight line**, or **linear motion**.

Recall that for some displacement function $s(t)$ the velocity function is $v(t) = s'(t)$.

So, given a velocity function we can determine the displacement function by the integral

$$s(t) = \int v(t)\, dt$$

Using the displacement function we can quickly determine the displacement in a time interval $a \leqslant t \leqslant b$.

$$\textbf{Displacement} = s(b) - s(a) = \int_a^b v(t)\, dt$$

We can also determine the **total distance travelled** in some time interval $a \leqslant t \leqslant b$.

Consider the following example:

A particle moves in a straight line with velocity function $v(t) = t - 3 \text{ cm s}^{-1}$.

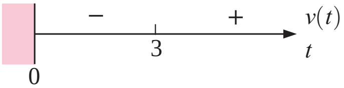
How far does it travel in the first 4 seconds of motion?

Now $s(t) = \int (t-3)\, dt = \frac{t^2}{2} - 3t + c$ for some constant c.

Clearly, the *displacement* of the particle in the first 4 seconds is

$$s(4) - s(0) = c - 4 - c = -4 \text{ cm}.$$

However, $v(t)$ has sign diagram:



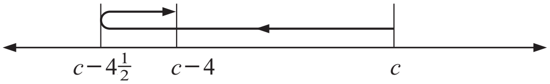
Since the velocity function changes sign at $t = 3$ seconds, the particle **reverses direction** at this time.

We need to remember this reversal of direction at $t = 3$ when calculating the total distance travelled.

We find the positions of the particle at $t = 0$, $t = 3$ and $t = 4$:

$$s(0) = c, \qquad s(3) = c - 4\tfrac{1}{2}, \qquad s(4) = c - 4.$$

We can hence draw a diagram of the motion:



The total distance travelled is $(4\frac{1}{2} + \frac{1}{2})$ cm $= 5$ cm.

Summary:

To find the total distance travelled given a velocity function $v(t) = s'(t)$ on $a \leqslant t \leqslant b$:

- Draw a sign diagram for $v(t)$ so we can determine any changes of direction.
- Determine $s(t)$ by integration, including a constant of integration.
- Find $s(a)$ and $s(b)$. Also find $s(t)$ at each time the direction changes.
- Draw a motion diagram.
- Determine the total distance travelled from the motion diagram.

Example 8 **Self Tutor**

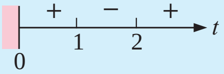
A particle P moves in a straight line with velocity function $v(t) = t^2 - 3t + 2$ m s^{-1}.

a How far does P travel in the first 4 seconds in motion?

b Find the displacement of P after 4 seconds.

a $v(t) = s'(t) = t^2 - 3t + 2$
$\quad = (t-1)(t-2)$

$\therefore$ sign diagram of v is:



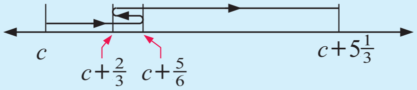
Since the signs change, P reverses direction at $t = 1$ and $t = 2$ seconds.

Now $s(t) = \displaystyle\int (t^2 - 3t + 2)dt = \frac{t^3}{3} - \frac{3t^2}{2} + 2t + c$

Now $s(0) = c$ $\qquad s(1) = \frac{1}{3} - \frac{3}{2} + 2 + c = c + \frac{5}{6}$

$s(2) = \frac{8}{3} - 6 + 4 + c = c + \frac{2}{3}$ $\qquad s(4) = \frac{64}{3} - 24 + 8 + c = c + 5\frac{1}{3}$

Motion diagram:



$\therefore$ total distance $= \left(c + \frac{5}{6} - c\right) + \left(c + \frac{5}{6} - \left[c + \frac{2}{3}\right]\right) + \left(c + 5\frac{1}{3} - \left[c + \frac{2}{3}\right]\right)$
$= \frac{5}{6} + \frac{5}{6} - \frac{2}{3} + 5\frac{1}{3} - \frac{2}{3}$
$= 5\frac{2}{3}$ m

b Displacement $=$ final position $-$ original position
$= s(4) - s(0)$
$= c + 5\frac{1}{3} - c$
$= 5\frac{1}{3}$ m

So, the displacement is $5\frac{1}{3}$ m to the right.

VELOCITY AND ACCELERATION FUNCTIONS

We know that the acceleration function is the derivative of velocity, so $a(t) = v'(t)$.

So, given an acceleration function, we can determine the velocity function by the integral

$$\boldsymbol{v(t) = \int a(t)\, dt.}$$

EXERCISE 22B.2

1 A particle has velocity function $v(t) = 1 - 2t$ cm s^{-1} as it moves in a straight line.

a Find the total distance travelled in the first second of motion.

b Find the displacement of the particle at the end of one second.

2 Particle P has velocity $v(t) = t^2 - t - 2$ cm s^{-1}.

a Find the total distance travelled in the first 3 seconds of motion.

b Find the displacement of the particle at the end of three seconds.

3 A particle moves along the x-axis with velocity function $x'(t) = 16t - 4t^3$ units per second. Find the total distance travelled in the time interval:

- **a** $0 \leqslant t \leqslant 3$ seconds
- **b** $1 \leqslant t \leqslant 3$ seconds.

4 A particle moves in a straight line with velocity function $v(t) = \cos t$ m s^{-1}.

- **a** Show that the particle oscillates between two points.
- **b** Find the distance between the two points in **a**.

5 The velocity of a particle travelling in a straight line is given by $v(t) = 50 - 10e^{-0.5t}$ m s^{-1}, where $t \geqslant 0$, t in seconds.

- **a** State the initial velocity of the particle.
- **b** Find the velocity of the particle after 3 seconds.
- **c** How long will it take for the particle's velocity to increase to 45 m s^{-1}?
- **d** Discuss $v(t)$ as $t \to \infty$.
- **e** Show that the particle's acceleration is always positive.
- **f** Draw the graph of $v(t)$ against t.
- **g** Find the total distance travelled by the particle in the first 3 seconds of motion.

6 A train moves along a straight track with acceleration $\frac{t}{10} - 3$ m s^{-2}. If the initial velocity of the train is 45 m s^{-1}, determine the total distance travelled in the first minute.

7 An object has initial velocity 20 m s^{-1} as it moves in a straight line with acceleration function $4e^{-\frac{t}{20}}$ m s^{-2}.

- **a** Show that as t increases the object approaches a limiting velocity.
- **b** Find the total distance travelled in the first 10 seconds of motion.

C PROBLEM SOLVING BY INTEGRATION

When we studied differential calculus, we saw how to find the rate of change of a function by differentiation.

In practical situations it is sometimes easier to measure the rate of change of a variable, for example, the rate of water flow through a pipe. In such situations we can use integration to find a function for the quantity concerned.

Example 9 **Self Tutor**

The marginal cost of producing x urns per week is given by
$C'(x) = 2.15 - 0.02x + 0.000\,36x^2$ dollars per urn provided $0 \leqslant x \leqslant 120$.

The initial costs before production starts are \$185. Find the total cost of producing 100 urns per day.

The marginal cost is $C'(x) = 2.15 - 0.02x + 0.000\,36x^2$ dollars per urn

$$\therefore \quad C(x) = \int (2.15 - 0.02x + 0.000\,36x^2)\,dx$$

$$\therefore \quad C(x) = 2.15x - 0.02\frac{x^2}{2} + 0.000\,36\frac{x^3}{3} + c$$

$$= 2.15x - 0.01x^2 + 0.000\,12x^3 + c$$

But $C(0) = 185$, so $c = 185$

$$\therefore \quad C(x) = 2.15x - 0.01x^2 + 0.000\,12x^3 + 185$$

$$\therefore \quad C(100) = 2.15(100) - 0.01(100)^2 + 0.000\,12(100)^3 + 185$$

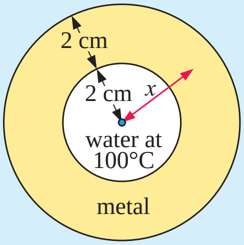
$$= 420$$ So, the total cost is \$420.

Example 10 **Self Tutor**

A metal tube has an annulus cross-section as shown. The outer radius is 4 cm and the inner radius is 2 cm. Within the tube, water is maintained at a temperature of 100^{o}C.

Within the metal the temperature drops off from inside to outside according to $\frac{dT}{dx} = -\frac{10}{x}$ where x is the distance from the central axis and $2 \leqslant x \leqslant 4$.

Find the temperature of the outer surface of the tube.



tube cross-section

$$\frac{dT}{dx} = \frac{-10}{x} \quad \text{so} \quad T = \int \frac{-10}{x}\,dx$$

$$\therefore \quad T = -10\ln x + c \qquad \{\text{since } x > 0\}$$

But when $x = 2$, $T = 100$

$$\therefore \quad 100 = -10\ln 2 + c$$

$$\therefore \quad c = 100 + 10\ln 2$$

Thus $T = -10\ln x + 100 + 10\ln 2$

$$T = 100 + 10\ln\left(\tfrac{2}{x}\right)$$

When $x = 4$, $T = 100 + 10\ln\left(\tfrac{1}{2}\right) \approx 93.1$

$\therefore$ the outer surface temperature is 93.1^{o}C.

EXERCISE 22C

1 The marginal cost per day of producing x gadgets is $C'(x) = 3.15 + 0.004x$ euros per gadget. What is the total cost of daily production of 800 gadgets given that the fixed costs before production commences are €450 per day?

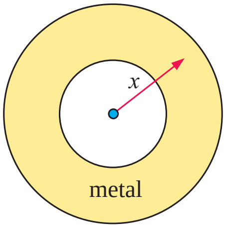
2 The marginal profit for producing x dinner plates per week is given by $P'(x) = 15 - 0.03x$ dollars per plate. If no plates are made a loss of \$650 each week occurs.

- **a** Find the profit function.
- **b** What is the maximum profit and when does it occur?
- **c** What production levels enable a profit to be made?

3 Jon needs to bulk up for the football season. His energy needs t days after starting his weight gain program are given by $E'(t) = 350(80+0.15t)^{0.8} - 120(80+0.15t)$ calories per day. Find Jon's total energy needs over the first week of the program.

4 The tube cross-section shown has inner radius of 3 cm and outer radius 6 cm. Within the tube, water is maintained at a temperature of 100^oC. Within the metal the temperature falls off at the rate $\frac{dT}{dx} = \frac{-20}{x^{0.63}}$ where x is the distance from the central axis and $3 \leqslant x \leqslant 6$.

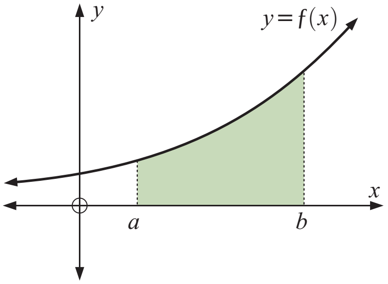
Find the temperature of the outer surface of the tube.

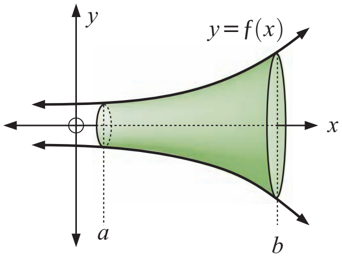


D SOLIDS OF REVOLUTION

Consider the curve $y = f(x)$ for $a \leqslant x \leqslant b$.

If the shaded part is **rotated about the x-axis** through 360^o, a 3-dimensional solid will be formed. This solid is called a **solid of revolution**.





VOLUME OF REVOLUTION

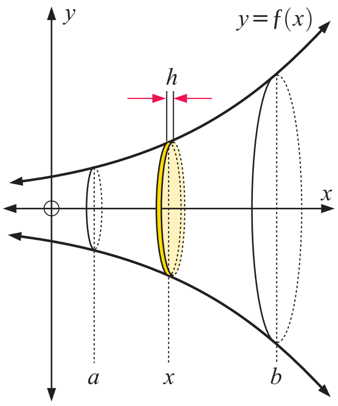
We can use integration to find volumes of revolution between $x = a$ and $x = b$.

The solid can be thought to be made up of an infinite number of thin cylindrical discs.

Since the volume of a cylinder $= \pi r^2 h$,

the left-most disc has approximate volume $\pi[f(a)]^2 h$, and

the right-most disc has approximate volume $\pi[f(b)]^2 h$.



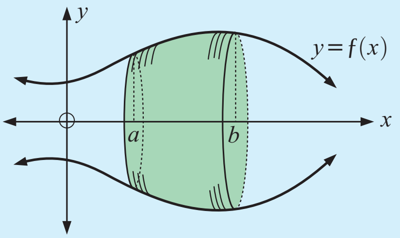
In general, $\pi[f(x)]^2 h$ is the approximate volume for the illustrated disc.

As there are infinitely many discs, we let $h \to 0$.

$$\therefore \quad V = \lim_{h \to 0} \sum_{x=a}^{x=b} \pi[f(x)]^2\, h = \int_a^b \pi[f(x)]^2\, dx = \pi \int_a^b y^2\, dx$$

When the region enclosed by $y = f(x)$, the x-axis, and the vertical lines $x = a$, $x = b$ is rotated about the x-axis to generate a solid, the volume of the solid is given by

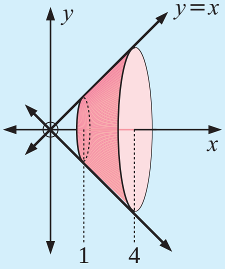
$$\textbf{Volume of revolution} = \pi \int_a^b y^2\, dx\,.$$



Example 11

Self Tutor

Use integration to find the volume of the solid generated when the line $y = x$ for $1 \leqslant x \leqslant 4$ is revolved around the x-axis.



$$\begin{aligned}\text{Volume of revolution } &= \pi \int_a^b y^2\, dx \\ &= \pi \int_1^4 x^2\, dx \\ &= \pi \left[\frac{x^3}{3}\right]_1^4 \\ &= \pi\left(\tfrac{64}{3} - \tfrac{1}{3}\right) \\ &= 21\pi \quad \text{cubic units}\end{aligned}$$

The volume of a cone is

$V_{\text{cone}} = \frac{1}{3}\pi r^2 h$

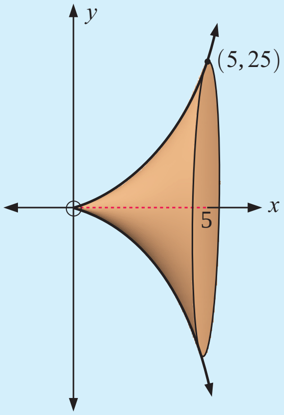
So, in this example

$$\begin{aligned}V &= \tfrac{1}{3}\pi 4^2(4) - \tfrac{1}{3}\pi 1^2(1) \\ &= \tfrac{64\pi}{3} - \tfrac{\pi}{3} \\ &= 21\pi \quad \text{which checks} \quad ✓\end{aligned}$$

Example 12

Self Tutor

Find the volume of the solid formed when the graph of the function $y = x^2$ for $0 \leqslant x \leqslant 5$ is revolved about the x-axis.



$$\begin{aligned}\text{Volume of revolution} &= \pi \int_a^b y^2\, dx \\ &= \pi \int_0^5 (x^2)^2\, dx \\ &= \pi \int_0^5 x^4\, dx \\ &= \pi \left[\frac{x^5}{5}\right]_0^5 \\ &= \pi(625 - 0) \\ &= 625\pi \quad \text{cubic units}\end{aligned}$$

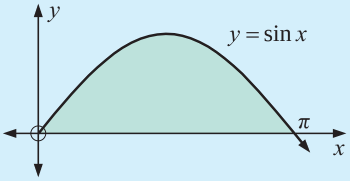
TI-*n*spire
TI-84
Casio

Example 13

Self Tutor

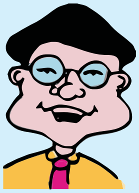
One arch of $y = \sin x$ is rotated about the x-axis.

What is the volume of revolution?



$$\begin{aligned}
\text{Volume} &= \pi \int_a^b y^2\,dx \\
&= \pi \int_0^{\pi} \sin^2 x\,dx \\
&= \pi \int_0^{\pi} \left[\tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\right]\,dx \\
&= \pi \left[\tfrac{x}{2} - \tfrac{1}{2}\left(\tfrac{1}{2}\right)\sin(2x)\right]_0^{\pi} \\
&= \pi \left[\left(\tfrac{\pi}{2} - \tfrac{1}{4}\sin(2\pi)\right) - \left(0 - \tfrac{1}{4}\sin 0\right)\right] \\
&= \pi \times \tfrac{\pi}{2} \\
&= \tfrac{\pi^2}{2}\ \text{units}^3
\end{aligned}$$

Remember
$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$
$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$



EXERCISE 22D.1

1 Find the volume of the solid formed when the following are revolved about the x-axis:

a $y = 2x$ for $0 \leqslant x \leqslant 3$

b $y = \sqrt{x}$ for $0 \leqslant x \leqslant 4$

c $y = x^3$ for $1 \leqslant x \leqslant 2$

d $y = x^{\frac{3}{2}}$ for $1 \leqslant x \leqslant 4$

e $y = x^2$ for $2 \leqslant x \leqslant 4$

f $y = \sqrt{25 - x^2}$ for $0 \leqslant x \leqslant 5$

g $y = \dfrac{1}{x-1}$ for $2 \leqslant x \leqslant 3$

h $y = x + \dfrac{1}{x}$ for $1 \leqslant x \leqslant 3$

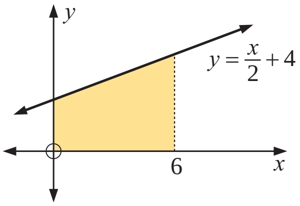
2 Use technology to find, correct to 3 significant figures, the volume of the solid of revolution formed when these functions are rotated through 360^o about the x-axis:

a $y = \dfrac{x^3}{x^2+1}$ for $1 \leqslant x \leqslant 3$

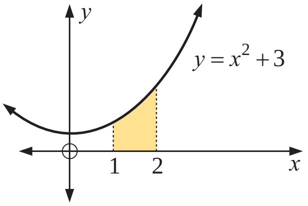
b $y = e^{\sin x}$ for $0 \leqslant x \leqslant 2$.

3 Find the volume of revolution when the shaded region is revolved about the x-axis.

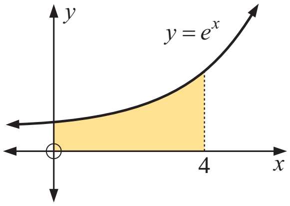
a



b



c



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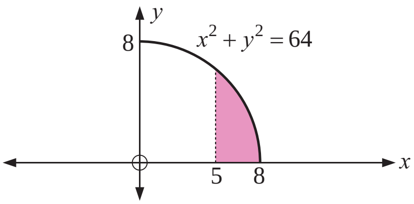
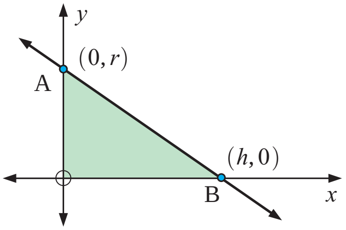


The shaded region is rotated about the x-axis.

- **a** Find the volume of revolution.
- **b** A hemispherical bowl of radius 8 cm contains water to a depth of 3 cm. What is the volume of water?

5

- **a** What is the name of the solid of revolution when the shaded region is revolved about the x-axis?
- **b** Find the equation of the line segment [AB] in the form $y = ax + b$.
- **c** Find a formula for the volume of the solid using $\pi \int_a^b y^2\,dx$.



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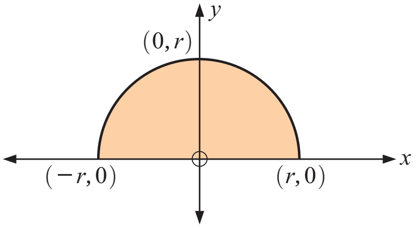


A circle with centre (0, 0) and radius r units has equation $x^2 + y^2 = r^2$.

- **a** If the shaded region is revolved about the x-axis, what solid is formed?
- **b** Use integration to show that the volume of revolution is $\frac{4}{3}\pi r^3$.

7 Find the volume of revolution when these regions are rotated about the x-axis:

- **a** $y = \cos x$ for $0 \leqslant x \leqslant \frac{\pi}{2}$
- **b** $y = \cos(2x)$ for $0 \leqslant x \leqslant \frac{\pi}{4}$

8

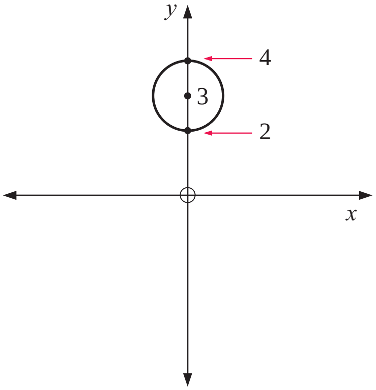
- **a** Sketch the graph of $y = \sin x + \cos x$ for $0 \leqslant x \leqslant \frac{\pi}{2}$.
- **b** Hence, find the volume of revolution of the shape bounded by $y = \sin x + \cos x$, the x-axis, $x = 0$ and $x = \frac{\pi}{4}$ when it is rotated about the x-axis.

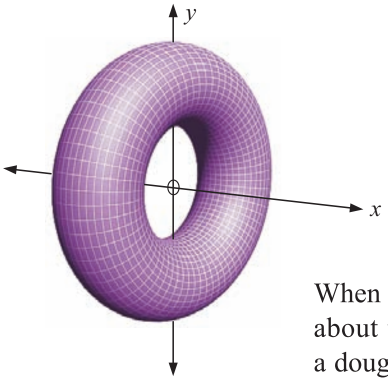
9

- **a** Sketch the graph of $y = 4\sin(2x)$ from $x = 0$ to $x = \frac{\pi}{4}$.
- **b** Hence, find the volume of revolution of the shape bounded by $y = 4\sin(2x)$, the x-axis, $x = 0$ and $x = \frac{\pi}{4}$ when it is rotated about the x-axis.

VOLUMES FOR TWO DEFINING FUNCTIONS

Consider the circle with centre (0, 3) and radius 1 unit.





DEMO

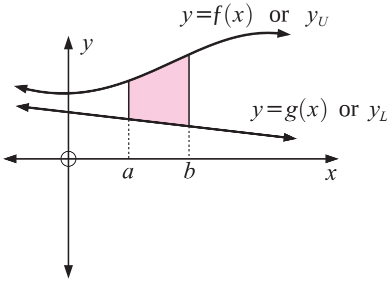
When this circle is revolved about the x-axis, we obtain a doughnut or *torus*.

In general, if the region bounded by $y_U = f(x)$ (on top) and $y_L = g(x)$ and the lines $x = a$, $x = b$ is revolved about the x-axis, then its volume of revolution is given by:

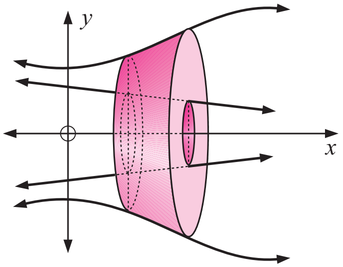
$$V = \pi \int_a^b [f(x)]^2 \, dx - \pi \int_a^b [g(x)]^2 \, dx$$

$$V = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2 \right) dx \quad \text{or} \quad V = \pi \int_a^b \left({y_U}^2 - {y_L}^2 \right) dx$$

Rotating



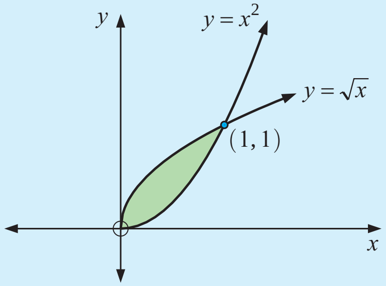
about the x-axis gives



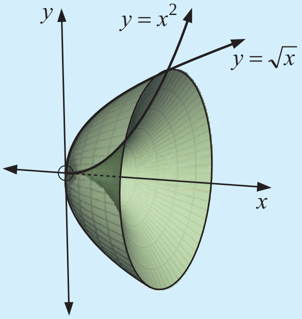
Example 14

Self Tutor

Find the volume of revolution generated by revolving the region between $y = x^2$ and $y = \sqrt{x}$ about the x-axis.



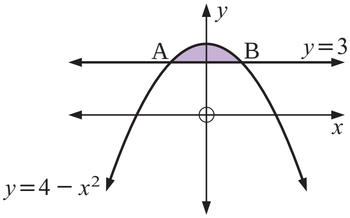
$$\begin{aligned}
\text{Volume} &= \pi \int_0^1 \left({y_U}^2 - {y_L}^2 \right) dx \\
&= \pi \int_0^1 \left(\left(\sqrt{x}\right)^2 - (x^2)^2 \right) dx \\
&= \pi \int_0^1 (x - x^4) \, dx \\
&= \pi \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 \\
&= \pi \left(\left(\tfrac{1}{2} - \tfrac{1}{5} \right) - (0) \right) \\
&= \tfrac{3\pi}{10} \text{ units}^3
\end{aligned}$$



EXERCISE 22D.2

1 The shaded region between $y = 4 - x^2$ and $y = 3$ is revolved about the x-axis.

- **a** What are the coordinates of A and B?
- **b** Find the volume of revolution.



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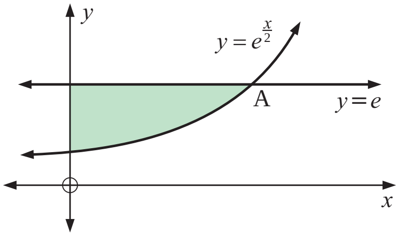
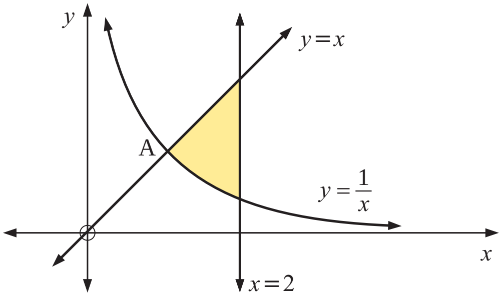


The shaded region is revolved about the x-axis.

- **a** Find the coordinates of A.
- **b** Find the volume of revolution.

3 The shaded region between $y = x$, $y = \dfrac{1}{x}$ and $x = 2$ is revolved about the x-axis.

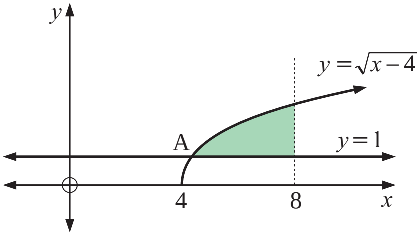
- **a** Find the coordinates of A.
- **b** Find the volume of revolution.



4 Find exactly the volume of the solid of revolution generated by rotating the region enclosed by $y = x^2 - 4x + 6$ and $x + y = 6$ through 360^o about the x-axis.

5 The shaded region is revolved about the x-axis.

- **a** State the coordinates of A.
- **b** Find the volume of revolution.



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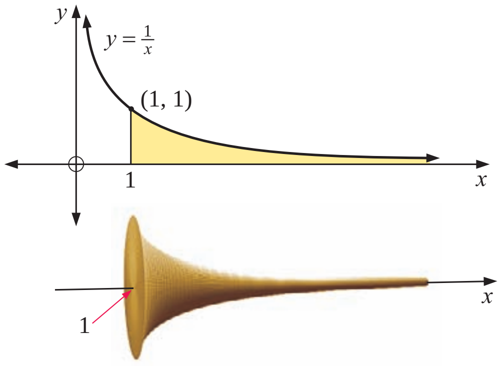


Prove that the shaded area from $x = 1$ to infinity is infinite whereas its volume of revolution is finite.

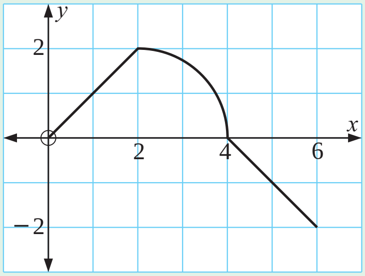
REVIEW SET 22A NON-CALCULATOR

1 The function $y = f(x)$ is graphed.
Find:

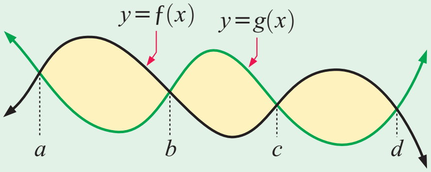
a $\displaystyle\int_0^4 f(x)\,dx$

b $\displaystyle\int_4^6 f(x)\,dx$

c $\displaystyle\int_0^6 f(x)\,dx$



2 Write an expression for the total shaded area.



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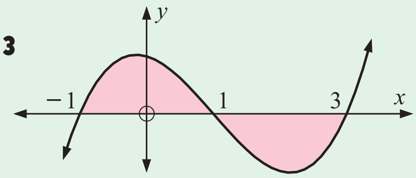
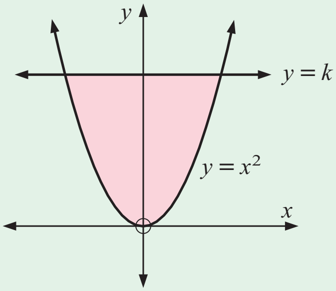


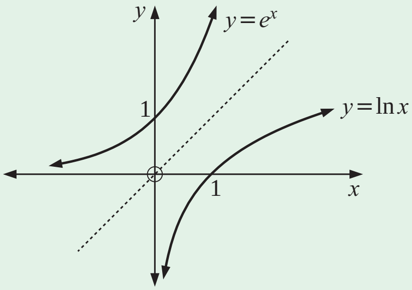
Is it true that $\displaystyle\int_{-1}^{3} f(x)\,dx$ represents the area of the shaded region?
Explain your answer briefly.

4 Determine k if the enclosed region has area $5\frac{1}{3}$ units^2.



5 By appealing only to geometrical evidence, explain why

$$\int_0^1 e^x\,dx + \int_1^e \ln x\,dx = e.$$



6 Find the area of the region enclosed by $y = x^2 + 4x + 1$ and $y = 3x + 3$.

7 A particle moves in a straight line with velocity $v(t) = t^2 - 6t + 8\ \text{m s}^{-1}$, $t \geqslant 0$.

a Draw a sign diagram for $v(t)$.

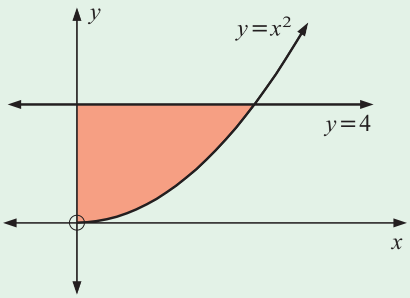
b Explain exactly what happens to the particle in the first 5 seconds of motion.

c After 5 seconds, how far is the particle from its original position?

d Find the total distance travelled in the first 5 seconds of motion.

8 Determine the area enclosed by the axes and $y = 4e^x - 1$.

9 Find the volume of revolution generated by rotating the shaded region through 360^o about the x-axis:



REVIEW SET 22B CALCULATOR

1 A particle moves in a straight line with velocity $v(t) = 2t - 3t^2$ m s^{-1}.
Find the distance travelled in the first second of motion.

2 Consider $f(x) = \dfrac{x}{1+x^2}$.

- **a** Find the position and nature of all turning points of $y = f(x)$.
- **b** Discuss $f(x)$ as $x \to \infty$ and as $x \to -\infty$.
- **c** Sketch the graph of $y = f(x)$.
- **d** Find, using technology, the area enclosed by $y = f(x)$, the x-axis, and the vertical line $x = -2$.

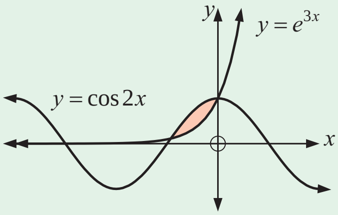
3 A particle moves in a straight line with velocity given by $v(t) = \sin t$ metres per second. Find the total distance travelled by the particle in the first 4 seconds of motion.

4 A boat travelling in a straight line has its engine turned off at time $t = 0$. Its velocity at time t seconds thereafter is given by $v(t) = \dfrac{100}{(t+2)^2}$ m s^{-1}.

- **a** Find the initial velocity of the boat, and its velocity after 3 seconds.
- **b** Discuss $v(t)$ as $t \to \infty$.
- **c** Sketch the graph of $v(t)$ against t.
- **d** Find how long it takes for the boat to travel 30 metres.
- **e** Find the acceleration of the boat at any time t.
- **f** Show that $\dfrac{dv}{dt} = -kv^{\frac{3}{2}}$, and find the value of the constant k.

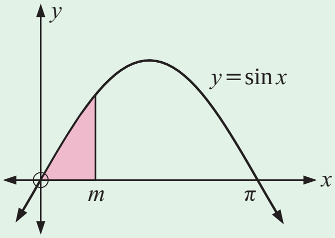
5 The figure shows the graphs of $y = \cos(2x)$ and $y = e^{3x}$ for $-\pi \leqslant x \leqslant \frac{\pi}{2}$.
Find correct to 4 decimal places:

- **a** the x-coordinates of their points of intersection
- **b** the area of the shaded region.



6 The marginal cost per day of producing x items is $C'(x) = 2 + 8e^{-x}$ pounds per item. What is the total cost of daily production of 80 items given that the fixed costs before production commences are 240 pounds per day?

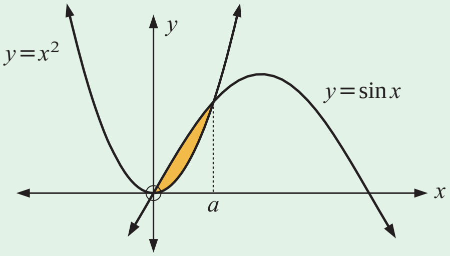
7 The shaded region has area $\frac{1}{2}$ unit^2. Find the value of m.



8 Find, correct to 4 decimal places:

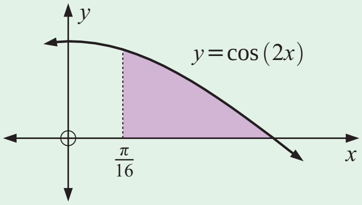
a the value of a

b the area of the shaded region.

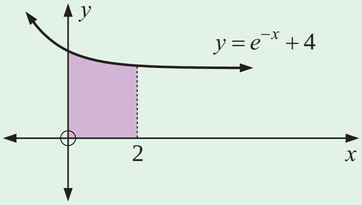


9 Find the volume of the solid of revolution formed when the shaded region is rotated through 360^o about the x-axis:

a



b



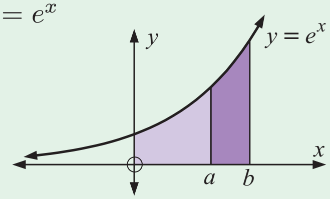
REVIEW SET 22C

1 At time $t = 0$ a particle passes through the origin with velocity 27 cm s^{-1}. Its acceleration t seconds later is $6t - 30$ cm s^{-2}.
Find the total distance that the particle has travelled when it momentarily comes to rest for the *second* time.

2 **a** Sketch the graphs of $y = \sin^2 x$ and $y = \sin x$ on the same set of axes for $0 \leqslant x \leqslant \pi$.

b Find the exact value of the area enclosed by these curves for $0 \leqslant x \leqslant \frac{\pi}{2}$.

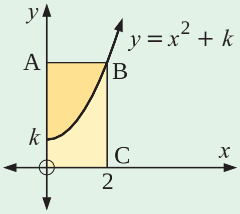
3 Find a given that the area of the region between $y = e^x$ and the x-axis from $x = 0$ to $x = a$ is 2 units^2.
Hence determine b such that the area of the region from $x = a$ to $x = b$ is also 2 units^2.



4 Determine the area of the region enclosed by $y = x$, $y = \sin x$ and $x = \pi$.

5 Determine the area enclosed by $y = \frac{2}{\pi}x$ and $y = \sin x$.

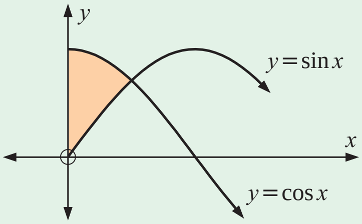
6 OABC is a rectangle and the two shaded regions are equal in area. Find k.



7 Find the volume of the solid of revolution formed when the following are rotated about the x-axis:

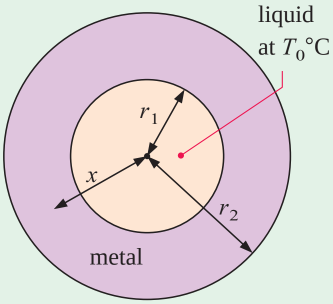
- **a** $y = x$ between $x = 4$ and $x = 10$
- **b** $y = x + 1$ between $x = 4$ and $x = 10$
- **c** $y = \sin x$ between $x = 0$ and $x = \pi$
- **d** $y = 1 - \cos x$ between $x = 0$ and $x = \frac{\pi}{2}$.

8



Find the volume of revolution if the shaded region is rotated through 360^o about the x-axis:

9



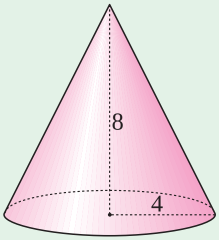
A metal tube has an annulus cross-section with radii r_1 and r_2 as shown.

Within the tube a liquid is maintained at temperature T_0 ^{o}C.

Within the metal, the temperature drops from inside to outside according to $\dfrac{dT}{dx} = \dfrac{k}{x}$ where k is a negative constant and x is the distance from the central axis.

Show that the outer surface has temperature $T_0 + k \ln\left(\dfrac{r_2}{r_1}\right)$.

10
- **a** Use $V = \frac{1}{3}\pi r^2 h$ to find the volume of this cone.
- **b** Check your answer to **a** by integration.



Chapter 23

Statistical distributions of discrete random variables

Syllabus reference: 6.9, 6.10

Contents:

A

DISCRETE RANDOM VARIABLES

RANDOM VARIABLES

In previous work we have described events mainly by using words. Where possible, it is far more convenient to use numbers.

> A **random variable** represents in number form the possible outcomes which could occur for some random experiment.

A **discrete random variable** X has possible values $x_1, x_2, x_3, ...$

For example:
- the number of houses in your suburb which have a 'power safety switch'
- the number of new bicycles sold each year by a bicycle store
- the number of defective light bulbs in the purchase order of a city store.

To determine the value of a discrete random variable we need to **count**.

A **continuous random variable** X could take possible values in some interval on the number line.

For example:
- the heights of men could all lie in the interval $50 < X < 250$ cm
- the volume of water in a rainwater tank during a given month could lie in the interval $0 < X < 100$ m^3.

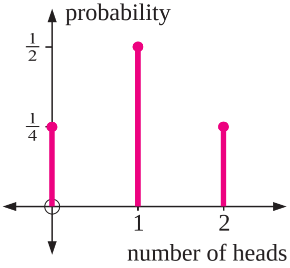
To determine the value of a continuous random variable we need to **measure**.

PROBABILITY DISTRIBUTIONS

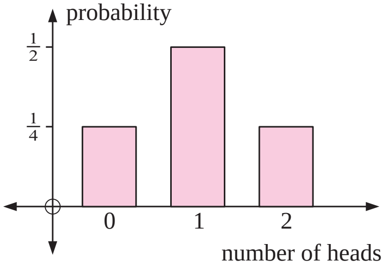
For any random variable there is a corresponding **probability distribution** which describes the probability that the variable will take any particular value.

The probability that the variable X takes value x is written as $\text{P}(X = x)$. For continuous distributions we can also sometimes write a probability distribution as a function $P(x)$.

For example, when tossing two coins, the random variable X could be 0 heads, 1 head, or 2 heads, so $X = 0$, 1 or 2. The associated probability distribution is $\text{P}(X = 0) = \frac{1}{4}$, $\text{P}(X = 1) = \frac{1}{2}$, and $\text{P}(X = 2) = \frac{1}{4}$ with graph:



or



Example 1 **Self Tutor**

A supermarket has three checkout points A, B and C. A government inspector checks for accuracy of the weighing scales at each checkout. If a weighing scale is accurate then Y is recorded, and if not, N is recorded. Suppose the random variable X is the number of accurate weighing scales at the supermarket.

a List the possible outcomes.

b Describe, using X, the events of there being:

i one accurate scale **ii** at least one accurate scale.

---

a Possible outcomes:

A	B	C	X
N	N	N	0
Y	N	N	1
N	Y	N	1
N	N	Y	1
N	Y	Y	2
Y	N	Y	2
Y	Y	N	2
Y	Y	Y	3

b **i** $X = 1$

ii $X = 1$, 2 or 3

EXERCISE 23A

1 Classify the following random variables as continuous or discrete:

- **a** the quantity of fat in a sausage.
- **b** the mark out of 50 for a geography test
- **c** the weight of a seventeen year old student
- **d** the volume of water in a cup of coffee
- **e** the number of trout in a lake
- **f** the number of hairs on a cat
- **g** the length of hairs on a horse
- **h** the height of a sky-scraper.

2 For each of the following:

- **i** identify the random variable being considered
- **ii** give possible values for the random variable
- **iii** indicate whether the variable is continuous or discrete.

- **a** To measure the rainfall over a 24-hour period in Singapore, the height of water collected in a rain gauge (up to 200 mm) is used.
- **b** To investigate the stopping distance for a tyre with a new tread pattern, a braking experiment is carried out.
- **c** To check the reliability of a new type of light switch, switches are repeatedly turned off and on until they fail.

3 A supermarket has four checkouts A, B, C and D. Management checks the weighing devices at each checkout. If a weighing device is accurate a Y is recorded; otherwise, N is recorded. The random variable being considered is the number of weighing devices which are accurate.

- **a** Suppose X is the random variable. What values can X have?
- **b** Tabulate the possible outcomes and the corresponding values for X.
- **c** Describe, using X, the events of:
  - **i** 2 devices being accurate
  - **ii** at least two devices being accurate.

4 Consider tossing three coins simultaneously. The random variable under consideration is the number of heads that could result.

- **a** List the possible values of X.
- **b** Tabulate the possible outcomes and the corresponding values of X.
- **c** Find the values of $\mathrm{P}(X = x)$, the probability of each x value occurring.
- **d** Graph the probability distribution $\mathrm{P}(X = x)$ against x as a probability histogram.

B DISCRETE PROBABILITY DISTRIBUTIONS

For each random variable there is a **probability distribution**.

The probability p_i of any given outcome lies between 0 and 1 (inclusive), so $0 \leqslant p_i \leqslant 1$.

If there are n possible outcomes then $\sum_{i=1}^{n} p_i = 1$,

or in other words $p_1 + p_2 + p_3 + ... + p_n = 1$.

The **probability distribution** of a **discrete random variable** can be given:

- in table form
- in graphical form
- in functional form as a **probability mass function** $P(x)$.

It provides us with all possible values of the variable and the probability of the occurrence of each value.

Example 2 **Self Tutor**

A magazine store recorded the number of magazines purchased by its customers in one week. 23% purchased one magazine, 38% purchased two, 21% purchased three, 13% purchased four, and 5% purchased five.

- **a** What is the random variable?
- **b** Make a probability table for the random variable.
- **c** Graph the probability distribution using a spike graph.

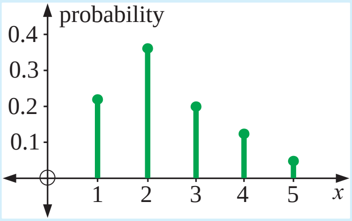
a The random variable X is the number of magazines sold.

So, $X = 1, 2, 3, 4$ or 5.

b

x	1	2	3	4	5
$\text{P}(X = x)$	0.23	0.38	0.21	0.13	0.05

c



Example 3 **Self Tutor**

Show that the following are probability distribution functions:

a $P(x) = \dfrac{x^2+1}{34}, \quad x = 1, 2, 3, 4$

b $P(x) = \binom{3}{x}(0.6)^x(0.4)^{3-x}, \quad x = 0, 1, 2, 3$

a $P(1) = \frac{2}{34} \qquad P(2) = \frac{5}{34} \qquad P(3) = \frac{10}{34} \qquad P(4) = \frac{17}{34}$

All of these obey $0 \leqslant P(x_i) \leqslant 1$, and $\sum P(x_i) = \frac{2}{34} + \frac{5}{34} + \frac{10}{34} + \frac{17}{34} = 1$

$\therefore$ $P(x)$ is a probability distribution function.

b For $P(x) = \binom{3}{x}(0.6)^x(0.4)^{3-x}$,

$$\begin{aligned} P(0) &= \tbinom{3}{0}(0.6)^0(0.4)^3 &&= 1 \times 1 \times (0.4)^3 &&= 0.064 \\ P(1) &= \tbinom{3}{1}(0.6)^1(0.4)^2 &&= 3 \times (0.6) \times (0.4)^2 &&= 0.288 \\ P(2) &= \tbinom{3}{2}(0.6)^2(0.4)^1 &&= 3 \times (0.6)^2 \times (0.4) &&= 0.432 \\ P(3) &= \tbinom{3}{3}(0.6)^3(0.4)^0 &&= 1 \times (0.6)^3 \times 1 &&= 0.216 \\ & && \text{Total} && \quad 1.000 \end{aligned}$$

All probabilities lie between 0 and 1, and $\sum P(x_i) = 1$.

$\therefore$ $P(x)$ is a probability distribution function.

EXERCISE 23B

1 Find k in these probability distributions:

a

x	0	1	2
$P(x)$	0.3	k	0.5

b

x	0	1	2	3
$\text{P}(X = x)$	k	$2k$	$3k$	k

2 The probabilities of Jason scoring home runs in each game during his baseball career are given in the following table. X is the number of home runs per game.

x	0	1	2	3	4	5
$P(x)$	a	0.3333	0.1088	0.0084	0.0007	0.0000

a What is the value of $P(2)$?

b What is the value of a? Explain what this number means.

  - **c** What is the value of $P(1) + P(2) + P(3) + P(4) + P(5)$? Explain what this represents.
  - **d** Draw a probability distribution spike graph of $P(x)$ against x.

3 Explain why the following are not valid probability distributions:

a

x	0	1	2	3
$P(x)$	0.2	0.3	0.4	0.2

b

x	2	3	4	5
$P(x)$	0.3	0.4	0.5	-0.2

4 Sally's number of hits in each softball match has the following probability distribution:

x	0	1	2	3	4	5
$P(x)$	0.07	0.14	k	0.46	0.08	0.02

  - **a** State clearly what the random variable represents.
  - **b** Find k.
  - **c** Find: **i** $P(X \geqslant 2)$ **ii** $P(1 \leqslant X \leqslant 3)$

5 A die is rolled twice.

  - **a** Draw a grid which shows the sample space.
  - **b** Suppose X denotes the sum of the results for the two rolls. Find the probability distribution of X.
  - **c** Draw a probability distribution histogram for this situation.

6 Find k for the following probability distributions:

  - **a** $P(x) = k(x+2)$ for $x = 1, 2, 3$
  - **b** $P(x) = \dfrac{k}{x+1}$ for $x = 0, 1, 2, 3$

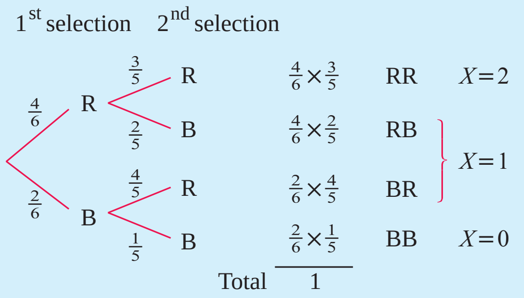
7 A discrete random variable X has probability distribution given by:
$P(X = x) = k\left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{4-x}$ where $x = 0, 1, 2, 3, 4$.

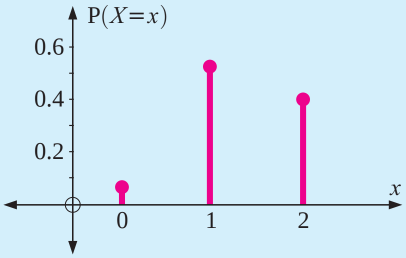
  - **a** Find $P(X = x)$ for $x = 0, 1, 2, 3$ and 4.
  - **b** Find k and hence find $P(X \geqslant 2)$.

Example 4

Self Tutor

A bag contains 4 red and 2 blue marbles. Two marbles are randomly selected without replacement. If X denotes the number of reds selected, find the probability distribution of X.





x	0	1	2
$P(X = x)$	$\frac{2}{30}$	$\frac{16}{30}$	$\frac{12}{30}$

8 Electrical components are produced and packed into boxes of 10. It is known that 4% of the components may be faulty. The random variable X denotes the number of faulty items in the box, and has a probability distribution

$$P(x) = \binom{10}{x} (0.04)^x (0.96)^{10-x}, \quad x = 0, 1, 2, ..., 10.$$

a Find the probability that a randomly selected box will contain no faulty component.

b Find the probability that a randomly selected box will contain at least one faulty component.

9 A bag contains 5 blue and 3 green tickets. Two tickets are randomly selected without replacement. We let X denote the number of blue tickets selected.

a Find the probability distribution of X.

b Suppose instead that three tickets are randomly selected without replacement. Find the probability distribution of X for $X = 0, 1, 2, 3$.

10 When a pair of dice is rolled, D denotes the sum of the top faces.

a Display the possible results in a table.

b Find $\text{P}(D = 7)$.

c Find the probability distribution of D.

d Find $\text{P}(D \geqslant 8 \mid D \geqslant 6)$.

11 When a pair of dice is rolled, N denotes the difference between the numbers on the top faces.

a Display the possible results in a table.

b Construct a probability distribution table for the possible values of N.

c Find $\text{P}(N = 3)$.

d Find $\text{P}(N \geqslant 3 \mid N \geqslant 1)$.

C EXPECTATION

Consider the following problem:

> A die is to be rolled 120 times. On how many occasions should we *expect* the result to be a "six"?

In order to answer this question we must first consider all possible outcomes of rolling the die. The possibilities are 1, 2, 3, 4, 5 and 6, and each of these is equally likely to occur.

Therefore, we would expect $\frac{1}{6}$ of them to be a "six".

$\frac{1}{6}$ of 120 is 20, so we expect 20 of the 120 rolls of the die to yield a "six".

However, this does not mean that we *will* get 20 sixes when we roll a die 120 times.

If there are n trials of an experiment, and an event has probability p of occurring in each of the trials, then the number of times we **expect** the event to occur is np.

We can also talk about the *expected* outcome from one trial of an experiment.

The expected outcome for the random variable X is the **mean** result μ.

In general, the **expectation** of the random variable X is

$$\mathbf{E}(\boldsymbol{X}) = \boldsymbol{\mu} = \sum_{i=1}^{n} \boldsymbol{x_i p_i} \quad \text{or} \quad \sum_{i=1}^{n} \boldsymbol{x_i}\, \mathbf{P}(\boldsymbol{X} = \boldsymbol{x_i}).$$

Example 5 **Self Tutor**

Each time a footballer kicks for goal he has a $\frac{3}{4}$ chance of being successful.

In a particular game he has 12 kicks for goal. How many goals would you expect him to kick?

$p = \text{P(goal)} = \frac{3}{4}$ $\quad \therefore$ the expected number of goals is $np = 12 \times \frac{3}{4} = 9$

Example 6 **Self Tutor**

Find the mean of the data of **Example 2**.

The probability table is:

x_i	1	2	3	4	5
p_i	0.23	0.38	0.21	0.13	0.05

Now $\mu = \sum x_i p_i$

$= 1(0.23) + 2(0.38) + 3(0.21) + 4(0.13) + 5(0.05)$

$= 2.39$

In the long run, the average number of magazines purchased per customer is 2.39.

FAIR GAMES

In gambling, we say that the **expected gain** of the player from each game is the expected return or payout from the game, less the amount it cost them to play.

The game will be **fair** if the expected gain is zero.

Suppose X represents the gain of a player from each game.
The game is **fair** if $\text{E}(X) = 0$.

Example 7 **Self Tutor**

In a game of chance, a player spins a square spinner labelled 1, 2, 3, 4. The player wins the amount of money shown in the table alongside, depending on which number comes up. Determine:

Number	1	2	3	4
Winnings	\$1	\$2	\$5	\$8

a the expected return for one spin of the spinner

b the expected *gain* of the player if it cost \$5 to play each game

c whether you would recommend playing this game.

a Let Y denote the return or payout from each spin.

As each number is equally likely, the probability for each number is $\frac{1}{4}$

$\therefore$ expected return $= \text{E}(Y) = \frac{1}{4} \times 1 + \frac{1}{4} \times 2 + \frac{1}{4} \times 5 + \frac{1}{4} \times 8 = \$4.$

b Let X denote the *gain* of the player from each game.

Since it costs \$5 to play the game, the expected gain $= \text{E}(X) = \text{E}(Y) - \$5$

$= \$4 - \$5$

$= -\$1$

c Since $\text{E}(X) \neq 0$, the game is not fair. In particular, since $\text{E}(X) = -\$1$, we expect on average the player to lose \$1 with each spin. We would not recommend that a person play the game.

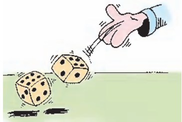
EXERCISE 23C

1 In a particular region, the probability that it will rain on any one day is 0.28. On how many days of the year would you expect it to rain?

2 **a** If 3 coins are tossed, what is the chance that they all fall heads?

b If the 3 coins are tossed 200 times, on how many occasions would you expect them all to fall heads?

3 If two dice are rolled simultaneously 180 times, on how many occasions would you expect to get a double?



4 A single coin is tossed once. If a head appears you win \$2, and if a tail appears you lose \$1. How much would you expect to win when playing this game three times?

5 During the snow season there is a $\frac{3}{7}$ probability of snow falling on any particular day. If Udo skis for five weeks, on how many days could he expect to see snow falling?

6 A goalkeeper has probability $\frac{3}{10}$ of saving a penalty attempt. How many goals would he expect to save from 90 attempts?

7 In a random survey of her electorate, politician A discovered the residents' voting intentions in relation to herself and her two opponents B and C. The results are indicated alongside:

A	B	C
165	87	48

a Estimate the probability that a randomly chosen voter in the electorate will vote for:

i A **ii** B **iii** C.

b If there are 7500 people in the electorate, how many of these would you expect to vote for:

i A **ii** B **iii** C?

8 A person rolls a normal six-sided die and wins the number of euros shown on the face.

a Find the expected return from one roll of the die.

b Find the expected *gain* of the player if it costs €4 to play the game. Would you advise the person to play several games?

c Suppose it costs €k to play the game. What value(s) of k will result in:

i a fair game **ii** a profit being made by the vendor?

9 A charity fundraiser gets a licence to run the following gambling game: A die is rolled and the returns to the player are given in the 'pay table' alongside. To play the game costs \$4. A result of getting a 6 wins \$10, so in fact you are ahead by \$6 if you get a 6 on the first roll.

Result	*Wins*
6	\$10
4, 5	\$4
1, 2, 3	\$1

a What are your chances of playing one game and winning:

i \$10 **ii** \$4 **iii** \$1?

b Your expected return from throwing a 6 is $\frac{1}{6} \times \$10$. What is your expected return from throwing:

i a 4 or 5 **ii** a 1, 2 or 3 **iii** a 1, 2, 3, 4, 5 or 6?

c What is your overall expected result at the end of one game?

d What is your overall expected result at the end of 100 games?

10 A person plays a game with a pair of coins. If two heads appear then £10 is won. If a head and a tail appear then £3 is won. If two tails appear then £5 is lost.

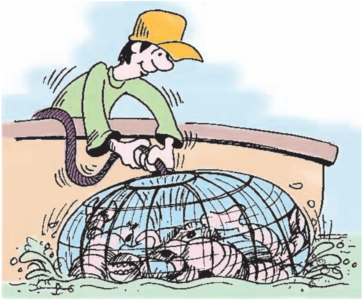
a How much would a person expect to win playing this game once?

b If the organiser of the game is allowed to make an average of £1 per game, how much should be charged to play the game once?

11 A country exports crayfish to overseas markets. The buyers are prepared to pay high prices when the crayfish arrive still alive.

If X is the number of deaths per dozen crayfish, the probability distribution for X is given by:

x_i	0	1	2	3	4	5	> 5
$P(x_i)$	0.54	0.26	0.15	k	0.01	0.01	0.00



a Find k.

b Over a long period, what is the mean number of deaths per dozen crayfish?

12 A random variable X has probability distribution given by

$P(x) = \dfrac{x^2 + x}{20}$ for $x = 1, 2, 3$. Calculate the mean μ for this distribution.

13 A pair of dice is rolled and the random variable M is the larger of the two numbers that are shown uppermost.

- **a** In table form, obtain the probability distribution of M.
- **b** Find the mean of the M-distribution.

14 At a charity event there is a money-raising game involving a pair of ordinary dice. The game costs $\$a$ to play. When the two dice are rolled, their sum is described by the variable X. The organisers decide that a sum which is less than 4 or between 7 and 9 inclusive is a loss of $\$\frac{a}{3}$, a result between 4 and 6 inclusive gives a return of \$7 and a result of 10 or more gives a return of \$21.

- **a** Determine $P(X \leqslant 3)$, $P(4 \leqslant X \leqslant 6)$, $P(7 \leqslant X \leqslant 9)$ and $P(X \geqslant 10)$.
- **b** Show that the expected gain of a player is given by $\frac{1}{6}(35 - 7a)$ dollars.
- **c** What value would a need to be for the game to be 'fair'?
- **d** Explain why the organisers would not let a be 4.
- **e** If the organisers let a be 6 and the game was played 2406 times, estimate the amount of money raised by this game.

D THE BINOMIAL DISTRIBUTION

Thus far in the chapter we have considered properties of general discrete random variables.

We now examine a special discrete random variable which is applied to **sampling with replacement**. The probability distribution associated with this variable is the **binomial probability distribution**.

For **sampling without replacement** the **hypergeometric probability distribution** is the model used, but that distribution is not part of this course.

BINOMIAL EXPERIMENTS

Consider an experiment for which there are two possible results: **success** if some event occurs, or **failure** if the event does not occur.

If we repeat this experiment in a number of **independent trials**, we call it a **binomial experiment**.

The probability of a success p must be constant for all trials. Since success and failure are complementary events, the probability of a failure is $1 - p$.

The random variable X is the total number of successes in n trials.

THE BINOMIAL PROBABILITY DISTRIBUTION

Suppose a spinner has three blue edges and one white edge. Clearly, for each spin we will get either a blue or a white.

The chance of finishing on blue is $\frac{3}{4}$ and on white is $\frac{1}{4}$.

If we call a blue result a 'success' and a white result a 'failure', then we have a binomial experiment.

We let p be the probability of getting a blue, so $p = \frac{3}{4}$. The probability of getting a white is $1 - p = \frac{1}{4}$.

Consider twirling the spinner $n = 3$ times.

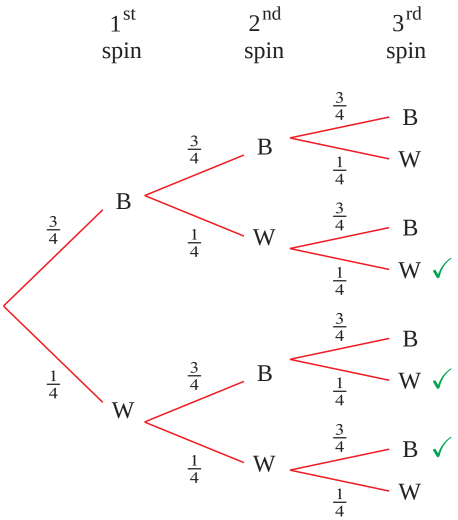
Let the random variable X be the number of 'successes' or blue results, so $X = 0$, 1, 2 or 3.

$$\begin{aligned}
\mathrm{P}(X=0) &= \mathrm{P}(\text{all 3 are white}) \\
&= \tfrac{1}{4} \times \tfrac{1}{4} \times \tfrac{1}{4} \\
&= \left(\tfrac{1}{4}\right)^3
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}(X=1) &= \mathrm{P}(\text{1 blue and 2 white}) \\
&= \mathrm{P}(\text{BWW or WBW or WWB}) \\
&= \left(\tfrac{3}{4}\right)\left(\tfrac{1}{4}\right)^2 \times 3 \quad \{\text{the 3 branches } \checkmark\}
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}(X=2) &= \mathrm{P}(\text{2 blue and 1 white}) \\
&= \mathrm{P}(\text{BBW or BWB or WBB}) \\
&= \left(\tfrac{3}{4}\right)^2\left(\tfrac{1}{4}\right) \times 3
\end{aligned}$$

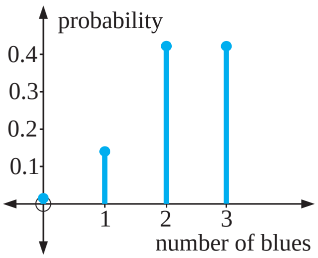
$$\begin{aligned}
\mathrm{P}(X=3) &= \mathrm{P}(\text{3 blues}) \\
&= \left(\tfrac{3}{4}\right)^3
\end{aligned}$$



The coloured factor 3 is the number of ways of getting one success in three trials, which is $\binom{3}{1}$.

We note that:

$$\begin{aligned}
\mathrm{P}(X=0) &= \left(\tfrac{1}{4}\right)^3 &&= \binom{3}{0}\left(\tfrac{3}{4}\right)^0\left(\tfrac{1}{4}\right)^3 \approx 0.0156 \\
\mathrm{P}(X=1) &= 3\left(\tfrac{1}{4}\right)^2\left(\tfrac{3}{4}\right)^1 &&= \binom{3}{1}\left(\tfrac{3}{4}\right)^1\left(\tfrac{1}{4}\right)^2 \approx 0.1406 \\
\mathrm{P}(X=2) &= 3\left(\tfrac{1}{4}\right)^1\left(\tfrac{3}{4}\right)^2 &&= \binom{3}{2}\left(\tfrac{3}{4}\right)^2\left(\tfrac{1}{4}\right)^1 \approx 0.4219 \\
\mathrm{P}(X=3) &= \left(\tfrac{3}{4}\right)^3 &&= \binom{3}{3}\left(\tfrac{3}{4}\right)^3\left(\tfrac{1}{4}\right)^0 \approx 0.4219
\end{aligned}$$



This suggests that: $\mathrm{P}(X = x) = \binom{3}{x}\left(\frac{3}{4}\right)^x\left(\frac{1}{4}\right)^{3-x}$ where $x = 0, 1, 2, 3$.

Notice that $\left(\frac{1}{4}\right)^3 + 3\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right) + 3\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3$ is the binomial expansion for $\left(\frac{1}{4} + \frac{3}{4}\right)^3$.

Consider a binomial experiment for which p is the probability of a *success* and $1-p$ is the probability of a *failure*.

If there are n independent trials then the probability that there are r *successes* and $n-r$ *failures* is given by

$\mathrm{P}(X=r) = \binom{n}{r} p^r (1-p)^{n-r}$ where $r = 0, 1, 2, 3, 4, ..., n$.

$\mathrm{P}(X=r)$ is the **binomial probability distribution function**.

The **expected** or **mean** outcome of the experiment is $\mu = \mathrm{E}(X) = np$.

If X is the random variable of a binomial experiment with parameters n and p, then we write $\boldsymbol{X \sim \mathrm{B}(n, p)}$ where $\sim$ reads "*is distributed as*".

We can quickly calculate binomial probabilities using a graphics calculator. For example:

- To find the probability $\mathrm{P}(X = r)$ that the variable takes the value r, we use the **binomial probability distribution function**.
- To find the probability $\mathrm{P}(X \leqslant r)$ that the variable takes a value which is *at most* r, we use the **binomial cumulative distribution function**.

For help using your calculator, consult the chapter of instructions at the front of the book.

Example 8 **Self Tutor**

72% of union members are in favour of a certain change to their conditions of employment. A random sample of five members is taken. Find:

a the probability that three members are in favour of the change in conditions

b the probability that at least three members are in favour of the changed conditions

c the expected number of members in the sample that are in favour of the change.

Let X denote the number of members in the sample in favour of the changes.

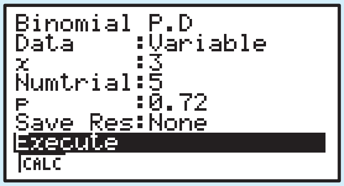
$n = 5$, so $X = 0, 1, 2, 3, 4$ or 5, and $p = 72\% = 0.72$

$\therefore \quad X \sim \mathrm{B}(5, 0.72)$.

a $\mathrm{P}(X=3)$

$= \binom{5}{3} (0.72)^3 (0.28)^2$

≈ 0.293



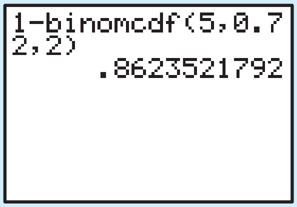
```
Binomial P.D
Data    :Variable
x       :3
Numtrial:5
p       :0.72
Save Res:None
Execute
CALC
```

b $\mathrm{P}(X \geqslant 3)$

$= 1 - \mathrm{P}(X \leqslant 2)$

$= 1 - 0.1376$

≈ 0.862



```
1-binomcdf(5,0.7
2,2)
      .8623521792
```

c $\mathrm{E}(X) = np = 5 \times 0.72 = 3.6$ members

EXERCISE 23D

1 For which of these probability experiments does the binomial distribution apply? Justify your answers, using a full sentence.

- **a** A coin is thrown 100 times. The variable is the number of heads.
- **b** One hundred coins are each thrown once. The variable is the number of heads.
- **c** A box contains 5 blue and 3 red marbles. I draw out 5 marbles, replacing the marble each time. The variable is the number of red marbles drawn.
- **d** A box contains 5 blue and 3 red marbles. I draw out 5 marbles without replacement. The variable is the number of red marbles drawn.
- **e** A large bin contains ten thousand bolts, 1% of which are faulty. I draw a sample of 10 bolts from the bin. The variable is the number of faulty bolts.

2 At a manufacturing plant, 35% of the employees work night-shift. If 7 employees were selected at random, find the probability that:

- **a** exactly 3 of them work night-shift
- **b** less than 4 of them work night-shift
- **c** at least 4 of them work night-shift.

3 Records show that 6% of the items assembled on a production line are faulty.
A random sample of 12 items is selected at random, with replacement.
Find the probability that:

- **a** none will be faulty
- **b** at most one will be faulty
- **c** at least two will be faulty
- **d** less than four will be faulty.

4 There is a 5% chance that any apple in a crate will have a blemish. If a sample of 25 apples is taken, find:

- **a** the probability that exactly 2 of these have blemishes
- **b** the probability that at least one has a blemish
- **c** the expected number of apples that will have a blemish.

5 The local bus service does not have a good reputation. It is known that the 8 am bus will run late on average two days out of every five. For any week of the year taken at random, find the probability of the 8 am bus being on time:

- **a** all 7 days
- **b** only on Monday
- **c** on any 6 days
- **d** on at least 4 days.

6 An infectious flu virus is spreading through a school. The probability of a randomly selected student having the flu next week is 0.3.

- **a** Calculate the probability that out of a class of 25 students, 2 or more will have the flu next week.
- **b** If more than 20% of the students are away with the flu next week, a class test will have to be cancelled. What is the probability that the test will be cancelled?
- **c** If the school has 350 students, find the expected number that will be absent from school next week.

7 During a season, a basketball player has a 94% success rate in shooting from the free throw line. In one match the basketballer has 20 shots from the free throw line.

- **a** Find the probability that the basketballer is successful on:
  - **i** all 20 throws
  - **ii** at least 18 throws.
- **b** Find the expected number of successful throws.

REVIEW SET 23A NON-CALCULATOR

1 $f(x) = \dfrac{a}{x^2+1}$, $x = 0, 1, 2, 3$ is a probability distribution function.

- **a** Find a.
- **b** Hence, find $\text{P}(X \geqslant 1)$.

2 A random sample of 120 toothbrushes is made with replacement from a very large batch where 4% are known to be defective. Find the mean number of defectives in the sample.

3 A random variable X has probability distribution function given by:

x	0	1	2	3	4
$P(x)$	0.10	0.30	0.45	0.10	k

- **a** Find k.
- **b** Find the mean μ for the distribution of X.

4 Three green balls and two yellow balls are placed in a hat. Two balls are randomly drawn without replacement. Let X be the number of green balls drawn.

- **a** Find the probability that: **i** $X = 0$ **ii** $X = 1$ **iii** $X = 2$
- **b** Find $\text{E}(X)$.

5 Lakshmi rolls a normal six-sided die. She wins twice the number of pounds as the number shown on the face.

- **a** How much does Lakshmi expect to win from one roll of the die?
- **b** If it costs £8 to play the game, would you advise Lakshmi to play several games? Explain your answer.

REVIEW SET 23B CALCULATOR

1 A discrete random variable X has probability distribution function $P(x) = k\left(\frac{3}{4}\right)^x\left(\frac{1}{4}\right)^{3-x}$ where $x = 0, 1, 2, 3$ and k is a constant.

- **a** Find k.
- **b** Find $\text{P}(X \geqslant 1)$.
- **c** Find the expected outcome for the variable X.

2 A manufacturer finds that 18% of the items produced from its assembly lines are defective. During a floor inspection, the manufacturer randomly selects ten items. Find the probability that the manufacturer finds:

- **a** one defective
- **b** two defective
- **c** at least two defective items.

3 From data over the last fifteen years it is known that the chance of a netballer with a knee injury needing major knee surgery in any one season is 0.0132. In 2007 there were 487 cases of knee injuries. Find the expected number of major knee surgeries.

4 An X-ray has probability of 0.96 of showing a fracture in the arm. If four different X-rays are taken of a particular fracture, find the probability that:

- **a** all four show the fracture
- **b** the fracture does not show up
- **c** at least three X-rays show the fracture
- **d** only one X-ray shows the fracture.

5 24% of visitors to a museum make voluntary donations upon entry. On a certain day the museum has 175 visitors. Find:

- **a** the expected number of donations
- **b** the probability that less than 40 visitors make a donation.

6 A school basketball team has 8 players, each of whom has a 75% chance of turning up for any given game. The team needs at least 5 players to avoid forfeiting the game.

- **a** Find the probability that, on a randomly chosen game, the team will:
  - **i** have all of its players
  - **ii** have to forfeit the game.
- **b** The team plays 30 games for the season. How many games would you expect the team to forfeit?

REVIEW SET 23C

1 Find k for:

- **a** the probability distribution function $P(x) = \frac{k}{2x}, \quad x = 1, 2, 3$
- **b** the probability distribution

x	0	1	2	3
$P(x)$	$\frac{k}{2}$	0.2	k^2	0.3

2 A random variable X has probability distribution function

$\mathrm{P}(X = x) = \binom{4}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{4-x}$ for $x = 0, 1, 2, 3, 4.$

- **a** Find $\mathrm{P}(X = x)$ for $x = 0, 1, 2, 3, 4.$
- **b** Find the mean μ for this distribution.

3 A die is biased such that the probability of obtaining a 6 is $\frac{2}{5}$. The die is rolled 1200 times. Let X be the number of sixes obtained. Find the mean of X.

4 Only 40% of young trees that are planted will survive the first year. The Botanical Gardens buys five young trees. Assuming independence, calculate the probability that during the first year:

- **a** exactly one tree will survive
- **b** at most one tree will survive
- **c** at least one tree will survive.

5 In a game, the numbers from 1 to 20 are written on tickets and placed in a bag. A player draws out a number at random. He or she wins \$3 if the number is even, \$6 if the number is a square number, and \$9 if the number is both even and square.

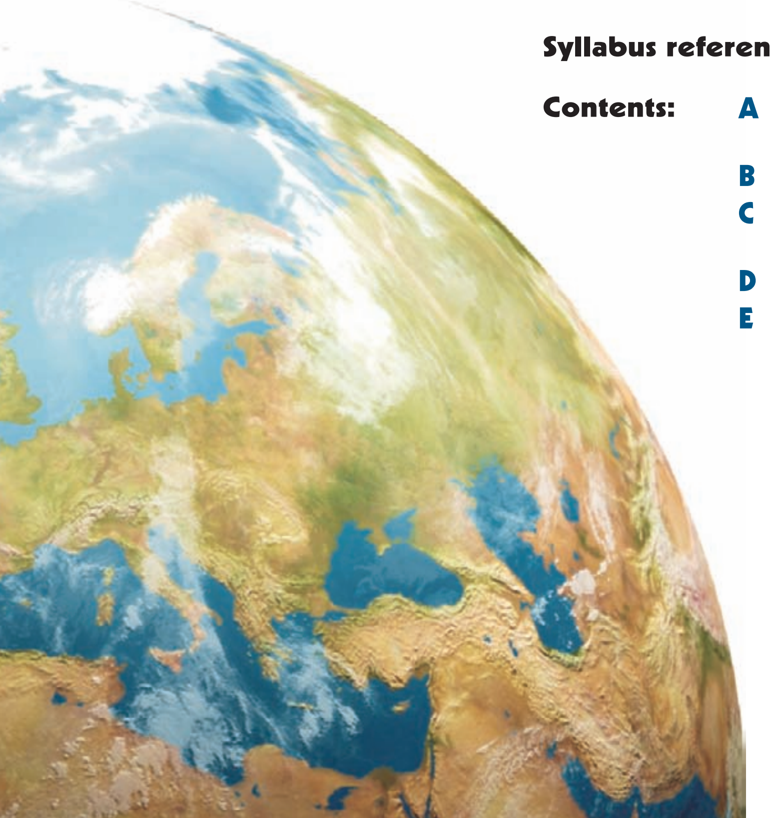
- **a** Calculate the probability that the player wins: **i** \$3 **ii** \$6 **iii** \$9
- **b** How much should be charged to play the game so that it is a fair game?

Chapter 24

Statistical distributions of continuous random variables

Syllabus reference: 6.11

Contents:



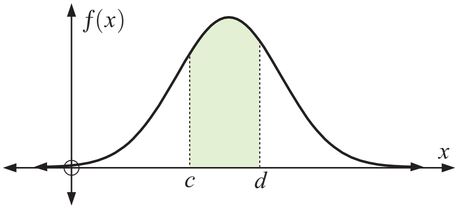
A CONTINUOUS PROBABILITY DENSITY FUNCTIONS

In the previous chapter we looked at discrete random variables and examined some probability distributions where the random variable X could take the non-negative integer values $x = 0, 1, 2, 3, 4, ...$

For a **continuous random variable** X, x can take any real value.

Consequently, a function is used to specify the probability distribution, and that function is called the **probability density function**.

Probabilities are found by calculating areas under the probability density function for a particular interval.



A **continuous probability density function (pdf)** is a function $f(x)$ such that $f(x) \geqslant 0$ on a given interval $a \leqslant x \leqslant b$ and $\int_a^b f(x)\,dx = 1$.

The probability that the variable X lies in the interval $c \leqslant X \leqslant d$ is

$$\mathrm{P}(c \leqslant X \leqslant d) = \int_c^d f(x)\,dx.$$

Since the variable is continuous, it is important to recognise that the probability of X taking any *exact* value is zero. Instead, we can only measure the probability of X being in a particular interval. A consequence of this is that

$$\mathrm{P}(c \leqslant X \leqslant d) = \mathrm{P}(c \leqslant X < d) = \mathrm{P}(c < X \leqslant d) = \mathrm{P}(c < X < d)$$

The **mean** μ or $\mathrm{E}(X)$ of a continuous probability density function is defined as

$$\mu = \int_a^b x\,f(x)\,dx.$$

Example 1 **Self Tutor**

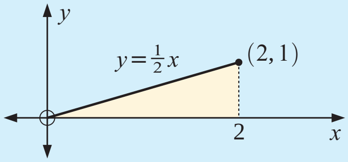
$f(x) = \begin{cases} \frac{1}{2}x & \text{on } 0 \leqslant x \leqslant 2 \\ 0 & \text{elsewhere} \end{cases}$ is a probability density function.

a Check that the above statement is true.

b Find $\mathrm{P}(\frac{1}{2} \leqslant X \leqslant 1)$.

c Find the mean of the distribution.

a



Area $= \frac{1}{2} \times 2 \times 1 = 1$ ✓

or $\int_0^2 \frac{1}{2}x\,dx = \left[\frac{1}{4}x^2\right]_0^2 = 1$ ✓

b $$\begin{aligned}P(\tfrac{1}{2} \leqslant X \leqslant 1) &= \int_{\frac{1}{2}}^{1} f(x)\,dx \\ &= \int_{\frac{1}{2}}^{1} \tfrac{1}{2}x\,dx \\ &= \left[\tfrac{1}{4}x^2\right]_{\frac{1}{2}}^{1} \\ &= \tfrac{1}{4} - \tfrac{1}{16} \\ &= \tfrac{3}{16}\end{aligned}$$

c $$\begin{aligned}\mu &= \int_0^2 x\,f(x)\,dx \\ &= \int_0^2 \tfrac{1}{2}x^2\,dx \\ &= \left[\tfrac{1}{6}x^3\right]_0^2 \\ &= 1\tfrac{1}{3}\end{aligned}$$

EXERCISE 24A

1 $f(x) = \begin{cases} 1 - \frac{1}{2}x, & 0 \leqslant x \leqslant 2 \\ 0 & \text{elsewhere} \end{cases}$ is a probability density function.

a Check that the above statement is true. **b** Find the mean of the distribution.

2 $f(x) = \begin{cases} ax(x-4), & 0 \leqslant x \leqslant 4 \\ 0 & \text{elsewhere} \end{cases}$ is a continuous probability density function.

a Find a.
b Sketch the graph of $y = f(x)$.
c Find $P(0 \leqslant X \leqslant 1)$.
d Find the mean of the distribution.

3 $f(x) = \begin{cases} kx^2(x-6), & 0 \leqslant x \leqslant 5 \\ 0 & \text{elsewhere} \end{cases}$ is a probability density function. Find:

a k **b** $E(X)$ **c** $P(3 \leqslant X \leqslant 5)$.

4 The continuous random variable X has the probability density function

$$f(x) = ax^4, \quad 0 \leqslant x \leqslant k.$$

Given that $P\left(X \leqslant \frac{2}{3}\right) = \frac{1}{243}$, find a and k.

5 The time taken in hours to perform a particular task has the probability density function: $f(x) = \begin{cases} k\sqrt{x} & 0 \leqslant x \leqslant 4 \\ 0 & \text{otherwise.} \end{cases}$

a Find k.
b Find the mean time for performing the task.
c Find $P(1 \leqslant X \leqslant 2.5)$.
d Find a such that $P(0 \leqslant X \leqslant a) = P(a \leqslant X \leqslant 4)$.

B NORMAL DISTRIBUTIONS

The normal distribution is the most important distribution for a continuous random variable. Many naturally occurring phenomena have a distribution that is normal, or approximately normal. Some examples are:

- physical attributes of a population such as height, weight, and arm length
- crop yields
- scores for tests taken by a large population

If X is **normally distributed** then its **probability density function** is given by

$$\boldsymbol{f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}} \quad \text{for} \quad -\infty < x < \infty$$

where μ is the mean and σ^2 is the variance of the distribution.

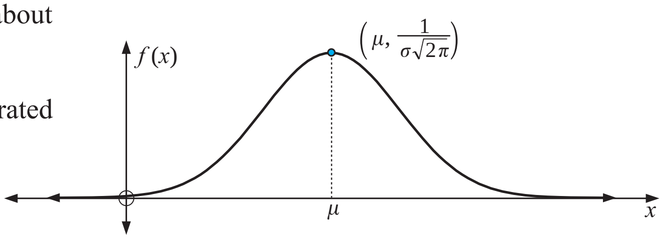
Each member of the family is specified by the **parameters** μ and σ^2, and we can write $\boldsymbol{X \sim \mathrm{N}(\mu,\, \sigma^2)}$.

This probability density function for the normal distribution represents a family of **bell-shaped normal curves**.

These curves are all symmetrical about the vertical line $x = \mu$.

A typical normal curve is illustrated alongside.

Notice that $f(\mu) = \dfrac{1}{\sigma\sqrt{2\pi}}$.



CHARACTERISTICS OF THE NORMAL PROBABILITY DENSITY FUNCTION

- The curve is symmetrical about the vertical line $x = \mu$.
- As $x \to \pm\infty$ the normal curve approaches its asymptote, the x-axis.
- $f(x) > 0$ for all x.
- The area under the curve is one unit^2, and so $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$.

  This is necessary since the total probability must be 1.
- More scores are distributed closer to the mean than further away. This results in the typical **bell shape**.

HOW THE NORMAL DISTRIBUTION ARISES

Consider the oranges picked from an orange tree. They do not all have the same weight. The variation may be due to several factors, including:

- genetics
- different times when the flowers were fertilised
- different amounts of sunlight reaching the leaves and fruit
- different weather conditions such as the prevailing winds.

The result is that most of the fruit will have weights close to the mean, while there are far fewer oranges that are *much* heavier or *much* lighter. This results in the bell-shaped distribution.

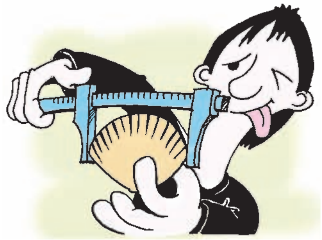
Once a normal model has been established, we can use it to make predictions about a distribution and to answer other relevant questions.

A TYPICAL NORMAL DISTRIBUTION

A large sample of cockle shells was collected and the maximum distance across each shell was measured. Click on the video clip icon to see how a histogram of the data is built up. Then click on the demo icon to observe the effect of changing the class interval lengths for normally distributed data.

VIDEO CLIP

DEMO



THE GEOMETRICAL SIGNIFICANCE OF μ AND σ

Differentiating $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ we obtain

$$f'(x) = \frac{-1}{\sigma^2\sqrt{2\pi}}\left(\frac{x-\mu}{\sigma}\right) e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$\therefore \quad f'(x) = 0$ only when $x = \mu$.

This corresponds to the point on the graph when $f(x)$ is a maximum.

Differentiating again, we obtain $f''(x) = \dfrac{-1}{\sigma^2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\left[\dfrac{1}{\sigma} - \dfrac{(x-\mu)^2}{\sigma^3}\right]$

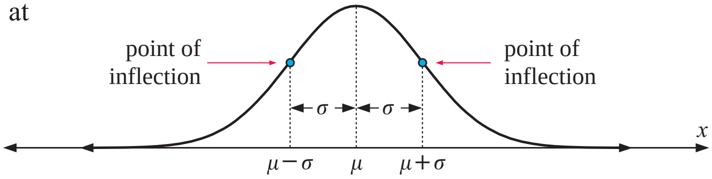
$\therefore \quad f''(x) = 0$ when $\dfrac{(x-\mu)^2}{\sigma^3} = \dfrac{1}{\sigma}$

$$\therefore \quad (x-\mu)^2 = \sigma^2$$

$$\therefore \quad x - \mu = \pm\sigma$$

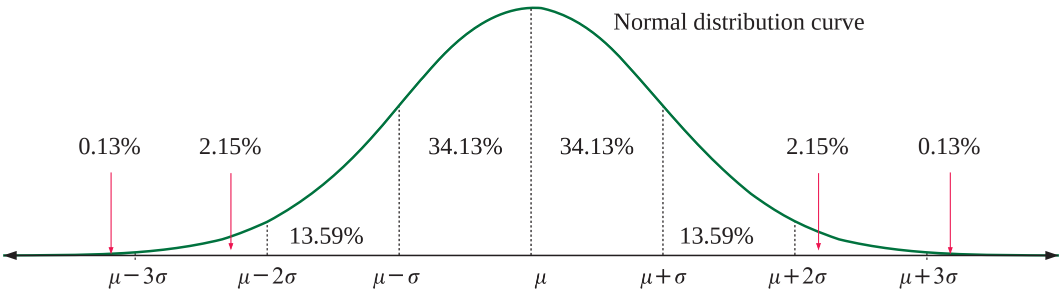
$$\therefore \quad x = \mu \pm \sigma$$

So, the points of inflection are at $x = \mu + \sigma$ and $x = \mu - \sigma$.



For a given normal curve, the standard deviation is uniquely determined as the horizontal distance from the vertical line $x = \mu$ to a point of inflection.

For a normal distribution with mean μ and standard deviation σ, the proportional breakdown of where the random variable could lie is given below.



Notice that:
- $\approx 68.26\%$ of values lie between $\mu - \sigma$ and $\mu + \sigma$
- $\approx 95.44\%$ of values lie between $\mu - 2\sigma$ and $\mu + 2\sigma$
- $\approx 99.74\%$ of values lie between $\mu - 3\sigma$ and $\mu + 3\sigma$.

INVESTIGATION 1 STANDARD DEVIATION SIGNIFICANCE

The purpose of this investigation is to check the proportions of normal distribution data which lie within σ, 2σ and 3σ of the mean.

What to do:

1 Click on the icon to start the demonstration in Microsoft® Excel.

2 Take a random sample of size $n = 1000$ from a normal distribution.

3 Find:
- **a** $\overline{x}$ and s
- **b** $\overline{x} - s$, $\overline{x} + s$
- **c** $\overline{x} - 2s$, $\overline{x} + 2s$
- **d** $\overline{x} - 3s$, $\overline{x} + 3s$

4 Count all values between:
- **a** $\overline{x} - s$ and $\overline{x} + s$
- **b** $\overline{x} - 2s$ and $\overline{x} + 2s$
- **c** $\overline{x} - 3s$ and $\overline{x} + 3s$

5 Determine the percentage of data values in these intervals. Do these confirm the theoretical percentages given above?

6 Repeat the procedure several times.

Example 2 **Self Tutor**

The chest measurements of 18 year old male footballers are normally distributed with a mean of 95 cm and a standard deviation of 8 cm.

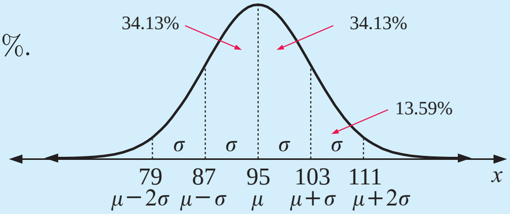
a Find the percentage of footballers with chest measurements between:
  i 87 cm and 103 cm
  ii 103 cm and 111 cm

b Find the probability that the chest measurement of a randomly chosen footballer is between 87 cm and 111 cm.

c Find the value of k such that 16% of chest measurements are below k cm.

a **i** We need the percentage between $\mu - \sigma$ and $\mu + \sigma$. This is $\approx 68.3\%$.

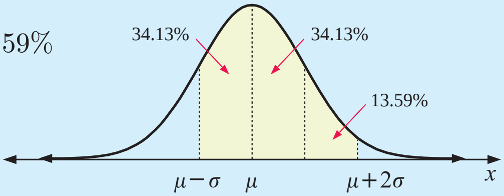
ii We need the percentage between $\mu + \sigma$ and $\mu + 2\sigma$. This is $\approx 13.6\%$.



b This is between $\mu - \sigma$ and $\mu + 2\sigma$.

The percentage is $2(34.13\%) + 13.59\%$
$\approx 81.9\%$.

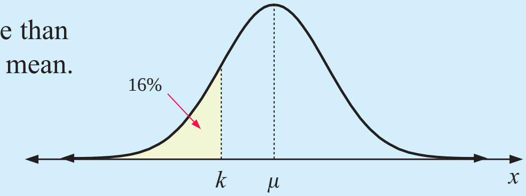
So, the probability is ≈ 0.819.



c Approximately 16% of data lies more than one standard deviation lower than the mean.

$\therefore$ k is σ below the mean μ

$\therefore$ $k = 95 - 8$
$= 87$ cm



EXERCISE 24B.1

1 Draw each of the following normal distributions accurately on one set of axes.

Distribution	*mean (mL)*	*standard deviation (mL)*
A	25	5
B	30	2
C	21	10

2 Explain why it is likely that the distributions of the following variables will be normal:

  a the volume of soft drink in cans

  b the diameter of bolts immediately after manufacture.

3 State the probability that a randomly selected, normally distributed value lies between:

  a σ below the mean and σ above the mean

  b the mean and the value 2σ above the mean.

4 When a specific type of radish is grown without fertiliser, the weights of the radishes produced are normally distributed with a mean of 40 g and a standard deviation of 10 g. When the same type of radish is grown in the same way except for the inclusion of fertiliser, the weights of the radishes produced are also normally distributed, but with a mean of 140 g and a standard deviation of 40 g. Determine the proportion of radishes grown:

- **a** without fertiliser with weights less than 50 grams
- **b** with fertiliser with weights less than 60 grams
- **c** **i** with and **ii** without fertiliser with weights between 20 and 60 g inclusive
- **d** **i** with and **ii** without fertiliser with weights greater than or equal to 60 g.

5 Given $X \sim \mathrm{N}(3, 0.1^2)$, find:

- **a** the probability that a randomly selected value lies within 2 standard deviations of the mean
- **b** the value of X which is 1.7 standard deviations below the mean.

6 The weights of Jason's oranges are normally distributed. 84% of the crop weigh more than 152 grams and 16% weigh more than 200 grams.

- **a** Find μ and σ for the crop.
- **b** What proportion of the oranges weigh between 152 grams and 224 grams?

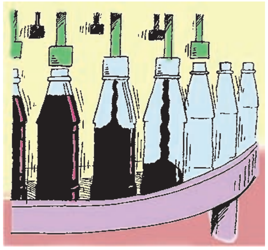
7 The height of male students is normally distributed with a mean of 170 cm and a standard deviation of 8 cm.

- **a** Find the percentage of male students whose height is:
  - **i** between 162 cm and 170 cm
  - **ii** between 170 cm and 186 cm.
- **b** Find the probability that a randomly chosen student from this group has a height:
  - **i** between 178 cm and 186 cm
  - **ii** less than 162 cm
  - **iii** less than 154 cm
  - **iv** greater than 162 cm.
- **c** Find the value of k such that 16% of the students are taller than k cm.

8 The heights of 13 year old boys are normally distributed. 97.72% of them are above 131 cm and 2.28% are above 179 cm.

- **a** Find μ and σ for the height distribution.
- **b** A 13 year old boy is randomly chosen. What is the probability that his height lies between 143 cm and 191 cm?

9 A bottle filling machine fills an average of 20 000 bottles a day with a standard deviation of 2000. Assuming that production is normally distributed and the year comprises 260 working days, calculate the approximate number of working days on which:



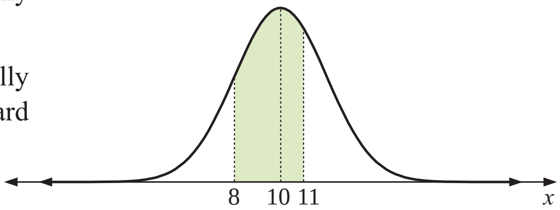
- **a** under 18 000 bottles are filled
- **b** over 16 000 bottles are filled
- **c** between 18 000 and 24 000 bottles (inclusive) are filled.

PROBABILITIES BY GRAPHICS CALCULATOR

We can use a graphics calculator to quickly find probabilities for a normal distribution.

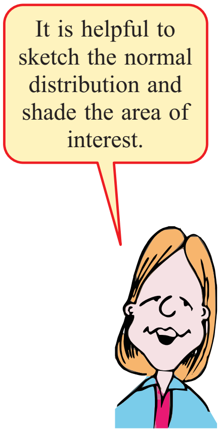
Suppose $X \sim \text{N}(10,\ 2^2)$, so X is normally distributed with mean 10 and standard deviation 2.

How do we find $\text{P}(8 \leqslant X \leqslant 11)$?
How do we find a if $\text{P}(X \leqslant a) = 0.479$?



Instructions for these tasks can be found in the graphics calculator chapter at the start of the book.

Use a calculator to answer the questions in the following exercise.



EXERCISE 24B.2

1 X is a random variable that is distributed normally with mean 70 and standard deviation 4. Find:

a $\text{P}(70 \leqslant X \leqslant 74)$ **b** $\text{P}(68 \leqslant X \leqslant 72)$ **c** $\text{P}(X \leqslant 65)$

2 X is a random variable that is distributed normally with mean 60 and standard deviation 5. Find:

a $\text{P}(60 \leqslant X \leqslant 65)$ **b** $\text{P}(62 \leqslant X \leqslant 67)$

c $\text{P}(X \geqslant 64)$ **d** $\text{P}(X \leqslant 68)$

e $\text{P}(X \leqslant 61)$ **f** $\text{P}(57.5 \leqslant X \leqslant 62.5)$

3 Given that $X \sim \text{N}(23,\ 5^2)$, find a such that:

a $\text{P}(X < a) = 0.378$ **b** $\text{P}(X \geqslant a) = 0.592$ **c** $\text{P}(23-a < X < 23+a) = 0.427$

C THE STANDARD NORMAL DISTRIBUTION (Z-DISTRIBUTION)

Every normal X-distribution can be **transformed** into the **standard normal distribution** or **Z-distribution** using the transformation $z = \dfrac{x-\mu}{\sigma}$.

In the following investigation we determine the mean and standard deviation of this Z-distribution.

INVESTIGATION 2 PROPERTIES OF $z = \dfrac{x-\mu}{\sigma}$

Suppose a random variable X is **normally distributed** with mean μ and standard deviation σ.

For each value of x we can calculate a **z-value** using the algebraic transformation $z = \dfrac{x-\mu}{\sigma}$.

What to do:

1 Consider the x-values: $1,\ 2,\ 2,\ 3,\ 3,\ 3,\ 3,\ 4,\ 4,\ 4,\ 4,\ 4,\ 5,\ 5,\ 5,\ 5,\ 6,\ 6,\ 7.$

- **a** Draw a graph of the distribution to check that it is approximately normal.
- **b** Find the mean μ and standard deviation σ for the distribution of x-values.
- **c** Use the transformation $z = \dfrac{x-\mu}{\sigma}$ to convert each x-value into a z-value.
- **d** Find the mean and standard deviation for the distribution of z-values.

2 Click on the icon to load a large sample drawn from a normal population. By clicking appropriately we can repeat the four steps of question **1** .

DEMO

3 Write a brief report of your findings.

You should have discovered that for a Z-distribution the mean is 0 and the standard deviation is 1. We can write $Z \sim \mathrm{N}(0,\ 1)$. This is true for *all* Z-distributions generated by transformation of a normal distribution, and this is why we call it the *standard* normal distribution.

For a normal X-distribution we know the probability density function is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

Substituting $x = z$, $\mu = 0$ and $\sigma = 1$, we find:

The **probability density function** for the Z-distribution is

$$\boldsymbol{f(z) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2}, \quad -\infty < z < \infty.}$$

Notice that the normal distribution function $f(x)$ has two parameters μ and σ, whereas the standard normal distribution function $f(z)$ has no parameters.

This means that a unique table of values can be constructed for $f(z)$, and we can use this table to compare normal distributions.

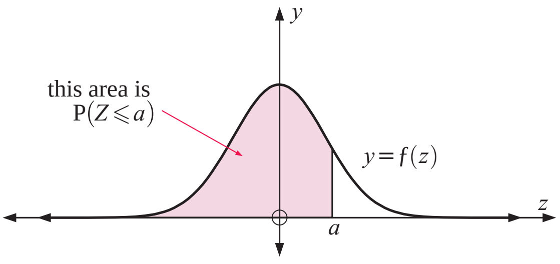
Before graphics calculators and computer packages, the standard normal distribution table was used exclusively for normal probability calculations such as those which follow.

The **cumulative distribution function** for the Z-distribution is $\boldsymbol{\Phi(a) = \mathrm{P}(Z \leqslant a)}$.

Since z is continuous,

$$\mathrm{P}(Z \leqslant a) = \mathrm{P}(Z < a) = \Phi(a)$$

and $$\mathrm{P}(Z \leqslant a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2}\, dz.$$

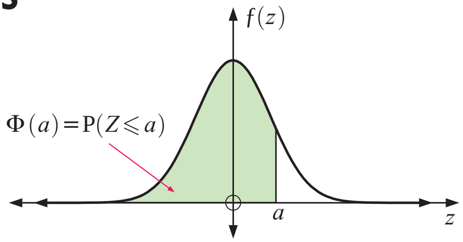


The table of curve areas on page **655** enables us to find $\mathrm{P}(Z \leqslant a)$ for $a \geqslant 0$.

For $a < 0$ we need to use the symmetry of the standard normal curve. In particular, we notice that

$$\begin{aligned} \mathrm{P}(Z \leqslant a) &= 1 - \mathrm{P}(Z \geqslant a) \\ &= 1 - \mathrm{P}(Z \leqslant -a) \quad \{\text{by symmetry}\} \end{aligned}$$

STANDARD NORMAL CURVE AREAS



	the second decimal digit of a									
a	**.00**	**.01**	**.02**	**.03**	**.04**	**.05**	**.06**	**.07**	**.08**	**.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998

USING A GRAPHICS CALCULATOR TO FIND PROBABILITIES

For a **TI-84 plus**:	To find	$\mathrm{P}(Z \leqslant a)$ or $\mathrm{P}(Z < a)$	use normalcdf($-$E99, a).
	To find	$\mathrm{P}(Z \geqslant a)$ or $\mathrm{P}(Z > a)$	use normalcdf(a, E99).
	To find	$\mathrm{P}(a \leqslant Z \leqslant b)$ or $\mathrm{P}(a < Z < b)$	use normalcdf(a, b).

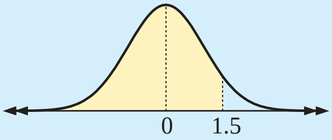
Consult the graphics calculator instructions at the start of the book for help using other calculator models.

Example 3 Self Tutor

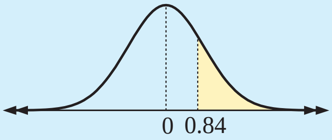
If Z is a standard normal variable, find without using technology:

a $P(Z \leqslant 1.5)$ **b** $P(Z > 0.84)$ **c** $P(-0.41 \leqslant Z \leqslant 0.67)$

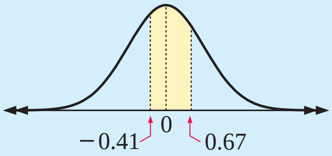
a $P(Z \leqslant 1.5)$
≈ 0.933



b $P(Z > 0.84)$
$= 1 - P(Z \leqslant 0.84)$
$\approx 1 - 0.7995$
≈ 0.200



c $P(-0.41 \leqslant Z \leqslant 0.67)$
$= P(Z \leqslant 0.67) - P(Z \leqslant -0.41)$
$= P(Z \leqslant 0.67) - (1 - P(Z \leqslant 0.41))$
$\approx 0.7486 - (1 - 0.6591)$
≈ 0.408



INTERPRETING Z-VALUES

If x is an observation from a normal distribution with mean μ and standard deviation σ, the Z-score of x is the number of standard deviations x is from the mean.

For example:

- If $z_1 = 1.84$ then z_1 is 1.84 standard deviations to the right of the mean.
- If $z_2 = -0.273$ then z_2 is 0.273 standard deviations to the left of the mean.

Z-values are therefore useful when comparing results from two or more different distributions.

Example 4 Self Tutor

Kelly scored 73% in History where the class mean was 68% and the standard deviation was 10.2%. In Mathematics she scored 66%, the class mean was 62%, and the standard deviation was 6.8%.

In which subject did Kelly perform better compared with the rest of her class?

Assume the scores for both subjects were normally distributed.

$$\text{Kelly's } Z\text{-score for History} = \frac{73-68}{10.2} \approx 0.490$$

$$\text{Kelly's } Z\text{-score for Mathematics} = \frac{66-62}{6.8} \approx 0.588$$

So, Kelly's result in Mathematics was 0.588 standard deviations above the mean, whereas her result in History was 0.490 standard deviations above the mean.

$\therefore$ Kelly's result in Mathematics was better, even though her score was lower.

EXERCISE 24C.1

1 If Z has standard normal distribution, find *using tables* and a sketch:

- **a** $P(Z \leqslant 1.2)$
- **b** $P(Z \geqslant 0.86)$
- **c** $P(Z \leqslant -0.52)$
- **d** $P(Z \geqslant -1.62)$
- **e** $P(-0.86 \leqslant Z \leqslant 0.32)$

2 If Z has standard normal distribution, find *using technology*:

- **a** $P(Z \geqslant 0.837)$
- **b** $P(Z \leqslant 0.0614)$
- **c** $P(Z \geqslant -0.876)$
- **d** $P(-0.3862 \leqslant Z \leqslant 0.2506)$
- **e** $P(-2.367 \leqslant Z \leqslant -0.6503)$

3 If Z has standard normal distribution, find:

- **a** $P(-0.5 < Z < 0.5)$
- **b** $P(-1.960 < Z < 1.960)$

4 Find a if Z has standard normal distribution and:

- **a** $P(Z \leqslant a) = 0.95$
- **b** $P(Z \geqslant a) = 0.90$

5 The table alongside shows Sergio's results in his mid-year examinations, along with the class means and standard deviations.

- **a** Find Sergio's Z-value for each subject.
- **b** Arrange Sergio's performances in each subject in order from 'best' to 'worst'.

	Sergio	μ	σ
Physics	83%	78%	10.8%
Chemistry	77%	72%	11.6%
Mathematics	84%	74%	10.1%
German	91%	86%	9.6%
Biology	72%	62%	12.2%

6 Pedro is studying Algebra and Geometry. He sits for the mid-year exams in each subject.

He is told that his Algebra mark is 56%, whereas the class mean and standard deviation are 50.2% and 15.8% respectively.

In Geometry he is told that the class mean and standard deviation are 58.7% and 18.7% respectively.

What percentage does Pedro need to have scored in Geometry to have an equivalent result to his Algebra mark?

STANDARDISING ANY NORMAL DISTRIBUTION

To find probabilities for a normally distributed random variable X:

Step 1: Convert x-values to z-values using $z = \dfrac{x-\mu}{\sigma}$.

Step 2: Sketch a standard normal curve and shade the required region.

Step 3: Use the standard normal tables or a graphics calculator to find the probability.

Example 5 **Self Tutor**

Given that X is a normal variable with mean 62 and standard deviation 7, find without using technology:

a $P(X \leqslant 69)$ **b** $P(58.5 \leqslant X \leqslant 71.8)$

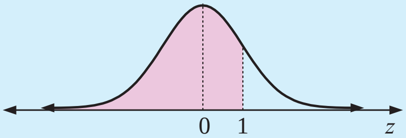
Interpret each result.

a $P(X \leqslant 69)$

$= P\left(\frac{X-62}{7} \leqslant \frac{69-62}{7}\right)$

$= P(Z \leqslant 1)$

≈ 0.841



There is an 84.1% chance that a randomly selected X-value is 69 or less.

b $P(58.5 \leqslant X \leqslant 71.8)$

$= P\left(\frac{58.5-62}{7} \leqslant \frac{X-62}{7} \leqslant \frac{71.8-62}{7}\right)$

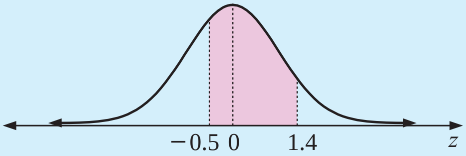
$= P(-0.5 \leqslant Z \leqslant 1.4)$

$= P(Z \leqslant 1.4) - P(Z \leqslant -0.5)$

$= P(Z \leqslant 1.4) - (1 - P(Z \leqslant 0.5))$

$= 0.9192 - (1 - 0.6915)$

≈ 0.611



There is a 61.1% chance that a randomly selected X-value is between 58.5 and 71.8 inclusive.

These probabilities can also be found using a graphics calculator without actually converting to standard normal Z-scores. Instructions for this can be found in the graphics calculator chapter at the front of the book.

EXERCISE 24C.2

1 A random variable X is normally distributed with mean 70 and standard deviation 4. By converting to the standard variable Z and then using the tabled probability values for Z, find:

a $P(X \geqslant 74)$ **b** $P(X \leqslant 68)$ **c** $P(60.6 \leqslant X \leqslant 68.4)$

2 A random variable X is normally distributed with mean 58.3 and standard deviation 8.96. By converting to the standard variable Z and then using your graphics calculator, find:

a $P(X \geqslant 61.8)$ **b** $P(X \leqslant 54.2)$ **c** $P(50.67 \leqslant X \leqslant 68.92)$

3 The length L of a nail is normally distributed with mean 50.2 mm and standard deviation 0.93 mm. Find:

a $P(L \geqslant 50)$ **b** $P(L \leqslant 51)$ **c** $P(49 \leqslant L \leqslant 50.5)$

D QUANTILES OR k-VALUES

Consider a population of crabs where the length of a shell, X mm, is normally distributed with mean 70 mm and standard deviation 10 mm.

A biologist wants to protect the population by allowing only the largest 5% of crabs to be harvested. He therefore asks the question: "95% of the crabs have lengths less than what?".

To answer this question we need to find the value of k such that $P(X \leqslant k) = 0.95$.

The number k is known as a **quantile**, and in this case is the 95% quantile.

When given a probability we need to find the corresponding measurement. This is the *inverse* of finding probabilities, and to do this we can either:

- read the value from an **inverse standard normal table**.
  The table on page **660** includes probabilities which are $\geqslant 0.5$ only.
  For other probabilities we again rely on the symmetry of the normal distribution. For the probability p where $p < 0.5$, we look up the Z-score corresponding to $1 - p$, then take its negative.
- use the **inverse normal function** on our graphics calculator.
  For instructions on using your calculator, see the graphics calculator chapter at the start of the book.

Example 6 **Self Tutor**

Find k for which $P(X \leqslant k) = 0.95$ given that $X \sim N(70, 10^2)$ and X is measured in mm.

$$P(X \leqslant k) = 0.95$$

$$\therefore \quad P\left(\frac{X-70}{10} \leqslant \frac{k-70}{10}\right) = 0.95$$

$$\therefore \quad P\left(Z \leqslant \frac{k-70}{10}\right) = 0.95$$

Using the inverse normal table we find:

$$\frac{k-70}{10} \approx 1.6449$$

$$\therefore \quad k \approx 86.4$$

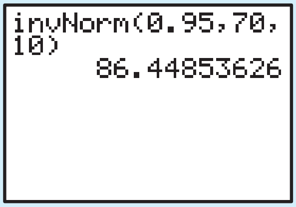
So, approximately 95% of the values are expected to be 86.4 mm or less.

or

Using technology:

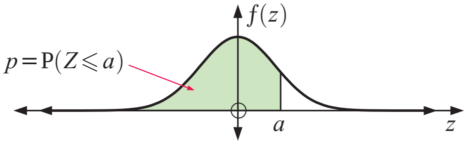
If $P(X \leqslant k) = 0.95$

then $k \approx 86.5$.



```
invNorm(0.95,70,
10)
        86.44853626
```

INVERSE NORMAL PROBABILITIES



p	0	.001	.002	.003	.004	.005	.006	.007	.008	.009
0.50	0.0000	0.0025	0.0050	0.0075	0.0100	0.0125	0.0150	0.0175	0.0201	0.0226
0.51	0.0251	0.0276	0.0301	0.0326	0.0351	0.0376	0.0401	0.0426	0.0451	0.0476
0.52	0.0502	0.0527	0.0552	0.0577	0.0602	0.0627	0.0652	0.0677	0.0702	0.0728
0.53	0.0753	0.0778	0.0803	0.0828	0.0853	0.0878	0.0904	0.0929	0.0954	0.0979
0.54	0.1004	0.1030	0.1055	0.1080	0.1105	0.1130	0.1156	0.1181	0.1206	0.1231
0.55	0.1257	0.1282	0.1307	0.1332	0.1358	0.1383	0.1408	0.1434	0.1459	0.1484
0.56	0.1510	0.1535	0.1560	0.1586	0.1611	0.1637	0.1662	0.1687	0.1713	0.1738
0.57	0.1764	0.1789	0.1815	0.1840	0.1866	0.1891	0.1917	0.1942	0.1968	0.1993
0.58	0.2019	0.2045	0.2070	0.2096	0.2121	0.2147	0.2173	0.2198	0.2224	0.2250
0.59	0.2275	0.2301	0.2327	0.2353	0.2378	0.2404	0.2430	0.2456	0.2482	0.2508
0.60	0.2533	0.2559	0.2585	0.2611	0.2637	0.2663	0.2689	0.2715	0.2741	0.2767
0.61	0.2793	0.2819	0.2845	0.2871	0.2898	0.2924	0.2950	0.2976	0.3002	0.3029
0.62	0.3055	0.3081	0.3107	0.3134	0.3160	0.3186	0.3213	0.3239	0.3266	0.3292
0.63	0.3319	0.3345	0.3372	0.3398	0.3425	0.3451	0.3478	0.3505	0.3531	0.3558
0.64	0.3585	0.3611	0.3638	0.3665	0.3692	0.3719	0.3745	0.3772	0.3799	0.3826
0.65	0.3853	0.3880	0.3907	0.3934	0.3961	0.3989	0.4016	0.4043	0.4070	0.4097
0.66	0.4125	0.4152	0.4179	0.4207	0.4234	0.4261	0.4289	0.4316	0.4344	0.4372
0.67	0.4399	0.4427	0.4454	0.4482	0.4510	0.4538	0.4565	0.4593	0.4621	0.4649
0.68	0.4677	0.4705	0.4733	0.4761	0.4789	0.4817	0.4845	0.4874	0.4902	0.4930
0.69	0.4959	0.4987	0.5015	0.5044	0.5072	0.5101	0.5129	0.5158	0.5187	0.5215
0.70	0.5244	0.5273	0.5302	0.5330	0.5359	0.5388	0.5417	0.5446	0.5476	0.5505
0.71	0.5534	0.5563	0.5592	0.5622	0.5651	0.5681	0.5710	0.5740	0.5769	0.5799
0.72	0.5828	0.5858	0.5888	0.5918	0.5948	0.5978	0.6008	0.6038	0.6068	0.6098
0.73	0.6128	0.6158	0.6189	0.6219	0.6250	0.6280	0.6311	0.6341	0.6372	0.6403
0.74	0.6433	0.6464	0.6495	0.6526	0.6557	0.6588	0.6620	0.6651	0.6682	0.6713
0.75	0.6745	0.6776	0.6808	0.6840	0.6871	0.6903	0.6935	0.6967	0.6999	0.7031
0.76	0.7063	0.7095	0.7128	0.7160	0.7192	0.7225	0.7257	0.7290	0.7323	0.7356
0.77	0.7388	0.7421	0.7454	0.7488	0.7521	0.7554	0.7588	0.7621	0.7655	0.7688
0.78	0.7722	0.7756	0.7790	0.7824	0.7858	0.7892	0.7926	0.7961	0.7995	0.8030
0.79	0.8064	0.8099	0.8134	0.8169	0.8204	0.8239	0.8274	0.8310	0.8345	0.8381
0.80	0.8416	0.8452	0.8488	0.8524	0.8560	0.8596	0.8633	0.8669	0.8706	0.8742
0.81	0.8779	0.8816	0.8853	0.8890	0.8927	0.8965	0.9002	0.9040	0.9078	0.9116
0.82	0.9154	0.9192	0.9230	0.9269	0.9307	0.9346	0.9385	0.9424	0.9463	0.9502
0.83	0.9542	0.9581	0.9621	0.9661	0.9701	0.9741	0.9782	0.9822	0.9863	0.9904
0.84	0.9945	0.9986	1.0027	1.0069	1.0110	1.0152	1.0194	1.0237	1.0279	1.0322
0.85	1.0364	1.0407	1.0451	1.0494	1.0537	1.0581	1.0625	1.0669	1.0714	1.0758
0.86	1.0803	1.0848	1.0893	1.0939	1.0985	1.1031	1.1077	1.1123	1.1170	1.1217
0.87	1.1264	1.1311	1.1359	1.1407	1.1455	1.1503	1.1552	1.1601	1.1650	1.1700
0.88	1.1750	1.1800	1.1850	1.1901	1.1952	1.2004	1.2055	1.2107	1.2160	1.2212
0.89	1.2265	1.2319	1.2372	1.2426	1.2481	1.2536	1.2591	1.2646	1.2702	1.2759
0.90	1.2816	1.2873	1.2930	1.2988	1.3047	1.3106	1.3165	1.3225	1.3285	1.3346
0.91	1.3408	1.3469	1.3532	1.3595	1.3658	1.3722	1.3787	1.3852	1.3917	1.3984
0.92	1.4051	1.4118	1.4187	1.4255	1.4325	1.4395	1.4466	1.4538	1.4611	1.4684
0.93	1.4758	1.4833	1.4909	1.4985	1.5063	1.5141	1.5220	1.5301	1.5382	1.5464
0.94	1.5548	1.5632	1.5718	1.5805	1.5893	1.5982	1.6072	1.6164	1.6258	1.6352
0.95	1.6449	1.6546	1.6646	1.6747	1.6849	1.6954	1.7060	1.7169	1.7279	1.7392
0.96	1.7507	1.7624	1.7744	1.7866	1.7991	1.8119	1.8250	1.8384	1.8522	1.8663
0.97	1.8808	1.8957	1.9110	1.9268	1.9431	1.9600	1.9774	1.9954	2.0141	2.0335
0.98	2.0537	2.0749	2.0969	2.1201	2.1444	2.1701	2.1973	2.2262	2.2571	2.2904
0.99	2.3263	2.3656	2.4089	2.4573	2.5121	2.5758	2.6521	2.7478	2.8782	3.0902

EXERCISE 24D

1 Z has a standard normal distribution. Illustrate, then find k using tabled values given:

a $\text{P}(Z \leqslant k) = 0.81$ **b** $\text{P}(Z \leqslant k) = 0.58$ **c** $\text{P}(Z \leqslant k) = 0.17$

2 Z has a standard normal distribution. Illustrate, then find k using technology given:

a $\text{P}(Z \leqslant k) = 0.384$ **b** $\text{P}(Z \leqslant k) = 0.878$ **c** $\text{P}(Z \leqslant k) = 0.1384$

3 **a** Show that $\text{P}(-k \leqslant Z \leqslant k) = 2\,\text{P}(Z \leqslant k) - 1$.

b Suppose Z has a standard normal distribution. Find k if:

i $\text{P}(-k \leqslant Z \leqslant k) = 0.238$ **ii** $\text{P}(-k \leqslant Z \leqslant k) = 0.7004$

4 **a** Find k if $\text{P}(X \leqslant k) = 0.9$ and $X \sim \text{N}(56, 18^2)$.

b Find k if $\text{P}(X \geqslant k) = 0.8$ and $X \sim \text{N}(38.7, 8.8^2)$.

E APPLICATIONS OF THE NORMAL DISTRIBUTION

Example 7 **Self Tutor**

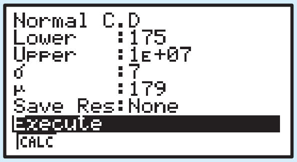
In 1972 the heights of rugby players were found to be normally distributed with mean 179 cm and standard deviation 7 cm. Find the probability that a randomly selected player in 1972 was:

a at least 175 cm tall **b** between 170 cm and 190 cm.

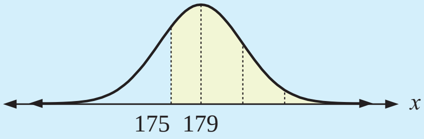
If X is the height of a player then X is normally distributed with $\mu = 179$, $\sigma = 7$.

a $\text{P}(X \geqslant 175)$

≈ 0.716 {using technology}

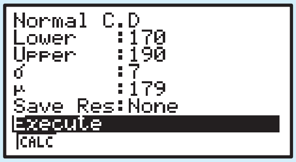


```
Normal C.D
Lower    :175
Upper    :1E+07
σ        :7
μ        :179
Save Res:None
Execute
CALC
```

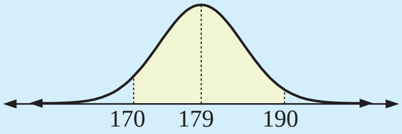


b $\text{P}(170 < X < 190)$

≈ 0.843 {using technology}



```
Normal C.D
Lower    :170
Upper    :190
σ        :7
μ        :179
Save Res:None
Execute
CALC
```



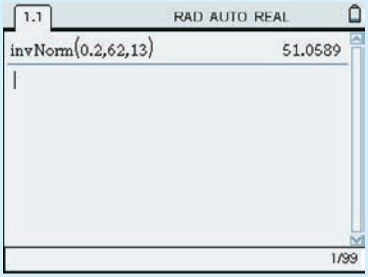
Example 8 **Self Tutor**

A university professor determines that 80% of this year's History candidates should pass the final examination. The examination results are expected to be normally distributed with mean 62 and standard deviation 13. Find the expected lowest score necessary to pass the examination.

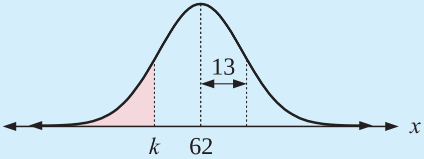
Let the random variable X denote the final examination result, so $X \sim \text{N}(62,\, 13^2)$.

We need to find k such that

$$\text{P}(X \geqslant k) = 0.8$$
$$\therefore \quad \text{P}(X \leqslant k) = 0.2$$
$$\therefore \quad k \approx 51.1$$



```
invNorm(0.2,62,13)                51.0589
```



So, the professor would decide between 51 and 52 for the minimum pass mark.

In the following example we **must** convert to Z-scores to answer the question.

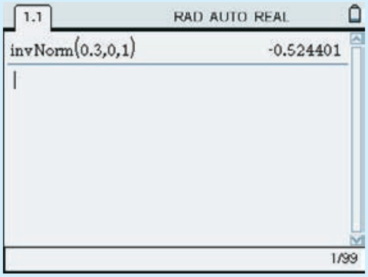
We always need to convert to Z-scores if we are trying to find an unknown mean μ or standard deviation σ.

Example 9 **Self Tutor**

Find the mean and standard deviation of a normally distributed random variable X if $\text{P}(X \leqslant 20) = 0.3$ and $\text{P}(X \geqslant 50) = 0.2$.

Let the unknown mean and standard deviation be μ and σ respectively.

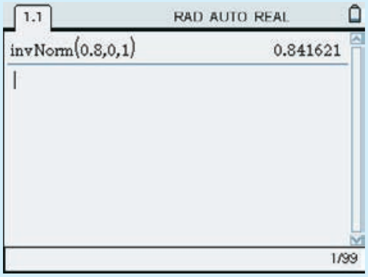
$$\text{Now} \quad \text{P}(X \leqslant 20) = 0.3$$
$$\text{and so} \quad \text{P}\left(\frac{X-\mu}{\sigma} \leqslant \frac{20-\mu}{\sigma}\right) = 0.3$$
$$\therefore \quad \text{P}\left(Z \leqslant \frac{20-\mu}{\sigma}\right) = 0.3$$



```
invNorm(0.3,0,1)                -0.524401
```

$$\therefore \quad \frac{20-\mu}{\sigma} \approx -0.5244$$
$$\therefore \quad 20-\mu \approx -0.5244\sigma \quad ..... \; (1)$$

$$\text{and} \quad \text{P}(X \geqslant 50) = 0.2$$
$$\therefore \quad \text{P}(X \leqslant 50) = 0.8$$
$$\text{and so} \quad \text{P}\left(\frac{X-\mu}{\sigma} \leqslant \frac{50-\mu}{\sigma}\right) = 0.8$$
$$\therefore \quad \text{P}\left(Z \leqslant \frac{50-\mu}{\sigma}\right) = 0.8$$



```
invNorm(0.8,0,1)                0.841621
```

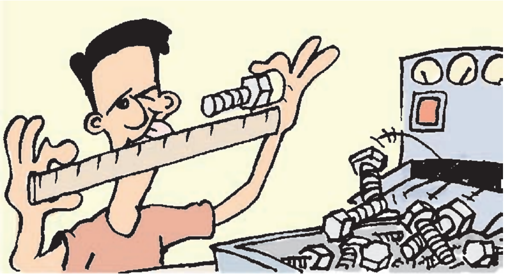
$$\therefore \quad \frac{50-\mu}{\sigma} \approx 0.8416$$
$$\therefore \quad 50-\mu \approx 0.8416\sigma \quad ..... \; (2)$$

Solving (1) and (2) simultaneously we get $\mu \approx 31.5$, $\sigma \approx 22.0$.

EXERCISE 24E

1 A machine produces metal bolts. The lengths of these bolts have a normal distribution with mean 19.8 cm and standard deviation 0.3 cm.

If a bolt is selected at random from the machine, find the probability that it will have a length between 19.7 cm and 20 cm.



2 Max's customers put money for charity into a collection box on the front counter of his shop. The average weekly collection is approximately normally distributed with mean \$40 and standard deviation \$6. What proportion of weeks would he expect to collect:

a between \$30.00 and \$50.00

b at least \$50.00?

3 The students of Class X sat a Physics test. The average score was 46 with a standard deviation of 25. The teacher decided to award an A to the top 7% of the students in the class. Assuming that the scores were normally distributed, find the lowest score that a student needed to obtain in order to achieve an A.

4 Eels are washed onto a beach after a storm. Their lengths have a normal distribution with mean 41 cm and variance 11 cm^2.

a If an eel is randomly selected, find the probability that it is at least 50 cm long.

b Find the proportion of eels measuring between 40 cm and 50 cm long.

c How many eels from a sample of 200 would you expect to measure at least 45 cm in length?

5 Find the mean and standard deviation of a normally distributed random variable X if $\mathrm{P}(X \geqslant 35) = 0.32$ and $\mathrm{P}(X \leqslant 8) = 0.26$.

6 **a** A random variable X is normally distributed. Find the mean and the standard deviation of X, given that $\mathrm{P}(X \geqslant 80) = 0.1$ and $\mathrm{P}(X \leqslant 30) = 0.15$.

b In the Mathematics examination at the end of the year, it was found that 10% of the students scored at least 80 marks, and no more than 15% scored less than 30 marks. Assuming the marks are normally distributed, what proportion of students scored more than 50 marks?

7 The IQ of students at a school is normally distributed with standard deviation 15. If 20% of the students have an IQ larger than 125, find the mean IQ of students at the school.

8 The distance an athlete can jump is normally distributed with mean 5.2 m. If 15% of the jumps by this athlete are less than 5 m, what is the standard deviation?

9 Circular metal tokens are used to operate a washing machine in a laundromat. The diameters of the tokens are normally distributed, and only tokens with diameters between 1.94 and 2.06 cm will operate the machine.

- **a** Find the mean and standard deviation of the distribution given that 2% of the tokens are too small, and 3% are too large.
- **b** Find the probability that at most one token out of a randomly selected sample of 20 will not operate the machine.

REVIEW SET 24A NON-CALCULATOR

1 The contents of a certain brand of soft drink can are normally distributed with mean 377 mL and standard deviation 4.2 mL.

- **a** Find the percentage of cans with contents:
  - **i** less than 368.6 mL
  - **ii** between 372.8 mL and 389.6 mL.
- **b** Find the probability that a randomly selected can has contents between 377 mL and 381.2 mL.

2 The edible part of a batch of Coffin Bay oysters is normally distributed with mean 38.6 grams and standard deviation 6.3 grams. If the random variable X is the mass of a Coffin Bay oyster:

- **a** find a if $\mathrm{P}(38.6-a \leqslant X \leqslant 38.6+a) = 0.6826$
- **b** find b if $\mathrm{P}(X \geqslant b) = 0.8413$.

3 $f(x) = \begin{cases} ax(x-3), & 0 \leqslant x \leqslant 2 \\ 0 & \text{elsewhere} \end{cases}$ is a continuous probability distribution function.

- **a** Find a.
- **b** Sketch the graph of $y = f(x)$.
- **c** Find the mean of the distribution.
- **d** Find $\mathrm{P}(1 \leqslant X \leqslant 2)$.

4 The continuous random variable Z is distributed such that $Z \sim \mathrm{N}(0,\,1)$.
Find the value of k if $\mathrm{P}(-k \leqslant Z \leqslant k) = 0.4$.

5 The distance that a 15 year old boy can throw a tennis ball is normally distributed with mean 35 m and standard deviation 4 m.
The distance that a 10 year old boy can throw a tennis ball is normally distributed with mean 25 m and standard deviation 3 m.
Jarrod is 15 years old and can throw a tennis ball 41 m. How far does his 10 year old brother Paul need to throw a tennis ball to perform as well as Jarrod?

6 State the probability that a randomly selected, normally distributed value lies between:

- **a** σ above the mean and 2σ above the mean
- **b** the mean and σ above the mean.

REVIEW SET 24B CALCULATOR

1 The arm lengths of 18 year old females are normally distributed with mean 64 cm and standard deviation 4 cm.

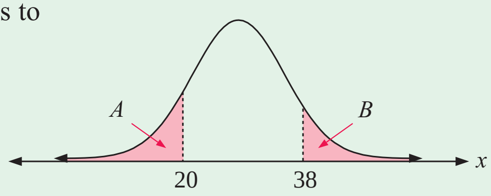
- **a** Find the percentage of 18 year old females whose arm lengths are:
  - **i** between 60 cm and 72 cm
  - **ii** greater than 60 cm.
- **b** Find the probability that a randomly chosen 18 year old female has an arm length in the range 56 cm to 64 cm.
- **c** The arm lengths of 70% of the 18 year old females are more than x cm. Find the value of x.

2 The length of steel rods produced by a machine is normally distributed with a standard deviation of 3 mm. It is found that 2% of all rods are less than 25 mm long. Find the mean length of rods produced by the machine.

3 The distribution curve shown corresponds to $X \sim \text{N}(\mu, \sigma^2)$.

Area A = Area $B = 0.2$.

- **a** Find μ and σ.
- **b** Calculate:
  - **i** $\text{P}(X \leqslant 35)$
  - **ii** $\text{P}(23 \leqslant X \leqslant 30)$



4 The marks of 2376 candidates in an IB examination are normally distributed with a mean of 49 marks and variance 225.

- **a** If the pass mark is 45, estimate the number of candidates who passed the examination.
- **b** If 7% of the candidates scored a '7', find the minimum mark required to obtain a '7'.

5 The life of a Xenon-brand battery is known to be normally distributed with a mean of 33.2 weeks and a standard deviation of 2.8 weeks.

- **a** Find the probability that a randomly selected battery will last at least 35 weeks.
- **b** For how many weeks can the manufacturer expect the batteries to last before 8% of them fail?

6 The random variable X is normally distributed with $\text{P}(X \leqslant 30) = 0.0832$ and $\text{P}(X \geqslant 90) = 0.101$.

- **a** Find the mean μ and standard deviation σ for X, correct to 3 decimal places.
- **b** Hence find $\text{P}(-7 \leqslant X - \mu \leqslant 7)$.

REVIEW SET 24C

1 A random variable X is normally distributed with mean 20.5 and standard deviation 4.3. Find:

a $\text{P}(X \geqslant 22)$ **b** $\text{P}(18 \leqslant X \leqslant 22)$ **c** k such that $\text{P}(X \leqslant k) = 0.3$.

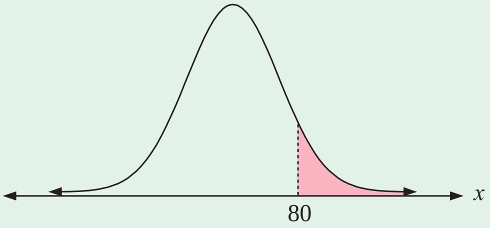
2 A factory has a machine designed to fill bottles of drink with a volume of 375 mL. It is found that the average amount of drink in each bottle is 376 mL, and that 2.3% of the drink bottles have a volume smaller than 375 mL. Assuming that the amount of drink in each bottle is distributed normally, find the standard deviation.

3 X is a continuous random variable where $X \sim \text{N}(\mu, 2^2)$.
Find $\text{P}(-0.524 < X - \mu < 0.524)$.

4 The lengths of metal rods produced in a manufacturing process are distributed normally with mean μ cm and standard deviation 6 cm. It is known that 5.63% of the rods have length greater than 89.52 cm. Find the mean length of the metal rods.

5 The random variable X is distributed normally with mean 50 and $\text{P}(X < 90) \approx 0.975$.

Find the shaded area in the given diagram which illustrates the probability density function for X.



6 The heights of 18 year old boys are normally distributed with a mean of 187 cm. Fifteen percent of all these boys have heights greater than 193 cm. Find the probability that two 18 year old boys chosen at random will have heights greater than 185 cm.

Chapter 25

Miscellaneous questions

Contents:

A NON-CALCULATOR QUESTIONS

EXERCISE 25A

1 A geometric sequence has $S_1 = 2$ and $S_2 = 8$. Find:

a the common ratio r **b** the twentieth term u_{20}.

2 Show that the sum of the first forty terms of the series
$\ln 2 + \ln 4 + \ln 8 + \ln 16 + ...$ is $820 \ln 2$.

3 If $f(x) = be^x$ and $g(x) = \ln(bx)$, find:

a $(f \circ g)(x)$ **b** $(g \circ f)(x)$

c x^*, in terms of b, such that $(f \circ g)(x^*) = (g \circ f)(x^*)$.

4 Let $f(x) = -2(x - b)^2 + 2$, a quadratic in x.

a State the coordinates of the vertex.

b Find the axes intercepts.

c The graph of function g is obtained by translating the graph of f vertically through b units. State the possible values of b such that the graph of g:

i has exactly one x-intercept **ii** has no x-intercepts

iii passes through the origin.

5 **a** Expand $(x - 2)^3$.

b Hence, find the coefficient of x^3 in $(3x^2 - 7)(x - 2)^3$.

6 Consider $f(x) = \sqrt{1 - 2x}$. Find:

a $f(0)$ **b** $f(-4)$ **c** the domain of $y = f(x)$ **d** the range of $y = f(x)$.

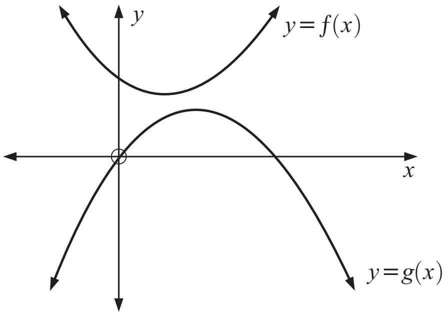
7 Let $a = \sin 20^o$ and $b = \tan 50^o$. In terms of a and b, write down expressions for:

a $\sin 160^o$ **b** $\tan(-50^o)$ **c** $\cos 70^o$ **d** $\tan 20^o$

8 Given $f(x) = \cos x$ and $g(x) = 2x$, solve:

a $(f \circ g)(x) = 1$ for $0 \leqslant x \leqslant 2\pi$ **b** $(g \circ f)(x) = 1$ for $0 \leqslant x \leqslant 2\pi$

9 Consider the functions
$f(x) = ax^2 + bx + c$ and
$g(x) = dx^2 + ex + h$ with graphs illustrated.
Copy and complete the following table by indicating whether the constant is positive, negative or zero:



Constant	a	b	c	d	e	h	Δ of $f(x)$	Δ of $g(x)$
Sign								

10 **a** If $\mathbf{A} = \begin{pmatrix} x & 1 \\ 0 & 3 \end{pmatrix}$ and $\mathbf{A}^2 = \begin{pmatrix} 25 & -2 \\ 0 & 9 \end{pmatrix}$, find x.

b If $\begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & 2 \\ -a & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, find a.

11 Suppose $(3\mathbf{A} + \mathbf{I})^{-1} = \mathbf{B}$ where $\mathbf{I}$ is the 2×2 identity matrix.

a Find an expression for $\mathbf{A}$ in terms of $\mathbf{B}$.

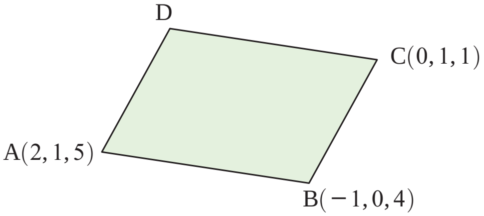
b Find $\mathbf{A}$ if $\mathbf{B} = \begin{pmatrix} -1 & -4 \\ 1 & 5 \end{pmatrix}$. **c** Find $\det \mathbf{A}$.

12 Consider $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + p\mathbf{j} - \mathbf{k}$.

a Find: **i** p if $\mathbf{u}$ is perpendicular to $\mathbf{v}$ **ii** $|\mathbf{u}|\,|\mathbf{v}|$

b Show that no value of p exists such that $\mathbf{v} - \mathbf{u}$ and $\mathbf{u}$ are parallel.

13 Consider parallelogram ABCD:



a Find $\overrightarrow{\text{BA}}$ and $\overrightarrow{\text{BC}}$.

b Hence find $|\overrightarrow{\text{BA}}|$ and $|\overrightarrow{\text{BC}}|$.

c What can be deduced about parallelogram ABCD?

d Calculate $\cos(\text{C}\widehat{\text{B}}\text{A})$.

e Find, in surd form, $\sin(\text{C}\widehat{\text{B}}\text{A})$.

f Hence find, in surd form, the area of ABCD.

14 a, b, c, d, e, f, g, h, i, j, k, l and m are 13 data values which have been arranged in *ascending* order.

a Which variable represents the median?

b Write down an algebraic expression for:

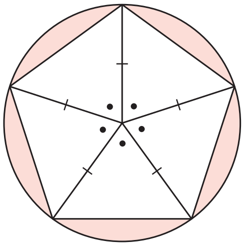
i the range **ii** the interquartile range.

15 The mean and standard deviation for the data set $\{a, b, c\}$ are 17.5 and 3.2 respectively. Copy and complete the table below by finding the mean and standard deviation of each new data set:

	New data set	*Mean*	*Standard deviation*
a	$\{2a, 2b, 2c\}$		
b	$\{a+2, b+2, c+2\}$		
c	$\{3a+5, 3b+5, 3c+5\}$		

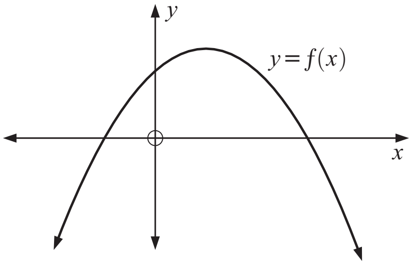
16 A dart is thrown at the dartboard shown. It is equally likely to land anywhere on the board. Given that the dart lands on the board, show that the probability of it landing on the shaded region is exactly

$$1 - \frac{5}{2\pi} \sin\left(\frac{2\pi}{5}\right).$$



17 For the function $y = f(x)$ with graph shown, sketch the graphs of:

a $y = f'(x)$ **b** $y = f''(x)$



18 Consider $g(x) = 3 - 2\cos(2x)$.

a Find $g'(x)$.

b Sketch $y = g'(x)$ for $-\pi \leqslant x \leqslant \pi$.

c Write down the number of solutions to $g'(x) = 0$ for $-\pi \leqslant x \leqslant \pi$.

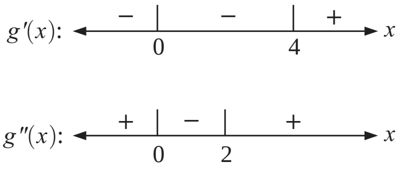
d Mark a point M on the sketch in **b** where $g'(x) = 0$ *and* $g''(x) > 0$.

19 A and B are mutually exclusive events where $\text{P}(A) = x$ and $\text{P}(B') = 0.43$.

a Write, in terms of x, an expression for $\text{P}(A \cup B)$.

b Find x given that $\text{P}(A \cup B) = 0.73$.

20 For the function $g(x)$, the sign diagrams for $g'(x)$ and $g''(x)$ are shown alongside. The points A(0, 2), B(2, 0) and C(4, -2) all lie on the graph of $y = g(x)$. Sketch the graph of $y = g(x)$, labelling all of the stationary points.



21 Consider $f(x) = xe^{1-2x}$.

a Show that $f'(x) = e^{1-2x}(1 - 2x)$.

b Find the point on the graph of $y = f(x)$ where the tangent is horizontal.

c Find values of x for which:

i $f(x) > 0$ **ii** $f'(x) > 0$

22 A particle moves in a straight line so that its position s at time t seconds is given by $s(t) = 3 - 4e^{2t} + kt$ metres, where k is a constant.

a Find the velocity function $v(t)$.

b If the particle is stationary when $t = \ln 3$, determine k.

23 Solve for x:

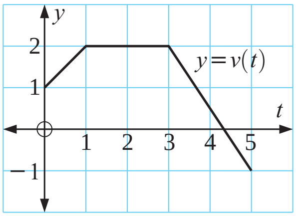
a $\log_3 27 = x$ **b** $\log_x 7 = 2$ **c** $e^{5-2x} = 8$

d $\ln(x^2 - 3) - \ln(2x) = 0$

24 The graph shows the velocity v m s^{-1} of an object at time t seconds, $t \geqslant 0$. Find and interpret:

a $v(0)$ **b** $v'(2)$

c $\displaystyle\int_1^3 v(t)\,dt$.



25 Suppose $\displaystyle\int_{-1}^{2} f(x)\,dx = 10$. Find the value of:

a $\displaystyle\int_{-1}^{2} (f(x) - 6)\,dx$ **b** k if $\displaystyle\int_{2}^{-1} kf(x)\,dx = -5$.

26 **a** Show that $(\sin\theta - \cos\theta)^2 = 1 - \sin 2\theta$.

b Hence calculate $\displaystyle\int_{0}^{\frac{\pi}{4}} (\sin\theta - \cos\theta)^2\,d\theta$.

27 The following table shows the probability distribution for a discrete random variable X.

x	-2	-1	0	1	2
$\mathrm{P}(X = x)$	0.1	a	0.25	b	0.15

Find a and b given that the expected value of X is 0.

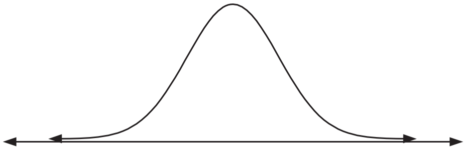
28 Consider the infinite geometric sequence: $e, \frac{1}{e}, \frac{1}{e^3}, \frac{1}{e^5}, ...$ Find, in terms of e:

a the common ratio **b** the 101st term **c** the sequence's sum.

29 Suppose $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ -1 & x & 4 \\ 5 & -6 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 8 \\ 4 \\ 14 \end{pmatrix}$.

Find x given that: **a** $\det \mathbf{A} = 12$ **b** $\mathbf{AB} = \mathbf{C}$

30 The random variable X is normally distributed with mean μ and standard deviation σ. Let k be such that $\mathrm{P}(X < k) = 0.7$.



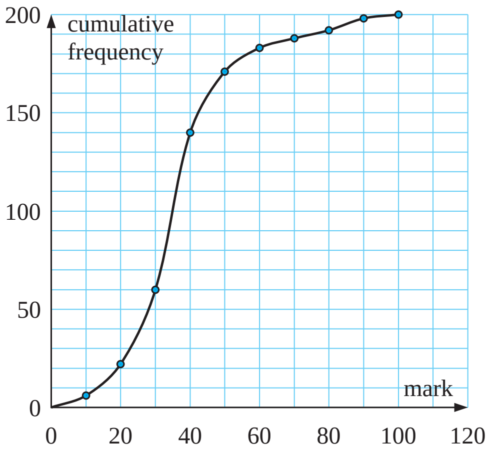
a Copy the normal distribution curve and mark on it μ and k.

b On the graph shade the region which illustrates $\mathrm{P}(X < k) = 0.7$.

c Find: **i** $\mathrm{P}(X > k)$ **ii** $\mathrm{P}(\mu < X < k)$ **iii** $\mathrm{P}(\mu - \sigma < X < k)$

d If $\mathrm{P}(X \geqslant t) = 0.2$, find $\mathrm{P}(k \leqslant X \leqslant t)$.

31



The examination marks for 200 students are displayed on the cumulative frequency graph shown. The pass mark for the examination was 30.

a What percentage of the students passed?

b A box-and-whisker-plot for the examination data is:

From the graph, estimate:

i m **ii** n **iii** p **iv** q

32 The line L has equation $y = (\tan 60^o)x$.

- **a** Find p given that the point $\text{A}(10,\ p)$ lies on L.
- **b** Find the equation of the line which passes through A and is perpendicular to L. Write your answer in the form $ax + by = c$.

33 The acceleration a cm s^{-2} of an object moving in a straight line is given by $a(t) = 1 - 3\cos\left(2t + \frac{\pi}{2}\right)$ where t is the time in seconds.
The object's initial velocity is 5 cm s^{-1}.

- **a** Find an expression for the object's velocity v in terms of t.
- **b** Find the velocity of the object at $t = \frac{\pi}{4}$ seconds.

34 Consider $f(x) = e^{3x-4} + 1$.

- **a** Show that $f^{-1}(x) = \dfrac{\ln(x-1)+4}{3}$.
- **b** Calculate $f^{-1}(8) - f^{-1}(3)$. Give your answer in the form $a \ln b$ where $a,\ b \in \mathbb{Q}^+$.

35 Given that $\sin A = \frac{2}{5}$ and $\frac{\pi}{2} \leqslant A \leqslant \pi$, find: **a** $\tan A$ **b** $\sin 2A$.

36 Consider the infinite geometric sequence $20,\ 10\sqrt{2},\ 10,\ 5\sqrt{2},\ ......$

- **a** Write the 12th term of the sequence in the form $k\sqrt{2}$ where $k \in \mathbb{Q}$.
- **b** Find the exact value of the sum of the infinite sequence in the form $a + b\sqrt{2}$ where $a,\ b \in \mathbb{Z}$.

37 A particle is initially located at $\text{P}(3,\ 1,\ -2)$, and it moves with fixed velocity in a straight line. After 2 seconds the particle is at $\text{Q}(1,\ 3,\ 4)$. Find:

- **a** $\overrightarrow{\text{PQ}}$ **b** the particle's speed **c** the equation of the straight line.

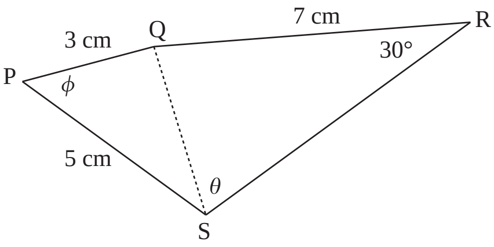
38 Suppose $\mathbf{P} = \begin{pmatrix} -1 & 3 \\ a & -4 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} -4 & -3 \\ -2 & b \end{pmatrix}$.

- **a** Find matrix $\mathbf{PQ}$ in terms of a and b. **b** Given that $\mathbf{PQ} = k\mathbf{I}$, find a, b and k.
- **c** Hence, find $\mathbf{P}^{-1}$ in terms of $\mathbf{Q}$.

39 Suppose $g(x) = e^{\frac{x}{4}}$ where $0 \leqslant x \leqslant 4$.

- **a** Sketch g on the given domain. **b** What is the range of the function g?
- **c** On the same set of axes used in **a**, sketch the graph of g^{-1}.
- **d** State the domain and range of g^{-1}. **e** Find algebraically $g^{-1}(x)$.

40 [QS] is a diagonal of quadrilateral PQRS where PQ = 3 cm, QR = 7 cm, PS = 5 cm and $\text{Q}\widehat{\text{R}}\text{S} = 30^o$.



- **a** Show that if $\text{S}\widehat{\text{P}}\text{Q} = \phi$, then $\text{QS} = \sqrt{34 - 30\cos\phi}$ cm.
- **b** If $\phi = 60^o$ and $\text{Q}\widehat{\text{S}}\text{R} = \theta$, show that $\sin\theta = \frac{7}{2\sqrt{19}}$.

c Find the *exact* length of [RS], given that θ is acute.

d Hence, find the exact perimeter and area of PQRS.

41 Suppose $f(x) = -\frac{1}{4}x^2 + 3x + 4$.

a **i** Write down $f'(x)$ in simplest form.

ii Find the equation of the normal to f at $(2, 9)$.

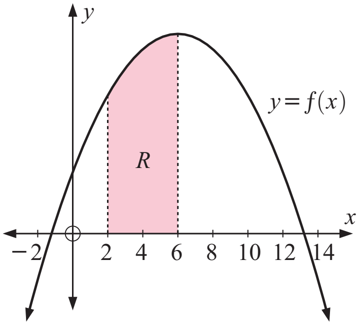
iii If this normal cuts f at another point A, find the coordinates of A.

b The graph of $y = f(x)$ is shown alongside.

i Write down an expression for the area of the shaded region R.

ii Calculate the exact area of R.

iii Suppose the region R is revolved about the x-axis through one revolution. Write down an expression for the volume of the solid formed.



42 **a** Consider the geometric sequence: $4, -12, 36, -108, ...$

i Write down the common ratio. **ii** Find the 14th term.

b Consider the sequence: $x, \ x-2, \ 2x-7, \ ......$

i Find x if the sequence is geometric.

ii Does the sequence have a sum to infinity? Explain your answer.

c Suppose the sequence $x, \ x-2, \ 2x-7, \ ......$ is arithmetic. Find:

i its 30th term **ii** the sum of its first 50 terms.

43 A and B have position vectors $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $2\mathbf{i} - \mathbf{j} - 8\mathbf{k}$ respectively.

a Find: **i** $\overrightarrow{AB}$ **ii** the unit vector $\mathbf{u}$ in the direction of $\overrightarrow{BA}$.

b Is $\mathbf{u}$ perpendicular to $\overrightarrow{OA}$?

c If C has position vector $\mathbf{i} + \mathbf{j} + a\mathbf{k}$, and $\overrightarrow{OC}$ is perpendicular to $a\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, find a.

d If M is the midpoint of [AB], find the position vector of M.

e Line L_1 passes through M and is parallel to $\overrightarrow{OA}$.
Write down the vector equation $\mathbf{r}_1$ of line L_1.

f Suppose line L_2 has vector equation $\mathbf{r}_2 = (m\mathbf{i} + \mathbf{j} - \mathbf{k}) + s(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$.

i Explain why L_1 is not parallel to L_2.

ii Find m if L_1 and L_2 intersect.

iii Find the position vector of P, the point of intersection of L_1 and L_2.

44 Suppose $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$.

a Find: **i** $\mathbf{A}^2$ **ii** $\mathbf{A}^3$ **iii** $\mathbf{A}^4$.

b Show that $\mathbf{A}^n = \begin{pmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{pmatrix}$ for $n = 1, 2, 3$ and 4.

c Assuming the result from **b** is also true for $n \geqslant 5$, find $\mathbf{A}^{10}$.

d Let $\mathbf{S}_n = \mathbf{A}^1 + \mathbf{A}^2 + \mathbf{A}^3 + \mathbf{A}^4 + \ldots\ldots + \mathbf{A}^n$.

  i Write down $\mathbf{S}_n$ as a sum of matrices.

  ii If $\mathbf{S}_n = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, explain why $r = 0$ and $s = n$.

  iii Show that $p = 2^{n+1} - 2$ and hence find q in similar form.

  iv Find $\mathbf{S}_{14}$ using the above results.

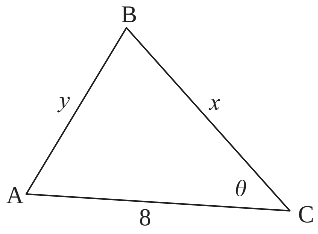
45 **a** Consider the quadratic $y = -x^2 + 12x - 20$.

  i Explain why this quadratic has a maximum value.

  ii What value of x gives this maximum value?

  iii What is the maximum value?

b



In ΔABC, AB $= y$, BC $= x$, AC $= 8$, and the perimeter of ABC is 20.

  i Write y in terms of x.

  ii Use the cosine rule to write y^2 in terms of x and $\cos\theta$.

  iii Hence, show that $\cos\theta = \dfrac{3x-10}{2x}$.

c If the area of the triangle in **b** is A, show that $A^2 = 16x^2\sin^2\theta$.

d Show that $A^2 = 20(-x^2 + 12x - 20)$.

e Hence, find the maximum area of ΔABC.

f Comment on the shape of the triangle when it has maximum area.

46 Suppose $f(x) = 4x - 3$ and $g(x) = x + 2$.

a Find $f^{-1}(x)$ and $g^{-1}(x)$, the inverse functions of f and g.

b Find $(f \circ g^{-1})(x)$.

c Find x such that $(f \circ g^{-1})(x) = f^{-1}(x)$.

d Suppose $H(x) = \dfrac{f(x)}{g(x)}$.

  i Sketch the graph of $H(x)$. Include its asymptotes and their equations.

  ii Find constants A and B such that $\dfrac{4x-3}{x+2} = A + \dfrac{B}{x+2}$

  iii Calculate the exact value of $\displaystyle\int_{-1}^{2} H(x)\,dx$.

  iv On your sketch in **i**, shade the region whose area is given by $\displaystyle\int_{1}^{3} H(x)\,dx$.

47 Hannah, Heidi and Holly have different sets of cards, but each set contains cards with the numbers 0, 1, 2, 3 or 4, one per card.

a Hannah wrongly states that the probability distribution of her set of cards is:

Why is Hannah wrong?

x	0	1	2	3	4
$\mathrm{P}(X = x)$	0.1	0.3	0.3	0.2	0.2

b Heidi correctly states that her probability distribution is:

x	0	1	2	3	4
$P(X = x)$	0.2	a	0.3	b	0.2

What can be deduced about a and b?

c Holly correctly states that the probability distribution for her set of cards is

$$P(X = x) = \frac{x(x+2)}{50}.$$

If one card is randomly chosen from Holly's set, find the probability that it is:

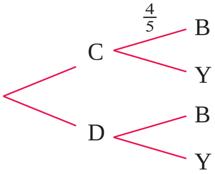
i a 2 **ii** not a 2.

48 An ordinary 6-sided die is used to select one of two bags C and D, and a ticket is drawn from that bag.

If a 1 or 2 is rolled, bag C is chosen; otherwise bag D is chosen.

Bag C contains 4 blue and 1 yellow ticket.

Bag D contains 2 blue and 3 yellow tickets.



a Copy and complete the tree diagram showing the possible outcomes.

b What is the probability that a yellow ticket is drawn from bag D?

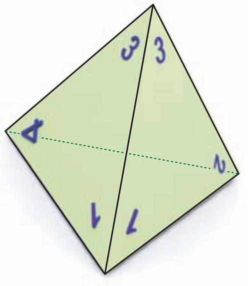
c Find the probability of drawing a yellow ticket from either bag.

d If a blue ticket is chosen, find the probability that it came from bag D.

e In a gambling game, a player wins \$6 for getting a blue ticket and \$9 for getting a yellow one. Find the player's expected return.

49 Two identical tetrahedral dice are rolled. Their four vertices are clearly labelled 1, 2, 3 and 4 as shown in the diagram.

The result when one die comes to rest is the number on the uppermost vertex. This is 3 in the diagram.



a Illustrate the sample space of 16 possible results when the two dice are rolled.

b Let X be the sum of the scores on the two dice. What are the possible values of X?

c Find: **i** $P(X = 4)$ **ii** $P(X > 4)$ **iii** $P(X = 6 \mid X > 4)$

d When Mimi uses the two dice to play a game she:

- wins €5 if the sum is 4,
- wins €1 if the sum is greater than 4,
- loses €d if the sum is less than 4.

What value should d have if Mimi's expected result is €0?

50 A particle moves in a straight line such that at time t seconds, $t \geqslant 0$, the acceleration is $a(t) = 3t - \sin t \text{ cm s}^{-2}$.

a What is the particle's acceleration at time $t = 0$ and $t = \frac{\pi}{2}$ seconds?

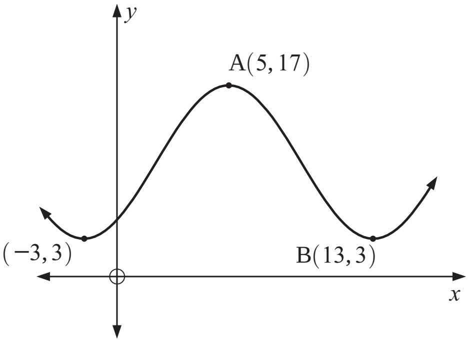
b If the initial velocity of the particle is 3 cm s^{-1}, find its velocity function $v(t)$.

c Find $\int_0^{\frac{\pi}{2}} v(t)\, dt$ and explain why the result is positive.

d Interpret the result in **c** with regard to the particle's motion.

51 The graph of $f(t) = a \sin b(t - c) + d$ is illustrated alongside.

A is a local maximum and B is a local minimum.



a Find the values of a, b, c and d.

b T is a transformation which consists of a translation of $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ followed by a vertical stretch with scale factor 2, x-axis invariant.

  i What are the coordinates of A′, the image of A under T?

  ii If $f(t)$ moves to $g(t)$ under T, find $g(t)$ in the form $g(t) = p \sin q(t-r)+s$.

c Describe fully the transformation which maps g back to f.

52 **a** Factorise $4^x - 2^x - 20$ in the form $(2^x + a)(2^x - b)$ where $a, b \in \mathbb{Z}^+$.

b Hence, find the exact solution of $2^x(2^x - 1) = 20$.

c Suppose $p = \log_5 2$.

  i Write the solution to **b** in terms of p.

  ii Find the solution to $8^x = 5^{1-x}$ in terms of p only.

53 Suppose $f(x) = a \cos 2x + b \sin^2 x$, $b < 2a$, $0 \leqslant x \leqslant 2\pi$.

a Show that $f'(x) = (b - 2a) \sin 2x$.

b Find the maximum value of $f'(x)$ and when this maximum occurs.

c Find the turning points of $y = f(x)$ on $0 \leqslant x \leqslant 2\pi$.

54 Suppose $S(x) = \frac{1}{2}(e^x - e^{-x})$ and $C(x) = \frac{1}{2}(e^x + e^{-x})$.

a Show that $[C(x)]^2 - [S(x)]^2 = 1$.

b Show that $\dfrac{d}{dx}[S(x)] = C(x)$.

c Find $\dfrac{d}{dx}[C(x)]$ in terms of $S(x)$.

d If $T(x) = \dfrac{S(x)}{C(x)}$, find $\dfrac{d}{dx}[T(x)]$ in terms of $C(x)$.

55 The size of a population at time t years is given by $P(t) = \dfrac{60\,000}{1 + 2e^{-\frac{t}{4}}}$, $t \geqslant 0$.

a Find $P'(t)$ and show that $P'(t) > 0$ for all $t \geqslant 0$.

b What is the significance of the result in **a**?

c Find $P''(t)$.

d Find the maximum growth rate of the population and the exact time when this occurs.

e Discuss $P(t)$ as $t \to \infty$, and find the value of $P(0)$.

f Using **a** to **e** only, sketch the population function.

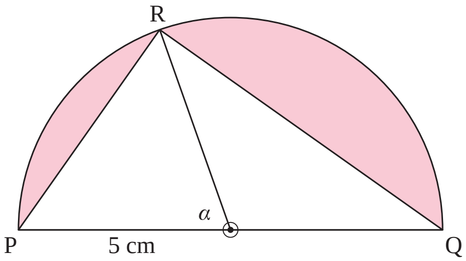
56 $P(x, y)$ lies on the curve $y = x^2$ and $A(0, 2)$ lies on the y-axis.

- **a** Show that $AP = \sqrt{x^4 - 3x^2 + 4}$.
- **b** Hence find the point(s) on $y = x^2$ such that AP is a minimum.

57 Suppose $f(x)$ is defined by $f : x \mapsto \cos^3 x$.

- **a** State the range of f.
- **b** For the interval $0 \leqslant x \leqslant 2\pi$, how many solutions does $8\cos^3 x = 1$ have?
- **c** Find $f'(x)$.
- **d** $h(x) = \sqrt{3}\cos x\sqrt{\sin x}$ is defined on $0 \leqslant x \leqslant \frac{\pi}{2}$.
  When $h(x)$ is revolved about the x-axis through one revolution, a solid is generated. Find the volume of this solid.

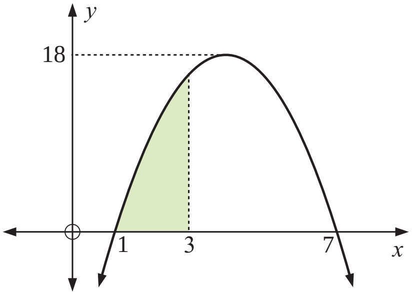
58



[PQ] is the diameter of a semi-circle and O is its centre. The radius [OP] has length 5 cm.

- **a** Find the area of triangle PQR.
- **b** Hence, find the shaded area A in terms of variable α.
- **c** Find the maximum and minimum values of the area A and the values of α when they occur.

59 The graph of the function $f(x) = a(x - h)^2 + k$ is shown alongside. It has x-intercepts 1 and 7, and a maximum value of 18.



- **a** Find the values of:
  - **i** h
  - **ii** k
  - **iii** a
- **b** Find the shaded area.

60

- **a** Find the exact value of x for which:
  - **i** $2^{1-2x} = 0.5$
  - **ii** $\log_x 7 = 5$
- **b** Solve for x: $25^x - 6(5^x) + 5 = 0$
- **c** If $2^x = 3^{1-x}$, show that $x = \log_6 3$.

61

- **a** Suppose $\dfrac{1 - \cos 2\theta}{\sin 2\theta} = \sqrt{3}$ where $0 < \theta < \frac{\pi}{2}$.
  - **i** Show that $\tan\theta = \sqrt{3}$ also.
  - **ii** Find θ.
- **b** If $\cos 2x = 2\cos x$, what is the value of $\cos x$?

62 The sum of the first n terms of a series is given by $S_n = n^3 + 2n - 1$. Find u_n, the nth term of the series.

63 Find the *exact* values of x for which $\sin^2 x + \sin x - 2 = 0$ and $-2\pi \leqslant x \leqslant 2\pi$.

64 If $f : x \mapsto \ln x$ and $g : x \mapsto 3 + x$ find:

- **a** $f^{-1}(2) \times g^{-1}(2)$
- **b** $(f \circ g)^{-1}(2)$.

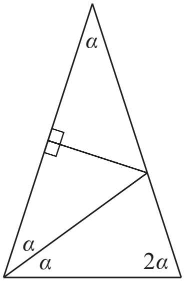
65 Find the coordinates of the point on the line L that is nearest to the origin if the equation of L is $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} + t(-\mathbf{i} + \mathbf{j} - \mathbf{k})$, $t \in \mathbb{R}$.

66 The function $f(x)$ satisfies the following criteria: $f'(x) > 0$ and $f''(x) < 0$ for all x, $f(2) = 1$, and $f'(2) = 2$.

- **a** Find the equation of the tangent to $f(x)$ at $x = 2$ and sketch it on a graph.
- **b** Sketch the graph of $y = f(x)$ on the same axes.
- **c** Explain why $f(x)$ has exactly one zero.
- **d** Estimate an interval in which the zero of $f(x)$ lies.

67 Solve simultaneously: $x = 16y$ and $\log_y x - \log_x y = \frac{8}{3}$.

68 Use the figure alongside to show that $\cos 36^o = \dfrac{1+\sqrt{5}}{4}$.



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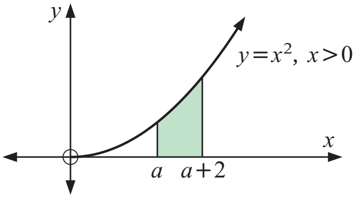


Find a given that the shaded region has area $5\frac{1}{6}$ units^2.

70 What can be deduced if $A \cap B$ and $A \cup B$ are independent events?

71 Solve the equation $\sin\theta\cos\theta = \frac{1}{4}$ for the interval $-\pi \leqslant \theta \leqslant \pi$.

72 f is defined by $x \mapsto \ln(x(x-2))$.

- **a** State the domain of f.
- **b** Find $f'(x)$.
- **c** Find the equation of the tangent to f at the point where $x = 3$.

73 Hat 1 contains three green and four blue tickets. Hat 2 contains four green and three blue tickets. One ticket is randomly selected from each hat.

- **a** What is the probability that the tickets are the same colour?
- **b** Given that the tickets are different colours, what is the probability that the green ticket came from Hat 2?

74 A normally distributed random variable X has a mean of 90.
The probability $\mathrm{P}(X < 85) \approx 0.16$.

- **a** Find $\mathrm{P}(90 < X < 95)$.
- **b** *Estimate* the standard deviation for the random variable X.

75 A discrete random variable X has a probability function given by the rule
$\mathrm{P}(X = x) = a\left(\frac{2}{5}\right)^x$, $x = 0, 1, 2, 3, \ldots\ldots$ Find the value of a.

76 Given that $x = \log_3 y^2$, express $\log_y 81$ in terms of x.

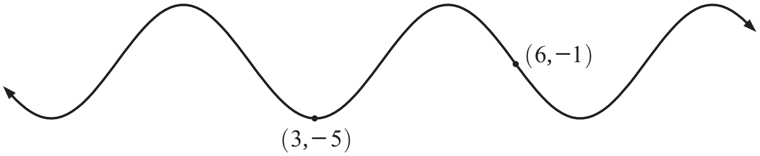
77 The ratio of the zeros of $x^2 + ax + b$ is 2 : 1. Find a relationship between a and b.

78 The point A$(-2, 3)$ lies on the graph of $y = f(x)$. Give the coordinates of the point that A moves to under the following transformations:

a $y = f(x-2)+1$ **b** $y = 2f(x-2)$ **c** $y = f(2x-3)$

d $y = \dfrac{1}{f(x)}$ **e** $y = f^{-1}(x)$

79 Find a trigonometric equation of the form $y = a\sin(b(x+c)) + d$ that represents the following graph with the information given below.
You may assume that $(3, -5)$ is a minimum point and $(6, -1)$ lies on the principal axis.



80 Suppose A and B are events such $\text{P}(A) = 0.3+x$, $\text{P}(B) = 0.2+x$ and $\text{P}(A \cap B) = x$.

a Find x if A and B are mutually exclusive events.

b Calculate the possible values of x if A and B are independent events.

81 Find a and b if the matrix $\mathbf{A} = \begin{pmatrix} a & -1 \\ b & 2 \end{pmatrix}$ is its own inverse. Hence find $\mathbf{A}^{11}$.

82 A function f is defined by $f(x) = \dfrac{x^2+1}{(x+1)^2}$.

a Write down the equations of the asymptotes of the graph of $y = f(x)$.

b Find $f'(x)$ and hence find the position and nature of any stationary points.

c Find $f''(x)$ and hence find the coordinates of all points of inflection.

d Sketch the graph of $y = f(x)$ showing all the above features.

83 If $f : x \mapsto 2x+1$ and $g : x \mapsto \dfrac{x+1}{x-2}$, find: **a** $(f \circ g)(x)$ **b** $g^{-1}(x)$.

84 A and B are two events such that $\text{P}(A) = \frac{1}{3}$ and $\text{P}(B) = \frac{2}{7}$.

a Find $\text{P}(A \cup B)$ if A and B are: **i** mutually exclusive **ii** independent.

b Find $\text{P}(A \mid B)$ if $\text{P}(A \cup B) = \frac{3}{7}$.

85 Find x in terms of a if $a > 1$ and $\log_a(x+2) = \log_a x + 2$.

86 Consider the expansion $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

a Write down the binomial expansion for $(a-b)^5$.

b Simplify
$(0.4)^5 + 5(0.4)^4(0.6) + 10(0.4)^3(0.6)^2 + 10(0.4)^2(0.6)^3 + 5(0.4)(0.6)^4 + (0.6)^5$.

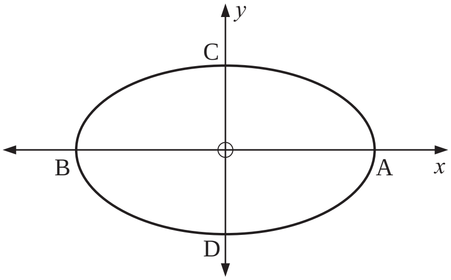
c Write $\left(2x + \dfrac{1}{x}\right)^5$ in simplified expanded form.

87 If $x + \dfrac{1}{x} = a$, find in terms of a:

a $x^2 + \dfrac{1}{x^2}$ **b** $x^3 + \dfrac{1}{x^3}$

88 The illustrated ellipse has equation

$$\frac{x^2}{16} + \frac{y^2}{4} = 1.$$



a Find the coordinates of points:

i A and B **ii** C and D.

b State the equation of the top half BCA of the ellipse.

c Write down a definite integral for the area of the ellipse.

d If the ellipse is rotated about the x-axis, a solid of revolution is generated. Find the exact volume of this solid.

89 Consider $f(x) = \sin^2 x$ for $0 \leqslant x \leqslant 2\pi$.

a Copy and complete the table of values:

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$f(x)$	0	$\frac{1}{2}$			0			$\frac{1}{2}$	

b Sketch the graph of $f(x) = \sin^2 x$ on the given domain.

c Check your graph by plotting the point where $x = \frac{\pi}{6}$.

d State the range of $y = f(x)$.

e Find the area enclosed by $y = f(x)$ and the x-axis for $0 \leqslant x \leqslant \pi$.

f Find the equation of the tangent to $y = f(x)$ at the point $\left(\frac{\pi}{4}, \frac{1}{2}\right)$.

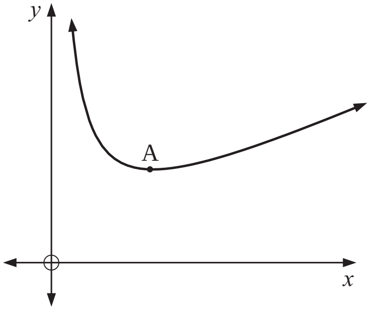
90 The random variable X has probability density function $f(x) = 2e^{-x}$, $0 \leqslant x \leqslant k$.

a Find $f(0)$. **b** Graph $y = f(x)$ for $x \geqslant 0$.

c Find the exact value of k. **d** Find the exact value of $\text{P}(0 \leqslant X \leqslant \frac{1}{2})$.

e Hence, show that $\text{P}(X \geqslant \frac{1}{2}) = \dfrac{2}{\sqrt{e}} - 1$.

91 The graph of $f(x) = x + \dfrac{1}{x}$ is given for $x > 0$.



a Find $f'(x)$ and solve the equation $f'(x) = 0$.

b Find the coordinates of the local minimum A.

c Copy and complete:
"the sum of a positive number and its reciprocal is at least ......"

d How many positive solutions would these equations have?

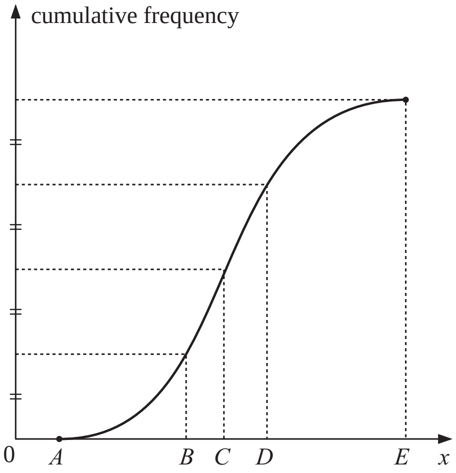
i $x + \dfrac{1}{x} = 1$ **ii** $x + \dfrac{1}{x} = 2$ **iii** $x + \dfrac{1}{x} = 3$ Give reasons.

92 A straight line passes through $A(2, 0, -3)$ and has direction vector $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

- **a** Write down the vector equation of the line.
- **b** Write down the parametric equations for the line.
- **c** What is represented by $P(2 + t, -t, -3 + 2t)$?
- **d** Find $\overrightarrow{BP}$ given $B(-1, 3, 5)$.
- **e** Find $\overrightarrow{BP} \bullet (\mathbf{i} - \mathbf{j} + 2\mathbf{k})$.
- **f** Use **e** to find t when $[BP]$ is perpendicular to the original line.
- **g** What point on the original line is closest to B?

93 A cumulative frequency graph for the random variable X is given alongside.

- **a** What is represented by:
  - **i** A **ii** B **iii** C
  - **iv** D **v** E?
- **b** What do these measure?
  - **i** $E - A$ **ii** $D - B$
- **c** Determine:
  - **i** $P(B \leqslant X \leqslant D)$
  - **ii** $P(X \geqslant B)$.
- **d** Draw an accurate box-and-whisker plot for the data set.



B CALCULATOR QUESTIONS

EXERCISE 25B

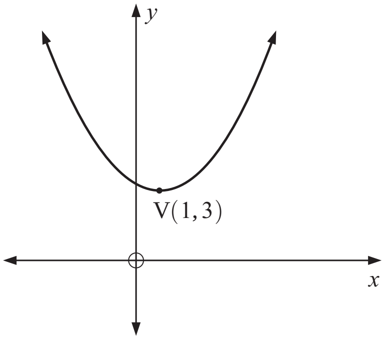
1 Find n given that $\sum_{k=1}^{n} (2k - 31) = 0$.

2 Consider the function $f(x) = 5\ln(x - 4) + 2$.

- **a** Graph the function $y = f(x)$. Clearly label the axes intercepts and asymptotes.
- **b** Solve the equation $f(x) = 1$.
- **c** Graph the function $y = f^{-1}(x)$ on the same set of axes. Clearly label the axes intercepts and asymptotes.
- **d** Find the equation of the normal to the curve $y = f(x)$ at the point where $x = 5$.

3 Find the constant term in the expansion of $\left(x - \dfrac{1}{5x^2}\right)^9$.

4 The diagram shows the graph of the quadratic function $f(x) = x^2 + mx + n$.



- **a** Determine the values of m and n.
- **b** Find k given that the graph passes through the point $(3, k)$.
- **c** State the vertex of $y = g(x)$ where $g(x) = f(x-1) + 2$.
- **d** Find the domain and range of $f(x)$ and $g(x)$.

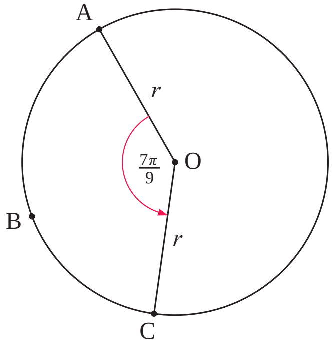
5 Consider the arithmetic sequence: $-900, -750, -600, -450, ...$

- **a** Find the 20th term of the sequence.
- **b** Find the sum of its first 20 terms.

6 Functions f and g are defined by $f : x \mapsto 2x$ and $g : x \mapsto 1 - 5x^2$. Solve:

- **a** $(f \circ g)(x) = -8$
- **b** $(g \circ f)(x) = -8$
- **c** $f'(x) = g'(x)$
- **d** $f^{-1}(x) = g(x)$

7



The perimeter of the sector OABC is 50π cm.

- **a** Find r.
- **b** Find the area of the sector OABC.
- **c** Calculate the length of the side of an equilateral triangle which has the same area as sector OABC.

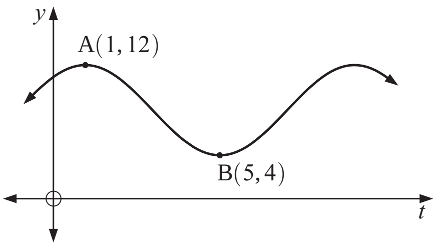
8 Let $f(x) = x\sin x - 3\cos x$ and $g(x) = \ln x$ for $-\pi \leqslant x \leqslant \pi$.

- **a** Sketch the graph of $y = f(x)$.
- **b** Hence, solve the equation $f(x) = 2$.
- **c** Sketch the graph of $y = g(x)$ on the same set of axes.
- **d** Let A be the area of the region enclosed by the graphs of f and g.
  - **i** Find an expression for A.
  - **ii** Calculate A correct to 3 significant figures.

9

- **a** Write down the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 0 \\ -1 & -3 & 5 \end{pmatrix}$.
- **b** Hence, solve the system of equations $\begin{cases} 2x - 3y - 5z = 0 \\ -x + 4y \qquad = 0 \\ -x - 3y + 5z = 5. \end{cases}$

10 The graph of a function of the form $f(t) = m\cos n(t-p) + r$ is shown.
It has a local maximum at A(1, 12) and a local minimum at B(5, 4).



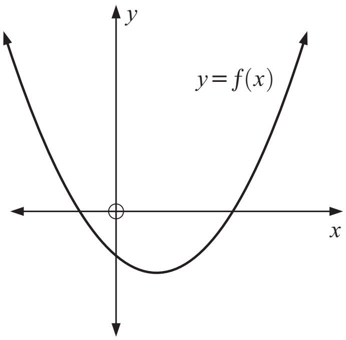
- **a** Determine the values of m, n, p and r.
- **b** Hence, determine:
  - **i** the value of $f(6)$
  - **ii** the smallest positive value of t such that $f(t) = 10$.

11 The lines L_1 and L_2 have equations:

$$\mathbf{r}_1 = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} \text{ and } \mathbf{r}_2 = \begin{pmatrix} -5 \\ 0 \\ 7 \end{pmatrix} + s\begin{pmatrix} 6 \\ -5 \\ 15 \end{pmatrix} \text{ respectively.}$$

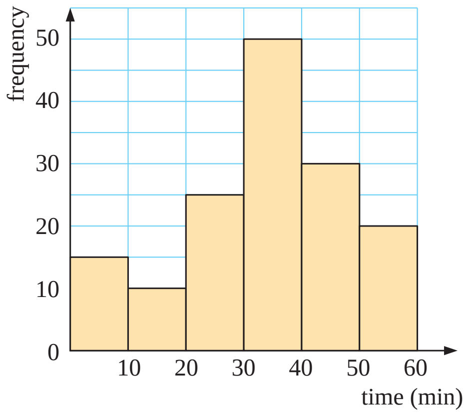
- **a** Determine the acute angle between L_1 and L_2.
- **b**
  - **i** Show that the point P$(-10, 3, -2)$ lies on L_1.
  - **ii** Does P lie on L_2? Give evidence for your answer.
- **c** Find the point of intersection of L_1 and L_2.
- **d** A third line L_3 has direction vector $\begin{pmatrix} a \\ 2 \\ 8 \end{pmatrix}$ and is perpendicular to L_1. Find a.

12 Consider the function $f(x) = (x+1)(x-\beta)$ where $\beta > 0$. A sketch of the function is shown alongside.



- **a** Determine the axes intercepts of the graph of $y = f(x)$.
- **b** Sketch the graphs of $f(x)$ and $g(x) = -f(x-1)$ on the same set of axes.
- **c** Hence, determine and label the axes intercepts of $y = g(x)$.

13



The frequency histogram illustrates the times taken by a group of people to solve a puzzle.

- **a** Construct a cumulative frequency graph for the data.
- **b** Hence estimate:
  - **i** the median time taken to solve the puzzle
  - **ii** the interquartile range of the data
  - **iii** the probability that a randomly selected person was able to complete the puzzle within 35 minutes.

14 A theatre has 30 rows of seats. There are 16 seats in the first row and each additional row has 2 more seats than the previous row. Seats are allocated randomly to all theatre patrons. Calculate the probability that a randomly chosen patron will be seated in the last row of the theatre.

15 Argentina's success rate at penalty shots is 86%. In a match against Brazil, Argentina takes 5 penalty shots.

- **a** Determine the probability that Argentina succeeds with all five of their penalty shots.
- **b** Max said that the probability of them scoring *exactly* three from their 5 shots taken is $(0.86)^3(0.14)^2$. Is Max correct? Give a reason for your answer.

16 One of the terms from the expansion of $(1+3x)^7$ is chosen at random. Calculate the probability that the coefficient of this term is greater than 1000.

17 In a particular race, Carl Lewis ran the 100 m in a time of 9.99 seconds. The mean time for all athletes in the race was 10.20 seconds and the standard deviation was 0.113 seconds.

In another race Carl ran the 200 m in a time of 17.30 seconds. The mean time for this race was 18.50 seconds and the standard deviation was 0.706 seconds.

- **a** Calculate Carl's z-scores for each event.
- **b** Based on the results of **a**, in which event did he perform better?

18 Let $f(x) = \sin(x^3)$ for $0 \leqslant x \leqslant \frac{\pi}{2}$.

- **a** Find the x-intercepts of the graph of f.
- **b** Sketch the graph of $y = f(x)$.
- **c** Find:
  - **i** the gradient of f at $x = \frac{\pi}{4}$
  - **ii** the equation of the tangent at $x = \frac{\pi}{4}$.
- **d** There exists a point P(x, y) on the interval $0 \leqslant x \leqslant \frac{\pi}{2}$ such that $f(x) > 0$, $f'(x) > 0$, and $f''(x) = 0$. Find the coordinates of P.

19 A comprehensive study of a new drug for treating epilepsy was conducted during 2006-2007. The results of the treatment are shown for two age groups in the table opposite.

	Under 35	Over 35
Treatment successful	951	257
Treatment unsuccessful	174	415

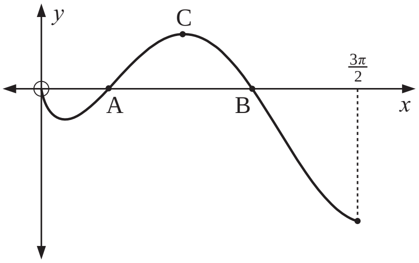
- **a** A patient from the study is selected at random. Calculate the probability that:
  - **i** the patient was successfully treated
  - **ii** the patient was over 35 given that his or her treatment was unsuccessful.
- **b** Ten patients from the study are selected at random. To calculate the probability that exactly 8 of them were successfully treated, Harry used a binomial probability distribution. Was Harry's method valid?

20 Consider $f(x) = 3e^{1-4x}$. Find:

- **a** $f'(x)$
- **b** $\int f(x)\,dx$
- **c** $\int_0^2 f(x)\,dx$ to 3 significant figures.

21 The graph of $y = (\ln x)\sin x$ for $0 \leqslant x \leqslant \frac{3\pi}{2}$ is shown.

The graph crosses the x-axis at A and B, and has a local maximum at C.



- **a** Write down the exact x-coordinates of A and B.
- **b** Find the coordinates of C, correct to 3 decimal places.
- **c** Find the coordinates of the point at which the gradient of the tangent takes the largest positive value.

  Give your answer correct to 3 decimal places.
- **d** What name is given to the point in **c**?

22 Mr O'Farrell deposits \$2000 into an account that pays interest at a rate of 4.5% p.a. compounded monthly. Calculate how long it will take for Mr O'Farrell's investment value to:

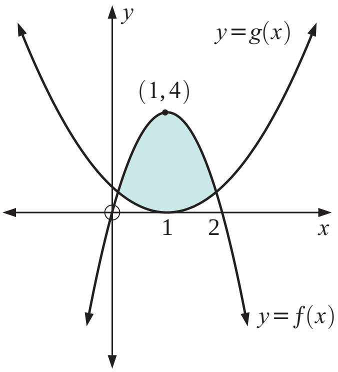
- **a** double
- **b** quadruple.

23
- **a** Find the remaining *three* terms of the expansion

$$(x^2+2)^5 = x^{10} + 10x^8 + 40x^6 + \ldots\ldots .$$

- **b** Hence, find $\displaystyle\int (x^2+2)^5\,dx$.

24 The diagram shows the graphs of the quadratic functions $y = f(x)$ and $g(x) = (x-1)^2$.



- **a** The graph of f has vertex $(1, 4)$ and x-intercepts 0 and 2.

  Determine the equation of f.
- **b** Let A be the area of the region enclosed by f and g.
  - **i** Write down an expression for A.
  - **ii** Calculate A.

25 Events A and B are independent with $\mathrm{P}(B) = 3\mathrm{P}(A)$ and $\mathrm{P}(A \cup B) = 0.68$.

- **a** Show that $[\mathrm{P}(B)]^2 - 4\mathrm{P}(B) + 2.04 = 0$
- **b** Hence, calculate $\mathrm{P}(B)$ and $\mathrm{P}(A)$.

26 The weights in kilograms of twelve students are:

63 76 99 65 63 51 52 95 63 71 65 83

- **a** Calculate the mean weight of the students.
- **b** When one student leaves the class, the mean weight of the remaining 11 students becomes 70 kg. Find the weight of the student who left.
- **c** Calculate the standard deviation of the 11 remaining students.
- **d** Hence, find the number of standard deviations that the heaviest student is from the mean.

27 In a mathematics quiz there are 30 multiple choice questions. There are 5 choices for each question, only one of which is correct. Assuming that you guess the answer to every question and you give an answer to all questions, find the probability of obtaining:

a exactly 10 correct answers **b** no more than 10 correct answers.

28 It is observed that 3% of all batteries produced by a company are defective.

a For a random sample of 20 batteries, calculate the probability that:

i none are defective **ii** at least one is defective.

b A random sample of n batteries is selected. Let X be the number of defectives in the sample.

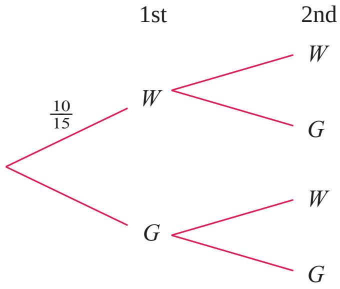
i Write down an expression for $\text{P}(X = 0)$.

ii Calculate the smallest value of n such that $\text{P}(X \geqslant 1) \geqslant 0.3$.

29 **a** Bag A contains 10 white and 5 green marbles. When a marble is randomly selected from the bag, we let W be the event a white marble is selected and G be the event a green marble is selected. Two marbles are selected without replacement from bag A.

i Copy and complete the tree diagram.

ii Calculate the probability that marbles of the same colour are selected.



b Bag B contains 5 white and n green marbles. Two marbles are selected without replacement from bag B. If the probability that both marbles are white is $\frac{2}{11}$, calculate n.

30 The masses of sea lions on an island are normally distributed with mean μ and standard deviation σ.

10% of them have mass greater than 900 kg and 15% of them have mass less than 500 kg. Find μ and σ.

31 A triathlon course begins with a 500 m swim on a bearing of 110° from the start S.

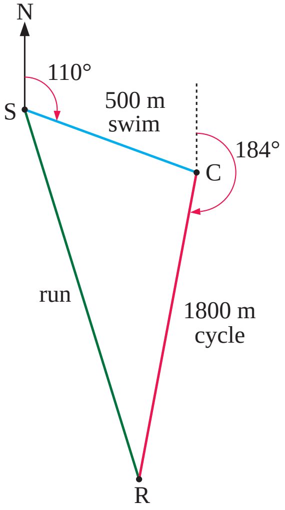
This is followed by a 1800 m cycling leg on a bearing of 184°.

The triathlon is completed with a run back to S.

A sketch of the course is shown. Calculate:

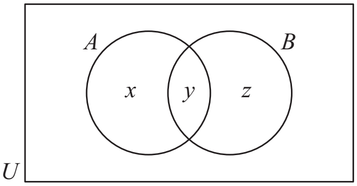
a the distance RS that a triathlete must run to complete the course

b the bearing from R to S.



32 A and B are two events such that $P(A \mid B) = 0.5$, $P(A \cup B) = 0.9$, and $P(A') = 0.2$.
For the Venn diagram given, find the probabilities:

a z **b** y **c** x.



33 Let $f(x) = \frac{1}{x}$, $x \neq 0$.

a The graph of f and the line $y = x + 2$ intersect at $x = m \pm \sqrt{n}$ where $m, n \in \mathbb{Z}$. Find m and n.

b The graph of f is transformed to the graph of g by a translation of $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ followed by a reflection in the x-axis.

i Find an expression for $g(x)$.
ii Write down the equations of the asymptotes of g.
iii What is the y-intercept of g?
iv Sketch the graph of $y = g(x)$.

34 The position vector of a moving object is $\mathbf{r}_1 = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ where t is the time in seconds, $t \geqslant 0$.

A second object has a position vector $\mathbf{r}_2 = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + t\begin{pmatrix} 9 \\ 4 \end{pmatrix}$.
All distances are in metres.

a For the first object, determine:

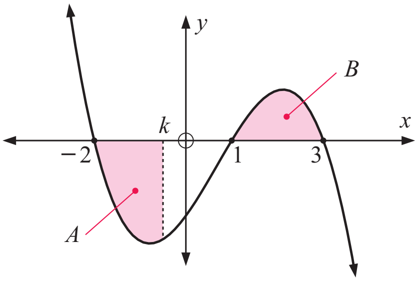
i the initial position **ii** the position after 10 seconds.

b Hence, find the distance the first object travels in the first 10 seconds.

c Show that the second object passes through the point (9, 8).

d Determine whether the objects collide at the point (9, 8).

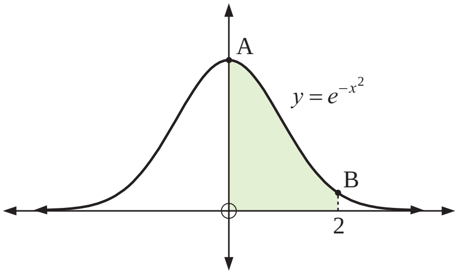
35 Consider the function
$$f(x) = -x^3 + 2x^2 + 5x - 6.$$
The graph of $y = f(x)$ is given alongside.
$x = k$ is a vertical line, where $k < 0$.
Given that the area of A equals the area of B, find the value of k.



36 The graph of $y = e^{-x^2}$ is illustrated.

a State the exact coordinates of points A and B.

b Find the area of the shaded region.



37 When a biased coin is tossed twice, the probability of getting two heads is 0.64.

a What is the probability of tossing a head with a single toss?

b If the coin is tossed 10 times, determine the probability of getting:

i exactly 6 heads **ii** at least 6 heads.

38 The heights of maize plants two months after planting are normally distributed with mean μ cm and standard deviation 6.8 cm. 75% of them are less than 45 cm high. Suppose X describes the height of a maize plant.

a Find the mean height μ cm. **b** Find $\text{P}(X < 25)$.

c Find a such that $\text{P}(X < 25) = \text{P}(X > a)$.

39 Suppose $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -2 & 2 \end{pmatrix}$.

a Find the matrix $\mathbf{A}^2 - 5\mathbf{A}$. **b** Show that $\mathbf{A}^{-1}$ exists.

c Use matrix algebra to show that $\mathbf{A}^3 = 17\mathbf{A} - 40\mathbf{I}$.

d If $\mathbf{A}^{-1} = a\mathbf{A} + b\mathbf{I}$ where $a, b \in \mathbb{Q}$, find a and b.

40 Let $f(x) = x\sin(2x)$, $0 < x < 3$.

a Sketch the graph of $y = f(x)$.

b Find the range of $y = f(x)$.

c Find the x-intercept b of the graph of $y = f(x)$ on the given domain.

d The region enclosed by $y = f(x)$ and the x-axis from 0 to b has area A units^2. Find A correct to 3 decimal places.

e If the region in **d** is revolved 360^o about the x-axis, a solid is formed. Find the volume of this solid, correct to 3 decimal places.

41 Consider $f(x) = e^{-3x}\sin x$, $-\frac{1}{2} \leqslant x \leqslant 3$.

a Show that $f'(x) = e^{-3x}(\cos x - 3\sin x)$.

b Find the equation of the tangent to the curve $y = f(x)$ at $\text{P}(\frac{\pi}{2}, e^{-\frac{3\pi}{2}})$.

c Find the area between the curve and the x-axis from $x = 0$ to $x = 1$.

42 Consider $f(x) = \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}$, $0 \leqslant x \leqslant \frac{\pi}{2}$.

a Show that $f(x) = \dfrac{2}{\sin 2x}$.

b Solve the equation $\sin 2x = 0$ and hence state the equations of any asymptote of $y = f(x)$ on $0 \leqslant x \leqslant \frac{\pi}{2}$.

c Without using calculus, find the least value of $f(x)$ and the corresponding value of x.

d If $\sin a = \frac{1}{3}$, find $f(2a)$ correct to 4 significant figures.

43 On an ostrich farm the weights of the birds are measured and found to be normally distributed. The weights of the females have mean 78.6 kg and standard deviation 5.03 kg. The weights of the males have mean 91.3 kg and standard deviation 6.29 kg.

a Find the probability that a randomly selected:

- **i** male will weigh less than 80 kg
- **ii** female will weigh less than 80 kg
- **iii** female will weigh between 70 and 80 kg.

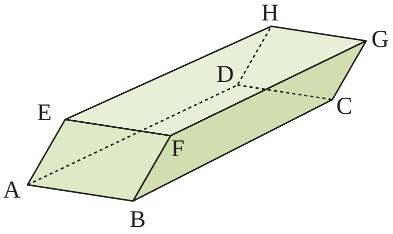
b If 20% of females weigh less than k kg, find k.

c The middle 90% of the males weigh between a kg and b kg. Find the values of a and b.

d In one field there are 82% females and 18% males. One of these ostriches is selected at random.

- **i** Calculate the probability that the ostrich weighs less than 80 kg.
- **ii** Given that the ostrich weighs less than 100 kg, what is the probability it is a male?

44 The solid figure shown is a parallelepiped. All six of its faces are parallelograms. A is $(3, 2, -1)$ and $\text{B}(1, -1, 4)$.



a Find $\overrightarrow{\text{BA}}$ and $|\overrightarrow{\text{BA}}|$.

b If C is at $(2, 0, 7)$, find the coordinates of D.

c If $\overrightarrow{\text{BF}} = \begin{pmatrix} 6 \\ -3 \\ -6 \end{pmatrix}$, find the coordinates of F.

d Calculate $\overrightarrow{\text{BA}} \bullet \overrightarrow{\text{BF}}$ and hence find $\cos \text{A}\widehat{\text{B}}\text{F}$.

e Hence, find the exact area of parallelogram ABFE.

45 A discrete random variable Y has the probability distribution shown:

y	1	2	3	4	5
$\text{P}(Y = y)$	$\frac{1}{10}$	$2t$	$\frac{3}{20}$	$2t^2$	$\frac{t}{2}$

a Find t.

b Find the expected value of the random variable Y.

c What is the significance of your result in **b**?

46 Consider $f(x) = 5x + e^{1-x^2} - 2$ for $-1 \leqslant x \leqslant 2$.

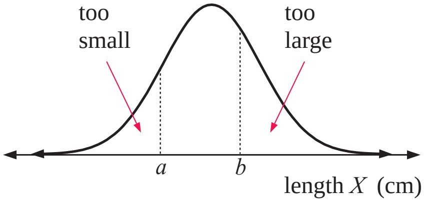
a Find the y-intercept of f.

b Sketch the graph of $y = f(x)$.

c Find any x-intercepts of f.

d Find the gradient of the tangent to $y = f(x)$ when $x = 1$.

47 The lengths of zucchinis are normally distributed with mean 24.3 cm and standard deviation 6.83 cm. A supermarket buying zucchinis in bulk finds that 15% of them are too small and 20% of them are too large for sale. The remainder, with lengths between a cm and b cm, are able to be sold.



a Find the values of a and b.

b A zucchini of saleable length is chosen at random. Find the probability that its length:

- **i** lies between 20 and 26 cm
- **ii** is less than 24.3 cm.

48 During March it rains more than 5 mm on 37% of days.

a During one week, what is the expected number of days of more than 5 mm rainfall?

b In a period of one week in March, what is the probability of rainfall of more than 5 mm on:

i exactly 3 days **ii** at least 2 days?

49 Line L_1 has vector equation $\mathbf{r}_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

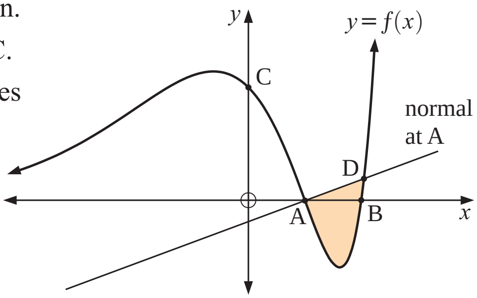
Line L_2 has vector equation $\mathbf{r}_2 = \begin{pmatrix} 6 \\ -3 \\ 4 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$.

a Point A lies on L_1 and has z-coordinate -2. Find the coordinates of A.

b Do lines L_1 and L_2 intersect? If they do, where do they meet?

c Find, correct to one decimal place, the angle between L_1 and L_2.

50 The graph of $f(x) = e^x(x^2 - 3x + 2)$ is shown.



a Find the coordinates of points A, B and C.

b Write down the equations of any asymptotes of $y = f(x)$.

c Show that $f'(x) = e^x(x^2 - x - 1)$.

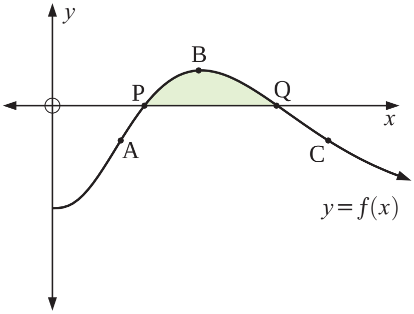
d Find the x-coordinates of the local maximum and local minimum.

e Show that the normal at A has equation $x - ey = 1$.

f Find the x-coordinate of D correct to 3 significant figures.

g Find the area of the shaded region correct to 3 significant figures.

51 The graph of $f(x) = e^{-x}x^3 - 1$ is shown.



a Find the y-intercept of f.

b Find the x-intercepts at P and Q correct to 3 decimal places.

c Find $f'(x)$ and hence find the coordinates of the local maximum at B.

d What are the exact x-coordinates of the points of inflection at A and C?

e Find the area of the shaded region.

52 When a pair of fair dice are rolled, E is the event of rolling *at least one* 6.

a Display the possible results on a 2-D graph.

b Determine $P(E)$.

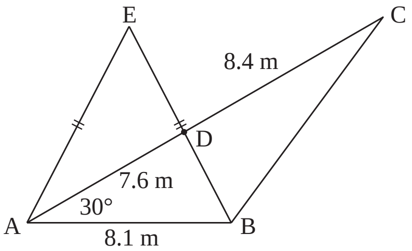
c The two dice are rolled 10 times. Determine the probability that event E occurs:

i exactly 2 times **ii** at most 3 times.

53 Points A and B have coordinates $(2, -1, 3)$ and $(-2, 4, -1)$ respectively.

- **a** Show that the line L_1 which passes through A and B has vector equation
$$\mathbf{r}_1 = (2 - 4t)\mathbf{i} + (5t - 1)\mathbf{j} + (3 - 4t)\mathbf{k}.$$
- **b** Given that $C(4, a, b)$ lies on the line through A and B, find a and b.
- **c** L_2 has equation $\mathbf{r}_2 = (3s - 5)\mathbf{i} + (s - 16)\mathbf{j} + (16 - s)\mathbf{k}$ and meets L_1 at point D. Find the coordinates of D.

54

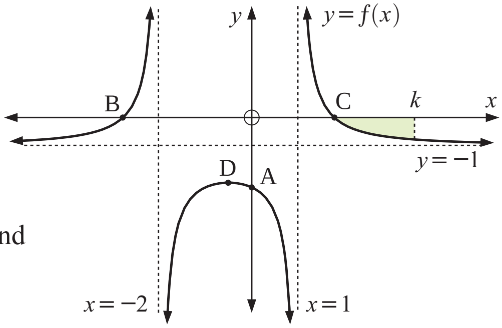


Consider the figure shown.

- **a** Find the lengths of [DB] and [BC].
- **b** Calculate the measures of angles $A\widehat{B}E$ and $D\widehat{B}C$.
- **c** Find the area of triangle BCD.
- **d** Calculate the length of [AE].

55 The graph of $f(x) = a + \dfrac{3}{(x-1)(x+b)}$ where $a, b \in \mathbb{Z}$ is shown.

- **a** State the values of a and b.
- **b** Find the y-intercept.
- **c** Find exactly the x-intercepts.
- **d** Show that $f'(x) = \dfrac{-3(2x+1)}{(x^2+x-2)^2}$ and hence find the coordinates of the local maximum at D.

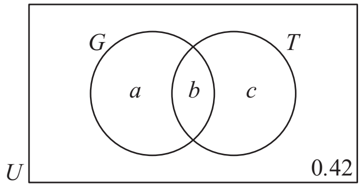


- **e** **i** Write down an expression for the shaded area as an integral.
  - **ii** If $k = 3$, find the shaded area.

56 **a** At a pet shop, let G be the event of a customer buying a goldfish and T be the event of buying a tortoise. It is known that $P(G) = 0.3$, $P(T) = 0.4$ and $P(\text{neither } G \text{ nor } T) = 0.42$.

  - **i** Show this information on a Venn diagram.
  - **ii** Find $P(G \cap T)$. Explain what this result means.
  - **iii** Are G and T independent events? Give reasons for your answer.

b At a second pet shop $P(T)$ is twice $P(G)$, and $P(\text{neither } G \text{ nor } T) = 0.42$. G and T are independent.



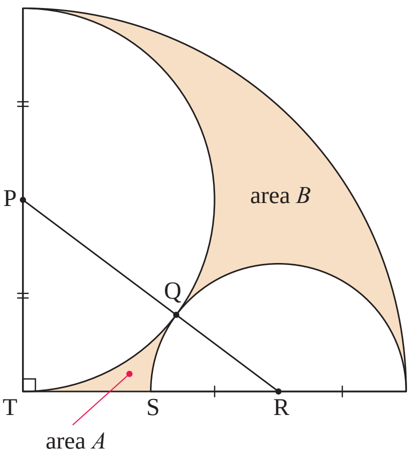
  - **i** Show that $2a + b = c$.
  - **ii** Show that $a = \sqrt{\dfrac{b}{2}} - b$.
  - **iii** Hence, find b and then a.
  - **iv** Determine $P(G)$.

57 The function f is defined by $f(x) = ax^3 + bx^2 + cx - 6$ where $a, b, c \in \mathbb{Z}$. The function passes through the points $(1, -10)$, $(3, 36)$ and $(-2, -4)$.

- **a** Show that $a + b + c = -4$.
- **b** Find two other equations connecting a, b and c.
- **c** Write the three equations from **a** and **b** in matrix form.
- **d** Solve for a, b and c.
- **e** Given that $f(x) = (x^2 - x - 2)(px + q)$, find p and q.

58 Two semi-circles touch each other within a quarter circle as shown. This means that P, Q and R are collinear. The radius of the quarter circle is 12 cm.

- **a** Use the theorem of Pythagoras to show that the radius of the smaller semi-circle is 4 cm.
- **b** Calculate, in radians, the measure of
  - **i** $\text{T}\widehat{\text{P}}\text{R}$ **ii** $\text{P}\widehat{\text{R}}\text{T}$.
- **c** Hence calculate the area of:
  - **i** A **ii** B



59 Solve the following equations:

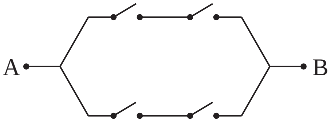
- **a** $\log_2(x^2 - 2x + 1) = 1 + \log_2(x - 1)$ **b** $3^{2x+1} = 5(3^x) + 2$

60 For a continuous function defined on the interval $a \leqslant x \leqslant b$, the length of the curve can be found using $L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$. Find the length of:

- **a** $y = x^2$ on the interval $0 \leqslant x \leqslant 1$ **b** $y = \sin x$ on the interval $0 \leqslant x \leqslant \pi$.

61 The diagram shows a simple electrical network.

Each symbol  represents a switch.



All four switches operate independently, and the probability of each one of them being closed is p.

- **a** In terms of p, find the probability that the current flows from A to B.
- **b** Find the least value of p for which the probability of current flow is at least 0.5.

62 In triangle ABC, the angle at A is double the angle at B.
If AC $= 5$ cm and BC $= 6$ cm, find:

- **a** the cosine of the angle at B **b** the length of [AB] using the cosine rule.
- **c** Are both solutions in **b** valid?

63 ABC is an equilateral triangle with sides 10 cm long. P is a point within the triangle which is 5 cm from A and 6 cm from B. How far is it from C?

64 A company manufactures computer chips, and it is known that 3% of them are faulty. In a batch of 500 such chips, find the probability that between 1 and 2 percent (inclusive) of the chips are faulty.

65 A random variable X is known to be distributed normally with standard deviation 2.83. Find the probability that a randomly selected score from X will differ from the mean by less than 4.

66 The Ferris wheel at the Royal Show turns one full circle every minute. The lowest point is 1 metre from the ground, and the highest point is 25 metres above the ground.

- **a** When riding on the Ferris wheel your height above ground level after t seconds is given by the model $h(t) = a + b\sin(c(t-d))$. Find the values of a, b, c and d given that you start your ride after entering your seat at the lowest point.
- **b** If the motor driving the Ferris wheel breaks down after 91 seconds, how high up would you be while waiting to be rescued?

67 Given $\mathbf{A} = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 2 & 1 \\ 0 & 6 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 7 & -3 \\ -1 & -2 & 1 \\ 6 & 12 & -5 \end{pmatrix}$, calculate **AB**.

Hence solve the system of equations $\begin{cases} 4a + 7b - 3c = -8 \\ -a - 2b + c = 3 \\ 6a + 12b - 5c = -15. \end{cases}$

68 The function f is defined by $f : x \mapsto e^{\sin^2 x}$, $0 \leqslant x \leqslant \pi$.

- **a** Use calculus to find the exact value(s) of x for which $f(x)$ has a maximum value.
- **b** Find $f''(x)$ and write down an equation that will enable you to find any points of inflection of $y = f(x)$.
- **c** Find the point(s) of inflection in the given domain.

69 The sum of an infinite geometric series is 49. The second term of the series is 10. Find the possible values for the sum of the first three terms of the series.

70 Let $f(x) = xe^{1-2x^2}$.

- **a** Find $f'(x)$ and $f''(x)$.
- **b** Find the exact coordinates of the stationary points of the function and determine their nature.
- **c** Find exactly the x-coordinates of the points of inflection of the function.
- **d** Discuss the behaviour of the function as $x \to \pm\infty$.
- **e** Sketch the function.

71 Consider the grouped data in the table. Estimate:

- **a** the mean value of X
- **b** the standard deviation of the X-distribution.

Score	*Frequency*
$9 < X \leqslant 11$	2
$11 < X \leqslant 13$	7
$13 < X \leqslant 15$	6
$15 < X \leqslant 17$	21
$17 < X \leqslant 19$	17
$19 < X \leqslant 21$	5

72 Consider the vectors $\mathbf{a} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \cos\theta \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} \sin\theta \\ -\sin\theta \\ \cos\theta \end{pmatrix}$.

- **a** Show that $|\mathbf{a}| = \sqrt{1+\cos^2\theta}$.
- **b** Explain why $|\mathbf{a}|$ satisfies the inequality $1 \leqslant |\mathbf{a}| \leqslant \sqrt{2}$
- **c** Find all values of θ in the interval $0 \leqslant \theta \leqslant 2\pi$ such that **a** and **b** are perpendicular.

73 Solve for x, $0 \leqslant x \leqslant 2\pi$:

- **a** $3 + 2\sin x = 0$
- **b** $3\cos\left(\frac{x}{2}\right) + 1 = 0$

74 A random variable X has the discrete probability density function $f(x) = \dfrac{x^2+kx}{50}$ for $x = 0, 1, 2, 3, 4$.

- **a** Find k.
- **b** Find μ, the mean of the distribution of X.
- **c** Find $\mathrm{P}(X \geqslant 2)$.

75 Consider $f(x) = \sin x\cos(2x)$ for $0 \leqslant x \leqslant \pi$.

- **a** Find $f'(x)$ in terms of $\cos x$ only.
- **b** Show that $f'(x) = 0$ when $\cos x = 0$ or $\pm\sqrt{\frac{5}{6}}$.
- **c** Hence, find the position and nature of the turning points of $y = f(x)$.
- **d** Graph $y = f(x)$, showing all key features.

ANSWERS

EXERCISE 1A

1 **a**, **d**, **e** **2** **a**, **b**, **c**, **e**, **g** **3** No, for example $x = 1$

4 $y = \pm\sqrt{9 - x^2}$

EXERCISE 1B

1 **a** 2 **b** 8 **c** -1 **d** -13 **e** 1

2 **a** 2 **b** 2 **c** -16 **d** -68 **e** $\frac{17}{4}$

3 **a** -3 **b** 3 **c** 3 **d** -3 **e** $\frac{15}{2}$

4 **a** $7 - 3a$ **b** $7 + 3a$ **c** $-3a - 2$ **d** $10 - 3b$
e $1 - 3x$ **f** $7 - 3x - 3h$

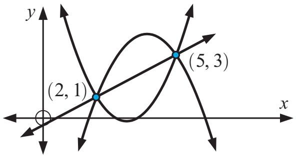
5 **a** $2x^2 + 19x + 43$ **b** $2x^2 - 11x + 13$
c $2x^2 - 3x - 1$ **d** $2x^4 + 3x^2 - 1$
e $2x^4 - x^2 - 2$ **f** $2x^2 + 4xh + 2h^2 + 3x + 3h - 1$

6 **a** **i** $-\frac{7}{2}$ **ii** $-\frac{3}{4}$ **iii** $-\frac{4}{9}$
b $x = 4$ **c** $\dfrac{2x+7}{x-2}$ **d** $x = \frac{9}{5}$

7 f is the function which converts x into $f(x)$ whereas $f(x)$ is the value of the function at any value of x.

8 **a** 6210 euros, value after 4 years
b $t = 4.5$, the time for the photocopier to reach a value of 5780 euros.
c 9650 euros

9



10 $f(x) = -2x + 5$

11 $a = 3,\ b = -2$

12 $a = 3,\ b = -1,\ c = -4,$
$T(x) = 3x^2 - x - 4$

EXERCISE 1C

1 **a** Domain $= \{x \mid x \geqslant -1\}$, Range $= \{y \mid y \leqslant 3\}$
b Domain $= \{x \mid -1 < x \leqslant 5\}$, Range $= \{y \mid 1 < y \leqslant 3\}$
c Domain $= \{x \mid x \neq 2\}$, Range $= \{y \mid y \neq -1\}$
d Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{y \mid 0 < y \leqslant 2\}$
e Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{y \mid y \geqslant -1\}$
f Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{y \mid y \leqslant \frac{25}{4}\}$
g Domain $= \{x \mid x \geqslant -4\}$, Range $= \{y \mid y \geqslant -3\}$
h Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{y \mid y > -2\}$
i Domain $= \{x \mid x \neq \pm 2\}$, Range $= \{y \mid y \leqslant -1 \text{ or } y > 0\}$

2 **a** $x < -6$ **b** $x = 0$ **c** $x \geqslant \frac{3}{2}$

3 **a** Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{y \mid y \in \mathbb{R}\}$
b Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{3\}$
c Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{y \mid y \geqslant 2\}$
d Domain $= \{x \mid x \leqslant -2,\ x \geqslant 2\}$, Range $= \{y \mid y \geqslant 0\}$
e Domain $= \{x \mid x \neq 2\}$, Range $= \{y \mid y \neq 0\}$
f Domain $= \{x \mid x \leqslant 2\}$, Range $= \{y \mid y \geqslant 0\}$
g Domain $= \{x \mid x > \frac{5}{2}\}$, Range $= \{y \mid y > 0\}$
h Domain $= \{x \mid x \neq 5\}$, Range $= \{y \mid y \neq 2\}$

4 **a** Domain $= \{x \mid x \geqslant 0\}$, Range $= \{y \mid y \geqslant 0\}$
b Domain $= \{x \mid x \neq 0\}$, Range $= \{y \mid y > 0\}$
c Domain $= \{x \mid x \leqslant 4\}$, Range $= \{y \mid y \geqslant 0\}$
d Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{y \mid y \geqslant -2\frac{1}{4}\}$
e Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{y \mid y \leqslant 2\frac{1}{12}\}$
f Domain $= \{x \mid x \neq 0\}$, Range $= \{y \mid y \leqslant -2 \text{ or } y \geqslant 2\}$
g Domain $= \{x \mid x \neq 2\}$, Range $= \{y \mid y \neq 1\}$
h Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{y \mid y \in \mathbb{R}\}$
i Domain $= \{x \mid x \neq -1 \text{ or } 2\}$,
Range $= \{y \mid y \leqslant \frac{1}{3} \text{ or } y \geqslant 3\}$
j Domain $= \{x \mid x \neq 0\}$, Range $= \{y \mid y \geqslant 2\}$
k Domain $= \{x \mid x \neq 0\}$, Range $= \{y \mid y \leqslant -2 \text{ or } y \geqslant 2\}$
l Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{y \mid y \geqslant -8\}$

EXERCISE 1D

1 **a** $5 - 2x$ **b** $-2x - 2$ **c** 11

2 $f(g(x)) = (2 - x)^2$, Domain $\{x \mid x \in \mathbb{R}\}$, Range $\{y \mid y \geqslant 0\}$
$g(f(x)) = 2 - x^2$, Domain $\{x \mid x \in \mathbb{R}\}$, Range $\{y \mid y \leqslant 2\}$

3 **a** $x^2 - 6x + 10$ **b** $2 - x^2$ **c** $x = \pm\frac{1}{\sqrt{2}}$

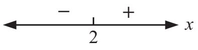
4 **a** Let $x = 0$, $\therefore\ b = d$ and so
$ax + b = cx + b$
$\therefore\ ax = cx$ for all x
Let $x = 1$, $\therefore\ a = c$

b $(f \circ g)(x) = [2a]x + [2b + 3] = 1x + 0$ for all x
$\therefore\ 2a = 1$ and $2b + 3 = 0$
$\therefore\ a = \frac{1}{2}$ and $b = -\frac{3}{2}$

c Yes, $\{(g \circ f)(x) = [2a]x + [3a + b]\}$

EXERCISE 1E

1 **a**



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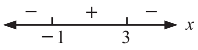

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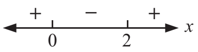

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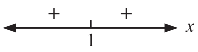

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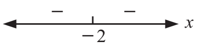

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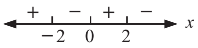

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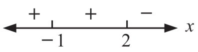

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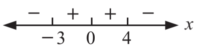

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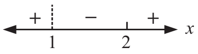

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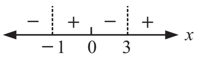

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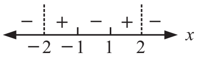
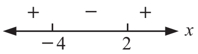


2 **a**



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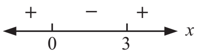

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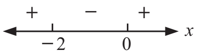

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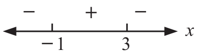

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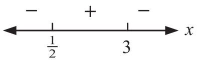

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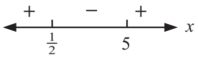

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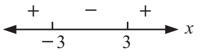

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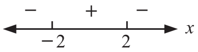

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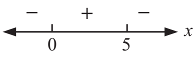

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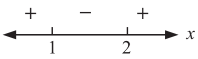

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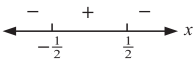

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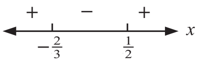

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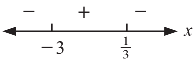

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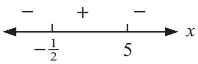

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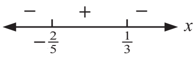

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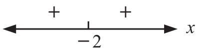

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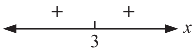

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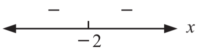

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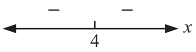

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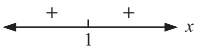

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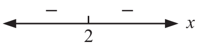

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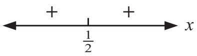

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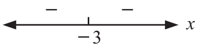

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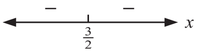
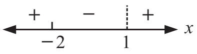


4 **a**



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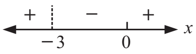

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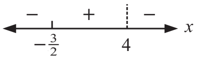

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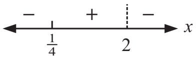

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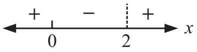

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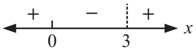

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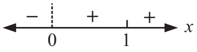

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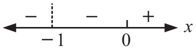

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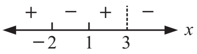

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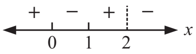

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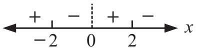

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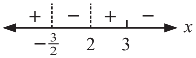
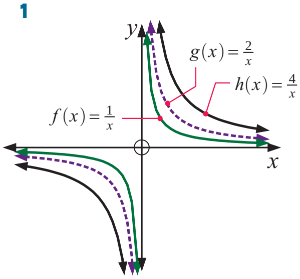


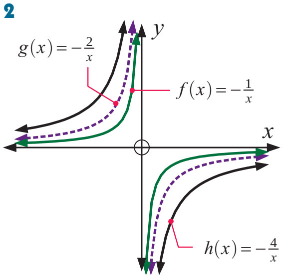
EXERCISE 1F

1



All have vert. asympt. $x = 0$ and horiz. asympt. $y = 0$. They are positive for $x > 0$ and negative for $x < 0$.

2



All have vert. asympt. $x = 0$ and horiz. asympt. $y = 0$. They are positive for $x < 0$ and negative for $x > 0$.

EXERCISE 1G

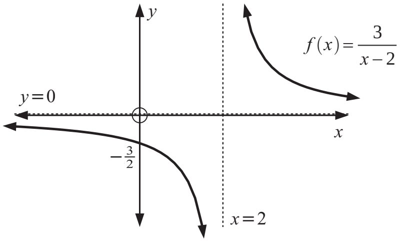
1 **a** **i** vertical asymptote $x = 2$, horizontal asymptote $y = 0$

ii as $x \to 2^-$, $y \to -\infty$ as $x \to \infty$, $y \to 0^+$

as $x \to 2^+$, $y \to \infty$ as $x \to -\infty$, $y \to 0^-$

iii no x-intercept, y-intercept $-\frac{3}{2}$

iv



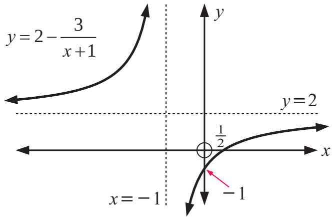
b **i** vertical asymptote $x = -1$, horizontal asymptote $y = 2$

ii as $x \to -1^-$, $y \to \infty$ as $x \to \infty$, $y \to 2^-$

as $x \to -1^+$, $y \to -\infty$ as $x \to -\infty$, $y \to 2^+$

iii x-intercept $\frac{1}{2}$, y-intercept -1

iv



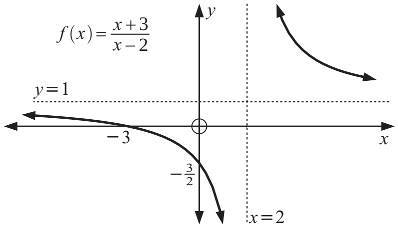
c **i** vertical asymptote $x = 2$, horizontal asymptote $y = 1$

ii as $x \to 2^-$, $y \to -\infty$ as $x \to \infty$, $y \to 1^+$

as $x \to 2^+$, $y \to \infty$ as $x \to -\infty$, $y \to 1^-$

iii x-intercept -3, y-intercept $-\frac{3}{2}$

iv



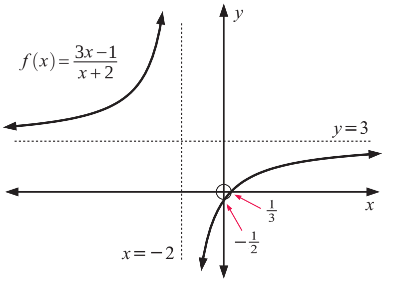
d **i** vertical asymptote $x = -2$, horizontal asymptote $y = 3$

ii as $x \to -2^-$, $y \to \infty$ as $x \to \infty$, $y \to 3^-$

as $x \to -2^+$, $y \to -\infty$ as $x \to -\infty$, $y \to 3^+$

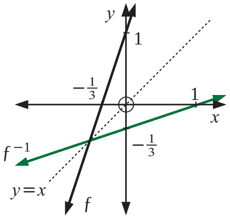
iii x-intercept $\frac{1}{3}$, y-intercept $-\frac{1}{2}$

iv



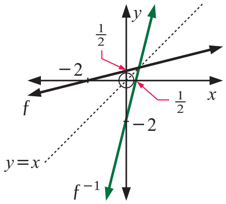
EXERCISE 1H

1 **a** **i**



ii, iii $f^{-1}(x) = \dfrac{x-1}{3}$

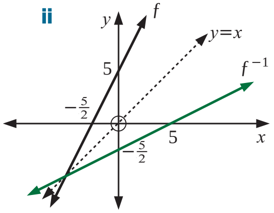
b **i**



ii, iii $f^{-1}(x) = 4x-2$

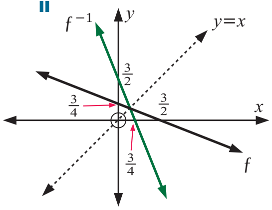
2 **a** **i** $f^{-1}(x) = \dfrac{x-5}{2}$

ii



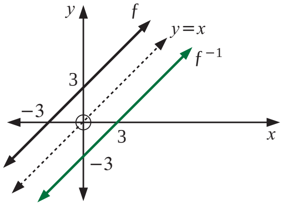
b **i** $f^{-1}(x) = -2x + \frac{3}{2}$

ii

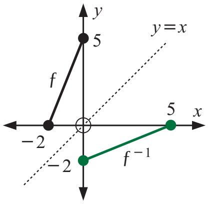


c **i** $f^{-1}(x)$
$= x-3$

ii



3 **a**



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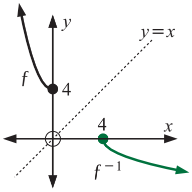
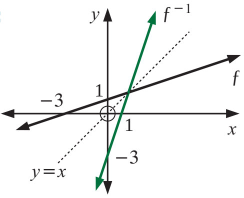
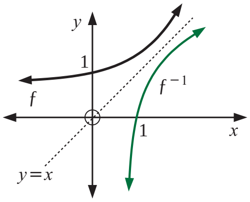


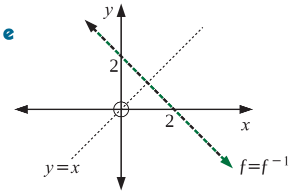
c



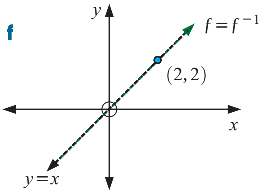
d



e



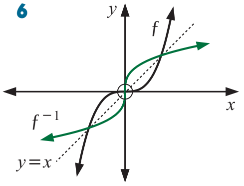
f



4 **a** $\{x \mid -2 \leqslant x \leqslant 0\}$
b $\{y \mid 0 \leqslant y \leqslant 5\}$
c $\{x \mid 0 \leqslant x \leqslant 5\}$
d $\{y \mid -2 \leqslant y \leqslant 0\}$

5 $\{y \mid -2 \leqslant y < 3\}$

6



7 **a** $f : x \mapsto \dfrac{1}{x}$, $x \neq 0$ satisfies both the vertical and horizontal line tests and so has an inverse function.

b $f^{-1}(x) = \dfrac{1}{x}$ and $f(x) = \dfrac{1}{x}$ $\therefore$ $f = f^{-1}$
$\therefore$ f is a self-inverse function

8 **a** $y = \dfrac{3x-8}{x-3}$ is symmetrical about $y = x$,
$\therefore$ f is a self-inverse function.

b $f^{-1}(x) = \dfrac{3x-8}{x-3}$ and $f(x) = \dfrac{3x-8}{x-3}$
$\therefore$ $f = f^{-1}$ $\therefore$ f is a self-inverse function

9 **a** $f^{-1}(x) = 2x+2$
b **i** $(f \circ f^{-1})(x) = x$ **ii** $(f^{-1} \circ f)(x) = x$

10 **a** 10 **b** $x = 3$

11 **a** **i** 25 **ii** 16 **b** $x = 1$

12 $(f^{-1} \circ g^{-1})(x) = \dfrac{x+3}{8}$ and $(g \circ f)^{-1}(x) = \dfrac{x+3}{8}$

13 **a** Is not **b** Is **c** Is **d** Is **e** Is

14 **b** **i** is the only one

REVIEW SET 1A

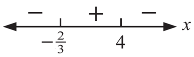
1 **a** 0 **b** -15 **c** $-\frac{5}{4}$ **2** $a = -6$, $b = 13$

3 **a** $x^2 - x - 2$ **b** $x^4 - 7x^2 + 10$

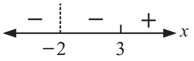
4 **a** **i** Range $= \{y \mid y \geqslant -5\}$, Domain $= \{x \mid x \in \mathbb{R}\}$
ii x-int -1, 5, y-int $-\frac{25}{9}$ **iii** is a function

b **i** Range $= \{y \mid y = 1$ or $-3\}$, Domain $= \{x \mid x \in \mathbb{R}\}$
ii no x-intercepts, y-intercept 1 **iii** is a function

5 **a**

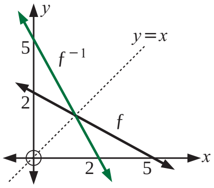


b



6 $a = 1$, $b = -1$

7 **a**



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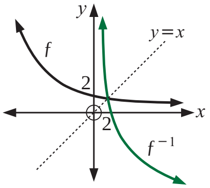


8 **a** $f^{-1}(x) = \dfrac{x-2}{4}$ **b** $f^{-1}(x) = \dfrac{3-4x}{5}$

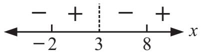
9 $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x) = x-2$

REVIEW SET 1B

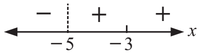
1 **a** Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{y \mid y \geqslant -4\}$
b Domain $= \{x \mid x \neq 0, 2\}$, Range $= \{y \mid y \leqslant -1$ or $y > 0\}$

2 **a** $2x^2 + 1$ **b** $4x^2 - 12x + 11$

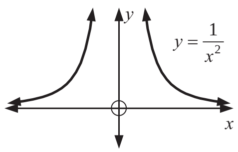
3 **a**



b



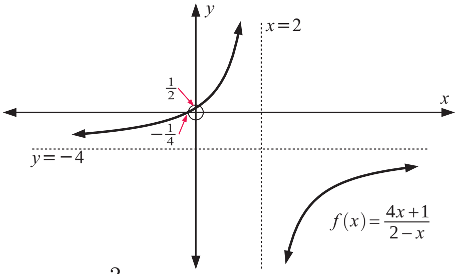
4 **a** $x = 0$ **b**



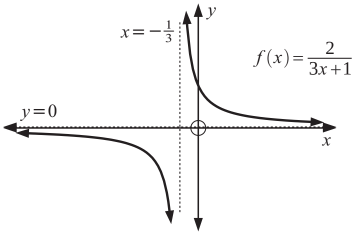
c Domain $= \{x \mid x \neq 0\}$, Range $= \{y \mid y > 0\}$

5 **a** $a = 2,\ \ b = -1$
b Domain $= \{x \mid x \neq 2\}$, Range $= \{y \mid y \neq -1\}$

6 **a** vertical asymptote $x = 2$, horizontal asymptote $y = -4$
b as $x \to 2^-$, $y \to \infty$ as $x \to \infty$, $y \to -4^-$
as $x \to 2^+$, $y \to -\infty$ as $x \to -\infty$, $y \to -4^+$
c x-intercept $-\frac{1}{4}$, y-intercept $\frac{1}{2}$
d

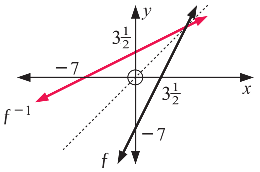


7 **a** $(g \circ f)(x) = \dfrac{2}{3x+1}$ **b** $x = -\frac{1}{2}$
c **i** vertical asymptote $x = -\frac{1}{3}$,
horizontal asymptote $y = 0$
ii



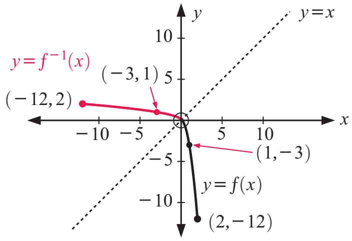
iii Domain $= \{x \mid x \neq -\frac{1}{3}\}$, Range $= \{y \mid y \neq 0\}$

8 **a**



b $f^{-1}(x) = \dfrac{x+7}{2}$

9 **a**



b Range $= \{y \mid 0 \leqslant y \leqslant 2\}$
c **i** $x \approx -1.83$ **ii** $x = -3$

REVIEW SET 1C

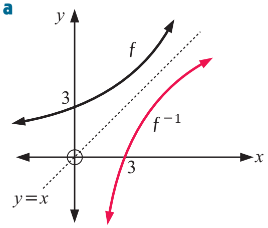
1 **a** Domain $= \{x \mid x \geqslant -2\}$, Range $= \{y \mid 1 \leqslant y < 3\}$
b Domain $= \{x \in \mathbb{R}\}$, Range $= \{y \mid y = -1,\ 1 \text{ or } 2\}$

2 **a** $10 - 6x$ **b** $x = 2$

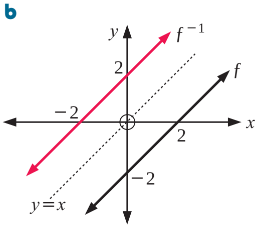
3 **a** $1 - 2\sqrt{x}$ **b** $\sqrt{1-2x}$

4 $a = 1,\ \ b = -6,\ \ c = 5$

5 **a**



b



6 **a** $f^{-1}(x) = \dfrac{7-x}{4}$ **b** $f^{-1}(x) = \dfrac{5x-3}{2}$

7 $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x) = \dfrac{4x+6}{15}$ **8** 16

EXERCISE 2A

1 **a** 4, 13, 22, 31 **b** 45, 39, 33, 27
c 2, 6, 18, 54 **d** 96, 48, 24, 12

2 **a** Starts at 8 and each term is 8 more than the previous term. Next two terms 40, 48.
b Starts at 2, each term is 3 more than the previous term; 14, 17.
c Starts at 36, each term is 5 less than the previous term; 16, 11.
d Starts at 96, each term is 7 less than the previous term; 68, 61.
e Starts at 1, each term is 4 times the previous term; 256, 1024.
f Starts at 2, each term is 3 times the previous term; 162, 486.
g Starts at 480, each term is half the previous term; 30, 15.
h Starts at 243, each term is $\frac{1}{3}$ of the previous term; 3, 1.
i Starts at 50 000, each term is $\frac{1}{5}$ of the previous term; 80, 16.

3 **a** Each term is the square of the term number; 25, 36, 49.
b Each term is the cube of the term number; 125, 216, 343.
c Each term is $n(n+1)$ where n is the term number; 30, 42, 56.

4 **a** 79, 75 **b** 1280, 5120 **c** 625, 1296
d 13, 17 **e** 16, 22 **f** 14, 18

EXERCISE 2B

1 **a** 2, 4, 6, 8, 10 **b** 4, 6, 8, 10, 12
c 1, 3, 5, 7, 9 **d** −1, 1, 3, 5, 7
e 5, 7, 9, 11, 13 **f** 13, 15, 17, 19, 21
g 4, 7, 10, 13, 16 **h** 1, 5, 9, 13, 17

2 **a** 2, 4, 8, 16, 32 **b** 6, 12, 24, 48, 96
c $3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}$ **d** −2, 4, −8, 16, −32

3 17, 11, 23, −1, 47

EXERCISE 2C

1 **a** 73 **b** 65 **c** $21\frac{1}{2}$

2 **a** 101 **b** −107 **c** $a + 14d$

3 **a** $u_1 = 6,\ d = 11$ **b** $u_n = 11n - 5$ **c** 545
d yes, u_{30} **e** no

4 **a** $u_1 = 87,\ d = -4$, **b** $u_n = 91 - 4n$ **c** −69 **d** u_{97}

5 **b** $u_1 = 1,\ d = 3$ **c** 169 **d** $u_{151} = 451$

6 **b** $u_1 = 32,\ d = -\frac{7}{2}$ **c** −227 **d** $n \geqslant 68$

7 **a** $k = 17\frac{1}{2}$ **b** $k = 4$ **c** $k = 4$ **d** $k = 0$
e $k = -2$ or 3 **f** $k = -1$ or 3

8 **a** $u_n = 6n - 1$ **b** $u_n = -\frac{3}{2}n + \frac{11}{2}$
c $u_n = -5n + 36$ **d** $u_n = -\frac{3}{2}n + \frac{1}{2}$

9 **a** $6\frac{1}{4}, 7\frac{1}{2}, 8\frac{3}{4}$ **b** $3\frac{5}{7}, 8\frac{3}{7}, 13\frac{1}{7}, 17\frac{6}{7}, 22\frac{4}{7}, 27\frac{2}{7}$

10 **a** $u_1 = 36,\ d = -\frac{2}{3}$ **b** 100 **11** 100 006

EXERCISE 2D.1

1 **a** $b = 18,\ c = 54$ **b** $b = 2\frac{1}{2},\ c = 1\frac{1}{4}$
c $b = 3,\ c = -1\frac{1}{2}$

2 **a** 96 **b** 6250 **c** 16

3 **a** 6561 **b** $\frac{19\,683}{64}$ **c** 16 **d** ar^8

4 **a** $u_1 = 5,\ r = 2$ **b** $u_n = 5 \times 2^{n-1},\ u_{15} = 81\,920$

5 **a** $u_1 = 12,\ r = -\frac{1}{2}$
b $u_n = 12 \times (-\frac{1}{2})^{n-1},\ u_{13} = \frac{3}{1024}$

6 $u_1 = 8,\ r = -\frac{3}{4},\ u_{10} = -0.600\,677\,49$

7 $u_1 = 8,\ r = \frac{1}{\sqrt{2}},\ u_n = 2^{\frac{7}{2} - \frac{n}{2}}$

8 **a** $k = \pm 14$ **b** $k = 2$ **c** $k = -2$ or 4

9 **a** $u_n = 3 \times 2^{n-1}$ **b** $u_n = 32 \times (-\frac{1}{2})^{n-1}$
c $u_n = 3 \times (\pm\sqrt{2})^{n-1}$ **d** $u_n = 10 \times (\pm\sqrt{2})^{1-n}$

10 **a** $u_9 = 13\,122$ **b** $u_{14} = 2916\sqrt{3} \approx 5050.66$
c $u_{18} \approx 0.000\,091\,55$

EXERCISE 2D.2

1 **a** \$3993.00 **b** \$993.00 **2** €11 470.39
3 **a** ¥43 923 **b** ¥13 923 **4** \$23 602.32
5 ¥148 024.43 **6** £51 249.06 **7** \$14 976.01
8 £11 477.02 **9** €19 712.33 **10** ¥19 522.47

EXERCISE 2D.3

1 **a** **i** 1550 ants **ii** 4820 ants **b** 12.2 weeks

2 **a** 278 animals **b** Year 2044

EXERCISE 2E.1

1 **a** **i** $S_n = 3 + 11 + 19 + 27 + ... + (8n - 5)$ **ii** 95
b **i** $S_n = 42 + 37 + 32 + ... + (47 - 5n)$ **ii** 160
c **i** $S_n = 12 + 6 + 3 + 1\frac{1}{2} + ... + 12(\frac{1}{2})^{n-1}$ **ii** $23\frac{1}{4}$
d **i** $S_n = 2 + 3 + 4\frac{1}{2} + 6\frac{3}{4} + ... + 2(\frac{3}{2})^{n-1}$ **ii** $26\frac{3}{8}$
e **i** $S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \dfrac{1}{2^{n-1}}$ **ii** $1\frac{15}{16}$
f **i** $S_n = 1 + 8 + 27 + 64 + ... + n^3$ **ii** 225

2 **a** 10 **b** 25 **c** 168 **d** 310 **3** $\displaystyle\sum_{n=1}^{20} (3n - 1) = 610$

EXERCISE 2E.2

1 **a** 820 **b** 3087.5 **c** -1460 **d** -740

2 **a** 1749 **b** 2115 **c** $1410\frac{1}{2}$

3 **a** 160 **b** -630 **c** 135 **4** 203 **5** -115.5 **6** 18

7 **a** 65 **b** 1914 **c** 47 850

8 **a** 14 025 **b** 71 071 **c** 3367

10 **a** $u_n = 2n - 1$ **c** $S_1 = 1,\ S_2 = 4,\ S_3 = 9,\ S_4 = 16$

11 56, 49 **12** 10, 4, -2 or -2, 4, 10

13 2, 5, 8, 11, 14 or 14, 11, 8, 5, 2

EXERCISE 2E.3

1 **a** $23.9766 \approx 24.0$ **b** $\approx 189\,134$
c ≈ 4.000 **d** ≈ 0.5852

2 **a** $S_n = \dfrac{3 + \sqrt{3}}{2}\left((\sqrt{3})^n - 1\right)$ **b** $S_n = 24(1 - (\frac{1}{2})^n)$
c $S_n = 1 - (0.1)^n$ **d** $S_n = \frac{40}{3}(1 - (-\frac{1}{2})^n)$

3 **a** $u_1 = 3$ **b** $r = \frac{1}{3}$ **c** $u_5 = \frac{1}{27}$

4 **a** 3069 **b** $\frac{4095}{1024} \approx 3.999$ **c** $-134\,217\,732$

5 **c** \$26 361.59

6 **a** $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}$ **b** $S_n = \dfrac{2^n - 1}{2^n}$
c $1 - (\frac{1}{2})^n = \dfrac{2^n - 1}{2^n}$ **d** as $n \to \infty,\ S_n \to 1$

EXERCISE 2E.4

1 **a** **i** $u_1 = \frac{3}{10}$ **ii** $r = 0.1$ **b** $S = \frac{1}{3}$

2 **a** $\frac{4}{9}$ **b** $\frac{16}{99}$ **c** $\frac{104}{333}$ **4** **a** 54 **b** 14.175

5 **a** 1 **b** $4\frac{2}{7}$ **6** $u_1 = 9,\ r = \frac{2}{3}$

7 $u_1 = 8,\ r = \frac{1}{5}$ and $u_1 = 2,\ r = \frac{4}{5}$

8 **b** $S_n = 1 + 18(1 - (0.9)^{n-1})$ **c** 19 seconds

REVIEW SET 2A

1 **a** arithmetic **b** arithmetic and geometric
c geometric **d** neither **e** arithmetic

2 $k = -\frac{11}{2}$ **3** $u_n = 33 - 5n,\ S_n = \frac{n}{2}(61 - 5n)$

4 $k = \pm\frac{2\sqrt{3}}{3}$ **5** $u_n = \frac{1}{6} \times 2^{n-1}$ or $-\frac{1}{6} \times (-2)^{n-1}$

6 21, 19, 17, 15, 13, 11

7 **a** $u_n = 89 - 3n$ **b** $u_n = \dfrac{2n + 1}{n + 3}$
c $u_n = 100(0.9)^{n-1}$

8 **a** $1 + 4 + 9 + 16 + 25 + 36 + 49$
b $\frac{4}{3} + \frac{5}{4} + \frac{6}{5} + \frac{7}{6} + \frac{8}{7} + \frac{9}{8} + \frac{10}{9} + \frac{11}{10}$

9 **a** $10\frac{4}{5}$ **b** $16 + 8\sqrt{2}$ **10** 18 metres

11 **a** $u_n = 3n + 1$

REVIEW SET 2B

1 **b** $u_1 = 6,\ r = \frac{1}{2}$ **c** 0.000 183

2 **a** 81 **b** $-1\frac{1}{2}$ **c** -486

3 **a** 1587 **b** $47\frac{253}{256} \approx 47.99$ **4** $u_{12} = 10\,240$

5 **a** €8415.31 **b** €8488.67 **c** €8505.75

6 **a** 42 **b** $u_{n+1} - u_n = 5$ **d** 1672

7 $u_n = \left(\frac{3}{4}\right) 2^{n-1}$ **a** 49 152 **b** 24 575.25

8 $u_{11} = \frac{8}{19\,683} \approx 0.000\,406$ **9** **a** 17 **b** $255\frac{511}{512} \approx 256.0$

10 **a** $\frac{1331}{2100} \approx 0.634$ **b** $6\frac{8}{15}$ **11** \$13 972.28

12 **a** 3470 **b** Year 2014

REVIEW SET 2C

1 **a** $d=-5$ **b** $u_1=63,\ d=-5$ **c** -117
d $u_{54}=-202$

2 **a** $u_1=3,\ r=4$ **b** $u_n=3\times 4^{n-1},\ u_9=196\,608$

3 $u_n=73-6n,\ u_{34}=-131$

4 **a** $\sum_{k=1}^{n}(7k-3)$ **b** $\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{k+1}$ **5** **a** 70 **b** ≈ 241.2

6 $\frac{64}{1875}$ **7** 12 **8** **a** £18 726.65 **b** £18 855.74

9 **a** $u_1=54,\ r=\frac{2}{3}$ and $u_1=150,\ r=-\frac{2}{5}$
b $|r|<1$ in both cases, so the series will converge.
For $u_1=54,\ r=\frac{2}{3},\ S=162$
For $u_1=150,\ r=-\frac{2}{5},\ S=107\frac{1}{7}$

10 **a** 35.5 km **b** 1183 km **11** **a** $0<x<1$ **b** $35\frac{5}{7}$

EXERCISE 3A

1 **a** $2^1=2,\ 2^2=4,\ 2^3=8,\ 2^4=16,\ 2^5=32,\ 2^6=64$
b $3^1=3,\ 3^2=9,\ 3^3=27,\ 3^4=81,\ 3^5=243,$
$3^6=729$
c $4^1=4,\ 4^2=16,\ 4^3=64,\ 4^4=256,\ 4^5=1024,$
$4^6=4096$

2 **a** $5^1=5,\ 5^2=25,\ 5^3=125,\ 5^4=625$
b $6^1=6,\ 6^2=36,\ 6^3=216,\ 6^4=1296$
c $7^1=7,\ 7^2=49,\ 7^3=343,\ 7^4=2401$

EXERCISE 3B

1 **a** -1 **b** 1 **c** 1 **d** -1 **e** 1 **f** -1
g -1 **h** -32 **i** -32 **j** -64 **k** 625 **l** -625

2 **a** 16 384 **b** 2401 **c** -3125 **d** -3125
e 262 144 **f** 262 144 **g** $-262\,144$
h 902.436 039 6 **i** $-902.436\,039\,6$
j $-902.436\,039\,6$

3 **a** $0.\overline{1}$ **b** $0.\overline{1}$ **c** $0.02\overline{7}$ **d** $0.02\overline{7}$
e 0.012 345 679 **f** 0.012 345 679 **g** 1 **h** 1

4 3 **5** 7

6 **a** Yes, $N=2^{s-1}$ **b** $2^{39}\approx 5.50\times 10^{11}$
c $2^{64}-1\approx 1.84\times 10^{19}$

EXERCISE 3C

1 **a** 5^{11} **b** d^8 **c** k^5 **d** $\frac{1}{7}$ **e** x^{10} **f** 3^{16}
g p^{-4} **h** n^{12} **i** 5^{3t} **j** 7^{x+2} **k** 10^{3-q} **l** c^{4m}

2 **a** 2^2 **b** 2^{-2} **c** 2^3 **d** 2^{-3} **e** 2^5 **f** 2^{-5}
g 2^1 **h** 2^{-1} **i** 2^6 **j** 2^{-6} **k** 2^7 **l** 2^{-7}

3 **a** 3^2 **b** 3^{-2} **c** 3^3 **d** 3^{-3} **e** 3^1 **f** 3^{-1}
g 3^4 **h** 3^{-4} **i** 3^0 **j** 3^5 **k** 3^{-5}

4 **a** 2^{a+1} **b** 2^{b+2} **c** 2^{t+3} **d** 2^{2x+2} **e** 2^{n-1}
f 2^{c-2} **g** 2^{2m} **h** 2^{n+1} **i** 2^1 **j** 2^{3x-1}

5 **a** 3^{p+2} **b** 3^{3a} **c** 3^{2n+1} **d** 3^{d+3} **e** 3^{3t+2}
f 3^{y-1} **g** 3^{1-y} **h** 3^{2-3t} **i** 3^{3a-1} **j** 3^3

6 **a** $4a^2$ **b** $27b^3$ **c** a^4b^4 **d** p^3q^3 **e** $\frac{m^2}{n^2}$
f $\frac{a^3}{27}$ **g** $\frac{b^4}{c^4}$ **h** 1 **i** $\frac{m^4}{81n^4}$ **j** $\frac{x^3y^3}{8}$

7 **a** $4a^2$ **b** $36b^4$ **c** $-8a^3$ **d** $-27m^6n^6$
e $16a^4b^{16}$ **f** $\frac{-8a^6}{b^6}$ **g** $\frac{16a^6}{b^2}$ **h** $\frac{9p^4}{q^6}$

8 **a** $\frac{a}{b^2}$ **b** $\frac{1}{a^2b^2}$ **c** $\frac{4a^2}{b^2}$ **d** $\frac{9b^2}{a^4}$ **e** $\frac{a^2}{bc^2}$
f $\frac{a^2c^2}{b}$ **g** a^3 **h** $\frac{b^3}{a^2}$ **i** $\frac{2}{ad^2}$ **j** $12am^3$

9 **a** a^{-n} **b** b^n **c** 3^{n-2} **d** a^nb^m **e** a^{-2n-2}

10 **a** 1 **b** $\frac{4}{7}$ **c** 6 **d** 27 **e** $\frac{9}{16}$ **f** $\frac{5}{2}$
g $\frac{27}{125}$ **h** $\frac{151}{5}$

11 **a** 3^{-2} **b** 2^{-4} **c** 5^{-3} **d** 3×5^{-1} **e** $2^2\times 3^{-3}$
f $2^{c-3}\times 3^{-2}$ **g** $3^{2k}\times 2^{-1}\times 5^{-1}$ **h** $2^p\times 3^{p-1}\times 5^{-2}$

12 **a** $5^3=21+23+25+27+29$
b $7^3=43+45+47+49+51+53+55$
c $12^3=133+135+137+139+141+143+145+147$
$+149+151+153+155$

EXERCISE 3D

1 **a** $2^{\frac{1}{5}}$ **b** $2^{-\frac{1}{5}}$ **c** $2^{\frac{3}{2}}$ **d** $2^{\frac{5}{2}}$ **e** $2^{-\frac{1}{3}}$
f $2^{\frac{4}{3}}$ **g** $2^{\frac{3}{2}}$ **h** $2^{\frac{3}{2}}$ **i** $2^{-\frac{4}{3}}$ **j** $2^{-\frac{3}{2}}$

2 **a** $3^{\frac{1}{3}}$ **b** $3^{-\frac{1}{3}}$ **c** $3^{\frac{1}{4}}$ **d** $3^{\frac{3}{2}}$ **e** $3^{-\frac{5}{2}}$

3 **a** $7^{\frac{1}{3}}$ **b** $3^{\frac{3}{4}}$ **c** $2^{\frac{4}{5}}$ **d** $2^{\frac{5}{3}}$ **e** $7^{\frac{2}{7}}$
f $7^{-\frac{1}{3}}$ **g** $3^{-\frac{3}{4}}$ **h** $2^{-\frac{4}{5}}$ **i** $2^{-\frac{5}{3}}$ **j** $7^{-\frac{2}{7}}$

4 **a** 2.28 **b** 1.83 **c** 0.794 **d** 0.435 **e** 1.68
f 1.93 **g** 0.523

5 **a** 8 **b** 32 **c** 8 **d** 125 **e** 4
f $\frac{1}{2}$ **g** $\frac{1}{27}$ **h** $\frac{1}{16}$ **i** $\frac{1}{81}$ **j** $\frac{1}{25}$

EXERCISE 3E.1

1 **a** $x^5+2x^4+x^2$ **b** $2^{2x}+2^x$ **c** $x+1$
d $7^{2x}+2(7^x)$ **e** $2(3^x)-1$ **f** x^2+2x+3
g $1+5(2^{-x})$ **h** 5^x+1 **i** $x^{\frac{3}{2}}+x^{\frac{1}{2}}+1$

2 **a** $4^x+2^{2+x}+3$ **b** $9^x+7(3^x)+10$
c $25^x-6(5^x)+8$ **d** $4^x+6(2^x)+9$
e $9^x-2(3^x)+1$ **f** $16^x+14(4^x)+49$
g $x-4$ **h** 4^x-9 **i** $x-x^{-1}$ **j** $x^2+4+\frac{4}{x^2}$
k $7^{2x}-2+7^{-2x}$ **l** $25-10(2^{-x})+4^{-x}$

EXERCISE 3E.2

1 **a** $5^x(5^x+1)$ **b** $10(3^n)$ **c** $7^n(1+7^{2n})$
d $5(5^n-1)$ **e** $6(6^{n+1}-1)$ **f** $16(4^n-1)$

2 **a** $(3^x+2)(3^x-2)$ **b** $(2^x+5)(2^x-5)$ **c** $(4+3^x)(4-3^x)$
d $(5+2^x)(5-2^x)$ **e** $(3^x+2^x)(3^x-2^x)$ **f** $(2^x+3)^2$
g $(3^x+5)^2$ **h** $(2^x-7)^2$ **i** $(5^x-2)^2$

3 **a** $(2^x+3)(2^x+6)$ **b** $(2^x+4)(2^x-5)$
c $(3^x+2)(3^x+7)$ **d** $(3^x+5)(3^x-1)$
e $(5^x+2)(5^x-1)$ **f** $(7^x-4)(7^x-3)$

4 **a** 2^n **b** 10^a **c** 3^b **d** $\frac{1}{5^n}$ **e** 5^x **f** $(\frac{3}{4})^a$
g 5 **h** 5^n

5 **a** 3^m+1 **b** $1+6^n$ **c** 4^n+2^n **d** 4^x-1
e 6^n **f** 5^n **g** 4 **h** 2^n-1 **i** $\frac{1}{2}$

6 **a** $n\,2^{n+1}$ **b** -3^{n-1}

EXERCISE 3F

1 **a** $x = 3$ **b** $x = 2$ **c** $x = 4$ **d** $x = 0$
e $x = -1$ **f** $x = -2$ **g** $x = -3$ **h** $x = 2$
i $x = -3$ **j** $x = -4$ **k** $x = 2$ **l** $x = 1$

2 **a** $x = \frac{5}{3}$ **b** $x = -\frac{3}{2}$ **c** $x = -\frac{3}{2}$ **d** $x = -\frac{1}{2}$
e $x = -\frac{2}{3}$ **f** $x = -\frac{5}{4}$ **g** $x = \frac{3}{2}$ **h** $x = \frac{5}{2}$
i $x = \frac{1}{8}$ **j** $x = \frac{9}{2}$ **k** $x = -4$ **l** $x = -4$
m $x = 0$ **n** $x = \frac{7}{2}$ **o** $x = -2$ **p** $x = -6$

3 **a** $x = \frac{1}{7}$ **b** has no solutions **c** $x = 2\frac{1}{2}$

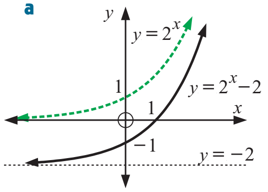
4 **a** $x = 3$ **b** $x = 3$ **c** $x = 2$
d $x = 2$ **e** $x = -2$ **f** $x = -2$

5 **a** $x = 1$ or 2 **b** $x = 1$ **c** $x = 1$ or 2
d $x = 1$ **e** $x = 2$ **f** $x = 0$

EXERCISE 3G

1 **a** 1.4 **b** 1.7 **c** 2.8 **d** 0.3 **e** 2.7 **f** 0.4

2 **a**



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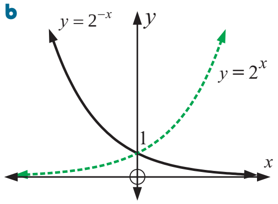

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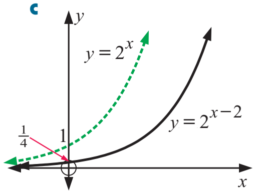

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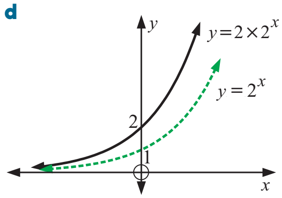
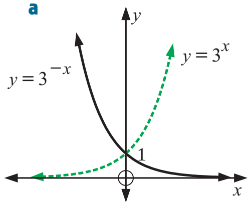


3 **a**



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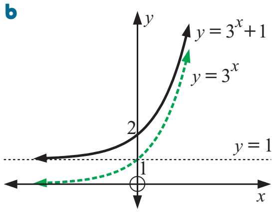

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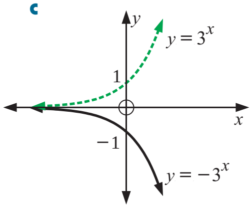

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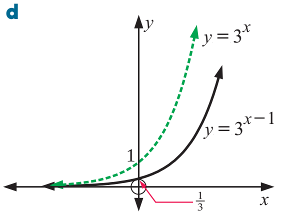
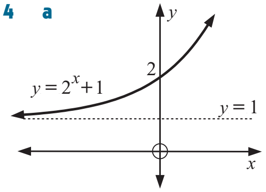


4 **a**



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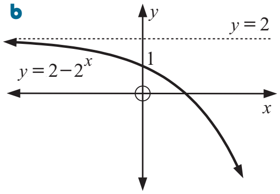

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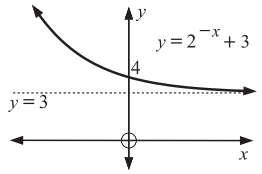
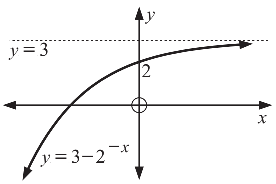


d

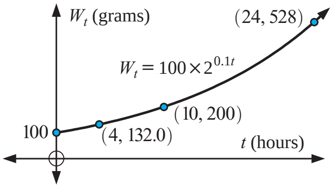


5 **a** $y \approx 3.67$ **b** $y \approx -0.665$ **c** $y \approx 3.38$
d $y \approx 2.62$

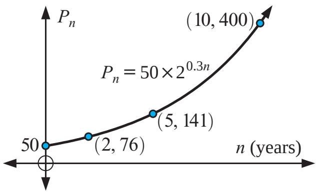
6 **a** as $x \to \infty$, $y \to \infty$
as $x \to -\infty$, $y \to 1$ (above) HA is $y = 1$
b as $x \to \infty$, $y \to -\infty$
as $x \to -\infty$, $y \to 2$ (below) HA is $y = 2$
c as $x \to \infty$, $y \to 3$ (above)
as $x \to -\infty$, $y \to \infty$ HA is $y = 3$
d as $x \to \infty$, $y \to 3$ (below)
as $x \to -\infty$, $y \to -\infty$ HA is $y = 3$

EXERCISE 3H.1

1 **a** 100 grams
b **i** 132 g
ii 200 g
iii 528 g
c



2 **a** 50
b **i** 76
ii 141
iii 400
c

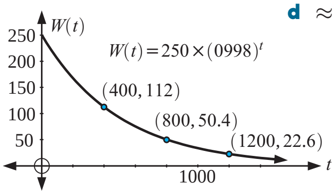


3 **a** V_0 **b** $2V_0$ **c** 100%
d 183% increase, percentage increase at 50°C compared with 20°C

4 **a** 12 bears **b** 146 bears **c** 248% increase

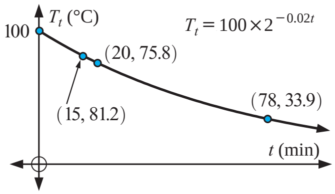
EXERCISE 3H.2

1 **a** 250 g **b** **i** 112 g **ii** 50.4 g **iii** 22.6 g
c

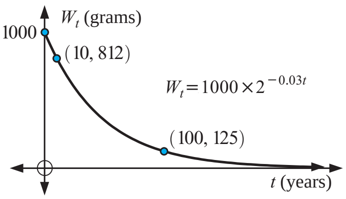


d ≈ 346 years

2 **a** 100°C
b **i** 81.2°C
ii 75.8°C
iii 33.9°C
c



3 **a** 1000 g
b **i** 812 g **ii** 125 g **iii** 9.31×10^{-7} g
c

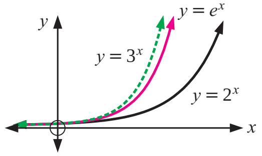


4 **a** W_0 **b** 12.9%

EXERCISE 3I

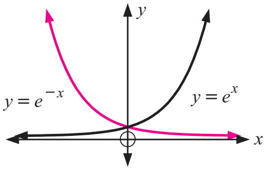
1 $e^1 \approx 2.718\,281\,828\ldots.$

2



The graph of $y = e^x$ lies between $y = 2^x$ and $y = 3^x$.

3 One is the other reflected in the y-axis.



4 a

5 **a** $e^x > 0$ for all x
b **i** 0.000 000 004 12 **ii** 970 000 000

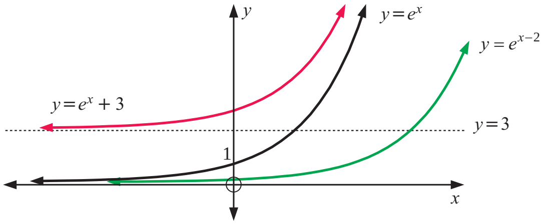
6 **a** ≈ 7.39 **b** ≈ 20.1 **c** ≈ 2.01 **d** ≈ 1.65 **e** ≈ 0.368

7 **a** $e^{\frac{1}{2}}$ **b** $e^{\frac{3}{2}}$ **c** $e^{-\frac{1}{2}}$ **d** e^{-2}

8 **a** $e^{0.18t}$ **b** $e^{0.004t}$ **c** $e^{-0.005t}$ **d** $\approx e^{-0.167t}$

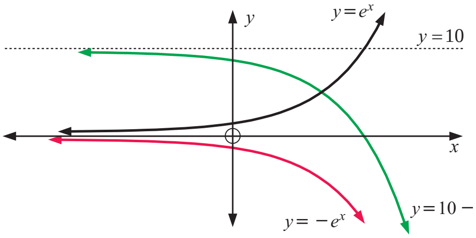
9 **a** 10.074 **b** 0.099 261 **c** 125.09 **d** 0.007 994 5 **e** 41.914 **f** 42.429 **g** 3540.3 **h** 0.006 342 4

10



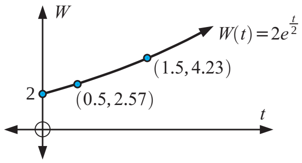
Domain of f, g and h is $\{x \mid x \in \mathbb{R}\}$
Range of f is $\{y \mid y > 0\}$, Range of g is $\{y \mid y > 0\}$
Range of h is $\{y \mid y > 3\}$

11

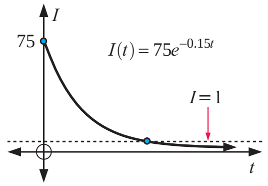


Domain of f, g and h is $\{x \mid x \in \mathbb{R}\}$
Range of f is $\{y \mid y > 0\}$, Range of g is $\{y \mid y < 0\}$
Range of h is $\{y \mid y < 10\}$

12 **a** **i** 2 g **ii** 2.57 g **iii** 4.23 g **iv** 40.2 g
b

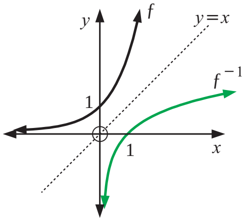


13 **a** **i** 64.6 amps **ii** 16.7 amps
b



c 28.8 seconds

14 $f^{-1}(x) = \log_e x$



REVIEW SET 3A

1 **a** -1 **b** 27 **c** $\frac{2}{3}$ **2** **a** a^6b^7 **b** $\dfrac{2}{3x}$ **c** $\dfrac{y^2}{5}$

3 **a** 2^{-4} **b** 2^{x+2} **c** 2^{2x-3}

4 **a** $\dfrac{1}{x^5}$ **b** $\dfrac{2}{a^2b^2}$ **c** $\dfrac{2a}{b^2}$ **5** **a** 3^{3-2a} **b** $3^{\frac{5}{2}-\frac{9}{2}x}$

6 **a** 4 **b** $\frac{1}{9}$ **7** **a** $\dfrac{m}{n^2}$ **b** $\dfrac{1}{m^3n^3}$ **c** $\dfrac{m^2p^2}{n}$ **d** $\dfrac{16n^2}{m^2}$

8 **a** $9 - 6(2^a) + 2^{2a}$ **b** $x - 4$ **c** $2^x + 1$

9 **a** $x = -2$ **b** $x = \frac{3}{4}$ **10** **a** $x = \frac{1}{3}$ **b** $x = -\frac{4}{5}$

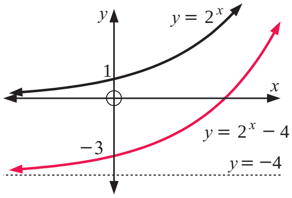
REVIEW SET 3B

1 **a** 2^{n+2} **b** $-\frac{6}{7}$ **c** $3\frac{3}{8}$ **d** $\dfrac{4}{a^2b^4}$

2 **a** 2.28 **b** 0.517 **c** 3.16 **3** **a** 3 **b** 24 **c** $\frac{3}{4}$

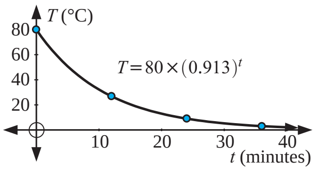
4 **a** 3^4 **b** 3^0 **c** 3^{-3} **d** 3^{-5}

5



a $y = 2^x$ has y-intercept 1 and horizontal asymptote $y = 0$
b $y = 2^x - 4$ has y-intercept -3 and horizontal asymptote $y = -4$

6 **a** 80°C
b **i** 26.8°C **ii** 9.00°C **iii** 3.02°C
c

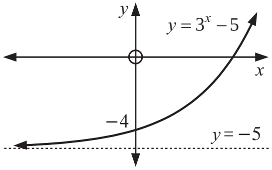


d ≈ 12.8 min

7 **a**

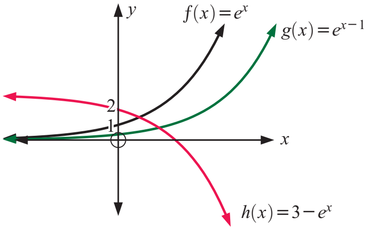
x	-2	-1	0	1	2
y	$-4\frac{8}{9}$	$-4\frac{2}{3}$	-4	-2	4

b as $x \to \infty$, $y \to \infty$; as $x \to -\infty$, $y \to -5$ (above)
c



d $y = -5$

8



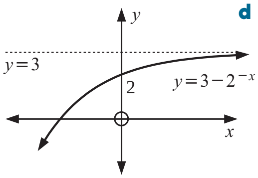
Domain of f, g and h is $\{x \mid x \in \mathbb{R}\}$
Range of f is $\{y \mid y > 0\}$ Range of g is $\{y \mid y > 0\}$
Range of h is $\{y \mid y < 3\}$

9 **a**

x	-2	-1	0	1	2
y	-1	1	2	$2\frac{1}{2}$	$2\frac{3}{4}$

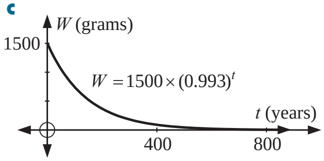
b as $x \to \infty$, $y \to 3$ (below); as $x \to -\infty$, $y \to -\infty$

c



d $y = 3$

10 **a** 1500 g
b **i** 90.3 g
ii 5.44 g
c



d 386 years

REVIEW SET 3C

1 **a** 8 **b** $-\frac{4}{5}$ **2** **a** a^{21} **b** p^4q^6 **c** $\dfrac{4b}{a^3}$

3 **a** 2^{-3} **b** 2^7 **c** 2^{12} **4** **a** $\dfrac{1}{b^3}$ **b** $\dfrac{1}{ab}$ **c** $\dfrac{a}{b}$

5 2^{2x} **6** **a** 5^0 **b** $5^{\frac{3}{2}}$ **c** $5^{-\frac{1}{4}}$ **d** 5^{2a+6}

7 **a** $1 + e^{2x}$ **b** $2^{2x} + 10(2^x) + 25$ **c** $x - 49$

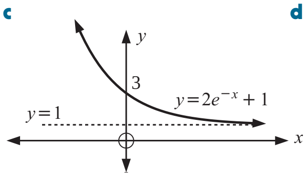
8 **a** $x = 5$ **b** $x = -4$ **9** **a** $x = 4$ **b** $x = -\frac{2}{5}$

10 **a**

x	-2	-1	0	1	2
y	15.8	6.44	3	1.74	1.27

b as $x \to \infty$, $y \to 1$ (above); as $x \to -\infty$, $y \to \infty$

c



d $y = 1$

EXERCISE 4A

1 **a** $10^4 = 10\,000$ **b** $10^{-1} = 0.1$ **c** $10^{\frac{1}{2}} = \sqrt{10}$
d $2^3 = 8$ **e** $2^{-2} = \frac{1}{4}$ **f** $3^{1.5} = \sqrt{27}$

2 **a** $\log_2 4 = 2$ **b** $\log_2(\frac{1}{8}) = -3$
c $\log_{10}(0.01) = -2$ **d** $\log_7 49 = 2$
e $\log_2 64 = 6$ **f** $\log_3(\frac{1}{27}) = -3$

3 **a** 5 **b** -2 **c** $\frac{1}{2}$ **d** 3 **e** 6 **f** 7 **g** 2
h 3 **i** -3 **j** $\frac{1}{2}$ **k** 2 **l** $\frac{1}{2}$ **m** 5 **n** $\frac{1}{3}$
o n **p** $\frac{1}{3}$ **q** -1 **r** $\frac{3}{2}$ **s** 0 **t** 1

4 **a** ≈ 2.18 **b** ≈ 1.40 **c** ≈ 1.87 **d** ≈ -0.0969
5 **a** $x = 8$ **b** $x = 2$ **c** $x = 3$ **d** $x = 14$
6 **a** 2 **b** 2 **c** -1 **d** $\frac{3}{4}$ **e** $-\frac{1}{2}$ **f** $\frac{5}{2}$ **g** $-\frac{3}{2}$ **h** $-\frac{3}{4}$

EXERCISE 4B

1 **a** 4 **b** -3 **c** 1 **d** 0 **e** $\frac{1}{2}$ **f** $\frac{1}{3}$ **g** $-\frac{1}{4}$ **h** $1\frac{1}{2}$
i $\frac{2}{3}$ **j** $1\frac{1}{2}$ **k** $1\frac{1}{3}$ **l** $3\frac{1}{2}$ **m** n **n** $a+2$ **o** $1-m$
p $a - b$

2 The following include calculator keys for the TI-84+:

a $\boxed{\text{log}}$ 10 000 $\boxed{\text{enter}}$, 4 **b** $\boxed{\text{log}}$ 0.001 $\boxed{\text{enter}}$, -3

c $\boxed{\text{log}}$ $\boxed{\text{2nd}}$ $\boxed{\sqrt{\ }}$ 10 $\boxed{)}$ $\boxed{)}$ $\boxed{\text{enter}}$, 0.5

d $\boxed{\text{log}}$ 10 $\boxed{\wedge}$ $\boxed{(}$ 1 $\boxed{\div}$ 3 $\boxed{)}$ $\boxed{)}$ $\boxed{\text{enter}}$, $0.\bar{3}$

e $\boxed{\text{log}}$ 100 $\boxed{\wedge}$ $\boxed{(}$ 1 $\boxed{\div}$ 3 $\boxed{)}$ $\boxed{)}$ $\boxed{\text{enter}}$, $0.\bar{6}$

f $\boxed{\text{log}}$ 10 $\boxed{\times}$ $\boxed{\text{2nd}}$ $\boxed{\sqrt{\ }}$ 10 $\boxed{)}$ $\boxed{)}$ $\boxed{\text{enter}}$, 1.5

g $\boxed{\text{log}}$ 1 $\boxed{\div}$ $\boxed{\text{2nd}}$ $\boxed{\sqrt{\ }}$ 10 $\boxed{)}$ $\boxed{)}$ $\boxed{\text{enter}}$, -0.5

h $\boxed{\text{log}}$ 1 $\boxed{\div}$ 10 $\boxed{\wedge}$ 0.25 $\boxed{)}$ $\boxed{\text{enter}}$, -0.25

3 **a** $10^{0.7782}$ **b** $10^{1.7782}$ **c** $10^{3.7782}$
d $10^{-0.2218}$ **e** $10^{-2.2218}$ **f** $10^{1.1761}$
g $10^{3.1761}$ **h** $10^{0.1761}$ **i** $10^{-0.8239}$
a $10^{-3.8239}$

4 **a** **i** 0.477 **ii** 2.477 **b** $\log 300 = \log(3 \times 10^2)$
5 **a** **i** 0.699 **ii** -1.301 **b** $\log 0.05 = \log(5 \times 10^{-2})$
6 **a** $x = 100$ **b** $x = 10$ **c** $x = 1$
d $x = \frac{1}{10}$ **e** $x = 10^{\frac{1}{2}}$ **f** $x = 10^{-\frac{1}{2}}$
g $x = 10\,000$ **h** $x = 0.000\,01$ **i** $x \approx 6.84$
j $x \approx 140$ **k** $x \approx 0.0419$ **l** $x \approx 0.000\,631$

EXERCISE 4C.1

1 **a** $\log 16$ **b** $\log 4$ **c** $\log 8$ **d** $\log 20$
e $\log 2$ **f** $\log 24$ **g** $\log 30$ **h** $\log 0.4$
i $\log 10$ **j** $\log 200$ **k** $\log 0.4$ **l** $\log 1$ or 0
m $\log 0.005$ **n** $\log 20$ **o** $\log 28$

2 **a** $\log 96$ **b** $\log 72$ **c** $\log 8$ **d** $\log\left(\frac{25}{8}\right)$
e $\log 6$ **f** $\log \frac{1}{2}$ **g** $\log 20$ **h** $\log 25$
i 1

3 **a** 2 **b** $\frac{3}{2}$ **c** 3 **d** $\frac{1}{2}$ **e** -2 **f** $-\frac{3}{2}$

5 **a** $p + q$ **b** $2p + 3q$ **c** $2q + r$ **d** $r + \frac{1}{2}q - p$
e $r - 5p$ **f** $p - 2q$

6 **a** $x + z$ **b** $z + 2y$ **c** $x + z - y$ **d** $2x + \frac{1}{2}y$
e $3y - \frac{1}{2}z$ **f** $2z + \frac{1}{2}y - 3x$

7 **a** 0.86 **b** 2.15 **c** 1.075

EXERCISE 4C.2

1 **a** $\log y = x \log 2$ **b** $\log y \approx 1.301 + 3 \log b$
c $\log M = \log a + 4 \log d$ **d** $\log T \approx 0.699 + \frac{1}{2} \log d$
e $\log R = \log b + \frac{1}{2} \log l$ **f** $\log Q = \log a - n \log b$
g $\log y = \log a + x \log b$ **h** $\log F \approx 1.30 - \frac{1}{2} \log n$
i $\log L = \log a + \log b - \log c$ **j** $\log N = \frac{1}{2} \log a - \frac{1}{2} \log b$
k $\log S \approx 2.30 + t \log 2$ **l** $\log y = m \log a - n \log b$

2 **a** $D = 2e$ **b** $F = \frac{5}{t}$ **c** $P = \sqrt{x}$ **d** $M = b^2c$
e $B = \frac{m^3}{n^2}$ **f** $N = \frac{1}{\sqrt[3]{p}}$ **g** $P = 10x^3$ **h** $Q = \frac{100}{x}$

3 **a** $x = 9$ **b** $x = 2$ or 4 **c** $x = 25\sqrt{5}$
d $x = 200$ **e** $x = 5$ **f** $x = 3$

EXERCISE 4D.1

1 **a** 3 **b** 0 **c** $\frac{1}{3}$ **d** -2

3 x does not exist such that $e^x = -2$ or 0

4 **a** a **b** $a + 1$ **c** $a + b$ **d** ab **e** $a - b$

5 **a** $e^{1.7918}$ **b** $e^{4.0943}$ **c** $e^{8.6995}$ **d** $e^{-0.5108}$
e $e^{-5.1160}$ **f** $e^{2.7081}$ **g** $e^{7.3132}$ **h** $e^{0.4055}$
i $e^{-1.8971}$ **j** $e^{-8.8049}$

6 **a** $x \approx 20.1$ **b** $x \approx 2.72$ **c** $x = 1$
d $x \approx 0.368$ **e** $x \approx 0.006\,74$ **f** $x \approx 2.30$
g $x \approx 8.54$ **h** $x \approx 0.0370$

EXERCISE 4D.2

1 **a** $\ln 45$ **b** $\ln 5$ **c** $\ln 4$ **d** $\ln 24$
e $\ln 1 = 0$ **f** $\ln 30$ **g** $\ln 4e$ **h** $\ln\left(\frac{6}{e}\right)$
i $\ln 20$ **j** $\ln 4e^2$ **k** $\ln\left(\frac{20}{e^2}\right)$ **l** $\ln 1 = 0$

2 **a** $\ln 972$ **b** $\ln 200$ **c** $\ln 1 = 0$ **d** $\ln 16$ **e** $\ln 6$
f $\ln\left(\frac{1}{3}\right)$ **g** $\ln\left(\frac{1}{2}\right)$ **h** $\ln 2$ **i** $\ln 16$

3 For example, for **a**, $\ln 27 = \ln 3^3 = 3 \ln 3$

5 **a** $D = ex$ **b** $F = \frac{e^2}{p}$ **c** $P = \sqrt{x}$
d $M = e^3y^2$ **e** $B = \frac{t^3}{e}$ **f** $N = \frac{1}{\sqrt[3]{g}}$
g $Q \approx 8.66x^3$ **h** $D \approx 0.518n^{0.4}$

EXERCISE 4E

1 **a** $x \approx 3.32$ **b** $x \approx 2.73$ **c** $x \approx 3.32$
d $x \approx 37.9$ **e** $x \approx -3.64$ **f** $x \approx -7.55$
g $x \approx 7.64$ **h** $x \approx 32.0$ **i** $x \approx 1150$

2 **a** $t \approx 6.340$ **b** $t \approx 74.86$ **c** $t \approx 8.384$
d $t \approx 132.9$ **e** $t \approx 121.5$ **f** $t \approx 347.4$

3 **a** $x \approx 2.303$ **b** $x \approx 6.908$ **c** $x \approx -4.754$
d $x \approx 3.219$ **e** $x \approx 15.18$ **f** $x \approx -40.85$
g $x \approx -14.63$ **h** $x \approx 137.2$ **i** $x \approx 4.868$

EXERCISE 4F

1 **a** ≈ 2.26 **b** ≈ -10.3 **c** ≈ -2.46 **d** ≈ 5.42

2 **a** $x \approx -4.29$ **b** $x \approx 3.87$ **c** $x \approx 0.139$

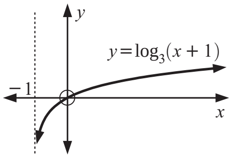
3 **a** $x \approx 0.683$ **b** $x \approx -1.89$

4 **a** $x = 16$ **b** $x \approx 1.71$

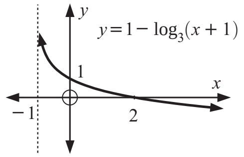
5 $x = \frac{\log 8}{\log 25}$ or $\log_{25} 8$

EXERCISE 4G

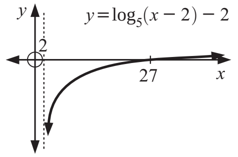
1 **a** **i** $x > -1,\ y \in \mathbb{R}$
ii VA is $x = -1$,
x and y-intercepts 0
iv $x = -\frac{2}{3}$
v $f^{-1}(x) = 3^x - 1$
iii



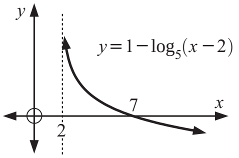
b **i** $x > -1,\ y \in \mathbb{R}$
ii VA is $x = -1$,
x-intercept 2,
y-intercept 1
iv $x = 8$
v $f^{-1}(x) = 3^{1-x} - 1$
iii



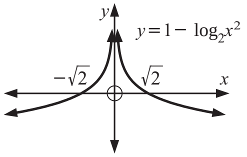
c **i** $x > 2,\ y \in \mathbb{R}$
ii VA is $x = 2$,
x-intercept 27,
no y-intercept
iv $x = 7$
v $f^{-1}(x) = 5^{2+x} + 2$
iii



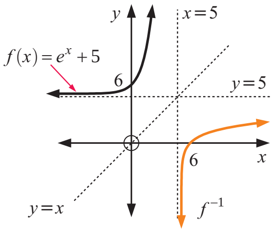
d **i** $x > 2,\ y \in \mathbb{R}$
ii VA is $x = 2$,
x-intercept 7,
no y-intercept
iv $x = 27$
v $f^{-1}(x) = 5^{1-x} + 2$
iii



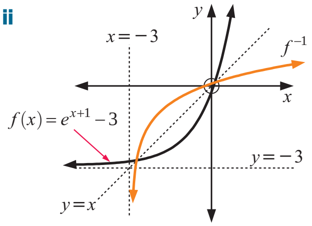
e **i** $x \in \mathbb{R},\ x \neq 0$,
$y \in \mathbb{R}$
ii VA is $x = 0$,
x-intercepts $\pm\sqrt{2}$,
no y-intercept
iv $x = \pm 2$
v if $x > 0$, $f^{-1}(x) = 2^{\frac{1-x}{2}}$
if $x < 0$, $f^{-1}(x) = -2^{\frac{1-x}{2}}$
iii



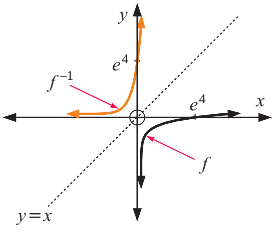
2 **a** **i** $f^{-1}(x)$
$= \ln(x - 5)$
iii domain of f is
$\{x \mid x \in \mathbb{R}\}$,
range is $\{y \mid y > 5\}$
domain of f^{-1} is
$\{x \mid x > 5\}$,
range is $\{y \mid y \in \mathbb{R}\}$
iv f has a HA $y = 5$,
f^{-1} has a VA $x = 5$
ii



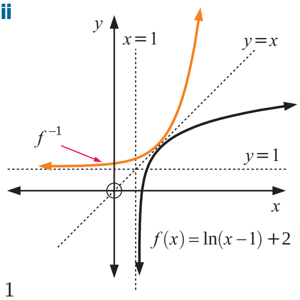
b **i** $f^{-1}(x)$
$= \ln(x + 3) - 1$
iii domain of f is
$\{x \mid x \in \mathbb{R}\}$,
range is
$\{y \mid y > -3\}$
domain of f^{-1} is
$\{x \mid x > -3\}$,
range is
$\{y \mid y \in \mathbb{R}\}$
iv f has a HA $y = -3$, f^{-1} has a VA $x = -3$
ii



c **i** $f^{-1}(x)$
$= e^{x+4}$
iii domain of f is
$\{x \mid x > 0\}$,
range of f is
$\{y \mid y \in \mathbb{R}\}$
domain of f^{-1} is
$\{x \mid x \in \mathbb{R}\}$,
range is
$\{y \mid y > 0\}$
iv f has a VA $x = 0$, f^{-1} has a HA $y = 0$
ii



d **i** $f^{-1}(x)$ $= 1 + e^{x-2}$ **ii**



iii domain of f is $\{x \mid x > 1\}$, range is $\{y \mid y \in \mathbb{R}\}$
domain of f^{-1} is $\{x \mid x \in \mathbb{R}\}$, range is $\{y \mid y > 1\}$

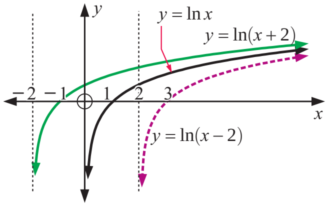
iv f has a VA $x = 1$, f^{-1} has a HA $y = 1$

3 $f^{-1}(x) = \frac{1}{2}\ln x$

a $(f^{-1} \circ g)(x) = \frac{1}{2}\ln(2x-1)$

b $(g \circ f)^{-1}(x) = \frac{1}{2}\ln\left(\frac{x+1}{2}\right)$

4 **a** A is $y = \ln x$ as its x-intercept is 1 **b**



c $y = \ln x$ has VA $x = 0$
$y = \ln(x-2)$ has VA $x = 2$
$y = \ln(x+2)$ has VA $x = -2$

5 $y = \ln(x^2) = 2\ln x$, so she is correct.
This is because the y-values are twice as large for $y = \ln(x^2)$ as they are for $y = \ln x$.

6 **a** $f^{-1} : x \mapsto \ln(x-2) - 3$

b **i** $x < -5.30$ **ii** $x < -7.61$ **iii** $x < -9.91$
iv $x < -12.2$ Conjecture HA is $y = 2$

c as $x \to -\infty$, $e^{x+3} \to 0$ and $y \to 2$ $\therefore$ HA is $y = 2$

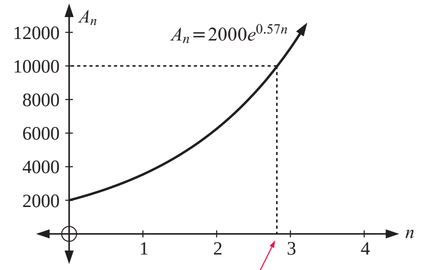
d VA of f^{-1} is $x = 2$, domain of f^{-1} is $\{x \mid x > 2\}$

EXERCISE 4H

1 **a** 3.90 h
b 15.5 h

2 **a** 6.93 h
b 13.9 h

3 **a** see graph alongside
b $n \approx 2.82$



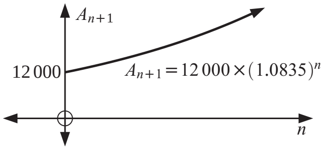
$\therefore$ approximately 2.8 weeks

4 In 6.17 years, or 6 years 62 days

5 8.65 years, or 8 years 237 days

6 **a** $\frac{8.4\%}{12} = 0.7\% = 0.007$ $r = 1 + 0.007 = 1.007$
b after 74 months

7 **a** €12 000 **b** $A_6 =$ €17 919.50
c $A_{3.25}$ is the value after 2 years 3 months
d 8.64 years **e**



8 **a** 17.3 years **b** 92.2 years **c** 115 years **9** 8.05 s

10 **a** 50.7 min **b** 152 min

11 **a** 25 years **b** 141 years **c** 166 years

12 **a** 10 000 years **b** 49 800 years

13 166 seconds **14** 12.9 seconds

REVIEW SET 4A

1 **a** 3 **b** 8 **c** -2 **d** $\frac{1}{2}$ **e** 0
f 1 **g** $\frac{1}{4}$ **h** -1 **i** $\frac{1}{3}$ **j** $\frac{1}{2}$

2 **a** $\frac{1}{2}$ **b** $-\frac{1}{3}$ **c** $a + b + 1$

3 **a** $\ln 144$ **b** $\ln\left(\frac{3}{2}\right)$ **c** $\ln\left(\frac{25}{e}\right)$ **d** $\ln 3$

4 **a** $\frac{3}{2}$ **b** -3 **c** $-\frac{3}{2}$

5 **a** $\log 144$ **b** $\log_2\left(\frac{16}{9}\right)$ **c** $\log_4 80$

6 **a** $\log P = \log 3 + x \log b$ **b** $\log m = 3\log n - 2\log p$

7 **a** $2x$ **b** $2 + x$ **c** $1 - x$

8 **a** $T = \dfrac{x^2}{y}$ **b** $K = n\sqrt{t}$

9 **a** $5\ln 2$ **b** $3\ln 5$ **c** $6\ln 3$

10 **a** $2A + 2B$ **b** $A + 3B$ **c** $3A + \frac{1}{2}B$
d $4B - 2A$ **e** $3A - 2B$

REVIEW SET 4B

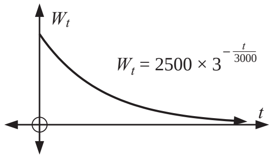
1 **a** $\approx 10^{1.51}$ **b** $\approx 10^{-2.89}$ **c** $\approx 10^{-4.05}$

2 **a** $x = \frac{1}{8}$ **b** $x \approx 82.7$ **c** $x \approx 0.0316$

3 **a** $k \approx 3.25 \times 2^x$ **b** $Q = P^3R$ **c** $A \approx \dfrac{B^5}{400}$

4 **a** $x \approx 1.209$ **b** $x \approx 1.822$

5 **a** 2500 g **d**
b 3290 years
c 42.3%



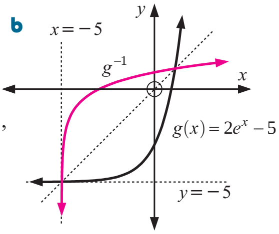
6 $x \approx 2.32$

7 **a** $x \approx 148$
b $x \approx 0.513$

8 **a** $x \approx 5.99$
b $x \approx 0.699$
c $x \approx 6.80$ **d** $x \approx 1.10$ or 1.39

9 **a** 3 years **b** 152%

10 **a** $g^{-1}(x)$ $= \ln\left(\frac{x+5}{2}\right)$ **b**



c domain of g is $\{x \mid x \in \mathbb{R}\}$, range is $\{y \mid y > -5\}$
domain of g^{-1} is $\{x \mid x > -5\}$, range is $\{y \mid y \in \mathbb{R}\}$

REVIEW SET 4C

1 **a** $\frac{3}{2}$ **b** $\frac{2}{3}$ **c** $a + b$

2 **a** 5 **b** $\frac{1}{2}$ **c** -1

3 **a** $\approx e^{3.00}$ **b** $\approx e^{8.01}$ **c** $\approx e^{-2.59}$

4 **a** $x = 1000$ **b** $x \approx 4.70$ **c** $x \approx 6.28$

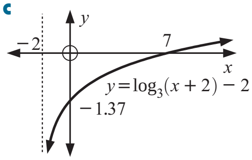
5 **a** $\ln 24$ **b** $\ln 3$ **c** $\ln 4$ **d** $\ln 125$

6 **a** $\log M = \log a + n \log b$ **b** $\log T = \log 5 - \frac{1}{2}\log l$
c $\log G = 2\log a + \log b - \log c$

7 **a** $x \approx 5.19$ **b** $x \approx 4.29$ **c** $x \approx -0.839$

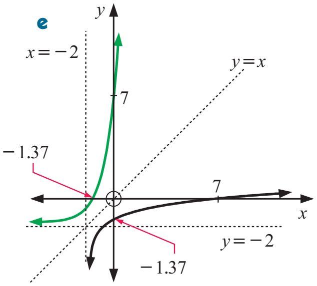
8 **a** $P = TQ^{1.5}$ **b** $M = \dfrac{e^{1.2}}{\sqrt{N}}$

9 **a** $x > -2,\ y \in \mathbb{R}$
b VA is $x = -2$,
x-intercept is 7,
y-intercept is ≈ -1.37
c



d $g^{-1}(x) = 3^{x+2} - 2$

e



10 **a** 13.9 weeks **b** 41.6 weeks **c** 138 weeks

EXERCISE 5A

1 **a** $2x$ **b** $x+2$ **c** $\dfrac{x}{2}$ **d** $2x+3$

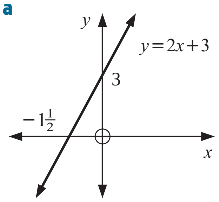
2 **a** $9x^2$ **b** $\dfrac{x^2}{4}$ **c** $3x^2$ **d** $2x^2 - 4x + 7$

3 **a** $64x^3$ **b** $4x^3$ **c** $x^3 + 3x^2 + 3x + 1$
d $2x^3 + 6x^2 + 6x - 1$

4 **a** 4^x **b** $2^{-x} + 1$ **c** $2^{x-2} + 3$ **d** $2^{x+1} + 3$

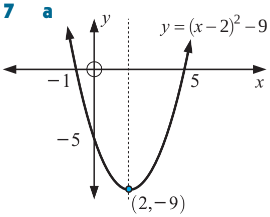
5 **a** $-\dfrac{1}{x}$ **b** $\dfrac{2}{x}$ **c** $\dfrac{2+3x}{x}$ **d** $\dfrac{2x+1}{x-1}$

6 **a**



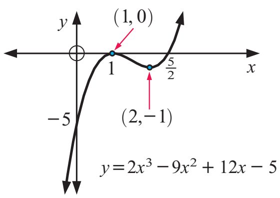
b **i** $-1\frac{1}{2}$ **ii** 3 **iii** 2

7 **a**

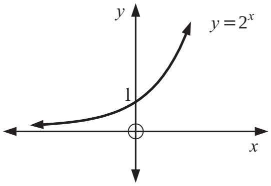


b x-ints are -1 and 5
y-int is -5

8 a,b

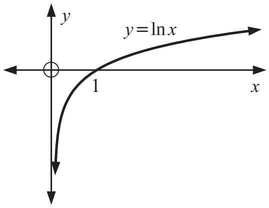


9



When $x = 0$,
$y = 2^0 = 1$ ✓
$2^x > 0$ for all x as the graph is always above the x-axis. ✓

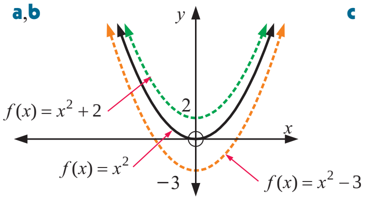
10



When $y = 0$,
$\ln x = 0$
$\therefore\ x = e^0 = 1.$

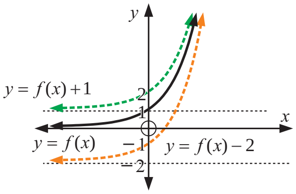
EXERCISE 5B.1

1 a,b

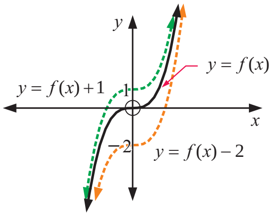


c **i** If $b > 0$, the function is translated vertically upwards through b units.
ii If $b < 0$, the function is translated vertically downwards $|b|$ units.

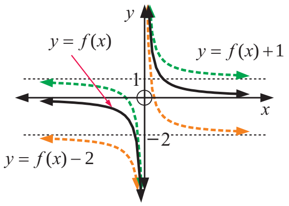
2 **a**



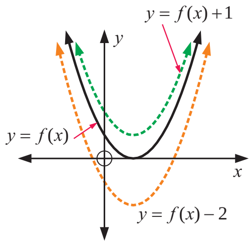
b



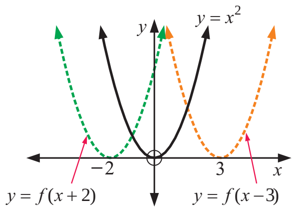
c



d

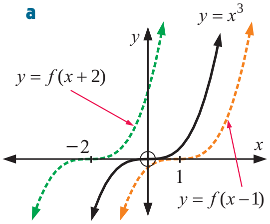


3 **a**

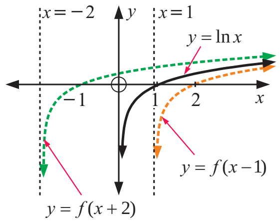


b **i** If $a > 0$, the graph is translated a units right.
ii If $a < 0$, the graph is translated $|a|$ units left.

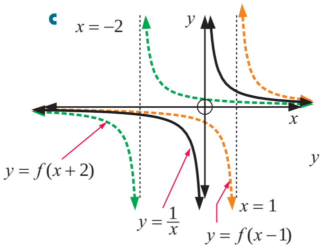
4 **a**



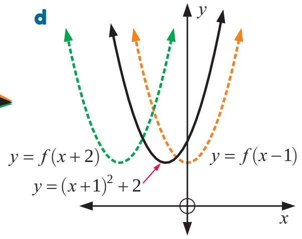
b



c

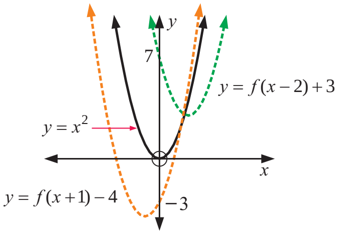


d

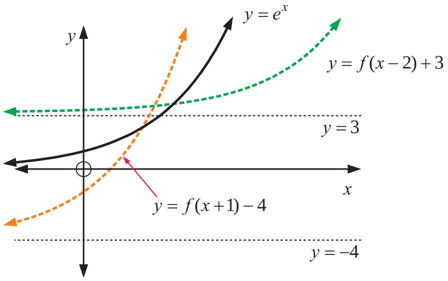


$y = f(x - a)$ is a horizontal translation of $y = f(x)$ through $\begin{pmatrix} a \\ 0 \end{pmatrix}$.

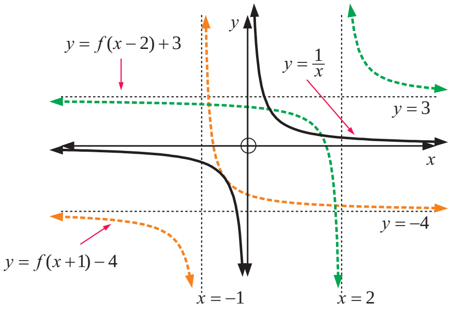
5 **a**



b

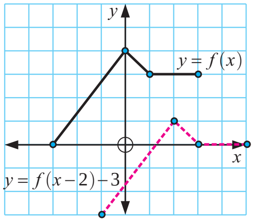


c

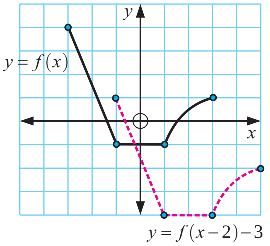


6 A translation of $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

a



b

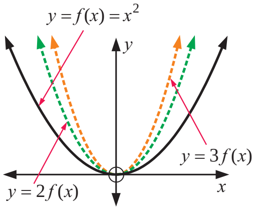


7 **a** **i** (3, 2) **ii** (0, 11) **iii** (5, 6)

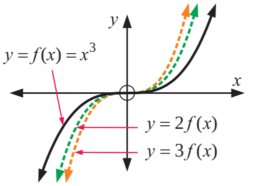
b **i** (−2, 4) **ii** (−5, 25) **iii** $(-1\frac{1}{2}, 2\frac{1}{4})$

EXERCISE 5B.2

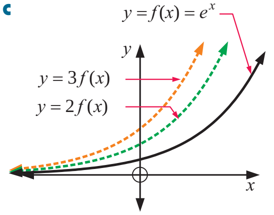
1 **a**



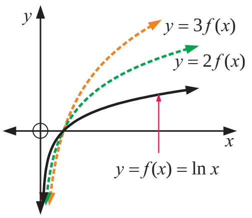
b



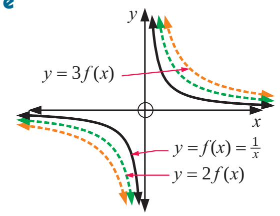
c



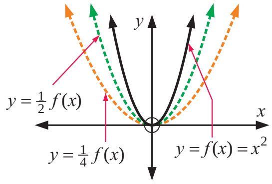
d



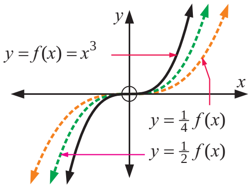
e



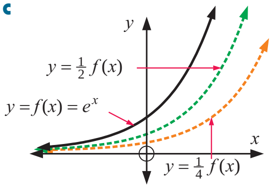
2 **a**



b

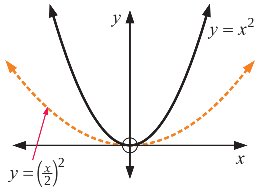


c

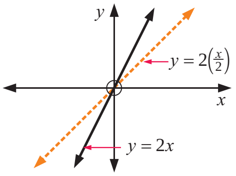


3 p affects the vertical stretching of the graph of $y = f(x)$ by a factor of p. If $p > 1$ the graph moves further away from the x-axis. If $0 < p < 1$ the graph moves closer to the x-axis.

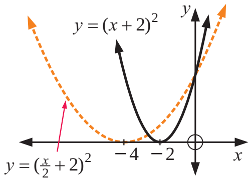
4 **a**



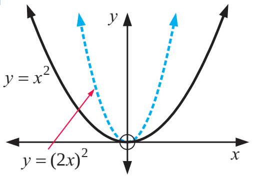
b



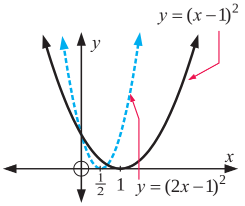
c



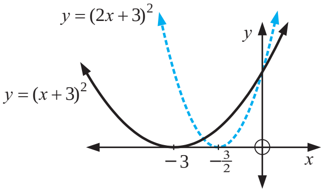
5 **a**



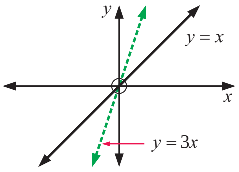
b



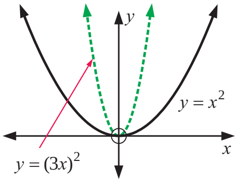
c



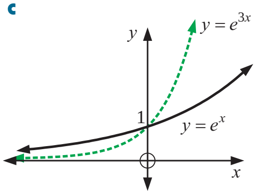
6 **a**



b

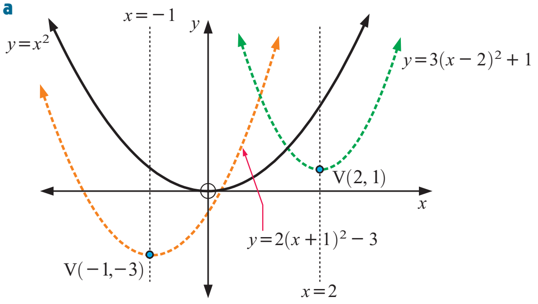


c

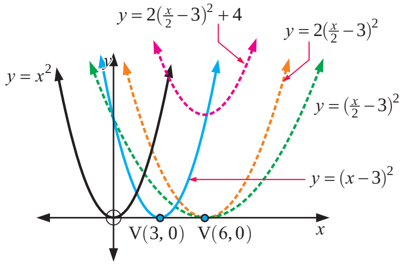


7 q affects the horizontal stretching of $y=f(x)$ by a factor of q.
If $q>1$ it moves further from the y-axis.
If $0<q<1$ it moves closer to the y-axis.

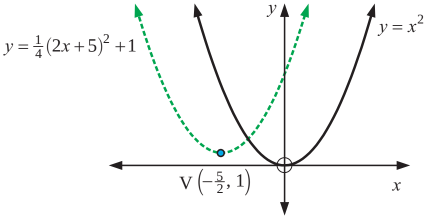
8 **a**



b



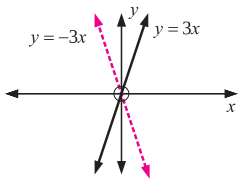
c



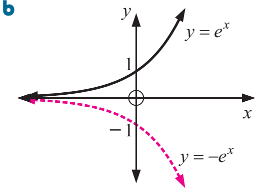
9 **a** **i** $(\frac{3}{2}, -15)$ **ii** $(\frac{1}{2}, 6)$ **iii** $(-1, 3)$
b **i** $(4, \frac{1}{3})$ **ii** $(-6, \frac{2}{3})$ **iii** $(-14, 1)$

EXERCISE 5B.3

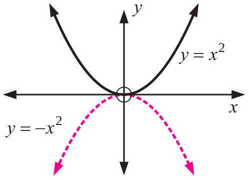
1 **a**



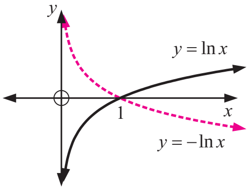
b



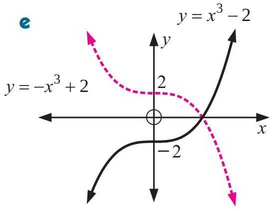
c



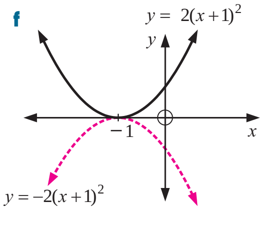
d



e



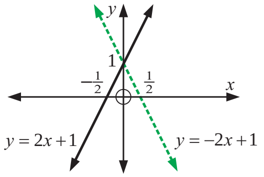
f



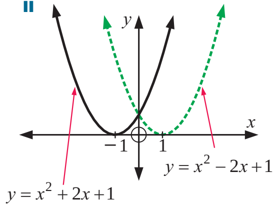
2 $y=-f(x)$ is the reflection of $y=f(x)$ in the x-axis.

3 **a** **i** $f(-x)=-2x+1$ **ii** $f(-x)=x^2-2x+1$
iii $f(-x)=-x^3$

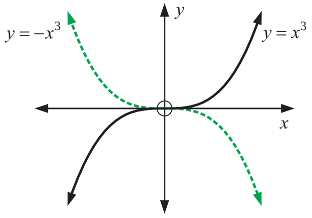
b **i**



ii



iii



4 $y = f(-x)$ is the reflection of $y = f(x)$ in the y-axis.

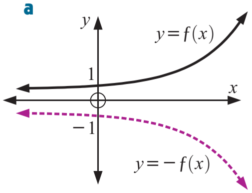
5 **a** **i** $(3, 0)$ **ii** $(2, 1)$ **iii** $(-3, -2)$
b **i** $(7, 1)$ **ii** $(-5, 0)$ **iii** $(-3, 2)$

6 **a** **i** $(-2, -1)$ **ii** $(0, 3)$ **iii** $(1, 2)$
b **i** $(-5, -4)$ **ii** $(0, 3)$ **iii** $(-2, 3)$

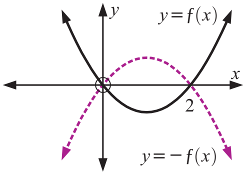
7 **a** A rotation about the origin through 180^o.
b $(-3, 7)$ **c** $(5, 1)$

EXERCISE 5B.4

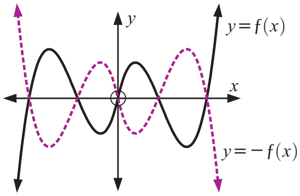
1 **a**



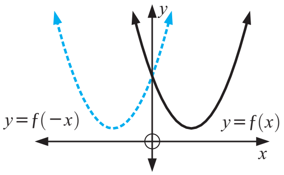
b



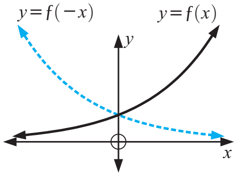
c



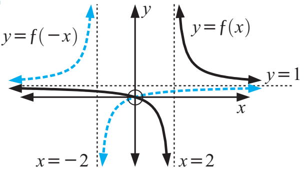
2 **a**



b

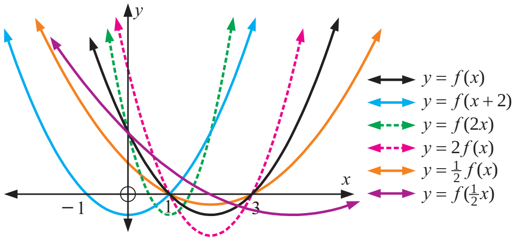


c

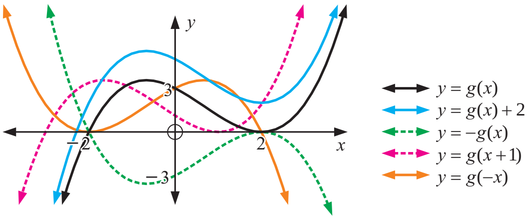


3 **a** **A** **b** **B** **c** **D** **d** **C**

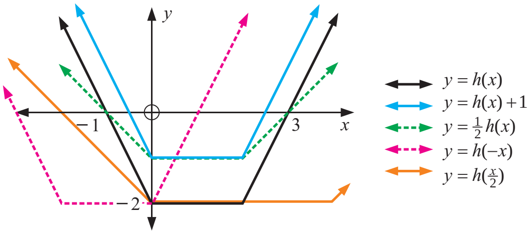
4



5



6

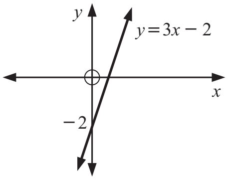


REVIEW SET 5A

1 **a** 3 **b** 8 **c** $4x^2-4x$ **d** x^2+2x **e** $3x^2-6x-2$

2 **a** -15 **b** 5 **c** $-x^2+x+5$
d $5-\frac{1}{2}x-\frac{1}{4}x^2$ **e** $-x^2-3x+5$

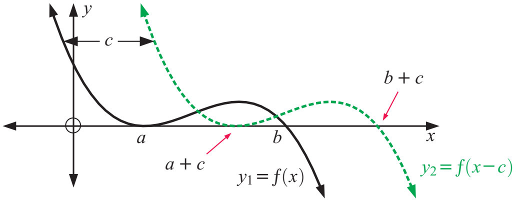
3 **a**



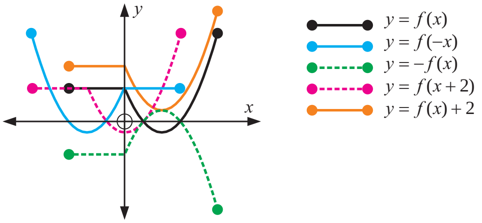
b **i** $\frac{2}{3}$ **ii** -2 **iii** 3
c **i** $y=-1.1$ **ii** $x=0.9$

4 $g(x)=3x^3-11x^2+14x-6$

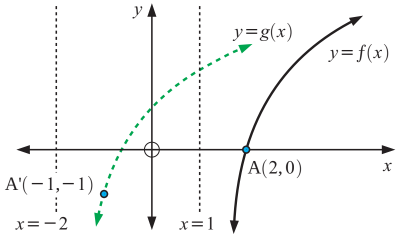
5



6

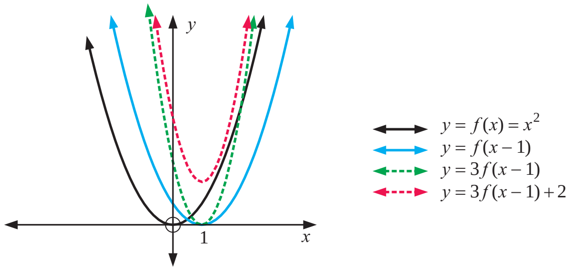


7 **a**



b $x=-2$ **c** $A'(-1, -1)$

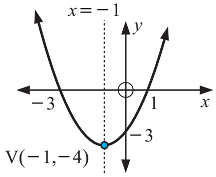
8



REVIEW SET 5B

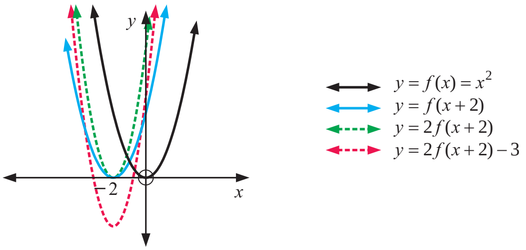
1 **a** 7 **b** $x^2 + 8x + 14$ **c** $3x^2 - 1$

2 **a**

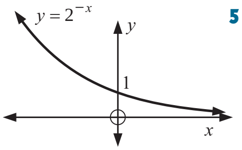


b **i** 1 and -3 **ii** -3 **c** V$(-1, -4)$

3



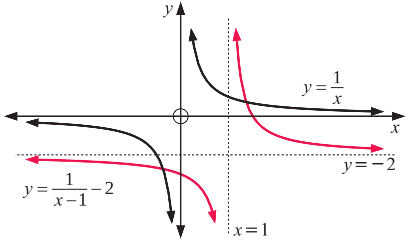
4 **a**



5 **a** $g(x) = (x-1)^2 + 8$
b $\{y \mid y \geqslant 4\}$
c $\{y \mid y \geqslant 8\}$

i true **ii** false
iii false **iv** true

6 **a** **i** $y = \dfrac{1}{x-1} - 2$
ii



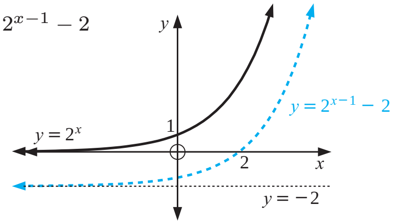
iii For $y = \dfrac{1}{x}$, VA is $x = 0$, HA is $y = 0$

For $y = \dfrac{1}{x-1} - 2$, VA is $x = 1$, HA is $y = -2$

iv For $y = \dfrac{1}{x}$, domain is $\{x \mid x \neq 0\}$,
range is $\{y \mid y \neq 0\}$

For $y = \dfrac{1}{x-1} - 2$, domain is $\{x \mid x \neq 1\}$,
range is $\{y \mid y \neq -2\}$

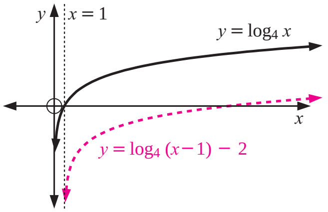
b **i** $y = 2^{x-1} - 2$
ii



iii For $y = 2^x$, HA is $y = 0$, no VA
For $y = 2^{x-1} - 2$, HA is $y = -2$, no VA

iv For $y = 2^x$, domain is $\{x \mid x \in \mathbb{R}\}$,
range is $\{y \mid y > 0\}$
For $y = 2^{x-1} - 2$, domain is $\{x \mid x \in \mathbb{R}\}$,
range is $\{y \mid y > -2\}$

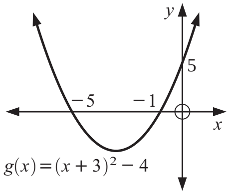
c **i** $y = \log_4(x-1) - 2$
ii



iii For $y = \log_4 x$, VA is $x = 0$, no HA
For $y = \log_4(x-1) - 2$, VA is $x = 1$, no HA

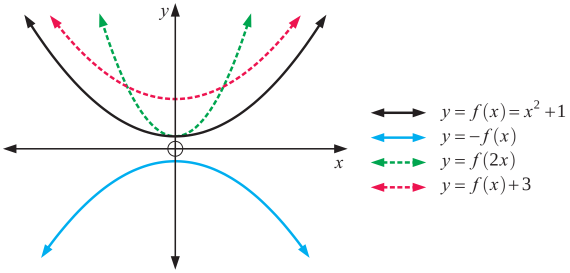
iv For $y = \log_4 x$, domain is $\{x \mid x > 0\}$,
range is $\{y \in \mathbb{R}\}$
For $y = \log_4(x-1) - 2$, domain is $\{x \mid x > 1\}$,
range is $\{y \in \mathbb{R}\}$

7 **a**



b x-intercepts -5 and -1,
y-intercept 5
c $(-3, -4)$

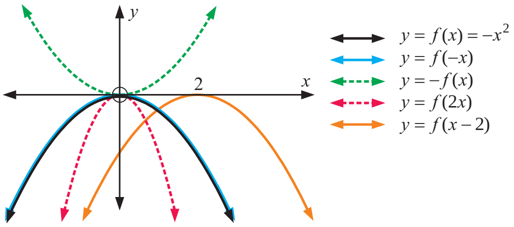
8



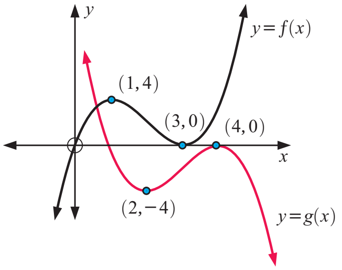
REVIEW SET 5C

1 **a** -1 **b** $\dfrac{2}{x}$ **c** $\dfrac{8}{x}$ **d** $\dfrac{10-3x}{x+2}$

2



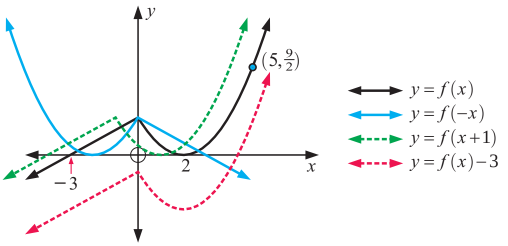
3 **a**



b (2, −4) and (4, 0)

4 $g(x) = -x^2 - 6x - 7$

5



6 $g(x) = x^3 + 6x^2 + 8x + 10$

7 **a** **i** $y = 3x + 8$ **ii** $y = 3x + 8$

b $f(x + k) = a(x + k) + b = ax + b + ka = f(x) + ka$

EXERCISE 6A.1

1 **a** $x = 0, -\frac{7}{4}$ **b** $x = 0, -\frac{1}{3}$ **c** $x = 0, \frac{7}{3}$

d $x = 0, \frac{11}{2}$ **e** $x = 0, \frac{8}{3}$ **f** $x = 0, \frac{3}{2}$

g $x = 3, 2$ **h** $x = 4, -2$ **i** $x = 3, 7$

j $x = 3$ **k** $x = -4, 3$ **l** $x = -11, 3$

2 **a** $x = \frac{2}{3}$ **b** $x = -\frac{1}{2}, 7$ **c** $x = -\frac{2}{3}, 6$

d $x = \frac{1}{3}, -2$ **e** $x = \frac{3}{2}, 1$ **f** $x = -\frac{2}{3}, -2$

g $x = -\frac{2}{3}, 4$ **h** $x = \frac{1}{2}, -\frac{3}{2}$ **i** $x = -\frac{1}{4}, 3$

j $x = -\frac{3}{4}, \frac{5}{3}$ **k** $x = \frac{1}{7}, -1$ **l** $x = -2, \frac{28}{15}$

3 **a** $x = 2, 5$ **b** $x = -3, 2$ **c** $x = 0, -\frac{3}{2}$

d $x = 1, 2$ **e** $x = \frac{1}{2}, -1$ **f** $x = 3$

EXERCISE 6A.2

1 **a** $x = -5 \pm \sqrt{2}$ **b** no real solns. **c** $x = 4 \pm 2\sqrt{2}$

d $x = 8 \pm \sqrt{7}$ **e** $x = -3 \pm \sqrt{5}$ **f** $x = 2 \pm \sqrt{6}$

g $x = -1 \pm \sqrt{10}$ **h** $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}$ **i** $x = \frac{1}{3} \pm \frac{\sqrt{7}}{3}$

2 **a** $x = 2 \pm \sqrt{3}$ **b** $x = -3 \pm \sqrt{7}$ **c** $x = 7 \pm \sqrt{3}$

d $x = 2 \pm \sqrt{7}$ **e** $x = -3 \pm \sqrt{2}$ **f** $x = 1 \pm \sqrt{7}$

g $x = -3 \pm \sqrt{11}$ **h** $x = 4 \pm \sqrt{6}$ **i** no real solns.

3 **a** $x = -1 \pm \frac{1}{\sqrt{2}}$ **b** $x = \frac{5}{2} \pm \frac{\sqrt{19}}{2}$ **c** $x = -2 \pm \sqrt{\frac{7}{3}}$

d $x = 1 \pm \sqrt{\frac{7}{3}}$ **e** $x = \frac{3}{2} \pm \sqrt{\frac{37}{20}}$ **f** $x = -\frac{1}{2} \pm \frac{\sqrt{6}}{2}$

EXERCISE 6A.3

1 **a** $x = 2 \pm \sqrt{7}$ **b** $x = -3 \pm \sqrt{2}$ **c** $x = 2 \pm \sqrt{3}$

d $x = -2 \pm \sqrt{5}$ **e** $x = 2 \pm \sqrt{2}$ **f** $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{7}$

g $x = -\frac{4}{9} \pm \frac{\sqrt{7}}{9}$ **h** $x = -\frac{7}{4} \pm \frac{\sqrt{97}}{4}$

2 **a** $x = -2 \pm 2\sqrt{2}$ **b** $x = -\frac{5}{8} \pm \frac{\sqrt{57}}{8}$ **c** $x = \frac{5}{2} \pm \frac{\sqrt{13}}{2}$

d $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{7}$ **e** $x = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$ **f** $x = \frac{3}{4} \pm \frac{\sqrt{17}}{4}$

EXERCISE 6B

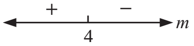
1 **a** 2 real distinct roots **b** 2 real distinct roots

c 2 real distinct roots **d** no real roots

e a repeated root

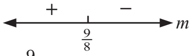
2 **a, c, d, f**

3 **a** $\Delta = 16 - 4m$



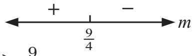
i $m = 4$ **ii** $m < 4$ **iii** $m > 4$

b $\Delta = 9 - 8m$



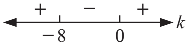
i $m = \frac{9}{8}$ **ii** $m < \frac{9}{8}$ **iii** $m > \frac{9}{8}$

c $\Delta = 9 - 4m$



i $m = \frac{9}{4}$ **ii** $m < \frac{9}{4}$ **iii** $m > \frac{9}{4}$

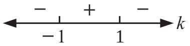
4 **a** $\Delta = k^2 + 8k$



i $k < -8$ or $k > 0$ **ii** $k \leqslant -8$ or $k \geqslant 0$

iii $k = -8$ or 0 **iv** $-8 < k < 0$

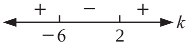
b $\Delta = 4 - 4k^2$



i $-1 < k < 1$ **ii** $-1 \leqslant k \leqslant 1$

iii $k = \pm 1$ **iv** $k < -1$ or $k > 1$

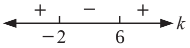
c $\Delta = k^2 + 4k - 12$



i $k < -6$ or $k > 2$ **ii** $k \leqslant -6$ or $k \geqslant 2$

iii $k = -6$ or 2 **iv** $-6 < k < 2$

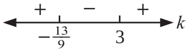
d $\Delta = k^2 - 4k - 12$



i $k < -2$ or $k > 6$ **ii** $k \leqslant -2$ or $k \geqslant 6$

iii $k = 6$ or -2 **iv** $-2 < k < 6$

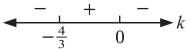
e $\Delta = 9k^2 - 14k - 39$



i $k < -\frac{13}{9}$ or $k > 3$ **ii** $k \leqslant -\frac{13}{9}$ or $k \geqslant 3$

iii $k = -\frac{13}{9}$ or 3 **iv** $-\frac{13}{9} < k < 3$

f $\Delta = -3k^2 - 4k$

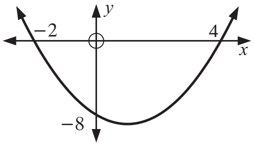


i $-\frac{4}{3} < k < 0$ **ii** $-\frac{4}{3} \leqslant k \leqslant 0$

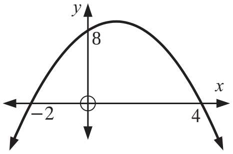
iii $k = -\frac{4}{3}$ or 0 **iv** $k < -\frac{4}{3}$ or $k > 0$

EXERCISE 6C.1

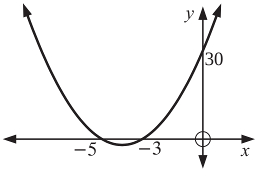
1 **a** $y = (x - 4)(x + 2)$



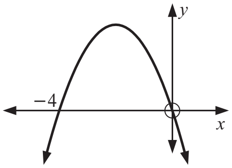
b $y = -(x - 4)(x + 2)$



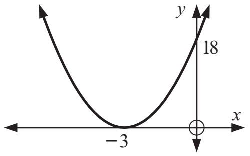
c $y = 2(x + 3)(x + 5)$



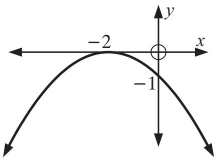
d $y = -3x(x + 4)$



e $y = 2(x+3)^2$



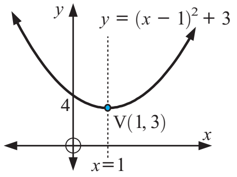
f $y = -\frac{1}{4}(x+2)^2$



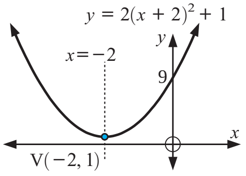
2 **a** $x = 1$ **b** $x = 1$ **c** $x = -4$
d $x = -2$ **e** $x = -3$ **f** $x = -2$

3 **a** C **b** E **c** B **d** F **e** G **f** H **g** A **h** D

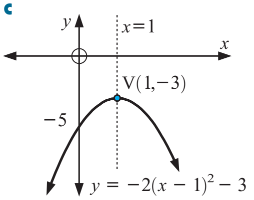
4 **a**



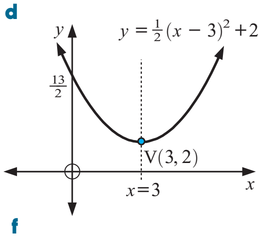
b



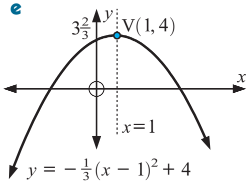
c



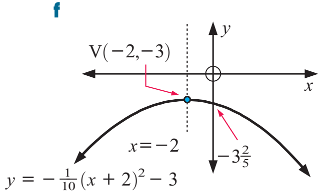
d



e



f

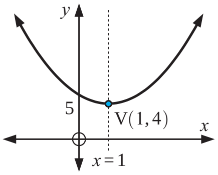


5 **a** G **b** A **c** E **d** B **e** I
f C **g** D **h** F **i** H

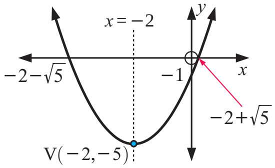
6 **a** $(2, -2)$ **b** $(-1, -4)$ **c** $(0, 4)$ **d** $(0, 1)$
e $(-2, -15)$ **f** $(-2, -5)$ **g** $(-\frac{3}{2}, -\frac{11}{2})$ **h** $(\frac{5}{2}, -\frac{19}{2})$
i $(1, -\frac{9}{2})$

7 **a** ± 3 **b** $\pm\sqrt{3}$ **c** -5 and -2
d 3 and -4 **e** 0 and 4 **f** -4 and -2
g -1 (touching) **h** 3 (touching) **i** $2 \pm \sqrt{3}$
j $-2 \pm \sqrt{7}$ **k** $3 \pm \sqrt{11}$ **l** $-4 \pm \sqrt{5}$

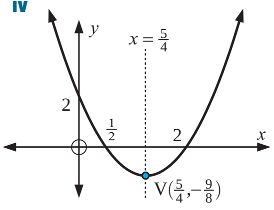
8 **a** **i** $x = 1$
ii $(1, 4)$
iii no x-intercept,
y-intercept 5
iv



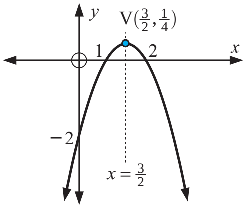
b **i** $x = -2$
ii $(-2, -5)$
iii x-int. $-2 \pm \sqrt{5}$,
y-intercept -1
iv



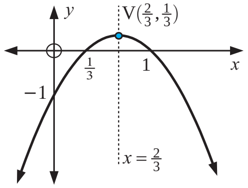
c **i** $x = \frac{5}{4}$
ii $(\frac{5}{4}, -\frac{9}{8})$
iii x-intercepts $\frac{1}{2}$, 2,
y-intercept 2
iv



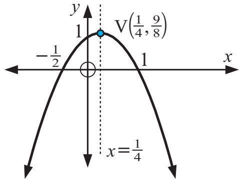
d **i** $x = \frac{3}{2}$
ii $(\frac{3}{2}, \frac{1}{4})$
iii x-intercepts 1, 2,
y-intercept -2
iv



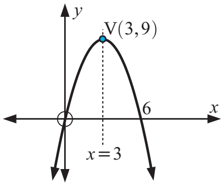
e **i** $x = \frac{2}{3}$
ii $(\frac{2}{3}, \frac{1}{3})$
iii x-intercepts $\frac{1}{3}$, 1,
y-intercept -1
iv



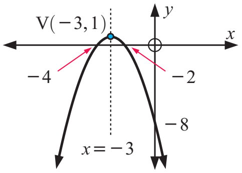
f **i** $x = \frac{1}{4}$
ii $(\frac{1}{4}, \frac{9}{8})$
iii x-intercepts $-\frac{1}{2}$, 1,
y-intercept 1
iv



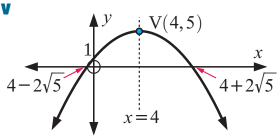
g **i** $x = 3$
ii $(3, 9)$
iii x-intercepts 0, 6,
y-intercept 0
iv



h **i** $x = -3$
ii $(-3, 1)$
iii x-int. -2, -4,
y-intercept -8
iv

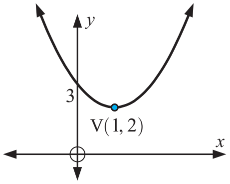


i **i** $x = 4$
ii $(4, 5)$
iii x-int. $4 \pm 2\sqrt{5}$,
y-intercept 1
iv

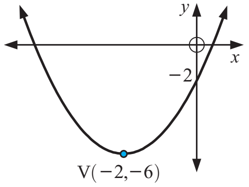


EXERCISE 6C.2

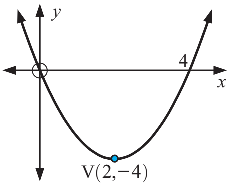
1 **a** $y = (x-1)^2 + 2$



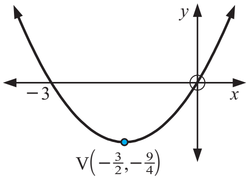
b $y = (x+2)^2 - 6$



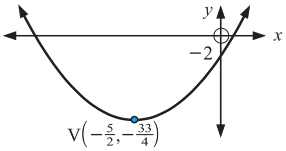
c $y = (x-2)^2 - 4$



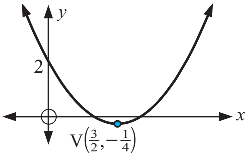
d $y = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4}$



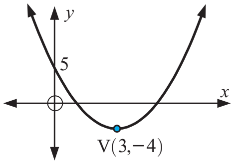
e $y = \left(x + \frac{5}{2}\right)^2 - \frac{33}{4}$



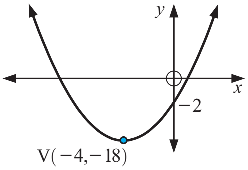
f $y = \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$



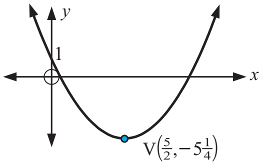
g $y = (x-3)^2 - 4$



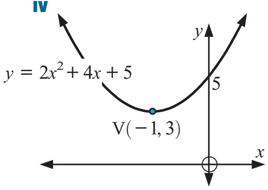
h $y = (x+4)^2 - 18$



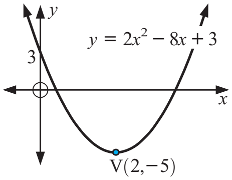
i $y = \left(x - \frac{5}{2}\right)^2 - 5\frac{1}{4}$



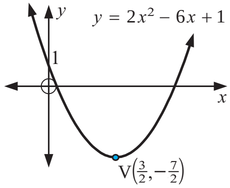
2 **a** **i** $y = 2(x+1)^2 + 3$
ii $(-1, 3)$ **iii** 5
iv



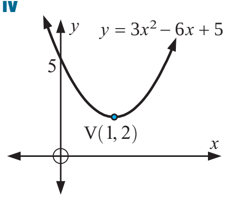
b **i** $y = 2(x-2)^2 - 5$
ii $(2, -5)$ **iii** 3
iv



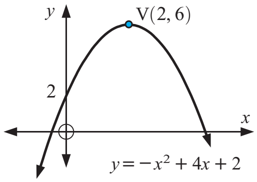
c **i** $y = 2(x - \frac{3}{2})^2 - \frac{7}{2}$
ii $(\frac{3}{2}, -\frac{7}{2})$ **iii** 1
iv



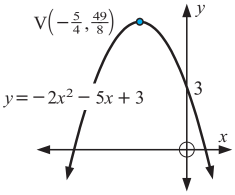
d **i** $y = 3(x-1)^2 + 2$
ii $(1, 2)$ **iii** 5
iv



e **i** $y = -(x-2)^2 + 6$
ii $(2, 6)$ **iii** 2
iv



f **i** $y = -2(x + \frac{5}{4})^2 + \frac{49}{8}$
ii $(-\frac{5}{4}, \frac{49}{8})$ **iii** 3
iv



3 **a** $y = (x-2)^2 + 3$
b $y = (x+3)^2 - 6$
c $y = -(x-2)^2 + 9$
d $y = 2\left(x + \frac{3}{2}\right)^2 - \frac{17}{2}$
e $y = -2\left(x + \frac{5}{2}\right)^2 + \frac{27}{2}$
f $y = 3\left(x - \frac{3}{2}\right)^2 - \frac{47}{4}$

EXERCISE 6C.3

1 **a** cuts x-axis twice
b touches x-axis
c cuts x-axis twice
d cuts x-axis twice
e cuts x-axis twice
f touches x-axis

2 **a** $a = 1$ which is > 0 and $\Delta = -15$ which is < 0
b $a = -1$ which is < 0 and $\Delta = -8$ which is < 0
c $a = 2$ which is > 0 and $\Delta = -40$ which is < 0
d $a = -2$ which is < 0 and $\Delta = -23$ which is < 0

3 $a = 3$ which is > 0 and $\Delta = k^2 + 12$ which is always > 0 {as $k^2 \geqslant 0$ for all k}

4 $-4 < k < 4$

EXERCISE 6D

1 **a** $y = 2(x-1)(x-2)$
b $y = 2(x-2)^2$
c $y = (x-1)(x-3)$
d $y = -(x-3)(x+1)$
e $y = -3(x-1)^2$
f $y = -2(x+2)(x-3)$

2 **a** $y = \frac{3}{2}(x-2)(x-4)$
b $y = -\frac{1}{2}(x+4)(x-2)$
c $y = -\frac{4}{3}(x+3)^2$

3 **a** $y = 3x^2 - 18x + 15$
b $y = -4x^2 + 6x + 4$
c $y = -x^2 + 6x - 9$
d $y = 4x^2 + 16x + 16$
e $y = \frac{3}{2}x^2 - 6x + \frac{9}{2}$
f $y = -\frac{1}{3}x^2 + \frac{2}{3}x + 5$

4 **a** $y = -(x-2)^2 + 4$
b $y = 2(x-2)^2 - 1$
c $y = -2(x-3)^2 + 8$
d $y = \frac{2}{3}(x-4)^2 - 6$
e $y = -2(x-2)^2 + 3$
f $y = 2(x - \frac{1}{2})^2 - \frac{3}{2}$

EXERCISE 6E

1 **a** (1, 7) and (2, 8)
b (4, 5) and (−3, −9)
c (3, 0) (touching)
d graphs do not meet

2 **a** (0.59, 5.59) and (3.41, 8.41) **b** (3, −4) touching
c graphs do not meet **d** (−2.56, −18.81) and (1.56, 1.81)

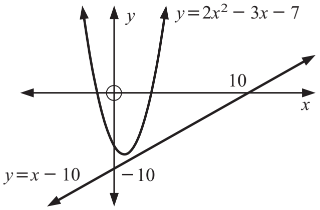
3 **a** (2, 4), (−1, 1)
b (1, 0), (−2, −3)
c (1, 4)
d (1, 4), (−4, −1)

5 $c = -9$

6 $m = 0$ or -8

7 -1 or 11

8 **a** $c < -9$
b example: $c = -10$



EXERCISE 6F

1 7 and −5 or −7 and 5
2 5 or $\frac{1}{5}$
3 14
4 18 and 20 or −18 and −20
5 15 and 17 or −15 and −17
6 15 sides
7 3.48 cm
8 **b** 6 cm by 6 cm by 7 cm
9 11.2 cm square
10 no
12 221 ha
13 2.03 m
14 52.1 km h^{-1}
15 554 km h^{-1}
16 61.8 km h^{-1}
17 32
18 No, tunnel is only 3.79 m wide 4.8 m above ground level.

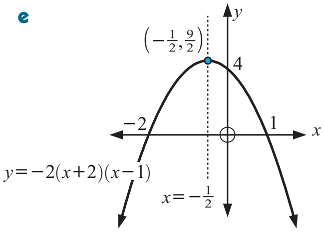
19 **a** $y = -\frac{1}{100}x^2 + 70$
b supports are 21 m, 34 m, 45 m, 54 m, 61 m, 66 m, 69 m

EXERCISE 6G

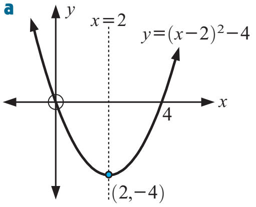
1 **a** min. -1, when $x = 1$ **b** max. 8, when $x = -1$
c max. $8\frac{1}{3}$, when $x = \frac{1}{3}$ **d** min. $-1\frac{1}{8}$, when $x = -\frac{1}{4}$
e min. $4\frac{15}{16}$ when $x = \frac{1}{8}$ **f** max. $6\frac{1}{8}$, when $x = \frac{7}{4}$
2 40 refrigerators, \$4000 **4** 500 m by 250 m
5 **c** 100 m by 112.5 m
6 **a** $41\frac{2}{3}$ m by $41\frac{2}{3}$ m **b** 50 m by $31\frac{1}{4}$ m **7** **b** $3\frac{1}{8}$ units
8 **a** $y = 6 - \frac{3}{4}x$ **b** 3 cm by 4 cm **9** 125 **10** 40 **11** 157

REVIEW SET 6A

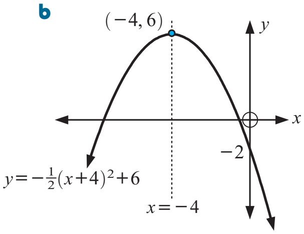
1 **a** $-2, 1$
b $x = -\frac{1}{2}$
c 4
d $(-\frac{1}{2}, \frac{9}{2})$
e



2 **a** $x = 0$ or 4 **b** $x = -\frac{5}{3}$ or 2 **c** $x = 15$ or -4
3 **a** $x = \frac{-5\pm\sqrt{13}}{2}$ **b** $x = \frac{-11\pm\sqrt{145}}{6}$
4 $x = -\frac{7}{2} \pm \frac{\sqrt{65}}{2}$
5 **a**



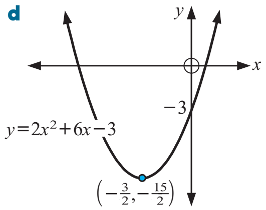
b



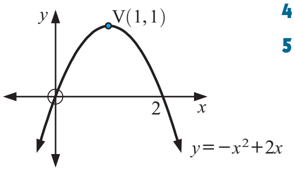
6 **a** $y = 3x^2 - 24x + 48$ **b** $y = \frac{2}{5}x^2 + \frac{16}{5}x + \frac{37}{5}$
7 $a = -2$ which is < 0 $\therefore$  a max.
max. $= 5$ when $x = 1$
8 $(4, 4)$ and $(-3, 18)$ **9** $k < -3\frac{1}{8}$
10 **a** $m = \frac{9}{8}$ **b** $m < \frac{9}{8}$ **c** $m > \frac{9}{8}$ **11** $\frac{6}{5}$ or $\frac{5}{6}$
13 **a** $m = -2$, $n = 4$ **b** $k = 7$ **c** $(2, 5)$
d $f(x)$ has domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y \geqslant 3\}$
$g(x)$ has domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y \geqslant 5\}$

REVIEW SET 6B

1 **a** $y = 2\left(x + \frac{3}{2}\right)^2 - \frac{15}{2}$
b $(-\frac{3}{2}, -\frac{15}{2})$
c -3
d

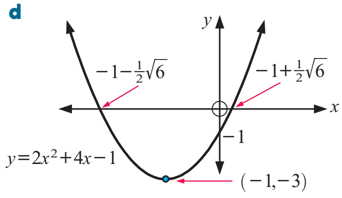


2 **a** $x \approx 0.586$ or 3.414 **b** $x \approx -0.186$ or 2.686
3



4 $x = \frac{4}{3}$, $V(\frac{4}{3}, 12\frac{1}{3})$
5 **a** two distinct rational roots
b a repeated root

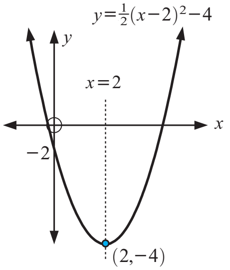
6 12.9 cm **7** **a** $c > -6$
b example: $c = -2$, $(-1, -5)$ and $(3, 7)$
8 **a** $x = -1$
b $(-1, -3)$
c y-intercept -1,
x-ints. $-1 \pm \frac{1}{2}\sqrt{6}$
d



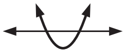
9 13.5 cm by 13.5 cm **10** touch at $(-2, 9)$
11 **a** min. $= 5\frac{2}{3}$ when $x = -\frac{2}{3}$
b max. $= 5\frac{1}{8}$ when $x = -\frac{5}{4}$
12 **b** $A = x\left(\frac{600 - 8x}{9}\right)$ **c** $37\frac{1}{2}$ m by $33\frac{1}{3}$ m **d** 1250 m^2
13 **a** $k = -12$ or 12 **b** $(0, 4)$
c horizontal translation of $\frac{4}{3}$ units

REVIEW SET 6C

1 **a** $x = 2$
b $(2, -4)$
c -2
d



2 **a** $x = \frac{5}{2} \pm \frac{\sqrt{37}}{2}$
b $x = \frac{7}{4} \pm \frac{\sqrt{73}}{4}$
3 **a** $x = \frac{7}{2} \pm \frac{\sqrt{37}}{2}$
b no real roots
4 **a** $y = \frac{20}{9}(x-2)^2 - 20$
b $y = -\frac{2}{7}(x-1)(x-7)$
c $y = \frac{2}{9}(x+3)^2$
5 **a** graph cuts x-axis twice



b graph cuts x-axis twice

6 **a** neither **b** positive definite
7 **a** $y = 3(x-3)(x+3)$ **b** $y = -6(x-2)^2 + 25$
8 17 cm **9** $\frac{1}{2}$ **10** $k < 1$
11 $y = -4x^2 + 4x + 24$ **12** $m = -5$ or 19
13 **a** **i** $A(-m, 0)$, $B(-n, 0)$ **ii** $x = \frac{-m-n}{2}$
b **i** positive **ii** negative

EXERCISE 7A

1 **a** $p^3 + 3p^2q + 3pq^2 + q^3$ **b** $x^3 + 3x^2 + 3x + 1$
c $x^3 - 9x^2 + 27x - 27$ **d** $8 + 12x + 6x^2 + x^3$
e $27x^3 - 27x^2 + 9x - 1$ **f** $8x^3 + 60x^2 + 150x + 125$
g $27x^3 - 9x^2 + x - \frac{1}{27}$ **h** $8x^3 + 12x + \frac{6}{x} + \frac{1}{x^3}$
2 **a** $1 + 4x + 6x^2 + 4x^3 + x^4$
b $p^4 - 4p^3q + 6p^2q^2 - 4pq^3 + q^4$
c $x^4 - 8x^3 + 24x^2 - 32x + 16$
d $81 - 108x + 54x^2 - 12x^3 + x^4$
e $1 + 8x + 24x^2 + 32x^3 + 16x^4$
f $16x^4 + 96x^3 + 216x^2 + 216x + 81$
g $x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$
h $16x^4 - 32x^2 + 24 - \frac{8}{x^2} + \frac{1}{x^4}$

3 **a** $x^5+10x^4+40x^3+80x^2+80x+32$
b $x^5-10x^4y+40x^3y^2-80x^2y^3+80xy^4-32y^5$
c $1+10x+40x^2+80x^3+80x^4+32x^5$
d $x^5-5x^3+10x-\dfrac{10}{x}+\dfrac{5}{x^3}-\dfrac{1}{x^5}$

4 **a** 1 6 15 20 15 6 1
b **i** $x^6+12x^5+60x^4+160x^3+240x^2+192x+64$
ii $64x^6-192x^5+240x^4-160x^3+60x^2-12x+1$
iii $x^6+6x^4+15x^2+20+\dfrac{15}{x^2}+\dfrac{6}{x^4}+\dfrac{1}{x^6}$

5 **a** $7+5\sqrt{2}$ **b** $161+72\sqrt{5}$ **c** $232-164\sqrt{2}$

6 **a** $64+192x+240x^2+160x^3+60x^4+12x^5+x^6$
b 65.944 160 601 201

7 $2x^5+11x^4+24x^3+26x^2+14x+3$

8 **a** 270 **b** 4320

EXERCISE 7B

1 **a** $1^{11}+\binom{11}{1}(2x)+\binom{11}{2}(2x)^2+....+\binom{11}{10}(2x)^{10}+(2x)^{11}$
b $(3x)^{15}+\binom{15}{1}(3x)^{14}\left(\frac{2}{x}\right)+\binom{15}{2}(3x)^{13}\left(\frac{2}{x}\right)^2+....$
$....+\binom{15}{14}(3x)\left(\frac{2}{x}\right)^{14}+\left(\frac{2}{x}\right)^{15}$
c $(2x)^{20}+\binom{20}{1}(2x)^{19}\left(-\frac{3}{x}\right)+\binom{20}{2}(2x)^{18}\left(-\frac{3}{x}\right)^2+....$
$....+\binom{20}{19}(2x)\left(-\frac{3}{x}\right)^{19}+\left(-\frac{3}{x}\right)^{20}$

2 **a** $T_6=\binom{15}{5}(2x)^{10}5^5$ **b** $T_4=\binom{9}{3}(x^2)^6y^3$
c $T_{10}=\binom{17}{9}x^8\left(-\frac{2}{x}\right)^9$ **d** $T_9=\binom{21}{8}(2x^2)^{13}\left(-\frac{1}{x}\right)^8$

3 **a** $\binom{10}{5}3^52^5$ **b** $\binom{6}{3}2^3(-3)^3$ **c** $\binom{6}{3}2^3(-3)^3$
d $\binom{12}{4}2^8(-1)^4$

4 **a** $\binom{15}{5}2^5$ **b** $\binom{9}{3}(-3)^3$

5 **a**
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

b sum
2
4
8
16
32

c The sum of the numbers in row n of Pascal's triangle is 2^n.
d After the first part let $x=1$.

6 **a** $\binom{8}{6}=28$ **b** $2\binom{9}{3}3^6-\binom{9}{4}3^5=91\,854$

7 $T_3=\binom{6}{2}(-2)^2x^8y^8$

8 **a** $84x^3$ **b** $n=6$ and $k=-2$ **9** $a=2$

REVIEW SET 7

1 **a** $x^3-6x^2y+12xy^2-8y^3$
b $81x^4+216x^3+216x^2+96x+16$

2 20 000 **3** 60

4 $(a+b)^6=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$
a $x^6-18x^5+135x^4-540x^3+1215x^2-1458x+729$
b $1+\dfrac{6}{x}+\dfrac{15}{x^2}+\dfrac{20}{x^3}+\dfrac{15}{x^4}+\dfrac{6}{x^5}+\dfrac{1}{x^6}$

5 $362+209\sqrt{3}$ **6** 64.964 808

7 $\binom{12}{6}\times2^6\times(-3)^6$ **8** $8\binom{6}{2}-6\binom{6}{1}=84$

9 $k=180$ **10** $c=3$

11 **a** 7 **b** $\binom{6}{4}\times3^2=135$

EXERCISE 8A

1 **a** $\frac{\pi}{2}^c$ **b** $\frac{\pi}{3}^c$ **c** $\frac{\pi}{6}^c$ **d** $\frac{\pi}{10}^c$ **e** $\frac{\pi}{20}^c$
f $\frac{3\pi}{4}^c$ **g** $\frac{5\pi}{4}^c$ **h** $\frac{3\pi}{2}^c$ **i** $2\pi^c$ **j** $4\pi^c$
k $\frac{7\pi}{4}^c$ **l** $3\pi^c$ **m** $\frac{\pi}{5}^c$ **n** $\frac{4\pi}{9}^c$ **o** $\frac{23\pi}{18}^c$

2 **a** 0.641^c **b** 2.39^c **c** 5.55^c **d** 3.83^c **e** 6.92^c

3 **a** 36^o **b** 108^o **c** 135^o **d** 10^o **e** 20^o
f 140^o **g** 18^o **h** 27^o **i** 150^o **j** 22.5^o

4 **a** 114.59^o **b** 87.66^o **c** 49.68^o **d** 182.14^o
e 301.78^o

5 **a**

Degrees	0	45	90	135	180	225	270	315	360
Radians	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π

b

Deg.	0	30	60	90	120	150	180	210	240	270	300	330	360
Rad.	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π

EXERCISE 8B

1 **a** **i** 49.5 cm **ii** 223 cm^2 **b** **i** 23.0 cm **ii** 56.8 cm^2

2 **a** 3.14 m **b** 9.30 m^2 **3** **a** 5.91 cm **b** 18.9 cm

4 **a** 0.686^c **b** 0.6^c

5 **a** 0.75^c, 24 cm^2 **b** 1.68^c, 21 cm^2 **c** 2.32^c, 126.8 cm^2

6 10 cm, 25 cm^2

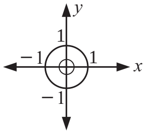
8 **a** 11.7 cm **b** 11.7 **c** 37.7 cm **d** 3.23^c

9 **a** $\alpha\approx18.43$ **b** $\theta\approx143.1$ **c** 387 m^2

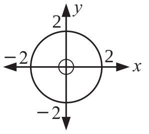
10 25.9 cm **11** **b** 2 h 24 min **12** 227 m^2

EXERCISE 8C.1

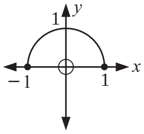
1 **a**



b



c



2 **a** **i** A($\cos26^o$, $\sin26^o$), B($\cos146^o$, $\sin146^o$), C($\cos199^o$, $\sin199^o$)
ii A(0.899, 0.438), B(-0.829, 0.559), C(-0.946, -0.326)
b **i** A($\cos123^o$, $\sin123^o$), B($\cos251^o$, $\sin251^o$), C($\cos(-35^o)$, $\sin(-35^o)$)
ii A(-0.545, 0.839), B(-0.326, -0.946), C(0.819, -0.574)

3

θ (degrees)	0^o	90^o	180^o	270^o	360^o	450^o
θ (radians)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$
sine	0	1	0	-1	0	1
cosine	1	0	-1	0	1	0
tangent	0	undef	0	undef	0	undef

4 **a** **i** $\frac{1}{\sqrt{2}}\approx0.707$ **ii** $\frac{\sqrt{3}}{2}\approx0.866$

b

θ (degrees)	30^o	45^o	60^o	135^o	150^o	240^o	315^o
θ (radians)	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{4\pi}{3}$	$\frac{7\pi}{4}$
sine	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$
cosine	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{2}}$
tangent	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	$\sqrt{3}$	-1

5 **a** **i** 0.985 **ii** 0.985 **iii** 0.866 **iv** 0.866
v 0.5 **vi** 0.5 **vii** 0.707 **viii** 0.707
b $\sin(180^o - \theta) = \sin\theta$
d **i** 135^o **ii** 129^o **iii** $\frac{2\pi}{3}$ **iv** $\frac{5\pi}{6}$

6 **a** **i** 0.342 **ii** -0.342 **iii** 0.5 **iv** -0.5
v 0.906 **vi** -0.906 **vii** 0.174 **viii** -0.174
b $\cos(180^o - \theta) = -\cos\theta$
d **i** 140^o **ii** 161^o **iii** $\frac{4\pi}{5}$ **iv** $\frac{3\pi}{5}$

7 **a** ≈ 0.6820 **b** ≈ 0.8572 **c** ≈ -0.7986
d ≈ 0.9135 **e** ≈ 0.9063 **f** ≈ -0.6691

8 **a**

Quadrant	*Degree measure*	*Radian measure*	$\cos\theta$	$\sin\theta$	$\tan\theta$
1	$0^o < \theta < 90^o$	$0 < \theta < \frac{\pi}{2}$	+ve	+ve	+ve
2	$90^o < \theta < 180^o$	$\frac{\pi}{2} < \theta < \pi$	−ve	+ve	−ve
3	$180^o < \theta < 270^o$	$\pi < \theta < \frac{3\pi}{2}$	−ve	−ve	+ve
4	$270^o < \theta < 360^o$	$\frac{3\pi}{2} < \theta < 2\pi$	+ve	−ve	−ve

b **i** 1 and 4 **ii** 2 and 3 **iii** 3 **iv** 2

9 **a** $A\widehat{O}Q = 180^o - \theta$
b [OQ] is a reflection of [OP] in the y-axis and so Q has coordinates $(-\cos\theta, \sin\theta)$.
c $\cos(180^o - \theta) = -\cos\theta$, $\sin(180^o - \theta) = \sin\theta$

EXERCISE 8C.2

1 **a** $\cos\theta = \pm\frac{\sqrt{3}}{2}$ **b** $\cos\theta = \pm\frac{2\sqrt{2}}{3}$ **c** $\cos\theta = \pm 1$
d $\cos\theta = 0$

2 **a** $\sin\theta = \pm\frac{3}{5}$ **b** $\sin\theta = \pm\frac{\sqrt{7}}{4}$ **c** $\sin\theta = 0$
d $\sin\theta = \pm 1$

3 **a** $\sin\theta = \frac{\sqrt{5}}{3}$ **b** $\cos\theta = -\frac{\sqrt{21}}{5}$ **c** $\cos\theta = \frac{4}{5}$
d $\sin\theta = -\frac{12}{13}$

4 **a** $-\frac{1}{2\sqrt{2}}$ **b** $-2\sqrt{6}$ **c** $\frac{1}{\sqrt{2}}$ **d** $-\frac{\sqrt{7}}{3}$

5 **a** $\sin x = \frac{2}{\sqrt{13}}$, $\cos x = \frac{3}{\sqrt{13}}$
b $\sin x = \frac{4}{5}$, $\cos x = -\frac{3}{5}$
c $\sin x = -\sqrt{\frac{5}{14}}$, $\cos x = -\frac{3}{\sqrt{14}}$
d $\sin x = -\frac{12}{13}$, $\cos x = \frac{5}{13}$

EXERCISE 8C.3

1

	a	**b**	**c**	**d**	**e**
$\sin\theta$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$
$\cos\theta$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$
$\tan\theta$	1	1	-1	0	1

2

	a	**b**	**c**	**d**	**e**
$\sin\beta$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\cos\beta$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\tan\beta$	$\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$

3 **a** $\frac{3}{4}$ **b** $\frac{1}{4}$ **c** 3 **d** $\frac{1}{4}$ **e** $-\frac{1}{4}$ **f** 1
g $\sqrt{2}$ **h** $\frac{1}{2}$ **i** $\frac{1}{2}$ **j** 2 **k** -1 **l** $-\sqrt{3}$

4 **a** 30^o, 150^o **b** 60^o, 120^o **c** 45^o, 315^o
d 120^o, 240^o **e** 135^o, 225^o **f** 240^o, 300^o

5 **a** $\frac{\pi}{4}, \frac{5\pi}{4}$ **b** $\frac{3\pi}{4}, \frac{7\pi}{4}$ **c** $\frac{\pi}{3}, \frac{4\pi}{3}$
d $0, \pi, 2\pi$ **e** $\frac{\pi}{6}, \frac{7\pi}{6}$ **f** $\frac{2\pi}{3}, \frac{5\pi}{3}$

6 **a** $\frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}$ **b** $\frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$ **c** $\frac{3\pi}{2}, \frac{7\pi}{2}$

7 **a** $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$ **b** $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$ **c** $\theta = \pi$
d $\theta = \frac{\pi}{2}$ **e** $\theta = \frac{3\pi}{4}, \frac{5\pi}{4}$ **f** $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$
g $\theta = 0, \pi, 2\pi$ **h** $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
i $\theta = \frac{5\pi}{6}, \frac{11\pi}{6}$ **j** $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

EXERCISE 8D

1 **a** $y = \sqrt{3}x$ **b** $y = x$ **c** $y = -\frac{1}{\sqrt{3}}x$

2 **a** $y = \sqrt{3}x + 2$ **b** $y = -\sqrt{3}x$ **c** $y = \frac{1}{\sqrt{3}}x - 2$

REVIEW SET 8A

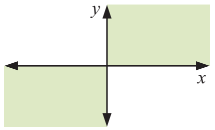
1 **a** $\frac{2\pi}{3}$ **b** $\frac{5\pi}{4}$ **c** $\frac{5\pi}{6}$ **d** 3π
2 **a** $\frac{\pi}{3}$ **b** 15^o **c** 84^o
3 **a** 0.358 **b** -0.035 **c** 0.259 **d** -0.731
4 **a** $1, 0$ **b** $-1, 0$
6 **a** $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$, $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$
b $\sin\left(\frac{8\pi}{3}\right) = \frac{\sqrt{3}}{2}$, $\cos\left(\frac{8\pi}{3}\right) = -\frac{1}{2}$ **7** $\frac{1}{\sqrt{15}}$
8 **a** -0.743 **b** -0.743 **c** 0.743 **d** -0.743
9 $\pm\frac{\sqrt{7}}{4}$ **10** **a** $\frac{\sqrt{3}}{2}$ **b** 0 **c** $\frac{1}{2}$ **11** **a** $-\frac{3}{\sqrt{13}}$ **b** $\frac{2}{\sqrt{13}}$

REVIEW SET 8B

1 **a** $(0.766, -0.643)$ **b** $(-0.956, 0.292)$
2 **a** 1.239^c **b** 2.175^c **c** -2.478^c
3 **a** 171.89^o **b** 83.65^o **c** 24.92^o **d** -302.01^o
4 111 cm^2
5 $M(\cos 73^o, \sin 73^o) \approx (0.292, 0.956)$
$N(\cos 190^o, \sin 190^o) \approx (-0.985, -0.174)$
$P(\cos 307^o, \sin 307^o) \approx (0.602, -0.799)$
6 $\approx 103^o$
7 **a** 150^o, 210^o **b** 45^o, 135^o **c** 120^o, 300^o
8 **a** $\theta = \pi$ **b** $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$
9 **a** 133^o **b** $\frac{14\pi}{15}$ **c** 174^o
10 perimeter $= 34.1$ cm, area $= 66.5$ cm^2
11 $r \approx 8.79$ cm, area ≈ 81.0 cm^2

REVIEW SET 8C

1 **a** 72^o **b** 225^o **c** 140^o **d** 330^o
2



3 **a** $0, -1$
b $0, -1$
4 **a** 0.961 **b** -0.961 **c** -0.961 **d** -0.961
5 **a** **i** $\theta = 60^o$ **ii** $\theta = \frac{\pi}{3}$ **b** $\frac{\pi}{3}$ units **6** 3
8 **a** $\frac{\sqrt{7}}{4}$ **b** $-\frac{\sqrt{7}}{3}$ **c** $-\frac{\sqrt{7}}{4}$ **9** **a** $2\frac{1}{2}$ **b** $1\frac{1}{2}$ **c** $-\frac{1}{2}$
10 **a** 0 **b** $\sin\theta$ **11** **a** $y = -\frac{1}{\sqrt{3}}x$ **b** $k = -2\sqrt{3}$

EXERCISE 9A

1 **a** 28.9 cm^2 **b** 384 km^2 **c** 28.3 cm^2 **2** $x \approx 19.0$

3 18.9 cm^2 **4** 137 cm^2 **5** 374 cm^2 **6** 7.49 cm

7 11.9 m **8** **a** 48.6^o or 131.4^o **b** 42.1^o or 137.9^o

9 $\frac{1}{4}$ is not covered

10 **a** 36.2 cm^2 **b** 62.8 cm^2 **c** 40.4 mm^2 **11** 4.69 cm^2

EXERCISE 9B

1 **a** 28.8 cm **b** 3.38 km **c** 14.2 m

2 $\widehat{A} \approx 52.0^o$, $\widehat{B} \approx 59.3^o$, $\widehat{C} \approx 68.7^o$ **3** 112^o

4 **a** 40.3^o **b** 107^o **5** **a** $\cos\theta = 0.65$ **b** $x \approx 3.81$

6 **a** $x = 3 + \sqrt{22}$ **b** $x = \dfrac{-3+\sqrt{73}}{2}$ **c** $x = \frac{5}{\sqrt{3}}$

7 **a** $x \approx 10.8$ **b** $x \approx 9.21$ **c** $x \approx 1.41$ or 7.78

EXERCISE 9C.1

1 **a** $x \approx 28.4$ **b** $x \approx 13.4$ **c** $x \approx 3.79$

2 **a** $a \approx 21.3$ cm **b** $b \approx 76.9$ cm **c** $c \approx 5.09$ cm

EXERCISE 9C.2

1 $\widehat{C} \approx 62.1^o$ or $\widehat{C} \approx 117.9^o$

2 **a** $\widehat{A} \approx 49.5^o$ **b** $\widehat{B} \approx 72.0^o$ or 108^o **c** $\widehat{C} \approx 44.3^o$

3 No, $\dfrac{\sin 85^o}{11.4} \neq \dfrac{\sin 27^o}{9.8}$ **4** $\text{A}\widehat{\text{B}}\text{C} = 66^o$, $\text{BD} \approx 4.55$ cm

5 $x \approx 17.7$, $y \approx 33.1$

6 **a** 91.3^o **b** 91.3^o

c .... *cosine rule* as it avoids the *ambiguous case*.

7 Area $\approx 25.1\text{ cm}^2$ **8** $x = 8 + \frac{11}{2}\sqrt{2}$

EXERCISE 9D

1 17.7 m **2** 207 m **3** 23.9^o **4** 77.5 m

5 **a** **i** 5.63 km **ii** 115^o **b** **i** Esko **ii** 3.68 min

c 295^o

6 9.38^o **7** 69.1 m **8** **a** 38.0 m **b** 94.0 m **9** 55.1^o

10 AC ≈ 11.7 km, BC ≈ 8.49 km

11 **a** 74.9 km^2 **b** 7490 hectares

12 9.12 km **13** 85.0 mm **14** 10.1 km **15** 29.2 m

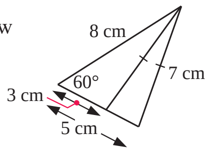
16 37.6 km

REVIEW SET 9A

1 14 km^2

2 If the unknown is an angle, use the cosine rule to avoid the ambiguous case.

3 **a** $x = 3$ or 5 **b** Kady can draw 2 triangles:



4 $\frac{12}{13}$ **5** 42 km

REVIEW SET 9B

1 **a** $x \approx 34.1$ **b** $x \approx 18.9$

2 AC ≈ 12.6 cm, $\widehat{A} \approx 48.6^o$, $\widehat{C} \approx 57.4^o$ **3** 113 cm^2

4 7.32 m **5** 204 m **6** 560 m, bearing 079.7^o

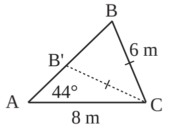
REVIEW SET 9C

1 **a** $x \approx 41.5$ **b** $x \approx 15.4$ **2** $x \approx 47.5$ or 132.5

3 $\text{E}\widehat{\text{D}}\text{G} \approx 74.4^o$ **4** **a** $10\,600\text{ m}^2$ **b** 1.06 ha

5 179 km, bearing 352^o

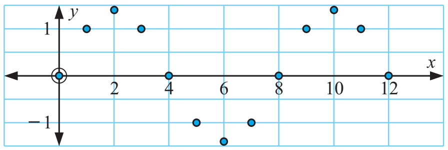
6 **a** The information given could give two triangles:



b $\approx 2.23\text{ m}^3$

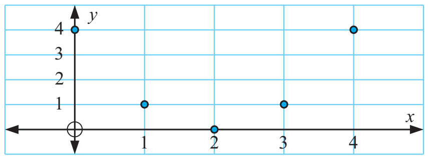
EXERCISE 10A

1 **a**



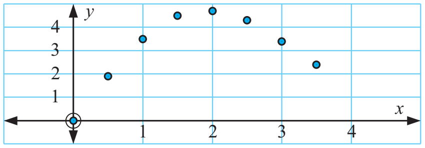
Data exhibits periodic behaviour.

b



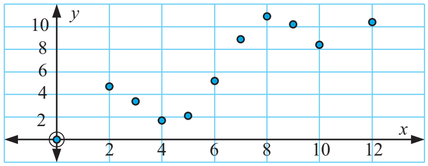
Not enough information to say data is periodic. It may in fact be quadratic.

c



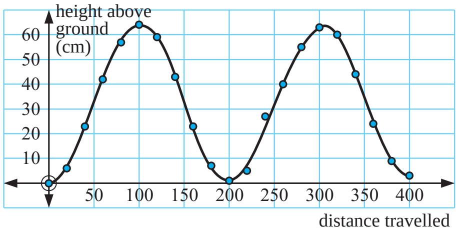
Not enough information to say data is periodic. It may in fact be quadratic.

d



Not enough information to say data is periodic.

2 **a**



b The data is periodic. **i** $y = 32$ (approx.) **ii** ≈ 64 cm

iii ≈ 200 cm **iv** ≈ 32 cm

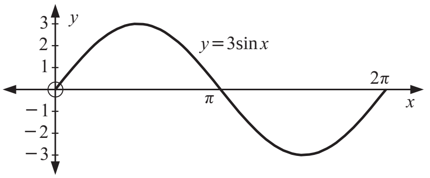
c A curve can be fitted to the data.

3 **a** periodic **b** periodic **c** periodic

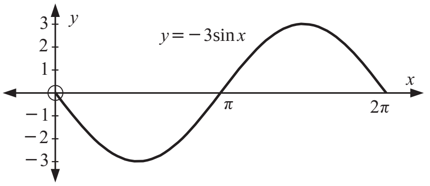
d not periodic **e** periodic **f** periodic

EXERCISE 10B.1

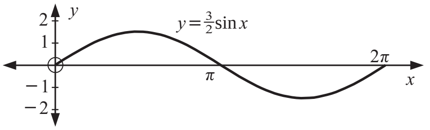
1 **a**



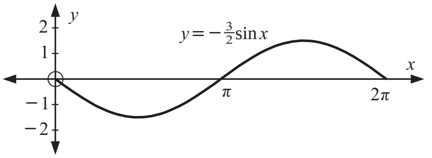
b



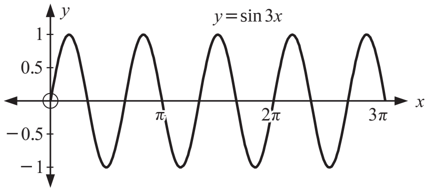
c



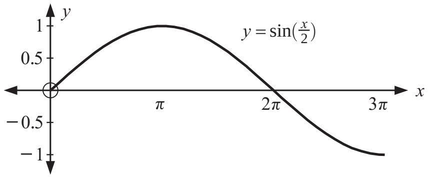
d



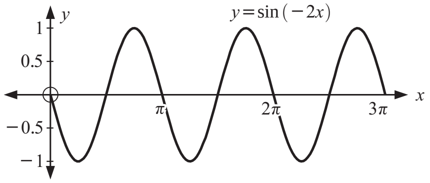
2 **a**



b



c

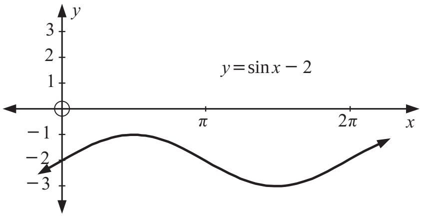


3 **a** $\frac{\pi}{2}$ **b** $\frac{\pi}{2}$ **c** 6π **d** $\frac{10\pi}{3}$

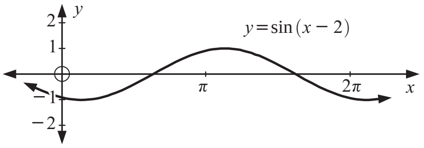
4 **a** $b = \frac{2}{5}$ **b** $b = 3$ **c** $b = \frac{1}{6}$ **d** $b = \frac{\pi}{2}$
e $b = \frac{\pi}{50}$

EXERCISE 10B.2

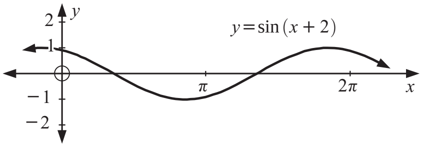
1 **a**



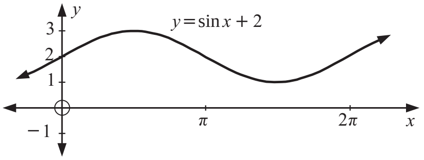
b



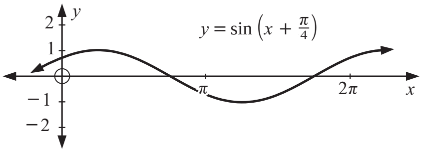
c



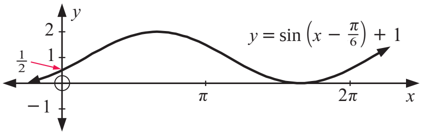
d



e



f



2 **a** $\frac{2\pi}{5}$ **b** 8π **c** π **3** **a** $\frac{2}{3}$ **b** 20 **c** $\frac{1}{50}$ **d** $\frac{\pi}{25}$

4 **a** vert. translation −1 **b** horiz. translation $\frac{\pi}{4}$ right
c vert. stretch, factor 2 **d** horiz. stretch, factor $\frac{1}{4}$
e vert. stretch, factor $\frac{1}{2}$ **f** horiz. stretch, factor 4
g reflection in the x-axis **h** translation $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$
i vert. stretch, factor 2, followed by a horiz. stretch, factor $\frac{1}{3}$
j translation $\begin{pmatrix} \frac{\pi}{3} \\ 2 \end{pmatrix}$

EXERCISE 10C

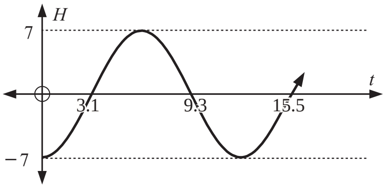
1 **a** $T \approx 6.5\sin\frac{\pi}{6}(t - 4.5) + 20.5$

2 **a** $T \approx 4.5\sin\frac{\pi}{6}(t - 10.5) + 11.5$

3 $T \approx 9.5\sin\frac{\pi}{6}(t - 10.5) - 9.5$

4 **a** $H \approx 7\sin 0.507(t - 3.1)$

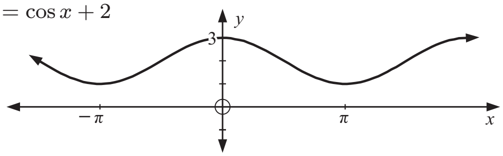
b



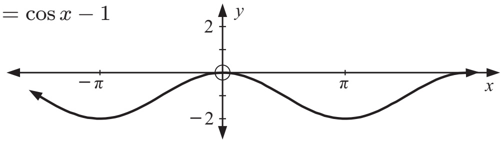
5 $H = 10\sin(\frac{\pi}{50}(t-25)) + 12$

EXERCISE 10D

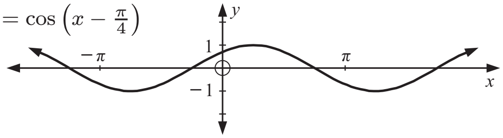
1 **a** $y = \cos x + 2$



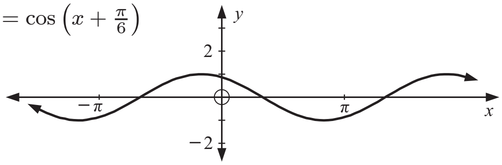
b $y = \cos x - 1$



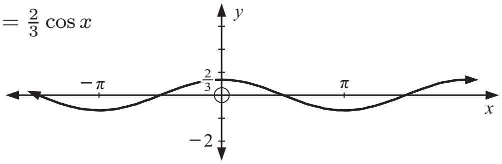
c $y = \cos\left(x - \frac{\pi}{4}\right)$



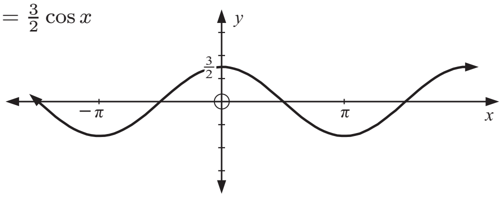
d $y = \cos\left(x + \frac{\pi}{6}\right)$



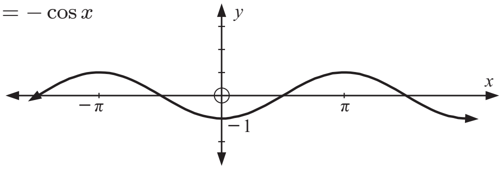
e $y = \frac{2}{3}\cos x$



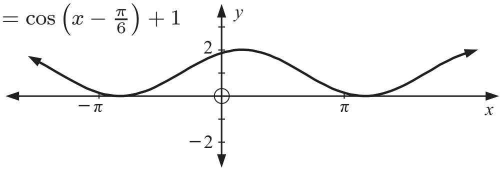
f $y = \frac{3}{2}\cos x$



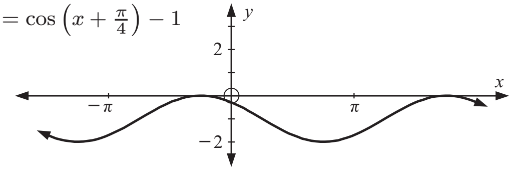
g $y = -\cos x$



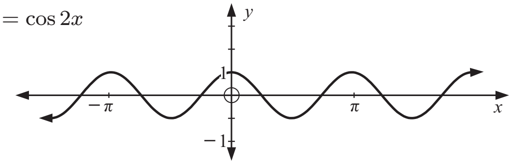
h $y = \cos\left(x - \frac{\pi}{6}\right) + 1$



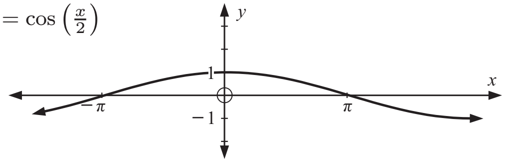
i $y = \cos\left(x + \frac{\pi}{4}\right) - 1$



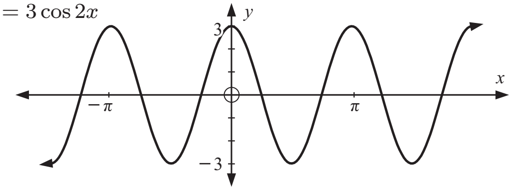
j $y = \cos 2x$



k $y = \cos\left(\frac{x}{2}\right)$



l $y = 3\cos 2x$



2 **a** $\frac{2\pi}{3}$ **b** 6π **c** 100

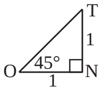
3 $|a|$ = amplitude, $b = \frac{2\pi}{\text{period}}$, c = horizontal translation, d = vertical translation

4 **a** $y = 2\cos 2x$ **b** $y = \cos\left(\frac{x}{2}\right) + 2$
c $y = -5\cos\left(\frac{\pi}{3}x\right)$

EXERCISE 10E.1

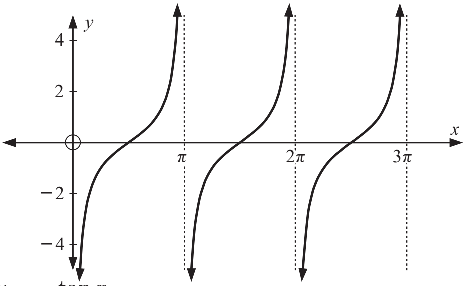
1 **a** 0 **b** ≈ 0.268 **c** ≈ 0.364
d ≈ 0.466 **e** ≈ 0.700 **f** 1
g ≈ 1.19 **h** ≈ 1.43

2 triangle TON is isosceles, ON = TN

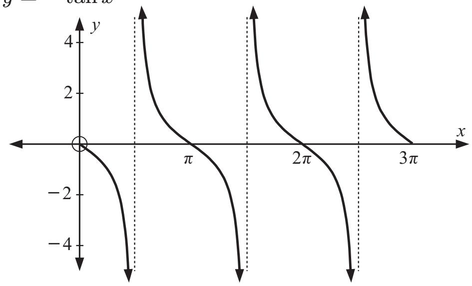


EXERCISE 10E.2

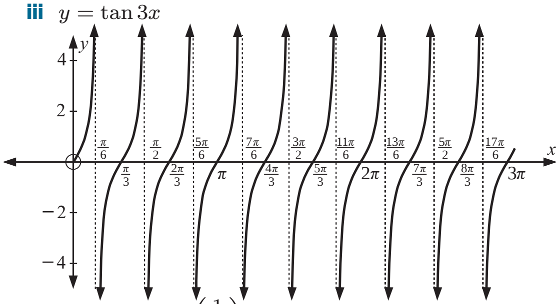
1 **a** **i** $y = \tan(x - \frac{\pi}{2})$



ii $y = -\tan x$



iii $y = \tan 3x$



2 **a** translation through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ **b** reflection in x-axis

c horizontal stretch, factor $q = 2$

3 **a** π **b** $\frac{\pi}{2}$ **c** $\frac{\pi}{n}$

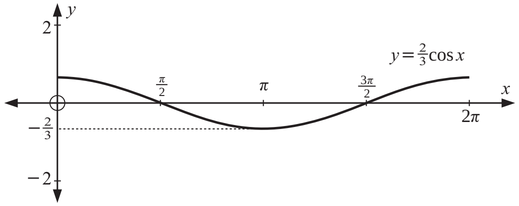
EXERCISE 10F

1 **a** 1 **b** undefined **c** 1

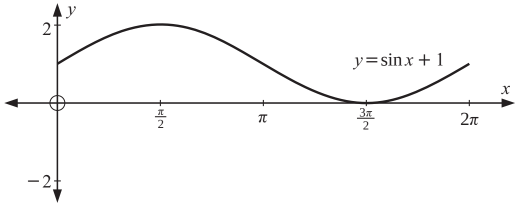
2 **a** π **b** 6π **c** π

3 **a** $b = 1$ **b** $b = 3$ **c** $b = 2$ **d** $b = \frac{\pi}{2}$

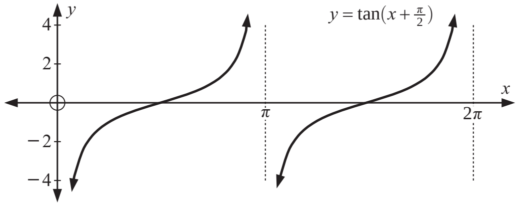
4 **a**



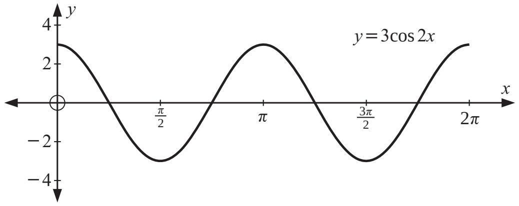
b



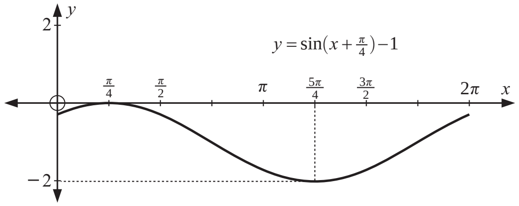
c



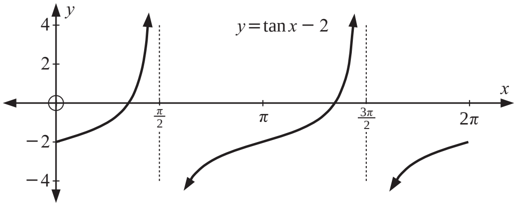
d



e



f



5

	a	**b**	**c**	**d**	**e**	**f**
maximum value	1	3	undef.	4	3	−2
minimum value	−1	−3	undef.	2	−1	−4

6 **a** vertical stretch, factor $\frac{1}{2}$ **b** horizontal stretch, factor 4

c reflection in the x-axis

d vertical translation down 2 units

e horizontally translate $\frac{\pi}{4}$ units to the left

f reflection in the y-axis

7 $m = 2, n = -3$ **8** $p = \frac{1}{2}, q = 1$

EXERCISE 10G.1

1 **a** $x \approx 0.3, 2.8, 6.6, 9.1, 12.9$ **b** $x \approx 5.9, 9.8, 12.2$

2 **a** $x \approx 1.2, 5.1, 7.4$ **b** $x \approx 4.4, 8.2, 10.7$

3 **a** $x \approx 0.4, 1.2, 3.5, 4.3, 6.7, 7.5, 9.8, 10.6, 13.0, 13.7$

b $x \approx 1.7, 3.0, 4.9, 6.1, 8.0, 9.3, 11.1, 12.4, 14.3, 15.6$

4 **a** **i** ≈ 1.6 **ii** ≈ -1.1

b **i** $x \approx 1.1, 4.2, 7.4$ **ii** $x \approx 2.2, 5.3$

EXERCISE 10G.2

1 **a** $x \approx 1.08, 4.35$ **b** $x \approx 0.666, 2.48$

c $x \approx 0.171, 4.92$ **d** $x \approx 1.31, 2.03, 2.85$

2 $x \approx -0.951, 0.234, 5.98$

EXERCISE 10G.3

1 **a** $x = \frac{\pi}{6}, \frac{13\pi}{6}, \frac{25\pi}{6}$ **b** $x = -\frac{\pi}{3}, \frac{5\pi}{3}$

c $x = -\frac{7\pi}{2}, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

d $x = \frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}, \frac{7\pi}{3}, \frac{17\pi}{6}, \frac{10\pi}{3}, \frac{23\pi}{6}$

2 **a** $x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{14\pi}{3}$

b $x = -330^o, -210^o, 30^o, 150^o$

c $x = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}$ **d** $x = -\frac{5\pi}{3}, -\pi, \frac{\pi}{3}, \pi$

e $x = -\frac{13\pi}{6}, -\frac{3\pi}{2}, -\frac{\pi}{6}, \frac{\pi}{2}, \frac{11\pi}{6}, \frac{5\pi}{2}$ **f** $x = 0, \frac{3\pi}{2}, 2\pi$

g $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ **h** $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

i $x = -\frac{8\pi}{9}, -\frac{4\pi}{9}, -\frac{2\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$

j $x = 0, \frac{\pi}{6}, \pi, \frac{7\pi}{6}, 2\pi$

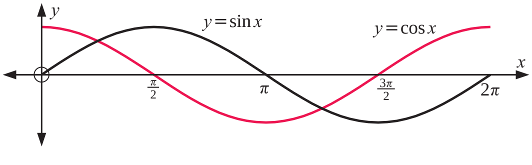
3 $X = \frac{\pi}{3} + k\pi, \quad k \in \mathbb{Z}$

a $x = \frac{\pi}{2}, \frac{3\pi}{2}$ **b** $x = \frac{\pi}{12}, \frac{\pi}{3}, \frac{7\pi}{12}, \frac{5\pi}{6}, \frac{13\pi}{12}, \frac{4\pi}{3}, \frac{19\pi}{12}, \frac{11\pi}{6}$

c $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

4 **a** $x = 0^o, 90^o, 180^o$ **b** $x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$

5 **a**



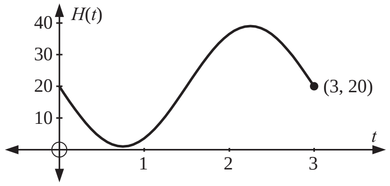
b $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$ **c** $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

6 **a** $x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$ **b** $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{7\pi}{4}$
c $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

EXERCISE 10H

1 **a** **i** 7500 grasshoppers **ii** 10 300 grasshoppers
b 10 500 grasshoppers, when $t = 4$ weeks
c **i** at $t = 1\frac{1}{3}$ wks and $6\frac{2}{3}$ wks **ii** at $t = 9\frac{1}{3}$ wks
d $2.51 \leqslant t \leqslant 5.49$

2 **a** 20 m **b** at $t = \frac{3}{4}$ minute **c** 3 minutes
d



3 **a** 400 water buffalo
b **i** 577 water buffalo **ii** 400 water buffalo
c 650, which is the maximum population.
d 150, after 3 years **e** $t \approx 0.26$ years

4 **a** $H(t) = 3\cos(\frac{\pi t}{2}) + 4$ **b** $t \approx 1.46$ s

5 **a** **i** true **ii** true **b** 116.8 cents L^{-1}
c on the 5th, 11th, 19th and 25th days
d 98.6 cents L^{-1} on the 1st and 15th day

EXERCISE 10I.1

1 **a** $2\sin\theta$ **b** $3\cos\theta$ **c** $2\sin\theta$ **d** $\sin\theta$
e $-2\cos\theta$ **f** $-3\cos\theta$

2 **a** 3 **b** -2 **c** -1 **d** $3\cos^2\theta$
e $4\sin^2\theta$ **f** $\cos\theta$ **g** $-\sin^2\theta$ **h** $-\cos^2\theta$
i $-2\sin^2\theta$ **j** 1 **k** $\sin\theta$ **l** $\sin\theta$

3 **a** $2\tan x$ **b** $-3\tan x$ **c** $\sin x$ **d** $\cos x$
e $5\sin x$ **f** $\dfrac{2}{\cos x}$

4 **a** $1 + 2\sin\theta + \sin^2\theta$ **b** $\sin^2\alpha - 4\sin\alpha + 4$
c $\tan^2\alpha - 2\tan\alpha + 1$ **d** $1 + 2\sin\alpha\cos\alpha$
e $1 - 2\sin\beta\cos\beta$ **f** $-4 + 4\cos\alpha - \cos^2\alpha$

EXERCISE 10I.2

1 **a** $(1 - \sin\theta)(1 + \sin\theta)$
b $(\sin\alpha + \cos\alpha)(\sin\alpha - \cos\alpha)$
c $(\tan\alpha + 1)(\tan\alpha - 1)$ **d** $\sin\beta(2\sin\beta - 1)$
e $\cos\phi(2 + 3\cos\phi)$ **f** $3\sin\theta(\sin\theta - 2)$
g $(\tan\theta + 3)(\tan\theta + 2)$ **h** $(2\cos\theta + 1)(\cos\theta + 3)$
i $(3\cos\alpha + 1)(2\cos\alpha - 1)$

2 **a** $1 + \sin\alpha$ **b** $\tan\beta - 1$ **c** $\cos\phi - \sin\phi$
d $\cos\phi + \sin\phi$ **e** $\dfrac{1}{\sin\alpha - \cos\alpha}$ **f** $\dfrac{\cos\theta}{2}$

EXERCISE 10J

1 **a** $\frac{24}{25}$ **b** $-\frac{7}{25}$ **2** **a** $-\frac{7}{9}$ **b** $\frac{1}{9}$

3 **a** $\cos\alpha = \frac{-\sqrt{5}}{3}$, $\sin 2\alpha = \frac{4\sqrt{5}}{9}$
b $\sin\beta = \frac{-\sqrt{21}}{5}$, $\sin 2\beta = \frac{-4\sqrt{21}}{25}$

4 **a** $\frac{1}{3}$ **b** $\frac{2\sqrt{2}}{3}$ **5** $\frac{3}{2}$

6 **a** $\sin 2\alpha$ **b** $2\sin 2\alpha$ **c** $\frac{1}{2}\sin 2\alpha$ **d** $\cos 2\beta$
e $-\cos 2\phi$ **f** $\cos 2N$ **g** $-\cos 2M$ **h** $\cos 2\alpha$
i $-\cos 2\alpha$ **j** $\sin 4A$ **k** $\sin 6\alpha$ **l** $\cos 8\theta$
m $-\cos 6\beta$ **n** $\cos 10\alpha$ **o** $-\cos 6D$ **p** $\cos 4A$
q $\cos\alpha$ **r** $-2\cos 6P$

8 **a** $x = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$ **b** $x = \frac{\pi}{2}, \frac{3\pi}{2}$
c $x = 0, \pi, 2\pi$

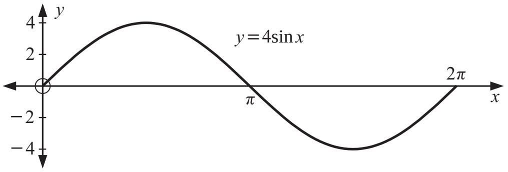
EXERCISE 10K

1 **a** $x = 0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}, 2\pi$ **b** $x = \frac{\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{3}$
c $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ **d** $x = \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$ **e** no solutions

2 **a** $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$ **b** $x = \frac{\pi}{3}, \frac{5\pi}{3}$ **c** $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$
d $x = 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}, 2\pi$ **e** $x = \frac{\pi}{4}$
f $x = \frac{\pi}{6}, \frac{5\pi}{6}$

REVIEW SET 10A

1



2 **a** minimum $= 0$, maximum $= 2$
b minimum $= -2$, maximum $= 2$

3 **a** $x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$ **b** $x = -\frac{7\pi}{4}, -\frac{5\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$

4 **a** $\frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}, \frac{17\pi}{9}$ **b** $\frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}$

5 $x = 0, \frac{3\pi}{2}, 2\pi, \frac{7\pi}{2}, 4\pi$

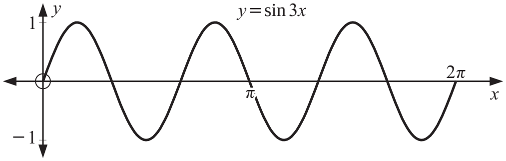
6 **a** $1 - \cos\theta$ **b** $\dfrac{1}{\sin\alpha + \cos\alpha}$ **c** $\dfrac{-\cos\alpha}{2}$

7 $\cos\alpha = -\frac{\sqrt{7}}{4}$, $\sin 2\alpha = \frac{3\sqrt{7}}{8}$ **9** **c** $x = \frac{16}{3}$

10 **a** 5000 **b** 3000, 7000 **c** $0.5 < t < 2.5$ and $6.5 < t \leqslant 8$

REVIEW SET 10B

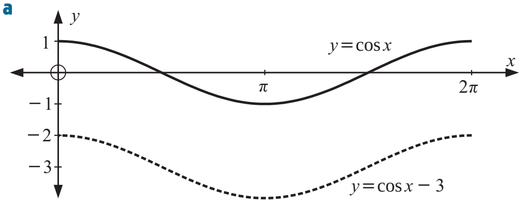
1



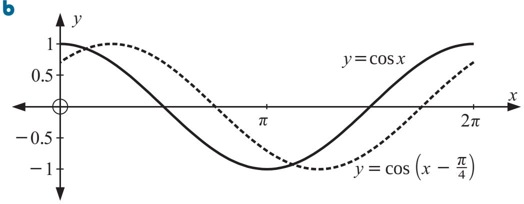
2 **a** 6π **b** $\frac{\pi}{4}$

3 **a** $x \approx 0.392, 2.75, 6.68$ **b** $x \approx 5.42$

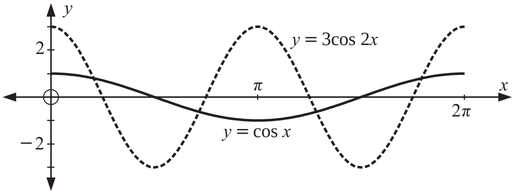
4 **a**



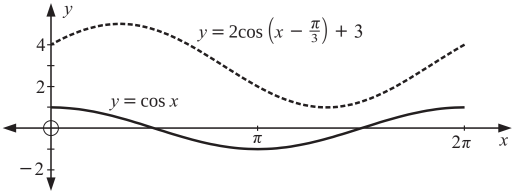
b



c



d



5 **a** $x \approx 1.12, 5.17, 7.40$ **b** $x \approx 0.184, 4.62$

6 **a** $\frac{120}{169}$ **b** $\frac{119}{169}$

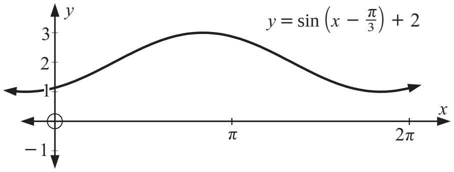
7 **a** **i** $x \approx 1.33, 4.47, 7.61$ **ii** $x \approx 5.30$
iii $x \approx 2.83, 5.97, 9.11$
b **i** $x = -\frac{\pi}{2}, \frac{\pi}{2}$ **ii** $x = -\frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}$
iii $x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$
c $x \approx 0.612, 3.754, 6.895$

8 **a** $T \approx 7.05\sin(\frac{\pi}{6}(t - 10.5)) + 24.75$

REVIEW SET 10C

1 **a** $x = \frac{3\pi}{2}$ **b** $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

2 **a**



b $1 \leqslant k \leqslant 3$

3 **a** $y = -4\cos 2x$ **b** $y = \cos\frac{\pi}{4}x + 2$

4 **a** $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ **b** $x = -\pi, -\frac{\pi}{3}, \pi, \frac{5\pi}{3}$

5 **a** $\cos\theta$ **b** $-\sin\theta$ **c** $5\cos^2\theta$ **d** $-\cos\theta$

6 **a** $4\sin^2\alpha - 4\sin\alpha + 1$ **b** $1 - \sin 2\alpha$

8 $\sin\theta = \frac{2}{\sqrt{13}}, \cos\theta = -\frac{3}{\sqrt{13}}$

9 **a** 28 milligrams per m^3 **b** 8.00 am Monday

EXERCISE 11A

1 **a** 1×4 **b** 2×1 **c** 2×2 **d** 3×3

2 **a** $\begin{pmatrix} 2 & 1 & 6 & 1 \end{pmatrix}$ **b** $\begin{pmatrix} 1.95 \\ 2.35 \\ 0.15 \\ 0.95 \end{pmatrix}$ **c** total cost of groceries

3 $\begin{pmatrix} 1000 & 1500 & 1250 \\ 1500 & 1000 & 1000 \\ 800 & 2300 & 1300 \\ 1200 & 1200 & 1200 \end{pmatrix}$ **4** $\begin{pmatrix} 40 & 50 & 55 & 40 \\ 25 & 65 & 44 & 30 \\ 35 & 40 & 40 & 35 \\ 35 & 40 & 35 & 50 \end{pmatrix}$

EXERCISE 11B.1

1 **a** $\begin{pmatrix} 9 & 1 \\ 3 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 6 & 8 \\ -1 & 1 \end{pmatrix}$
c $\begin{pmatrix} 3 & 4 \\ -6 & -1 \end{pmatrix}$ **d** $\begin{pmatrix} 0 & 0 \\ -11 & -3 \end{pmatrix}$

2 **a** $\begin{pmatrix} 20 & 1 & -8 \\ 8 & 10 & -2 \\ 1 & -5 & 18 \end{pmatrix}$ **b** $\begin{pmatrix} -14 & 9 & -14 \\ 12 & -6 & 14 \\ -5 & 3 & -4 \end{pmatrix}$
c $\begin{pmatrix} 14 & -9 & 14 \\ -12 & 6 & -14 \\ 5 & -3 & 4 \end{pmatrix}$

3 **a** Friday $\begin{pmatrix} 85 \\ 92 \\ 52 \end{pmatrix}$ Saturday $\begin{pmatrix} 102 \\ 137 \\ 49 \end{pmatrix}$ **b** $\begin{pmatrix} 187 \\ 229 \\ 101 \end{pmatrix}$

4 **a** **i** $\begin{pmatrix} 1.72 \\ 27.85 \\ 0.92 \\ 2.53 \\ 3.56 \end{pmatrix}$ **ii** $\begin{pmatrix} 1.79 \\ 28.75 \\ 1.33 \\ 2.25 \\ 3.51 \end{pmatrix}$ **c** $\begin{pmatrix} 0.07 \\ 0.90 \\ 0.41 \\ -0.28 \\ -0.05 \end{pmatrix}$
b subtract cost price from selling price

5 **a** $\begin{array}{c} \\ \text{fr} \\ \text{st} \\ \text{mi} \end{array} \begin{array}{c} \begin{array}{cc} \text{L} & \text{R} \end{array} \\ \begin{pmatrix} 23 & 19 \\ 17 & 29 \\ 31 & 24 \end{pmatrix} \end{array}$ **b** $\begin{array}{c} \\ \text{fr} \\ \text{st} \\ \text{mi} \end{array} \begin{array}{c} \begin{array}{cc} \text{L} & \text{R} \end{array} \\ \begin{pmatrix} 18 & 25 \\ 7 & 13 \\ 36 & 19 \end{pmatrix} \end{array}$ **c** $\begin{array}{c} \\ \text{fr} \\ \text{st} \\ \text{mi} \end{array} \begin{array}{c} \begin{array}{cc} \text{L} & \text{R} \end{array} \\ \begin{pmatrix} 41 & 44 \\ 24 & 42 \\ 67 & 43 \end{pmatrix} \end{array}$

6 **a** $x = -2, \ y = -2$ **b** $x = 0, \ y = 0$

7 **a** $\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1 & 3 \\ 5 & 2 \end{pmatrix}, \ \mathbf{B} + \mathbf{A} = \begin{pmatrix} 1 & 3 \\ 5 & 2 \end{pmatrix}$

8 **a** $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \begin{pmatrix} 6 & 3 \\ -1 & 6 \end{pmatrix}, \ \mathbf{A} + (\mathbf{B} + \mathbf{C}) = \begin{pmatrix} 6 & 3 \\ -1 & 6 \end{pmatrix}$

EXERCISE 11B.2

1 **a** $\begin{pmatrix} 12 & 24 \\ 48 & 12 \end{pmatrix}$ **b** $\begin{pmatrix} 2 & 4 \\ 8 & 2 \end{pmatrix}$
c $\begin{pmatrix} \frac{1}{2} & 1 \\ 2 & \frac{1}{2} \end{pmatrix}$ **d** $\begin{pmatrix} -3 & -6 \\ -12 & -3 \end{pmatrix}$

2 **a** $\begin{pmatrix} 3 & 5 & 6 \\ 2 & 8 & 7 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 1 & 4 \\ 0 & 4 & 1 \end{pmatrix}$
c $\begin{pmatrix} 5 & 8 & 11 \\ 3 & 14 & 11 \end{pmatrix}$ **d** $\begin{pmatrix} 5 & 7 & 14 \\ 2 & 16 & 9 \end{pmatrix}$

3 **a** $\begin{pmatrix} 12 \\ 24 \\ 120 \\ 60 \end{pmatrix}$ **b** $\begin{pmatrix} 3 \\ 6 \\ 30 \\ 15 \end{pmatrix}$ **c** $\begin{pmatrix} 9 \\ 18 \\ 90 \\ 45 \end{pmatrix}$

4 **a** $\begin{array}{c} \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\ \begin{pmatrix} 35 & 46 & 46 & 69 \\ 58 & 46 & 35 & 86 \\ 46 & 46 & 58 & 58 \\ 12 & 23 & 23 & 17 \end{pmatrix} \end{array}$ **b** $\begin{array}{c} \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\ \begin{pmatrix} 26 & 34 & 34 & 51 \\ 43 & 34 & 26 & 64 \\ 34 & 34 & 43 & 43 \\ 9 & 17 & 17 & 13 \end{pmatrix} \end{array}$

5 **a** $\begin{pmatrix} 75 \\ 27 \\ 102 \end{pmatrix} \begin{array}{l} \leftarrow \text{DVD} \\ \leftarrow \text{VHS} \\ \leftarrow \text{games} \end{array} \quad \begin{pmatrix} 136 \\ 43 \\ 129 \end{pmatrix} \begin{array}{l} \leftarrow \text{DVD} \\ \leftarrow \text{VHS} \\ \leftarrow \text{games} \end{array}$
b $\begin{pmatrix} 211 \\ 70 \\ 231 \end{pmatrix} \begin{array}{l} \leftarrow \text{DVD} \\ \leftarrow \text{VHS} \\ \leftarrow \text{games} \end{array}$ **c** total weekly average hirings

6 12**F**

EXERCISE 11B.3

1 **a** 3**A** **b** **O** **c** −**C** **d** **O** **e** 2**A** + 2**B**
f −**A** − **B** **g** −2**A** + **C** **h** 4**A** − **B**
i 3**B**

2 **a** $\mathbf{X}=\mathbf{A}-\mathbf{B}$ **b** $\mathbf{X}=\mathbf{C}-\mathbf{B}$ **c** $\mathbf{X}=2\mathbf{C}-4\mathbf{B}$
d $\mathbf{X}=\frac{1}{2}\mathbf{A}$ **e** $\mathbf{X}=\frac{1}{3}\mathbf{B}$ **f** $\mathbf{X}=\mathbf{A}-\mathbf{B}$
g $\mathbf{X}=2\mathbf{C}$ **h** $\mathbf{X}=\frac{1}{2}\mathbf{B}-\mathbf{A}$ **i** $\mathbf{X}=\frac{1}{4}(\mathbf{A}-\mathbf{C})$

3 **a** $\mathbf{X}=\begin{pmatrix}3&6\\9&18\end{pmatrix}$ **b** $\mathbf{X}=\begin{pmatrix}\frac{1}{2}&-\frac{1}{4}\\\frac{3}{4}&\frac{5}{4}\end{pmatrix}$

c $\mathbf{X}=\begin{pmatrix}-1&-6\\1&-\frac{1}{2}\end{pmatrix}$

EXERCISE 11B.4

1 **a** (11) **b** (22) **c** (16) **2** $\begin{pmatrix}w&x&y&z\end{pmatrix}\begin{pmatrix}\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\end{pmatrix}$

3 **a** $\mathbf{P}=\begin{pmatrix}27&35&39\end{pmatrix}$, $\mathbf{Q}=\begin{pmatrix}4\\3\\2\end{pmatrix}$

b total cost $=\begin{pmatrix}27&35&39\end{pmatrix}\begin{pmatrix}4\\3\\2\end{pmatrix}=\$291$

4 **a** $\mathbf{P}=\begin{pmatrix}10&6&3&1\end{pmatrix}$, $\mathbf{N}=\begin{pmatrix}3\\2\\4\\2\end{pmatrix}$

b total points $=\begin{pmatrix}10&6&3&1\end{pmatrix}\begin{pmatrix}3\\2\\4\\2\end{pmatrix}=56$ points

EXERCISE 11B.5

1 Number of columns in **A** does not equal number of rows in **B**.

2 **a** $m=n$ **b** 2×3 **c** **B** has 3 columns, **A** has 2 rows

3 **a** **i** does not exist **ii** $\begin{pmatrix}28&29\end{pmatrix}$

b **i** $\begin{pmatrix}8\end{pmatrix}$ **ii** $\begin{pmatrix}2&0&3\\8&0&12\\4&0&6\end{pmatrix}$

4 **a** $\begin{pmatrix}3&5&3\end{pmatrix}$ **b** $\begin{pmatrix}-2\\1\\1\end{pmatrix}$

5 **a** $\mathbf{C}=\begin{pmatrix}12.50\\9.50\end{pmatrix}$, $\mathbf{N}=\begin{pmatrix}2375&5156\\2502&3612\end{pmatrix}$

b $\begin{pmatrix}78\,669.50\\65\,589\end{pmatrix}$ income from day 1
income from day 2 **c** \$144 258.50

6 **a** $\mathbf{R}=\begin{pmatrix}1&1\\1&2\\2&3\end{pmatrix}$ **b** $\mathbf{P}=\begin{pmatrix}7&3&19\\6&2&22\end{pmatrix}$

c $\begin{pmatrix}48&70\\52&76\end{pmatrix}$ **d** My costs at store A are €48, my friend's costs at store B are €76.

e store A

EXERCISE 11B.6

1 **a** $\begin{pmatrix}16&18&15\\13&21&16\\10&22&24\end{pmatrix}$ **b** $\begin{pmatrix}10&6&-7\\9&3&0\\4&-4&-10\end{pmatrix}$

c $\begin{pmatrix}22&0&132&176&198\\44&154&88&110&0\\176&44&88&88&132\end{pmatrix}$ **d** $\begin{pmatrix}115\\136\\46\\106\end{pmatrix}$

2 **a** $\begin{pmatrix}3&3&2\end{pmatrix}$ **b** $\begin{pmatrix}125&150&140\\44&40&40\\75&80&65\end{pmatrix}$

c $\begin{pmatrix}657&730&670\end{pmatrix}$ **d** $\begin{pmatrix}369&420&385\end{pmatrix}$

e $\begin{pmatrix}657&730&670\\369&420&385\end{pmatrix}$

3 \$224 660

4 **a**
$$\begin{pmatrix}125&195&225\end{pmatrix}\times\begin{pmatrix}15&12&13&11&14&16&8\\4&3&6&2&0&4&7\\3&1&4&4&3&2&0\end{pmatrix}$$
$$-\begin{pmatrix}85&120&130\end{pmatrix}\times\begin{pmatrix}15&12&13&11&14&16&8\\4&3&6&2&0&4&7\\3&1&4&4&3&2&0\end{pmatrix}$$
$=\$7125$

b
$$\begin{pmatrix}125&195&225\end{pmatrix}\times\begin{pmatrix}15&12&13&11&14&16&8\\4&3&6&2&0&4&7\\3&1&4&4&3&2&0\end{pmatrix}$$
$$-\begin{pmatrix}85&120&130\end{pmatrix}\times\begin{pmatrix}20&20&20&20&20&20&20\\15&15&15&15&15&15&15\\5&5&5&5&5&5&5\end{pmatrix}$$
$=-\$9030$, which is a loss of \$9030

c
$$\left(\begin{pmatrix}125&195&225\end{pmatrix}-\begin{pmatrix}85&120&130\end{pmatrix}\right)$$
$$\times\begin{pmatrix}15&12&13&11&14&16&8\\4&3&6&2&0&4&7\\3&1&4&4&3&2&0\end{pmatrix}$$

EXERCISE 11B.7

1 $\mathbf{AB}=\begin{pmatrix}-1&1\\-1&7\end{pmatrix}$, $\mathbf{BA}=\begin{pmatrix}0&2\\3&6\end{pmatrix}$, $\mathbf{AB}\neq\mathbf{BA}$

2 $\mathbf{AO}=\mathbf{OA}=\mathbf{O}$ **4** **b** yes, $\mathbf{I}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

5 **a** $\begin{pmatrix}7&0\\0&7\end{pmatrix}$ **b** $\begin{pmatrix}97&-59\\118&38\end{pmatrix}$

6 **a** $\mathbf{A}^2$ does not exist **b** when **A** is a square matrix

EXERCISE 11B.8

1 **a** $\mathbf{A}^2+\mathbf{A}$ **b** $\mathbf{B}^2+2\mathbf{B}$ **c** $\mathbf{A}^3-2\mathbf{A}^2+\mathbf{A}$
d $\mathbf{A}^3+\mathbf{A}^2-2\mathbf{A}$ **e** $\mathbf{AC}+\mathbf{AD}+\mathbf{BC}+\mathbf{BD}$
f $\mathbf{A}^2+\mathbf{AB}+\mathbf{BA}+\mathbf{B}^2$ **g** $\mathbf{A}^2-\mathbf{AB}+\mathbf{BA}-\mathbf{B}^2$
h $\mathbf{A}^2+2\mathbf{A}+\mathbf{I}$ **i** $9\mathbf{I}-6\mathbf{B}+\mathbf{B}^2$

2 **a** $\mathbf{A}^3=3\mathbf{A}-2\mathbf{I}$, $\mathbf{A}^4=4\mathbf{A}-3\mathbf{I}$
b $\mathbf{B}^3=3\mathbf{B}-2\mathbf{I}$, $\mathbf{B}^4=6\mathbf{I}-5\mathbf{B}$, $\mathbf{B}^5=11\mathbf{B}-10\mathbf{I}$
c $\mathbf{C}^3=13\mathbf{C}-12\mathbf{I}$, $\mathbf{C}^5=121\mathbf{C}-120\mathbf{I}$

3 **a** **i** $\mathbf{I}+2\mathbf{A}$ **ii** $2\mathbf{I}-2\mathbf{A}$ **iii** $10\mathbf{A}+6\mathbf{I}$
b $\mathbf{A}^2+\mathbf{A}+2\mathbf{I}$
c **i** $-3\mathbf{A}$ **ii** $-2\mathbf{A}$ **iii** **A**

4 **a** $\mathbf{AB}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$ **b** $\mathbf{A}^2=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{pmatrix}$

c false as $\mathbf{A}(\mathbf{A}-\mathbf{I})=\mathbf{O}$ does not imply that $\mathbf{A}=\mathbf{O}$ or $\mathbf{A}-\mathbf{I}=\mathbf{O}$

d $\begin{pmatrix}0&0\\0&0\end{pmatrix}$, $\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $\begin{pmatrix}a&b\\\frac{a-a^2}{b}&1-a\end{pmatrix}$, $b\neq 0$

5 For example, $\mathbf{A}=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, gives $\mathbf{A}^2=\begin{pmatrix}0&0\\0&0\end{pmatrix}$

6 **a** $a=3$, $b=-4$ **b** $a=1$, $b=8$

7 $p=-2$, $q=1$ **a** $\mathbf{A}^3=5\mathbf{A}-2\mathbf{I}$ **b** $\mathbf{A}^4=-12\mathbf{A}+5\mathbf{I}$

EXERCISE 11C.1

1 **a** $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = 3\mathbf{I}$, $\begin{pmatrix} 1 & -2 \\ -\frac{2}{3} & \frac{5}{3} \end{pmatrix}$

b $\begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} = 10\mathbf{I}$, $\begin{pmatrix} 0.2 & 0.4 \\ -0.1 & 0.3 \end{pmatrix}$

2 **a** -2 **b** -1 **c** 0 **d** 1

3 **a** 26 **b** 6 **c** -1 **d** $a^2 + a$

4 **a** -3 **b** 9 **c** -12

5 **Hint:** Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

6 **a** $|\mathbf{A}| = ad - bc$, $|\mathbf{B}| = wz - xy$

b $\mathbf{AB} = \begin{pmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{pmatrix}$,

$|\mathbf{AB}| = (ad - bc)(wz - xy)$

7 **a** **i** -2 **ii** -8 **iii** -2 **iv** -9 **v** 2

8 **a** $\frac{1}{14}\begin{pmatrix} 5 & -4 \\ 1 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ **c** does not exist

d $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **e** does not exist **f** $-\frac{1}{15}\begin{pmatrix} 7 & -2 \\ -4 & -1 \end{pmatrix}$

g $\frac{1}{10}\begin{pmatrix} 2 & -4 \\ 1 & 3 \end{pmatrix}$ **h** $\begin{pmatrix} -3 & -1 \\ 2 & 1 \end{pmatrix}$

EXERCISE 11C.2

1 **a** $\begin{pmatrix} x + 2y \\ 3x + 4y \end{pmatrix}$ **b** $\begin{pmatrix} 2a + 3b \\ a - 4b \end{pmatrix}$

2 **a** $\begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$

b $\begin{pmatrix} 4 & -3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ -5 \end{pmatrix}$

c $\begin{pmatrix} 3 & -1 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$

3 **a** $x = \frac{32}{7}$, $y = \frac{22}{7}$ **b** $x = -\frac{37}{23}$, $y = -\frac{75}{23}$

c $x = \frac{17}{13}$, $y = -\frac{37}{13}$ **d** $x = \frac{59}{13}$, $y = -\frac{25}{13}$

e $x = -40$, $y = -24$ **f** $x = \frac{1}{34}$, $y = \frac{55}{34}$

4 **b** **i** $\mathbf{X} = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}$ **ii** $\mathbf{X} = \begin{pmatrix} \frac{13}{7} & \frac{3}{7} \\ -\frac{2}{7} & -\frac{8}{7} \end{pmatrix}$

5 **a** **i** $k = -3$ **ii** $\dfrac{1}{2k + 6}\begin{pmatrix} 2 & -1 \\ 6 & k \end{pmatrix}$, $k \neq -3$

b **i** $k = 0$ **ii** $\dfrac{1}{3k}\begin{pmatrix} k & 1 \\ 0 & 3 \end{pmatrix}$, $k \neq 0$

c **i** $k = -2$ or 1

ii $\dfrac{1}{(k + 2)(k - 1)}\begin{pmatrix} k & -2 \\ -1 & k + 1 \end{pmatrix}$, $k \neq -2$ or 1

6 **a** **i** $\begin{pmatrix} 2 & -3 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \end{pmatrix}$, $|\mathbf{A}| = 10$

ii Yes, $x = 2.5$, $y = -1$

b **i** $\begin{pmatrix} 2 & k \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \end{pmatrix}$, $|\mathbf{A}| = -2 - 4k$

ii $k \neq -\frac{1}{2}$, $x = \dfrac{8 + 11k}{2 + 4k}$, $y = \dfrac{5}{1 + 2k}$

iii $k = -\frac{1}{2}$, no solutions

EXERCISE 11C.3

1 $\mathbf{X} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ 1 & 0 \end{pmatrix}$

2 **b** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

3 **a** $\mathbf{A}^{-1} = \begin{pmatrix} 0 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, $(\mathbf{A}^{-1})^{-1} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$

b $(\mathbf{A}^{-1})^{-1}(\mathbf{A}^{-1}) = (\mathbf{A}^{-1})(\mathbf{A}^{-1})^{-1} = \mathbf{I}$

c $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$

4 **a** **i** $\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$ **ii** $\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$ **iii** $\begin{pmatrix} \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$

iv $\begin{pmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$ **v** $\begin{pmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$ **vi** $\begin{pmatrix} \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$

c $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ and $(\mathbf{BA})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$

d $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = (\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = \mathbf{I}$

$\mathbf{AB}$ and $\mathbf{B}^{-1}\mathbf{A}^{-1}$ are inverses

5 $(k\mathbf{A})\left(\dfrac{1}{k}\mathbf{A}^{-1}\right) = \left(\dfrac{1}{k}\mathbf{A}^{-1}\right)(k\mathbf{A}) = \mathbf{I}$

$k\mathbf{A}$ and $\dfrac{1}{k}\mathbf{A}^{-1}$ are inverses

6 **a** $\mathbf{X} = \mathbf{ABZ}$ **b** $\mathbf{Z} = \mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{X}$

7 $\mathbf{A}^2 = 2\mathbf{A} - \mathbf{I}$, $\mathbf{A}^{-1} = 2\mathbf{I} - \mathbf{A}$

8 **a** $\mathbf{A}^{-1} = 4\mathbf{I} - \mathbf{A}$ **b** $\mathbf{A}^{-1} = 5\mathbf{I} + \mathbf{A}$ **c** $\mathbf{A}^{-1} = \frac{3}{2}\mathbf{A} - 2\mathbf{I}$

10 If $\mathbf{A}^{-1}$ exists, so $|\mathbf{A}| \neq 0$.

EXERCISE 11D.1

1 **a** 41 **b** -8 **c** 0 **d** 6 **e** -6 **f** -12

2 **a** $x = 1$ or 5

b When $x = 1$ or 5, the matrix does not have an inverse.

3 **a** abc **b** 0 **c** $3abc - a^3 - b^3 - c^3$ **4** $k \neq -3$

5 for all values of k except $\frac{1}{2}$ or -9

6 **a** $k = 1$ or 4 **b** $k = \frac{5}{2}$ or 2

EXERCISE 11D.2

1 $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2\mathbf{I}$, $\begin{pmatrix} -\frac{11}{2} & \frac{9}{2} & \frac{15}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 4 & -3 & -5 \end{pmatrix}$

2 **a** $\begin{pmatrix} \frac{5}{4} & \frac{3}{4} & -\frac{7}{4} \\ -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ -\frac{3}{4} & -\frac{1}{4} & \frac{5}{4} \end{pmatrix}$ **b** $\begin{pmatrix} -5.5 & 4.5 & 7.5 \\ -0.5 & 0.5 & 0.5 \\ 4 & -3 & -5 \end{pmatrix}$

3 **a** $\begin{pmatrix} 0.050 & -0.011 & -0.066 \\ 0.000 & 0.014 & 0.028 \\ -0.030 & 0.039 & 0.030 \end{pmatrix}$

b $\begin{pmatrix} 1.596 & -0.996 & -0.169 \\ -3.224 & 1.925 & 0.629 \\ 2.000 & -1.086 & -0.396 \end{pmatrix}$

EXERCISE 11E

1 **a** $\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 3 \\ 9 & -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ -1 \end{pmatrix}$

b $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 13 \end{pmatrix}$

c $\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \\ -2 \end{pmatrix}$

2 $\mathbf{AB} = \mathbf{I}$, $a = 2$, $b = -1$, $c = 3$

3 $\mathbf{MN} = 4\mathbf{I}$, $u = -1$, $v = 3$, $w = 5$

4 **a** $x = 2.3$, $y = 1.3$, $z = -4.5$
b $x = -\frac{1}{3}$, $y = -\frac{95}{21}$, $z = \frac{2}{21}$
c $x = 2$, $y = 4$, $z = -1$

5 **a** $x = 2$, $y = -1$, $z = 5$ **b** $x = 4$, $y = -2$, $z = 1$
c $x = 4$, $y = -3$, $z = 2$ **d** $x = 4$, $y = 6$, $z = -7$
e $x = 3$, $y = 11$, $z = -7$
f $x \approx 0.33$, $y \approx 7.65$, $z \approx 4.16$

6 **a** x represents the cost per football in dollars,
y represents the cost per baseball in dollars,
z represents the cost per basketball in dollars
b 12 basketballs

7 **a** $2x + 3y + 8z = 352$
$x + 5y + 4z = 274$
$x + 2y + 11z = 351$
b $x = 42$, $y = 28$, $z = 23$
c €1 201 000

8 **a** Cashews \$12, Macadamias \$15, Brazil nuts \$10
b \$11.80 per kg

9 **a** $5p + 5q + 6r = 405$
$15p + 20q + 6r = 1050$
$15p + 20q + 36r = 1800$
b $p = 24$,
$q = 27$,
$r = 25$

10 **a** $a = 50\,000$, $b = 100\,000$, $c = 240\,000$ **b** yes
c 2009, $\approx$ £284 000, 2011, $\approx$ £377 000

REVIEW SET 11A

1 **a** $\begin{pmatrix} 4 & 2 \\ -2 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 9 & 6 \\ 0 & -3 \end{pmatrix}$ **c** $\begin{pmatrix} -2 & 0 \\ 4 & -8 \end{pmatrix}$

d $\begin{pmatrix} 2 & 2 \\ 2 & -5 \end{pmatrix}$ **e** $\begin{pmatrix} -5 & -4 \\ -2 & 6 \end{pmatrix}$ **f** $\begin{pmatrix} 7 & 6 \\ 4 & -11 \end{pmatrix}$

g $\begin{pmatrix} -1 & 8 \\ 2 & -4 \end{pmatrix}$ **h** $\begin{pmatrix} 3 & 2 \\ -6 & -8 \end{pmatrix}$ **i** $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ 0 & -1 \end{pmatrix}$

j $\begin{pmatrix} 9 & 4 \\ 0 & 1 \end{pmatrix}$ **k** $\begin{pmatrix} -3 & -10 \\ 6 & 8 \end{pmatrix}$ **l** $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{12} \end{pmatrix}$

2 **a** $a = 0$, $b = 5$, $c = 1$, $d = -4$
b $a = 2$, $b = -1$, $c = 3$, $d = 8$

3 **a** $\mathbf{Y} = \mathbf{B} - \mathbf{A}$ **b** $\mathbf{Y} = \frac{1}{2}(\mathbf{D} - \mathbf{C})$ **c** $\mathbf{Y} = \mathbf{A}^{-1}\mathbf{B}$
d $\mathbf{Y} = \mathbf{C}\mathbf{B}^{-1}$ **e** $\mathbf{Y} = \mathbf{A}^{-1}(\mathbf{C} - \mathbf{B})$ **f** $\mathbf{Y} = \mathbf{B}^{-1}\mathbf{A}$

4 $a = 3$ **5** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

6 **a** $\begin{pmatrix} 10 \end{pmatrix}$ **b** $\begin{pmatrix} 4 & 3 & 2 \\ 8 & 6 & 4 \\ 0 & 0 & 0 \end{pmatrix}$ **c** $\begin{pmatrix} 15 & 18 & 21 \end{pmatrix}$

d $\mathbf{CA}$ does not exist **e** $\begin{pmatrix} 5 \\ 7 \\ 5 \end{pmatrix}$

7 **b** $2\mathbf{A} - \mathbf{I}$ **8** $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, $\mathbf{A}^{-1} = \mathbf{B}$ **9** $k \neq -3$ or 1

10 $\mathbf{X} = \begin{pmatrix} 1 & 3 & 2 \\ -1 & 1 & 3 \end{pmatrix}$ **11** $m = 6$ or -3

13 $\mathbf{M} = \begin{pmatrix} 0 & -2 \\ 5 & 1 \end{pmatrix}$

14 **a** $a = 1$, $b = -1$,
$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & -1 \end{pmatrix}$
b $x = -5$, $y = 4$, $z = 7$

REVIEW SET 11B

1 $x = 1$, $y = -1$, $z = 2$

2 **a** $x = 0$, $y = -\frac{1}{2}$ **b** $x = \frac{12}{7}$, $y = \frac{13}{7}$

c $\mathbf{X} = \begin{pmatrix} -1 & 8 \\ -2 & 6 \end{pmatrix}$ **d** $\mathbf{X} = \begin{pmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{pmatrix}$

e $\mathbf{X} = \begin{pmatrix} \frac{14}{3} \\ \frac{1}{3} \end{pmatrix}$ **f** $\mathbf{X} = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$

3 **a** $\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{52} & \frac{5}{26} \\ \frac{1}{4} & -\frac{5}{52} & -\frac{1}{26} \\ \frac{1}{4} & \frac{7}{52} & -\frac{9}{26} \end{pmatrix}$ **b** $x = 2$, $y = 1$, $z = 3$

4 $x = -1$, 2 or -4

5 **a** $\begin{pmatrix} 10 & -12 \\ -10 & 4 \end{pmatrix}$ **b** $\begin{pmatrix} 2 & 6 & -3 \\ -4 & -2 & 11 \end{pmatrix}$

c not possible **d** $\begin{pmatrix} 2.9 & -0.3 \\ -0.3 & 2.1 \end{pmatrix}$

6 **a** **i** $|\mathbf{B}| \neq 0$ **ii** $\mathbf{AB} = \mathbf{BA}$ **b** $k \neq 3$, -2 or 2

7 $x = -2$, $y = 3$, $z = -1$

8 **a** $a = -3$, $b = 18$, $c = 48$ $\Rightarrow s(t) = -3t^2 + 18t + 48$
b 48 m **c** 8 seconds

9 $\mathbf{X} = \begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix}$ **10** **a** $d = 80$ **b** $a = 2$, $b = 8$, $c = 10$

12 **a** $3x + 2y + 5z = 267$
$2x + 3y + z = 145$
$x + 5y + 4z = 230$
b Opera €32
Play €18
Concert €27
c €200

REVIEW SET 11C

1 **a** $\begin{pmatrix} 4 & 8 \\ 0 & 2 \\ 6 & 4 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 2 \\ 0 & \frac{1}{2} \\ \frac{3}{2} & 1 \end{pmatrix}$ **c** $\begin{pmatrix} 11 & 12 \end{pmatrix}$

d **BA** does not exist.

2 **a** $\begin{pmatrix} 4 & 2 \\ 2 & 4 \\ 3 & 4 \end{pmatrix}$ **b** $\begin{pmatrix} 2 & -2 \\ 0 & 4 \\ -1 & -2 \end{pmatrix}$ **c** $\begin{pmatrix} -\frac{3}{2} & 3 \\ \frac{1}{2} & -4 \\ 2 & \frac{7}{2} \end{pmatrix}$

3 $k \neq \frac{3}{4}$ **4** $k = -6$

5 **a** $\begin{pmatrix} \frac{7}{2} & -4 \\ \frac{-5}{2} & 3 \end{pmatrix}$ **b** does not exist **c** $\begin{pmatrix} 1 & \frac{5}{3} \\ -2 & -\frac{11}{3} \end{pmatrix}$

6 **a** $\begin{pmatrix} -9 & 6 & 6 \\ 3 & -3 & 0 \end{pmatrix}$ **b** $\begin{pmatrix} -10 & -6 \\ 5 & 3 \end{pmatrix}$

c $\begin{pmatrix} -2 & 0 & 4 \\ 10 & -7 & -6 \\ -1 & 0 & 2 \end{pmatrix}$ **d** not possible

e $\begin{pmatrix} 0 & 22 \\ 7 & -12 \\ 0 & 11 \end{pmatrix}$

7 $x = 5$

8 **a** $\mathbf{A}(\frac{5}{3}\mathbf{A} - 2\mathbf{I}) = \mathbf{I}$ **b** $\mathbf{A}^{-1} = \frac{5}{3}\mathbf{A} - 2\mathbf{I}$

9 **a** $\begin{pmatrix} -1 & -26 \\ 0 & -27 \end{pmatrix}$ **b** $\mathbf{X} = \begin{pmatrix} 1 & 13 \\ -3 & 17 \end{pmatrix}$

10 $\mathbf{A}^3 = 27\mathbf{A} + 10\mathbf{I}$, $\mathbf{A}^4 = 145\mathbf{A} + 54\mathbf{I}$,
$\mathbf{A}^5 = 779\mathbf{A} + 290\mathbf{I}$, $\mathbf{A}^6 = 4185\mathbf{A} + 1558\mathbf{I}$

11 \$56.30

12 **a** $\mathbf{AB} = \begin{pmatrix} c & d-2 \\ ac & ad-b \end{pmatrix}$

b $\mathbf{A}^{-1} = \dfrac{1}{b-2a}\begin{pmatrix} b & -2 \\ -a & 1 \end{pmatrix}$

c $c = -1$, $d = 0$

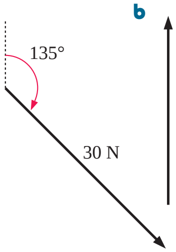
13 **a** $\mathbf{A}^3 = -\mathbf{I}$, $\mathbf{A}^4 = -\mathbf{A}$, $\mathbf{A}^5 = -\mathbf{A} + \mathbf{I}$, $\mathbf{A}^6 = \mathbf{I}$, $\mathbf{A}^7 = \mathbf{A}$,
$\mathbf{A}^8 = \mathbf{A} - \mathbf{I}$

b $\mathbf{A}^{6n+3} = (\mathbf{A}^6)^n\mathbf{A}^3 = -\mathbf{I}$, $\mathbf{A}^{6n+5} = -\mathbf{A} + \mathbf{I}$

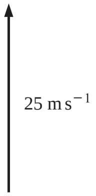
c $\mathbf{A}^{-1} = -\mathbf{A} + \mathbf{I}$

EXERCISE 12A.1

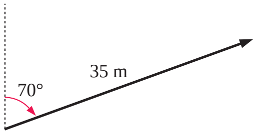
1 **a**



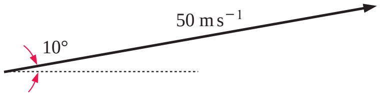
b



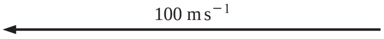
c



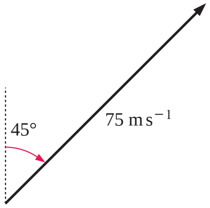
d



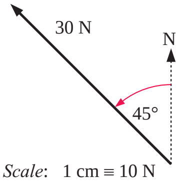
2 **a**



b

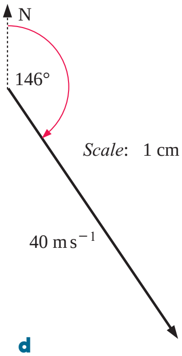


3 **a**



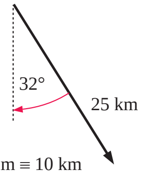
Scale: 1 cm ≡ 10 N

b



Scale: 1 cm ≡10 m s⁻¹

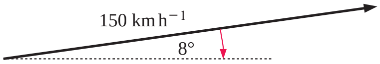
c



Scale: 1 cm ≡ 10 km

d

Scale: 1 cm ≡ 30 km h⁻¹



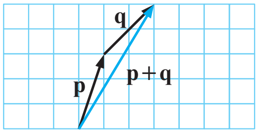
EXERCISE 12A.2

1 **a** **p**, **q**, **s**, **t** **b** **p**, **q**, **r**, **t** **c** **p** and **r**, **q** and **t** **d** **q**, **t**
e **p** and **q**, **p** and **t**

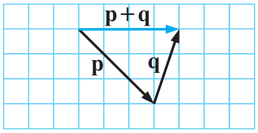
2 **a** true **b** true **c** false **d** false **e** true **f** false

EXERCISE 12B.1

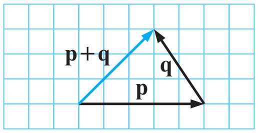
1 **a**



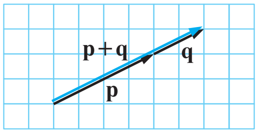
b



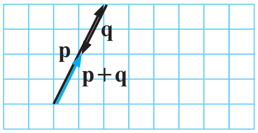
c



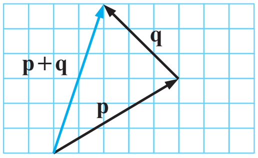
d



e

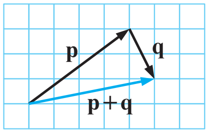


f

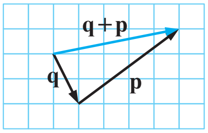


2 **a** $\overrightarrow{AC}$ **b** $\overrightarrow{BD}$ **c** $\overrightarrow{AD}$ **d** $\overrightarrow{AD}$

3 **a** **i**



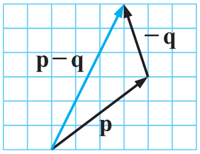
ii



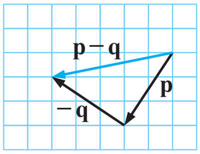
b yes

EXERCISE 12B.2

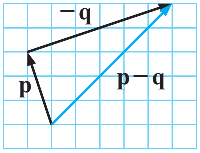
1 **a**



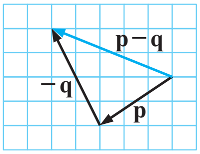
b



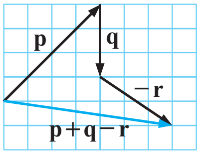
c



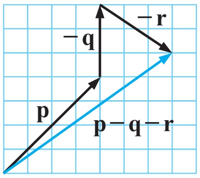
d



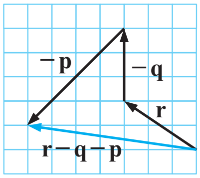
2 **a**



b



c



3 **a** $\overrightarrow{AB}$ **b** $\overrightarrow{AB}$ **c** $\mathbf{0}$ **d** $\overrightarrow{AD}$ **e** $\mathbf{0}$ **f** $\overrightarrow{AD}$

4 **a** $\mathbf{t} = \mathbf{r} + \mathbf{s}$ **b** $\mathbf{r} = -\mathbf{s} - \mathbf{t}$

c $\mathbf{r} = -\mathbf{p} - \mathbf{q} - \mathbf{s}$ **d** $\mathbf{r} = \mathbf{q} - \mathbf{p} + \mathbf{s}$

e $\mathbf{p} = \mathbf{t} + \mathbf{s} + \mathbf{r} - \mathbf{q}$ **f** $\mathbf{p} = -\mathbf{u} + \mathbf{t} + \mathbf{s} - \mathbf{r} - \mathbf{q}$

5 **a** **i** $\mathbf{r} + \mathbf{s}$ **ii** $-\mathbf{t} - \mathbf{s}$ **iii** $\mathbf{r} + \mathbf{s} + \mathbf{t}$

b **i** $\mathbf{p} + \mathbf{q}$ **ii** $\mathbf{q} + \mathbf{r}$ **iii** $\mathbf{p} + \mathbf{q} + \mathbf{r}$

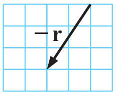
EXERCISE 12B.3

1 **a** 24.6 km h^{-1} **b** 9.93^o east of south

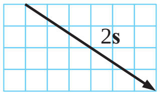
2 **a** 82.5 m **b** 23.3^o west of north **c** 48.4 seconds

EXERCISE 12B.4

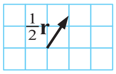
1 **a**



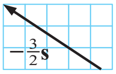
b



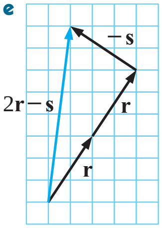
c



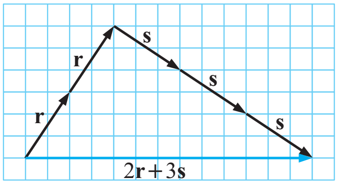
d



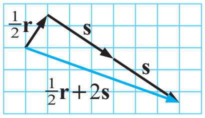
e



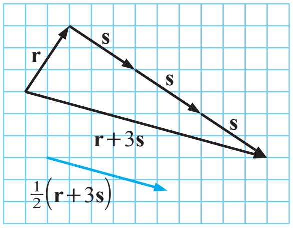
f



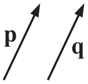
g



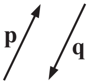
h



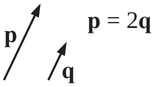
2 **a**



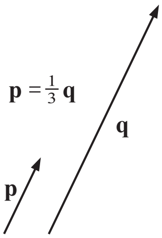
b



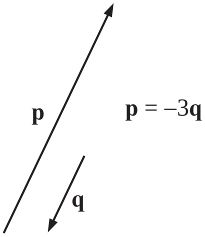
c



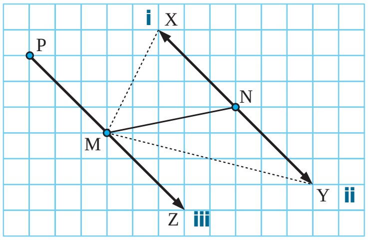
d



e



3 **a**

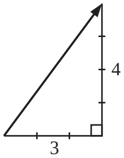


b a parallelogram

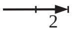
4 **a** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

EXERCISE 12C.1

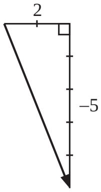
1 **a**



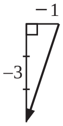
b



c



d



2 **a** $\begin{pmatrix}7\\3\end{pmatrix}$ **b** $\begin{pmatrix}-6\\0\end{pmatrix}$ **c** $\begin{pmatrix}2\\-5\end{pmatrix}$

d $\begin{pmatrix}0\\6\end{pmatrix}$ **e** $\begin{pmatrix}-6\\3\end{pmatrix}$ **f** $\begin{pmatrix}-5\\-5\end{pmatrix}$

EXERCISE 12C.2

1 **a** $\begin{pmatrix}-2\\6\end{pmatrix}$ **b** $\begin{pmatrix}-2\\6\end{pmatrix}$ **c** $\begin{pmatrix}-1\\-1\end{pmatrix}$ **d** $\begin{pmatrix}-1\\-1\end{pmatrix}$

e $\begin{pmatrix}-5\\-3\end{pmatrix}$ **f** $\begin{pmatrix}-5\\-3\end{pmatrix}$ **g** $\begin{pmatrix}-6\\4\end{pmatrix}$ **h** $\begin{pmatrix}-4\\1\end{pmatrix}$

2 **a** $\begin{pmatrix}-3\\7\end{pmatrix}$ **b** $\begin{pmatrix}-4\\-3\end{pmatrix}$ **c** $\begin{pmatrix}-8\\-1\end{pmatrix}$ **d** $\begin{pmatrix}-6\\9\end{pmatrix}$

e $\begin{pmatrix}0\\-5\end{pmatrix}$ **f** $\begin{pmatrix}6\\-9\end{pmatrix}$

3 **a** $\begin{pmatrix}-5\\4\end{pmatrix}$ **b** $\begin{pmatrix}1\\2\end{pmatrix}$ **c** $\begin{pmatrix}6\\-5\end{pmatrix}$

4 **a** $\begin{pmatrix}2\\4\end{pmatrix}$ **b** $\begin{pmatrix}-2\\5\end{pmatrix}$ **c** $\begin{pmatrix}3\\-3\end{pmatrix}$ **d** $\begin{pmatrix}1\\-5\end{pmatrix}$

e $\begin{pmatrix}6\\-5\end{pmatrix}$ **f** $\begin{pmatrix}1\\3\end{pmatrix}$

EXERCISE 12C.3

1 **a** $\begin{pmatrix}-3\\-15\end{pmatrix}$ **b** $\begin{pmatrix}-1\\2\end{pmatrix}$ **c** $\begin{pmatrix}0\\14\end{pmatrix}$ **d** $\begin{pmatrix}5\\-3\end{pmatrix}$

e $\begin{pmatrix}\frac{5}{2}\\\frac{11}{2}\end{pmatrix}$ **f** $\begin{pmatrix}-7\\7\end{pmatrix}$ **g** $\begin{pmatrix}5\\11\end{pmatrix}$ **h** $\begin{pmatrix}3\\\frac{17}{3}\end{pmatrix}$

2 **a** $\begin{pmatrix}8\\-1\end{pmatrix}$ **b** $\begin{pmatrix}8\\-1\end{pmatrix}$ **c** $\begin{pmatrix}8\\-1\end{pmatrix}$

EXERCISE 12C.4

1 **a** $\sqrt{13}$ units **b** $\sqrt{17}$ units **c** $5\sqrt{2}$ units **d** $\sqrt{10}$ units

e $\sqrt{29}$ units

2 **a** $\sqrt{10}$ units **b** $2\sqrt{10}$ units **c** $2\sqrt{10}$ units **d** $3\sqrt{10}$ units

e $3\sqrt{10}$ units **f** $2\sqrt{5}$ units **g** $8\sqrt{5}$ units **h** $8\sqrt{5}$ units

i $\sqrt{5}$ units **j** $\sqrt{5}$ units

4 **a** $\overrightarrow{AB} = \begin{pmatrix}1\\6\end{pmatrix}$, $AB = \sqrt{37}$ units

b $\overrightarrow{BA} = \begin{pmatrix}-1\\-6\end{pmatrix}$, $BA = \sqrt{37}$ units

c $\overrightarrow{BC} = \begin{pmatrix}-4\\-1\end{pmatrix}$, $BC = \sqrt{17}$ units

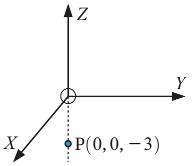
d $\overrightarrow{DC} = \begin{pmatrix}3\\7\end{pmatrix}$, $DC = \sqrt{58}$ units

e $\overrightarrow{CA} = \begin{pmatrix}3\\-5\end{pmatrix}$, $CA = \sqrt{34}$ units

f $\overrightarrow{DA} = \begin{pmatrix}6\\2\end{pmatrix}$, $DA = 2\sqrt{10}$ units

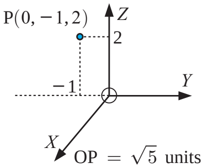
EXERCISE 12D

1 **a**



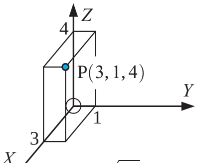
OP = 3 units

b



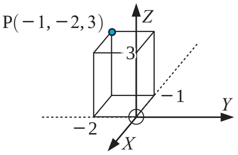
OP = $\sqrt{5}$ units

c



OP = $\sqrt{26}$ units

d



OP = $\sqrt{14}$ units

2 **a** **i** $\sqrt{14}$ units **ii** $(-\frac{1}{2}, \frac{1}{2}, 2)$

b **i** $\sqrt{14}$ units **ii** $(1, -\frac{1}{2}, \frac{3}{2})$

c **i** $\sqrt{21}$ units **ii** $(1, -\frac{1}{2}, 0)$

d **i** $\sqrt{14}$ units **ii** $(1, \frac{1}{2}, -\frac{3}{2})$

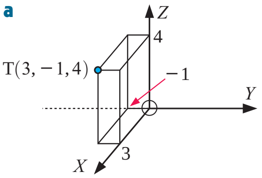
4 **a** isosceles **b** right angled **c** right angled

d straight line **5** (0, 3, 5), $r = \sqrt{3}$ units

6 **a** $(0, y, 0)$ **b** (0, 2, 0) and (0, −4, 0)

EXERCISE 12E

1 **a**



b $\overrightarrow{OT} = \begin{pmatrix}3\\-1\\4\end{pmatrix}$

c $OT = \sqrt{26}$ units

2 **a** $\overrightarrow{AB} = \begin{pmatrix}4\\-1\\-3\end{pmatrix}$, $\overrightarrow{BA} = \begin{pmatrix}-4\\1\\3\end{pmatrix}$

b $AB = \sqrt{26}$ units $BA = \sqrt{26}$ units

3 $\overrightarrow{OA} = \begin{pmatrix}3\\1\\0\end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix}-1\\1\\2\end{pmatrix}$, $\overrightarrow{AB} = \begin{pmatrix}-4\\0\\2\end{pmatrix}$

4 **a** $\overrightarrow{NM} = \begin{pmatrix}5\\-4\\-1\end{pmatrix}$ **b** $\overrightarrow{MN} = \begin{pmatrix}-5\\4\\1\end{pmatrix}$ **c** $MN = \sqrt{42}$ units

5 **a** $\overrightarrow{OA} = \begin{pmatrix}-1\\2\\5\end{pmatrix}$, $OA = \sqrt{30}$ units

b $\overrightarrow{AC} = \begin{pmatrix}-2\\-1\\-5\end{pmatrix}$, $AC = \sqrt{30}$ units

c $\overrightarrow{CB} = \begin{pmatrix}5\\-1\\3\end{pmatrix}$, $CB = \sqrt{35}$ units

6 **a** $\sqrt{13}$ units **b** $\sqrt{14}$ units **c** 3 units

7 **a** $a = 5,\ b = 6,\ c = -6$ **b** $a = 4,\ b = 2,\ c = 1$

8 **a** $a = \frac{1}{3},\ b = 2,\ c = 1$ **b** $a = 1,\ b = 2$

c $a = 1,\ b = -1,\ c = 2$

9 **a** $r = 2,\ s = 4,\ t = -7$ **b** $r = -4,\ s = 0,\ t = 3$

10 **a** $\overrightarrow{AB} = \begin{pmatrix}2\\-5\\-1\end{pmatrix}$, $\overrightarrow{DC} = \begin{pmatrix}2\\-5\\-1\end{pmatrix}$

b ABCD is a parallelogram

11 **a** S(−2, 8, −3)

EXERCISE 12F

1 **a** $\mathbf{x} = \frac{1}{2}\mathbf{q}$ **b** $\mathbf{x} = 2\mathbf{n}$ **c** $\mathbf{x} = -\frac{1}{3}\mathbf{p}$

d $\mathbf{x} = \frac{1}{2}(\mathbf{r} - \mathbf{q})$ **e** $\mathbf{x} = \frac{1}{5}(4\mathbf{s} - \mathbf{t})$ **f** $\mathbf{x} = 3(4\mathbf{m} - \mathbf{n})$

2 **a** $y = \begin{pmatrix}-1\\\frac{3}{2}\end{pmatrix}$ **b** $y = \begin{pmatrix}2\\4\end{pmatrix}$ **c** $y = \begin{pmatrix}\frac{3}{2}\\-\frac{1}{2}\end{pmatrix}$

d $y = \begin{pmatrix}\frac{5}{4}\\\frac{3}{4}\end{pmatrix}$

4 **a** B(−1, 10) **b** B(−2, −9) **c** B(7, 4)

5 **a** M(1, 4) **b** $\overrightarrow{CA} = \begin{pmatrix}7\\5\end{pmatrix}$, $\overrightarrow{CM} = \begin{pmatrix}5\\3\end{pmatrix}$, $\overrightarrow{CB} = \begin{pmatrix}3\\1\end{pmatrix}$

6 **a** $\mathbf{x} = \begin{pmatrix}4\\-6\\-5\end{pmatrix}$ **b** $\mathbf{x} = \begin{pmatrix}1\\-\frac{2}{3}\\\frac{5}{3}\end{pmatrix}$ **c** $\mathbf{x} = \begin{pmatrix}\frac{3}{2}\\-1\\\frac{5}{2}\end{pmatrix}$

7 $\overrightarrow{AB} = \begin{pmatrix}3\\4\\-2\end{pmatrix}$, $AB = \sqrt{29}$ units

9 C(5, 1, −8), D(8, −1, −13), E(11, −3, −18)

10 **a** parallelogram **b** parallelogram

c not parallelogram

11 **a** D(9, −1) **b** R(3, 1, 6) **c** X(2, −1, 0)

12 **a** $\overrightarrow{BD} = \frac{1}{2}\mathbf{a}$ **b** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ **c** $\overrightarrow{BA} = -\mathbf{b} + \mathbf{a}$

d $\overrightarrow{OD} = \mathbf{b} + \frac{1}{2}\mathbf{a}$ **e** $\overrightarrow{AD} = \mathbf{b} - \frac{1}{2}\mathbf{a}$ **f** $\overrightarrow{DA} = \frac{1}{2}\mathbf{a} - \mathbf{b}$

13 **a** $\begin{pmatrix}-1\\5\\-1\end{pmatrix}$ **b** $\begin{pmatrix}-3\\4\\-2\end{pmatrix}$ **c** $\begin{pmatrix}-3\\6\\-5\end{pmatrix}$

14 **a** $\begin{pmatrix}3\\1\\-2\end{pmatrix}$ **b** $\begin{pmatrix}1\\-3\\4\end{pmatrix}$ **c** $\begin{pmatrix}1\\4\\-9\end{pmatrix}$

d $\begin{pmatrix}2\\-4\\10\end{pmatrix}$ **e** $\begin{pmatrix}3\\2\\-5\end{pmatrix}$ **f** $\begin{pmatrix}-1\\\frac{3}{2}\\-\frac{7}{2}\end{pmatrix}$

g $\begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix}$ **h** $\begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$

15 **a** $\sqrt{11}$ units **b** $\sqrt{14}$ units **c** $\sqrt{38}$ units

d $\sqrt{3}$ units **e** $\begin{pmatrix} \sqrt{11} \\ -3\sqrt{11} \\ 2\sqrt{11} \end{pmatrix}$ **f** $\begin{pmatrix} -\frac{1}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \\ \frac{3}{\sqrt{11}} \end{pmatrix}$

16 **a** $r = 2,\ s = -5$ **b** $r = 4,\ s = -1$

EXERCISE 12G

1 $r = 3,\ s = -9$ **2** $a = -6,\ b = -4$

3 **a** $\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}$ or $\begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$ **b** $\begin{pmatrix} -\frac{4}{3} \\ -\frac{2}{3} \\ \frac{4}{3} \end{pmatrix}$ or $\begin{pmatrix} \frac{4}{3} \\ \frac{2}{3} \\ -\frac{4}{3} \end{pmatrix}$

4 **a** $\overrightarrow{AB} \parallel \overrightarrow{CD}$, AB = 3 CD

b $\overrightarrow{RS} \parallel \overrightarrow{KL}$, RS $= \frac{1}{2}$KL opposite direction

c A, B and C are collinear and AB = 2 BC

5 **a** $\overrightarrow{PR} = \begin{pmatrix} -1 \\ -3 \\ 3 \end{pmatrix}$, $\overrightarrow{QS} = \begin{pmatrix} -2 \\ -6 \\ 6 \end{pmatrix}$ **b** PR $= \frac{1}{2}$QS

EXERCISE 12H

1 **a** unit vector **b** unit vector **c** not a unit vector

d unit vector **e** not a unit vector

2 **a** $2\mathbf{i} - \mathbf{j}$ **b** $-3\mathbf{i} - 4\mathbf{j}$ **c** $-3\mathbf{i}$ **d** $7\mathbf{j}$ **e** $\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$

3 **a** $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ **b** $\begin{pmatrix} 5 \\ -4 \end{pmatrix}$ **c** $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$ **d** $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ **e** $\begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}$

4 **a** $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ $\sqrt{3}$ units **b** $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ $\sqrt{11}$ units **c** $\begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}$ $\sqrt{26}$ units **d** $\begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ $\frac{1}{\sqrt{2}}$ units

5 **a** $k = \pm 1$ **b** $k = \pm 1$ **c** $k = 0$

d $k = \pm\frac{\sqrt{11}}{4}$ **e** $k = \pm\frac{2}{3}$

6 **a** 5 units **b** $\sqrt{6}$ units **c** 3 units

d ≈ 6.12 units

7 **a** $\frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j})$ **b** $\frac{1}{\sqrt{13}}(2\mathbf{i} - 3\mathbf{k})$ **c** $\frac{1}{\sqrt{33}}(-2\mathbf{i} - 5\mathbf{j} - 2\mathbf{k})$

8 **a** $\frac{3}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ **b** $-\frac{2}{\sqrt{17}}\begin{pmatrix} -1 \\ -4 \end{pmatrix}$ **c** $\frac{6}{\sqrt{18}}\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$

d $-\frac{5}{3}\begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$

EXERCISE 12I

1 **a** 7 **b** 22 **c** 29 **d** 66 **e** 52 **f** 3 **g** 5 **h** 1

2 **a** 2 **b** 2 **c** 14 **d** 14 **e** 4 **f** 4

3 **a** -1 **b** 94.1^o **4** **a** 1 **b** 1 **c** 0

5 **a** 5 **b** -9

7 **a** $t = 6$ **b** $t = -8$ **c** $t = 0$ or 2 **d** $t = -\frac{3}{2}$

8 **a** $t = -\frac{3}{2}$ **b** $t = -\frac{6}{7}$ **c** $t = \frac{-1 \pm \sqrt{5}}{2}$ **d** impossible

9 Show $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{c} = \mathbf{a} \bullet \mathbf{c} = 0$ **10** **b** $t = -\frac{5}{6}$

11 **a** $\widehat{BAC}$ is a right angle **b** not right angled

c $\widehat{BAC}$ is a right angle **d** $\widehat{ACB}$ is a right angle

12 $\overrightarrow{AB} \bullet \overrightarrow{AC} = 0$, $\therefore$ $\widehat{BAC}$ is a right angle

13 **b** $|\overrightarrow{AB}| = \sqrt{14}$ units, $|\overrightarrow{BC}| = \sqrt{14}$ units, ABCD is a rhombus

c 0, the diagonals of a rhombus are perpendicular

14 **a** 78.7^o **b** 63.4^o **c** 63.4^o **d** 71.6^o

15 **a** $k\begin{pmatrix} -2 \\ 5 \end{pmatrix}$, $k \neq 0$ **b** $k\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $k \neq 0$

c $k\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $k \neq 0$ **d** $k\begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $k \neq 0$

e $k\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $k \neq 0$

16 $\widehat{ABC} \approx 62.5^o$, the exterior angle 117.5^o

17 **a** 54.7^o **b** 60^o **c** 35.3^o

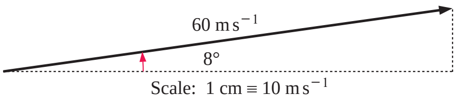
18 **a** 30.3^o **b** 54.2^o **19** **a** $M(\frac{3}{2}, \frac{5}{2}, \frac{3}{2})$ **b** 51.5^o

20 **a** $t = 0$ or -3 **b** $r = -2,\ s = 5,\ t = -4$

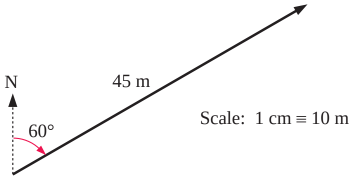
21 **a** 74.5^o **b** 72.5^o

REVIEW SET 12A

1 **a**



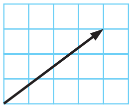
b

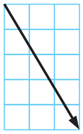


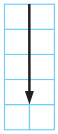
2 **a** $\overrightarrow{AC}$ **b** $\overrightarrow{AD}$

3 **a** $\mathbf{q} = \mathbf{p} + \mathbf{r}$ **b** $\mathbf{l} = \mathbf{k} - \mathbf{j} + \mathbf{n} - \mathbf{m}$

4 **a** $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ **b** $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ **c** $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$







5 $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ **6** **a** $\mathbf{p} + \mathbf{q}$ **b** $\frac{3}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$ **7** $m = 5,\ n = -\frac{1}{2}$

8 $\begin{pmatrix} 8 \\ -8 \\ 7 \end{pmatrix}$ **9** **a** -13 **b** -36

11 **a** $\mathbf{x} = \begin{pmatrix} -11 \\ 5 \\ -10 \end{pmatrix}$ **b** $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ **12** $k = 6$

13 $k\begin{pmatrix} 5 \\ 4 \end{pmatrix}$, $k \neq 0$ **14** **a** **i** $\mathbf{p} + \mathbf{q}$ **ii** $\frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$

15 $\mathbf{a} \bullet \mathbf{b} = -4$, $\mathbf{b} \bullet \mathbf{c} = 10$, $\mathbf{a} \bullet \mathbf{c} = -10$ **16** $a = -2,\ b = 0$

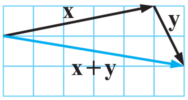
17 **a** $\mathbf{q} + \mathbf{r}$ **b** $\mathbf{r} + \mathbf{q}$, DB = AC, [DB] ∥ [AC]

18 **a** $t = -4$ **b** $\overrightarrow{LM} = \begin{pmatrix} 5 \\ -3 \\ -4 \end{pmatrix}$, $\overrightarrow{KM} = \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix}$

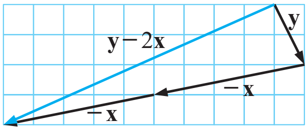
So, $\overrightarrow{LM} \bullet \overrightarrow{KM} = 0$ $\therefore$ $\widehat{M} = 90^o$

REVIEW SET 12B

1 **a**



b



2 4.84 km, 208^o

3 AB = AC = $\sqrt{53}$ units and BC = $\sqrt{46}$ units $\therefore$ Δ is isosceles

4 **a** $\sqrt{17}$ units **b** $\sqrt{13}$ units **c** $\sqrt{10}$ units **d** $\sqrt{109}$ units

5 80.3^o **6** $r = 4,\ \ s = 7$

7 **a** $\begin{pmatrix} -6 \\ 1 \\ 3 \end{pmatrix}$ **b** $\sqrt{46}$ units **c** $(-1, 3\frac{1}{2}, \frac{1}{2})$

8 **a** -1 **b** $\begin{pmatrix} 4 \\ -1 \\ 7 \end{pmatrix}$ **c** 60^o

9 $c = \frac{50}{3}$ **10** **a** $\begin{pmatrix} 7 \\ -12 \\ -7 \end{pmatrix}$ **b** $\begin{pmatrix} 1 \\ -\frac{5}{3} \\ -\frac{2}{3} \end{pmatrix}$ **11** 64.0^o

12 (0, 0, 1) and (0, 0, 9) **13** $t = \frac{2}{3}$ or -3 **14** 72.3^o

15 **a** $\overrightarrow{AC} = -\mathbf{p} + \mathbf{r},\ \overrightarrow{BC} = -\mathbf{q} + \mathbf{r}$ **16** **a** 8 **b** 62.2^o

17 **a** $r = -2,\ \ s = \frac{15}{2}$ **b** $\pm\frac{4}{\sqrt{14}}(3\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ **18** 16.1^o

REVIEW SET 12C

1 **a** $\overrightarrow{PQ}$ **b** $\overrightarrow{PR}$

2 **a** $\begin{pmatrix} 3 \\ -3 \\ 11 \end{pmatrix}$ **b** $\begin{pmatrix} 7 \\ -3 \\ -26 \end{pmatrix}$ **c** $\sqrt{74}$ units

3 **a** AB = $\frac{1}{2}$CD, [AB] $\parallel$ [CD] **b** C is the midpoint of [AB].

4 **a** $\overrightarrow{PQ} = \begin{pmatrix} -3 \\ 12 \\ 3 \end{pmatrix}$ **b** $\sqrt{162}$ units **c** $\sqrt{61}$ units

5 **a** $\mathbf{r} + \mathbf{q}$ **b** $-\mathbf{p} + \mathbf{r} + \mathbf{q}$ **c** $\mathbf{r} + \frac{1}{2}\mathbf{q}$ **d** $-\frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{r}$

6 **a** $\begin{pmatrix} -4 \\ -2 \end{pmatrix}$ **b** $\begin{pmatrix} -1 \\ -13 \end{pmatrix}$ **c** $\begin{pmatrix} -4 \\ 8 \end{pmatrix}$

7 **a** $\mathbf{X} = \begin{pmatrix} -1 \\ \frac{1}{3} \end{pmatrix}$ **b** $\mathbf{X} = \begin{pmatrix} 1 \\ -10 \end{pmatrix}$ **8** $\mathbf{v} \bullet \mathbf{w} = \pm 6$

9 $t = 2 \pm \sqrt{2}$ **10** $\widehat{K} \approx 123.7^o,\ \widehat{L} \approx 11.3^o,\ \widehat{M} = 45^o$

11 **a** $k = \pm\frac{7}{\sqrt{33}}$ **b** $k = \pm\frac{1}{\sqrt{3}}$ **12** 40.7^o

14 $\widehat{K} \approx 64.4^o,\ \widehat{L} \approx 56.9^o,\ \widehat{M} \approx 58.7^o$

15 $\overrightarrow{OT} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$ or $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$

16 **a** $k = \pm\frac{1}{2}$ **b** $-\frac{5}{\sqrt{14}}\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ **17** **a** 10 **b** 61.6^o

18 $r = 3,\ s = -\frac{5}{2},\ t = \frac{1}{4}$ **19** $\sin\theta = \frac{2}{\sqrt{5}}$

EXERCISE 13A.1

1 **a** **i** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \end{pmatrix},\ t \in \mathbb{R}$ **ii** $4x - y = 16$

b **i** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 8 \end{pmatrix},\ t \in \mathbb{R}$ **ii** $4x - y = 18$

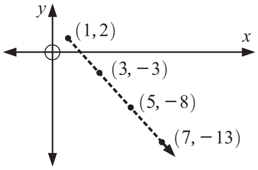
c **i** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 7 \end{pmatrix},\ t \in \mathbb{R}$ **ii** $7x - 3y = -42$

d **i** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 11 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \end{pmatrix},\ t \in \mathbb{R}$ **ii** $x + 2y = 21$

2 $x = -1 + 2t,\quad y = 4 - t,\quad t \in \mathbb{R}$
Points are: (−1, 4), (1, 3), (5, 1), (−3, 5), (−9, 8)

3 **a** When $t = 1,\ x = 3,\ y = -2$ $\therefore$ yes **b** $k = -5$

4 **a** (1, 2) **b**



c $\sqrt{29}$ cm s^{-1}

5 $x = 1 - t,\ y = 5 + 3t$ or $3x + y = 8$

EXERCISE 13A.2

1 **a** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ $t \in \mathbb{R}$

b $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix},\ t \in \mathbb{R}$

c $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ t \in \mathbb{R}$

2 **a** $x = 5 - t,\ y = 2 + 2t,\ z = -1 + 6t,\ t \in \mathbb{R}$
b $x = 2t,\ y = 2 - t,\ z = -1 + 3t,\ t \in \mathbb{R}$
c $x = 3,\ y = 2,\ z = -1 + t,\ t \in \mathbb{R}$

3 **a** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix},\ t \in \mathbb{R}$

b $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix},\ t \in \mathbb{R}$

c $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + t\begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix},\ t \in \mathbb{R}$

d $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix},\ t \in \mathbb{R}$

4 **a** $(-\frac{1}{2}, \frac{9}{2}, 0)$ **b** (0, 4, 1) **c** (4, 0, 9)

5 (0, 7, 3) and $(\frac{20}{3}, -\frac{19}{3}, -\frac{11}{3})$

EXERCISE 13A.3

1 75.5^o **2** 75.7^o

3 $\begin{pmatrix} 5 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 10 \end{pmatrix} = 0,$ so the vectors are perpendicular.

4 **a** 28.6^o **b** $x = -\frac{48}{7}$

EXERCISE 13B.1

1 **a** **i** (−4, 3) **ii** $\begin{pmatrix} 12 \\ 5 \end{pmatrix}$ **iii** 13 m s^{-1}

b **i** (0, −6) **ii** $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ **iii** 5 m s^{-1}

c **i** (−2, −7) **ii** $\begin{pmatrix} -6 \\ -4 \end{pmatrix}$ **iii** $\sqrt{52}$ m s^{-1}

d **i** (5, −5) **ii** $\begin{pmatrix} 8 \\ 4 \end{pmatrix}$ **iii** $\sqrt{80}$ m s^{-1}

2 **a** $\begin{pmatrix}120\\-90\end{pmatrix}$ **b** $\begin{pmatrix}12\\3.5\end{pmatrix}$ **c** $\begin{pmatrix}20\sqrt{5}\\10\sqrt{5}\end{pmatrix}$ **3** $\begin{pmatrix}-12\\30\\-84\end{pmatrix}$

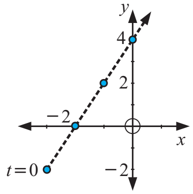
EXERCISE 13B.2

1 **a** $\begin{pmatrix}-3+2t\\-2+4t\end{pmatrix}$

b $\begin{pmatrix}2\\8\end{pmatrix}$

c **i** $t = 1.5$ s
ii $t = 0.5$ s

d



2 **a** A is at (4, 5), B is at (1, −8)

b For A it is $\begin{pmatrix}1\\-2\end{pmatrix}$. For B it is $\begin{pmatrix}2\\1\end{pmatrix}$.

c For A, speed is $\sqrt{5}$ km h^{-1}. For B, speed is $\sqrt{5}$ km h^{-1}.

d $\begin{pmatrix}1\\-2\end{pmatrix}\bullet\begin{pmatrix}2\\1\end{pmatrix} = 0$

3 **a** $\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}-5\\4\end{pmatrix} + t\begin{pmatrix}3\\-1\end{pmatrix}$ $\therefore$ $x_1(t) = -5 + 3t,$
$y_1(t) = 4 - t$

b speed $= \sqrt{10}$ km min^{-1}

c a minutes later, $(t - a)$ min have elapsed.

$\therefore$ $\begin{pmatrix}x_2\\y_2\end{pmatrix} = \begin{pmatrix}15\\7\end{pmatrix} + (t-a)\begin{pmatrix}-4\\-3\end{pmatrix}$

$\therefore$ $x_2(t) = 15 - 4(t - a),$ $y_2(t) = 7 - 3(t - a)$

d Torpedo is fired at 1:35:28 pm and the explosion occurs at 1:37:42 pm.

4 **a** $\begin{pmatrix}-3\\1\\-0.5\end{pmatrix}$ **b** ≈ 19.2 km h^{-1}

c $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}6\\9\\3\end{pmatrix} + t\begin{pmatrix}-3\\1\\-0.5\end{pmatrix}$, $t \in \mathbb{R}$ **d** 1 hour

EXERCISE 13B.3

1 **a** 6**i** − 6**j** **b** $\begin{pmatrix}6-6t\\-6+8t\end{pmatrix}$ **c** when $t = \frac{3}{4}$ hours

d $t = 0.84$ and position is (0.96, 0.72)

2 **a** $\begin{pmatrix}-120\\-40\end{pmatrix}$ **b** $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}200\\100\end{pmatrix} + t\begin{pmatrix}-120\\-40\end{pmatrix}$

c $\begin{pmatrix}80\\60\end{pmatrix}$ **d** $\left|\begin{pmatrix}80\\60\end{pmatrix}\right| = 100$ km

e at 1:45 pm and $d_{\min} \approx 31.6$ km **f** 2:30 pm

3 **a** A(18, 0) and B(0, 12) **b** R is at $\left(x, \dfrac{36-2x}{3}\right)$

c $\overrightarrow{PR} = \begin{pmatrix}x-4\\\frac{36-2x}{3}\end{pmatrix}$ and $\overrightarrow{AB} = \begin{pmatrix}-18\\12\end{pmatrix}$

d $\left(\frac{108}{13}, \frac{84}{13}\right)$ and distance ≈ 7.77 km

4 **a** A(3, −4) and B(4, 3)

b For A $\begin{pmatrix}-1\\2\end{pmatrix}$, for B $\begin{pmatrix}-3\\-2\end{pmatrix}$ **c** 97.1^o

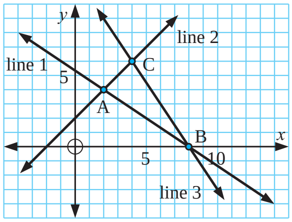
d at $t = 1.5$ hours

5 **a** (2, −1, 4) **b** $\sqrt{27}$ units

6 **a** $(2, \frac{1}{2}, \frac{5}{2})$ **b** $\sqrt{\frac{3}{2}}$ units

EXERCISE 13B.4

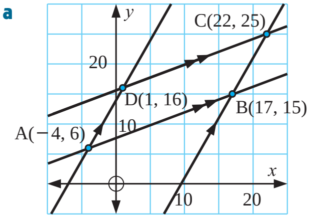
1 **a**



b A(2, 4),
B(8, 0),
C(4, 6)

c BC = BA
$= \sqrt{52}$ units
$\therefore$ isosceles △

2 **a**



b A(−4, 6),
B(17, 15),
C(22, 25),
D(1, 16)

3 **a** A(2, 3), B(8, 6), C(5, 0) **b** AB = BC = $\sqrt{45}$ units

4 **a** P(10, 4), Q(3, −1), R(20, −10)

b $\overrightarrow{PQ} = \begin{pmatrix}-7\\-5\end{pmatrix}$, $\overrightarrow{PR} = \begin{pmatrix}10\\-14\end{pmatrix}$, $\overrightarrow{PQ}\bullet\overrightarrow{PR} = 0$

c $\widehat{QPR} = 90^o$ **d** 74 units^2

5 **a** A is at (2, 5), B(18, 9), C(14, 25), D(−2, 21)

b $\overrightarrow{AC} = \begin{pmatrix}12\\20\end{pmatrix}$ and $\overrightarrow{DB} = \begin{pmatrix}20\\-12\end{pmatrix}$

i $\sqrt{544}$ units **ii** $\sqrt{544}$ units **iii** 0

c Diagonals are perpendicular and equal in length, and as their midpoints are both (8, 15), ABCD is a square.

EXERCISE 13C

1 **a** They intersect at (1, 2, 3), angle $\approx 10.9^o$.
b Lines are skew, angle $\approx 62.7^o$.
c They are parallel, $\therefore$ angle $= 0^o$.
d They are skew, angle $\approx 11.4^o$.
e They intersect at (−4, 7, −7), angle $\approx 40.2^o$.
f They are parallel, $\therefore$ angle $= 0^o$.

REVIEW SET 13A

1 **a** $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-6\\3\end{pmatrix} + t\begin{pmatrix}4\\-3\end{pmatrix}$

b $x = -6 + 4t,\ y = 3 - 3t,\ t \in \mathbb{R}$

2 $m = 10$ **3** $x = 3 + 2t,\ y = -3 + 5t$ *or* $5x - 2y = 21$

4 **a** $\overrightarrow{PQ} = \begin{pmatrix}1\\4\\-3\end{pmatrix}$ $\left|\overrightarrow{PQ}\right| = \sqrt{26}$ units, $\overrightarrow{QR} = \begin{pmatrix}-4\\-1\\4\end{pmatrix}$

b $x = 2 + t,\ y = 4t,\ z = 1 - 3t,\ t \in \mathbb{R}$

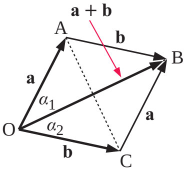
5 $\frac{1}{\sqrt{74}}\mathbf{i} + \frac{8}{\sqrt{74}}\mathbf{j} + \frac{3}{\sqrt{74}}\mathbf{k}$ or $-\frac{1}{\sqrt{74}}\mathbf{i} - \frac{8}{\sqrt{74}}\mathbf{j} - \frac{3}{\sqrt{74}}\mathbf{k}$

6 **a** A(5, 2), B(6, 5), C(8, 3)

b $\left|\overrightarrow{AB}\right| = \sqrt{10}$ units, $\left|\overrightarrow{BC}\right| = \sqrt{8}$ units, $\left|\overrightarrow{AC}\right| = \sqrt{10}$ units

c isosceles

7 **a** OABC is a rhombus. So, its diagonals bisect its angles.



b $x = 7,\ y = 3 + \frac{1}{3}t,$
$z = -4 + \frac{1}{3}t,\ t \in \mathbb{R}$

c $(7, 3\frac{3}{4}, -3\frac{1}{4})$

8 (4, 1, −3) and (1, −5, 0)

REVIEW SET 13B

1 $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\8\end{pmatrix}+t\begin{pmatrix}5\\4\end{pmatrix},\ t\in\mathbb{R}$

2 **a** $\overrightarrow{PQ}=\begin{pmatrix}5\\-2\\-4\end{pmatrix}$ **b** 41.8^o

3 **a** **i** $-6\mathbf{i}+10\mathbf{j}$ **ii** $-5\mathbf{i}-15\mathbf{j}$
iii $(-6-5t)\mathbf{i}+(10-15t)\mathbf{j}$
b $t=0.48$ h
c shortest dist. ≈ 8.85 km, so will miss reef.

4 **a** **i** $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2\\-3\end{pmatrix}+t\begin{pmatrix}4\\-1\end{pmatrix},\ t\in\mathbb{R}$
ii $x+4y=-10$
b **i** $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-1\\6\end{pmatrix}+t\begin{pmatrix}6\\-8\end{pmatrix},\ t\in\mathbb{R}$
ii $4x+3y=14$

5 **a** $(17,\ -9,\ 0)$ **b** $(0,\ \frac{7}{3},\ \frac{17}{3})$ **c** $(\frac{7}{2},\ 0,\ \frac{9}{2})$ **6** 8.13^o

7 **a** X23, $x_1=2+t,\ y_1=4-3t,\ t\geqslant 0$
b Y18, $x_2=13-t,\ y_2=3-2a+at,\ t\geqslant 2$
c interception occurred at 2:22:30 pm
d bearing $\approx 193^o$, ≈ 4.54 units per minute

8 **a** intersecting at $(4,\ 3,\ 1)$ angle $\approx 44.5^o$
b skew, angle $\approx 71.2^o$

REVIEW SET 13C

1 $2\sqrt{10}(3\mathbf{i}-\mathbf{j})$

2 **a** $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3\\2\\-1\end{pmatrix}+t\begin{pmatrix}-4\\0\\5\end{pmatrix},\ t\in\mathbb{R}$
b $(-5,\ 2,\ 9)$ or $(11,\ 2,\ -11)$

3 **a** $(-4,\ 3)$ **b** $(28,\ 27)$ **c** $10\ \text{m s}^{-1}$ **d** $\begin{pmatrix}8\\6\end{pmatrix}$

4 **a** (KL) is parallel to (MN) as $\begin{pmatrix}5\\-2\end{pmatrix}$ is parallel to $\begin{pmatrix}-5\\2\end{pmatrix}$
b (KL) is perpendicular to (NK) as $\begin{pmatrix}5\\-2\end{pmatrix}\bullet\begin{pmatrix}4\\10\end{pmatrix}=0$
and (NK) is perpendicular to (MN) as $\begin{pmatrix}4\\10\end{pmatrix}\bullet\begin{pmatrix}-5\\2\end{pmatrix}=0$
c K(7, 17), L(22, 11), M(33, −5), N(3, 7) **d** 261 units^2

5 30.5^o

6 **a** $\left|\overrightarrow{AB}\right|=\sqrt{22}$ units
b $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3\\-1\\1\end{pmatrix}+t\begin{pmatrix}-3\\3\\-2\end{pmatrix},\ t\in\mathbb{R}$

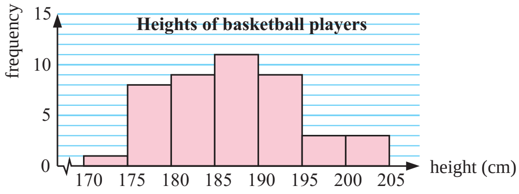
7 26.4^o

8 **a** Road A: $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-9\\2\end{pmatrix}+t\begin{pmatrix}4\\-3\end{pmatrix},\ t\in\mathbb{R}$
Road B: $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}6\\-18\end{pmatrix}+s\begin{pmatrix}5\\12\end{pmatrix},\ s\in\mathbb{R}$
b Road B, 13 km

9 **a** $\overrightarrow{AB}\bullet\overrightarrow{AC}=\begin{pmatrix}-2\\-1\\6\end{pmatrix}\bullet\begin{pmatrix}5\\2\\2\end{pmatrix}=0$
b $x=4-2t,\ y=2-t,\ z=-1+6t,\ t\in\mathbb{R}$
c $x=4+5s,\ y=2+2s,\ z=-1+2s,\ s\in\mathbb{R}$

EXERCISE 14A

1 **a** Heights can take any value from 170 cm to 205 cm, e.g., 181.37 cm.
b



c The modal class is $185\leqslant H<190$ cm, as this occurred the most frequently.
d slightly positively skewed

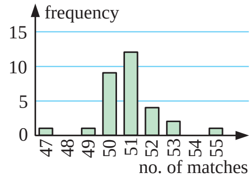
2 **a** Continuous numerical, but has been rounded to become discrete numerical data.
b

Stem	*Leaf*
0	3 6 8 8 8 8
1	0 0 0 0 0 2 2 2 4 4 4 4 5 5 5 5 6 6 6 6 7 8 8 8 8 9
2	0 0 0 1 2 4 5 5 5 6 7 7 8
3	1 2 2 2 3 4 5 7 8
4	0 2 5 5 5 6

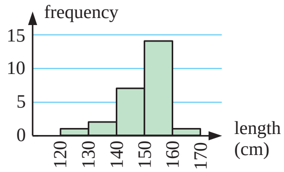
1 | 2 means 12 minutes

c positively skewed
d The modal travelling time was between 10 and 20 minutes.

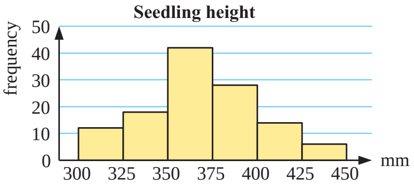
3 **a** column graph



b frequency histogram



4 **a**



b 20
c 58.3%
d **i** 1218
ii 512

EXERCISE 14B.1

1 **a** **i** 5.61 **ii** 6 **iii** 6
b **i** 16.3 **ii** 17 **iii** 18
c **i** 24.8 **ii** 24.9 **iii** 23.5

2 **a** A : 6.46 B : 6.85 **b** A : 7 B : 7
c The data sets are the same except for the last value, and the last value of A is less than the last value of B, so the mean of A is less than the mean of B.
d The middle value of the data sets is the same, so the median is the same.

3 **a** mean: \$29 300, median: \$23 500, mode: \$23 000
b The mode is the lowest value, so does not take the higher values into account.
c No, since the data is positively skewed, the median is not in the centre.

4 **a** mean: 3.19, median: 0, mode: 0
b The data is very positively skewed so the median is not in the centre.

c The mode is the lowest value so does not take the higher values into account.
d yes, 21 and 42 **e** no

5 **a** 44 **b** 44 **c** 40.2 **d** increase mean to 40.3

6 $185 604 **7** 3144 km **8** 116 **9** 17.25 goals per game

10 **a** mean = $163 770, median = $147 200 (differ by $16 570)
b **i** mean selling price **ii** median selling price

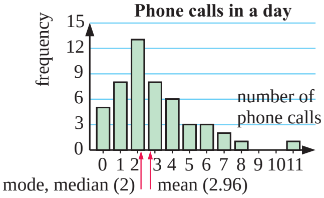
11 $x = 15$ **12** $a = 5$ **13** 37 **14** 14.8 **15** 6 and 12

16 9 and 7

EXERCISE 14B.2

1 **a** 1 **b** 1 **c** 1.43

2 **a** **i** 2.96 **ii** 2 **iii** 2 **b**
c positively skewed
d The mean takes into account the larger numbers of phone calls.
e the mean



3 **a** **i** 49 **ii** 49 **iii** 49.0 **b** no
c The sample of only 30 is not large enough. The company could have won its case by arguing that a larger sample would have found an average of 50 matches per box.

4 **a** **i** 2.61 **ii** 2 **iii** 2
b This school has more children per family than the average Australian family.
c positive
d The mean is larger than the median and the mode.

5 **a** **i** 69.1 **ii** 67 **iii** 73 **b** **i** 5.86 **ii** 5.8 **iii** 6.7

6 **a** $x = 5$ **b** 75%

7 **a** **i** 5.63 **ii** 6 **iii** 6 **b** **i** 6.81 **ii** 7 **iii** 7
c the mean **d** yes

8 **a** ≈ 70.9 g **b** ≈ 210 g **c** 139 g **9** 10.1 cm

10 **a** mean for A ≈ 50.7, mean for B ≈ 49.9
b No, as to the nearest match, A is 51 and B is 50.

11 **a** **i** €31 500 **ii** €28 000 **iii** €33 300 **b** The mean.

EXERCISE 14B.3

1 31.7

2 **a** 70 **b** ≈ 411 000 litres, ≈ 411 kL **c** ≈ 5870 L

3 **a** 125 people **b** ≈ 119 marks **c** $\frac{3}{25}$ **d** 137 marks

EXERCISE 14C.1

1 **a** **i** 6 **ii** $Q_1 = 4$, $Q_3 = 7$ **iii** 7 **iv** 3
b **i** 17.5 **ii** $Q_1 = 15$, $Q_3 = 19$ **iii** 14 **iv** 4
c **i** 24.9 **ii** $Q_1 = 23.5$, $Q_3 = 26.1$ **iii** 7.7 **iv** 2.6

2 **a** median = 2.45, $Q_1 = 1.45$, $Q_3 = 3.8$
b range = 5.2, IQR = 2.35
c **i** ..... greater than 2.45 min **ii** ...... less than 3.8 min
iii The minimum waiting time was 0 minutes and the maximum waiting time was 5.2 minutes. The waiting times were spread over 5.2 minutes.

3 **a** 3 **b** 42 **c** 20 **d** 13 **e** 29 **f** 39 **g** 16

4 **a** **i** 124 cm **ii** $Q_1 = 116$ cm, $Q_3 = 130$ cm
b **i** ...... 124 cm tall **ii** ...... 130 cm tall
c **i** 29 cm **ii** 14 cm **d** ...... over 14 cm

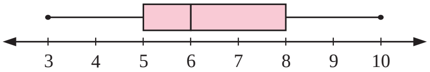
5 **a** **i** 7 **ii** 6 **iii** 5 **iv** 7 **v** 2
b **i** 10 **ii** 7 **iii** 6 **iv** 8 **v** 2

EXERCISE 14C.2

1 **a** **i** 35 **ii** 78 **iii** 13 **iv** 53 **v** 26
b **i** 65 **ii** 27

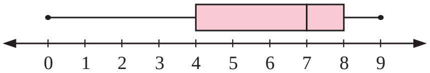
2 **a** ...... was 98, ... was 25
b .... greater than or equal to 70 **c** .... at least 85 marks
d ...... between 55 and 85 ...... **e** 73 **f** 30 **g** ≈ 67

3 **a** **i** min = 3, $Q_1 = 5$, median = 6, $Q_3 = 8$, max = 10
ii



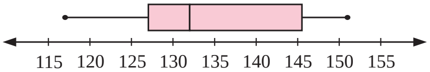
iii range = 7 **iv** IQR = 3

b **i** min = 0, $Q_1 = 4$, median = 7, $Q_3 = 8$, max = 9
ii



iii range = 9 **iv** IQR = 4

c **i** min = 117, $Q_1 = 127$, med. = 132, $Q_3 = 145.5$, max = 151
ii



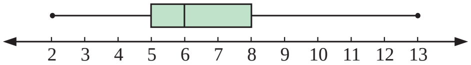
iii range = 34 **iv** IQR = 18.5

4 **a**

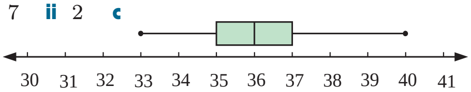
Statistic	*Year 9*	*Year 12*
min value	1	6
Q_1	5	10
median	7.5	14
Q_3	10	16
max value	12	17.5

b **i** Year 9: 11, Year 12: 11.5
ii Year 9: 5, Year 12: 6
c **i** true
ii true

5 **a** median = 6, $Q_1 = 5$, $Q_3 = 8$ **b** 3
c



6 **a** Min = 33, $Q_1 = 35$, $Q_2 = 36$, $Q_3 = 37$, Max = 40
b **i** 7 **ii** 2 **c**



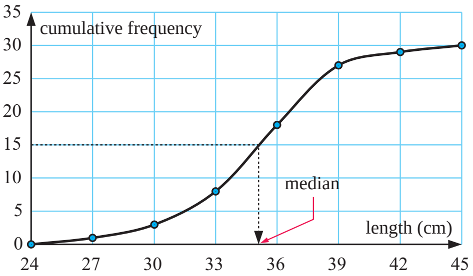
EXERCISE 14D

1 **a** 25.2 cm **b** **i** 40 **ii** 40

2 **a**

Length (x cm)	*Frequency*	*C. frequency*
$24 \leqslant x < 27$	1	1
$27 \leqslant x < 30$	2	3
$30 \leqslant x < 33$	5	8
$33 \leqslant x < 36$	10	18
$36 \leqslant x < 39$	9	27
$39 \leqslant x < 42$	2	29
$42 \leqslant x < 45$	1	30

b



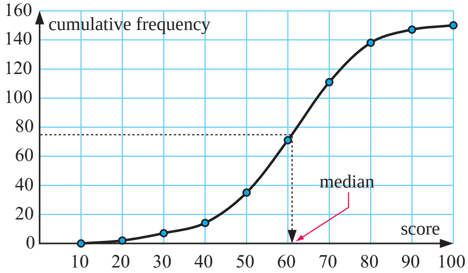
c median ≈ 35 cm

d actual median = 34.5, ∴ a good approx.

3 **a** 9 **b** ≈ 28.3% **c** 7 cm **d** IQR ≈ 2.4 cm

e 10 cm, which means that 90% of the seedlings have a height of 10 cm or less.

4



a ≈ 61 **b** ≈ 87 students **c** ≈ 76 students

d 24 (or 25) students **e** 76 marks

5 **a** 26 years **b** 36% **c** **i** 0.527 **ii** 0.0267

6 **a** 27 min **b** 29 min **c** 31.3 min

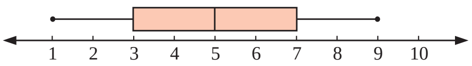
d IQR ≈ 4.3 min **e** 28 min 10 s

7 **a** 2270 h **b** ≈ 69% **c** 62 or 63

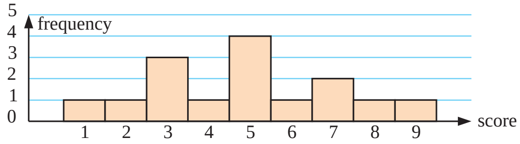
EXERCISE 14E

1 **a** $\overline{x} \approx 4.87$, Min = 1, $Q_1 = 3$, $Q_2 = 5$, $Q_3 = 7$, Max = 9

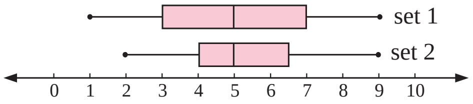
b



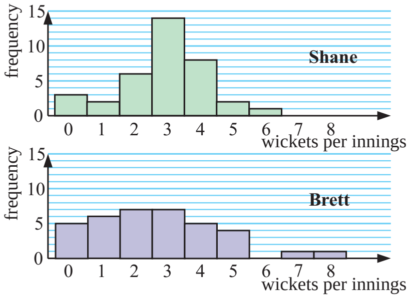
c



d $\overline{x} \approx 5.24$, Min = 2, $Q_1 = 4$, $Q_2 = 5$, $Q_3 = 6.5$, Max = 9



2 **a** discrete **c**



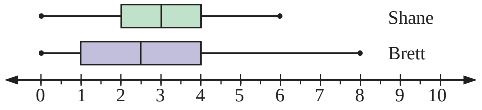
d There are no outliers for Shane.
Brett has outliers of 7 and 8 which must not be removed.

e Shane's distribution is reasonably symmetrical. Brett's distribution is positively skewed.

f Shane has a higher mean (≈ 2.89 wickets) compared with Brett (≈ 2.67 wickets). Shane has a higher median (3 wickets) compared with Brett (2.5 wickets). Shane's modal number of wickets is 3 (14 times) compared with Brett, who has two modal values of 2 and 3 (7 times each).

g Shane's range is 6 wickets, compared with Brett's range of 8 wickets. Shane's IQR is 2 wickets, compared with Brett's IQR of 3 wickets. Brett's wicket taking shows greater spread or variability.

h



i Generally, Shane takes more wickets than Brett and is a more consistent bowler.

3 **a** continuous

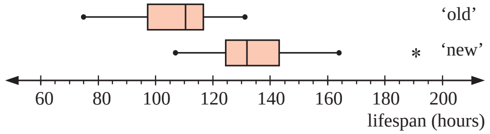
c For the 'new type' of globes, 191 hours could be considered an outlier. However, it could be a genuine piece of data, so we will include it in the analysis.

d The mean and median are ≈ 25% and ≈ 19% higher for the 'new type' of globe compared with the 'old type'.

	Old type	*New type*
Mean	107	134
Median	110.5	132
Range	56	84
IQR	19	18.5

The range is higher for the 'new type' of globe (but has been affected by the 191 hours).
The IQR for each type of globe is almost the same.

e



f For the 'old type' of globe, the data is bunched to the right of the median, hence the distribution is negatively skewed.
For the 'new type' of globe, the data is bunched to the left of the median, hence the distribution is positively skewed.

g The manufacturer's claim, that the 'new type' of globe has a 20% longer life than the 'old type' seems to be backed up by the 25% higher mean life and 19.5% higher median life.

EXERCISE 14F.1

1 **a** Sample A

b, **c**

	A	B
$\overline{x}$	8	8
s	2	1.06

2 **a**

	$\overline{x}$	s
Andrew	25	4.97
Brad	30.5	12.6

b Andrew

3 **a** Rockets: range = 11, $\overline{x} = 5.7$;
Bullets: range = 11, $\overline{x} = 5.7$

b We suspect the Rockets, they have two zeros.

c Rockets: $s = 3.9$ ⟵ greater variability
Bullets: $s \approx 3.29$

d standard deviation

4 **a** We suspect variability in standard deviation since the factors may change every day.

b **i** sample mean **ii** sample standard deviation

c less variability

5 **a** $\overline{x} = 69$, $s \approx 6.05$ **b** $\overline{x} = 79$, $s \approx 6.05$

c The distribution has simply shifted by 10 kg. The mean increases by 10 kg and the standard deviation remains the same.

6 **a** $\overline{x} = 1.01$ kg; $s = 0.17$ **b** $\overline{x} = 2.02$ kg; $s = 0.34$

c Doubling the values doubles the mean and the standard deviation.

7 $p = 6, q = 9$ **8** $a = 8, b = 6$

9 **a** 0.809 **b** 0.150
c the extreme value greatly increases the standard deviation

EXERCISE 14F.2

1 **a** $s_n \approx 6.77$ kg $\therefore$ $\sigma \approx 6.77$ kg **b** $\mu \approx 93.8$ kg

2 **a** $\overline{x} \approx 77.5$ g, $s_n \approx 7.44$ g **b** $\mu \approx 77.5$ g, $\sigma \approx 7.44$ g

EXERCISE 14F.3

1 **a** $\overline{x} \approx 1.72$ children, $s \approx 1.67$ children
b $\mu \approx 1.72$ children, $\sigma \approx 1.67$ children

2 **a** $\overline{x} \approx 14.5$ years, $s \approx 1.75$ years
b $\mu \approx 14.5$ years, $\sigma \approx 1.75$ years

3 **a** $\overline{x} \approx 37.3$ toothpicks, $s \approx 1.45$ toothpicks
b $\mu \approx 37.3$ toothpicks, $\sigma \approx 1.45$ toothpicks

4 **a** $\overline{x} \approx 48.3$ cm, $s \approx 2.66$ cm
b $\mu \approx 48.3$ cm, $\sigma \approx 2.66$ cm

5 **a** $\overline{x} \approx \$390.30$, $s \approx \$15.87$
b $\mu \approx \$390.30$, $\sigma \approx \$15.87$

EXERCISE 14G

1 **a** 16% **b** 84% **c** 97.4% **d** 0.15% **2** 3 times

3 **a** 5 **b** 32 **c** 136 **4** **a** 458 babies **b** 444 babies

REVIEW SET 14A

1 **a** Diameter of bacteria colonies

0	4 8 9
1	3 5 5 7
2	1 1 5 6 8 8
3	0 1 2 3 4 5 5 6 6 7 7 9
4	0 1 2 7 9

0 | 4 means 0.4 cm

b **i** 3.15 cm **ii** 4.5 cm
c The distribution is slightly negatively skewed.

2 $a = 8$, $b = 6$ or $a = 6$, $b = 8$

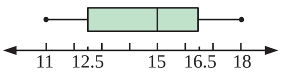
3 **a**

	Girls	*Boys*
shape	pos. skewed	approx. symm.
centre (median)	36.3 s	34.9 s
spread (range)	7.7 s	4.9 s

b The girls' distribution is positively skewed and boys' distribution is approximately symmetrical. The median swim times for boys is 1.4 seconds lower than for girls but the range of the girls' swim times is 2.8 seconds higher than for boys. The analysis supports the conjecture that boys generally swim faster than girls with less spread of times.

4 **a** 2.5% **b** 95% **c** 68%

5



6 **a** 58.5 s **b** 6 s **7** **a** 2.5% **b** 84% **c** 81.5%

8 **a** 88 students **b** $m \approx 24$

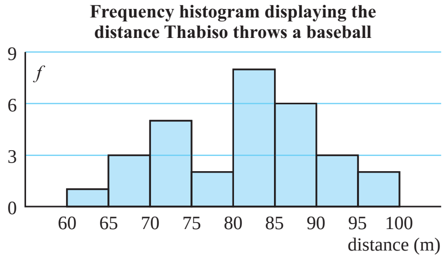
REVIEW SET 14B

1 **a** highest = 97.5 m, lowest = 64.6 m
b use groups $60 \leqslant d < 65$, $65 \leqslant d < 70$, ...

c **Distances thrown by Thabiso**

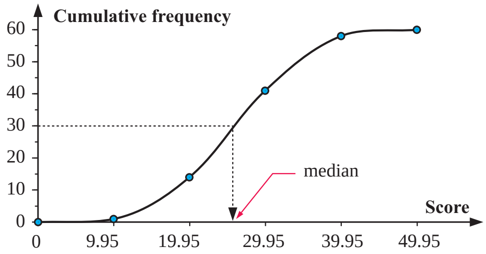
Distance (m)	*Tally*	*Freq. (f)*
$60 \leqslant d < 65$	|	1
$65 \leqslant d < 70$	|||	3
$70 \leqslant d < 75$	𝍸	5
$75 \leqslant d < 80$	||	2
$80 \leqslant d < 85$	𝍸 |||	8
$85 \leqslant d < 90$	𝍸 |	6
$90 \leqslant d < 95$	|||	3
$95 \leqslant d < 100$	||	2
	Total	30

d



e **i** ≈ 81.1 m **ii** ≈ 83.1 m

2 **a**



b ≈ 25.9 **c** ≈ 12.0 **d** $\overline{x} \approx 26.0$, $s \approx 8.31$

3 **a** **i** 101.5 **ii** 98 **iii** 105.5 **b** 7.5
c $\overline{x} = 100.2$, $s \approx 7.59$

4 **a** $\overline{x} \approx 33.6$ L, $s \approx 7.63$ L **b** $\mu \approx 33.6$ L, $\sigma \approx 7.63$ L

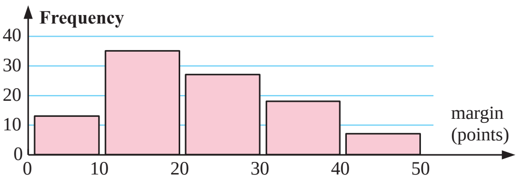
5 **a** $\overline{x} \approx 49.6$, $s \approx 1.60$
b Does not justify claim. Need a larger sample.

6 range = 19, lower quartile = 119, upper quartile = 130, $s \approx 6.38$

7 $a = 8$, $b = 4$

REVIEW SET 14C

1



2 **a** 68% **b** 95% **c** 81.5% **d** 13.5%

3 ≈ 414 customers

4 **a**

	A	*B*
Min	11	11.2
Q_1	11.6	12
Median	12	12.6
Q_3	12.6	13.2
Max	13	13.8

b

		A	*B*
i	Range	2	2.6
ii	IQR	1	1.2

c **i** We know the members of squad A generally ran faster because their median time is lower.
ii We know the times in squad B are more varied because their range and IQR is higher.

5 **a** $\overline{x} =$ €103.51, $s \approx$ €19.40 **b** $\mu =$ €103.51, $\sigma \approx$ €19.40

6 **a** mean is 18.8, standard deviation is 2.6 **b** 13.6 to 24.0

7 **a** 120 students **b** 65 marks **c** 54 and 75
d 21 marks **e** 73% of them **f** 81 marks

EXERCISE 15A

1 **a** 0.78 **b** 0.22 **2** **a** 0.487 **b** 0.051 **c** 0.731

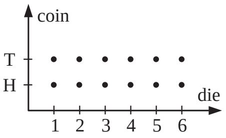
3 **a** 43 days **b** **i** ≈ 0.047 **ii** ≈ 0.186 **iii** 0.465

4 **a** ≈ 0.089 **b** ≈ 0.126

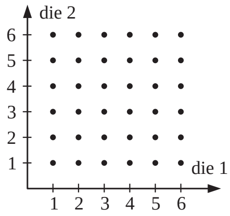
EXERCISE 15B

1 **a** {A, B, C, D} **b** {BB, BG, GB, GG}
c {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA}
d {GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB}

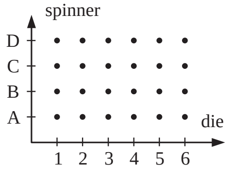
2 **a**



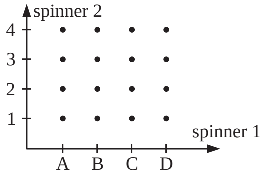
b



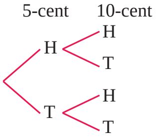
c



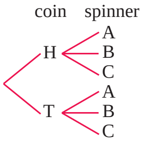
d



3 **a**



b



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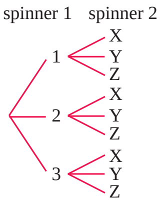
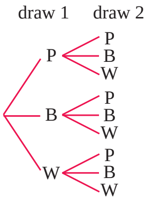


d



EXERCISE 15C.1

1 **a** $\frac{1}{5}$ **b** $\frac{1}{3}$ **c** $\frac{7}{15}$ **d** $\frac{4}{5}$ **e** $\frac{1}{5}$ **f** $\frac{8}{15}$

2 **a** 4 **b** **i** $\frac{2}{3}$ **ii** $\frac{1}{3}$

3 **a** $\frac{1}{4}$ **b** $\frac{1}{9}$ **c** $\frac{4}{9}$ **d** $\frac{1}{36}$ **e** $\frac{1}{18}$ **f** $\frac{1}{6}$
g $\frac{1}{12}$ **h** $\frac{1}{3}$

4 **a** $\frac{1}{7}$ **b** $\frac{2}{7}$ **c** $\frac{124}{1461}$ **d** $\frac{237}{1461}$ {remember leap years}

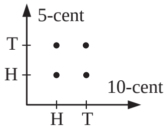
5 {AKN, ANK, KAN, KNA, NAK, NKA}
a $\frac{1}{3}$ **b** $\frac{1}{3}$ **c** $\frac{1}{3}$ **d** $\frac{2}{3}$

6 **a** {GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB}
b **i** $\frac{1}{8}$ **ii** $\frac{1}{8}$ **iii** $\frac{1}{8}$ **iv** $\frac{3}{8}$ **v** $\frac{1}{2}$ **vi** $\frac{7}{8}$

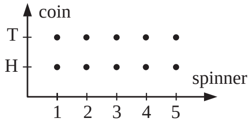
7 **a** {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA}
b **i** $\frac{1}{2}$ **ii** $\frac{1}{2}$ **iii** $\frac{1}{2}$ **iv** $\frac{1}{2}$

EXERCISE 15C.2

1



2 **a**



a $\frac{1}{4}$ **b** $\frac{1}{4}$ **c** $\frac{1}{2}$ **d** $\frac{3}{4}$

b 10 **c** **i** $\frac{1}{10}$ **ii** $\frac{1}{5}$
iii $\frac{3}{5}$ **iv** $\frac{3}{5}$

3 **a** $\frac{1}{36}$ **b** $\frac{1}{18}$ **c** $\frac{5}{9}$ **d** $\frac{11}{36}$ **e** $\frac{5}{18}$
f $\frac{25}{36}$ **g** $\frac{1}{6}$ **h** $\frac{5}{18}$ **i** $\frac{2}{9}$ **j** $\frac{13}{18}$

EXERCISE 15D

1 **a** 0.476 **b** 0.241 **c** 0.483 **d** 0.578 **e** 0.415

2 **a** 7510 **b** **i** 0.325 **ii** 0.653 **iii** 0.243

3 **a** 0.428 **b** 0.240 **c** 0.758 **d** 0.257 **e** 0.480

EXERCISE 15E.1

1 **a** $\frac{6}{7}$ **b** $\frac{36}{49}$ **c** $\frac{216}{343}$ **2** **a** $\frac{1}{8}$ **b** $\frac{1}{8}$

3 **a** 0.0096 **b** 0.8096 **4** **a** $\frac{1}{16}$ **b** $\frac{15}{16}$

5 **a** 0.56 **b** 0.06 **c** 0.14 **d** 0.24

6 **a** $\frac{8}{125}$ **b** $\frac{12}{125}$ **c** $\frac{27}{125}$

EXERCISE 15E.2

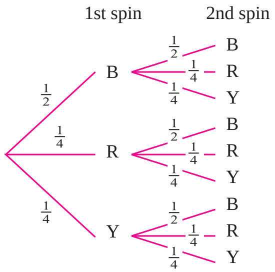
1 **a** $\frac{14}{55}$ **b** $\frac{1}{55}$ **2** **a** $\frac{7}{15}$ **b** $\frac{7}{30}$ **c** $\frac{7}{15}$

3 **a** $\frac{3}{100}$ **b** $\frac{3}{100} \times \frac{2}{99} \approx 0.0006$
c $\frac{3}{100} \times \frac{2}{99} \times \frac{1}{98} \approx 0.000\,006$ **d** $\frac{97}{100} \times \frac{96}{99} \times \frac{95}{98} \approx 0.912$

4 **a** $\frac{4}{7}$ **b** $\frac{2}{7}$

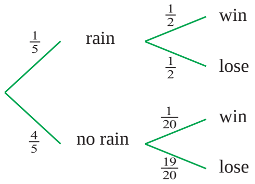
EXERCISE 15F

1 **a**



b $\frac{1}{4}$
c $\frac{1}{16}$
d $\frac{5}{8}$
e $\frac{3}{4}$

2



P(win) $= \frac{7}{50}$

3 0.032 **4** $\frac{17}{40}$ **5** $\frac{9}{38}$ **6** **a** $\frac{11}{30}$ **b** $\frac{19}{30}$

EXERCISE 15G

1 **a** $\frac{20}{49}$ **b** $\frac{10}{21}$ **2** **a** $\frac{3}{10}$ **b** $\frac{1}{10}$ **c** $\frac{3}{5}$

3 **a** $\frac{2}{9}$ **b** $\frac{5}{9}$ **4** **a** $\frac{1}{3}$ **b** $\frac{2}{15}$ **c** $\frac{4}{15}$ **d** $\frac{4}{15}$

These are all possibilities, so their probabilities must sum to 1.

5 **a** $\frac{1}{5}$ **b** $\frac{3}{5}$ **c** $\frac{4}{5}$ **6** $\frac{19}{45}$

7 **a** $\frac{2}{100} \times \frac{1}{99} \approx 0.0002$ **b** $\frac{98}{100} \times \frac{97}{99} \approx 0.9602$

c $1 - \frac{98}{100} \times \frac{97}{99} \approx 0.0398$

8 $\frac{7}{33}$ **9** 7 to start with

EXERCISE 15H

1 **a** $(p+q)^4 = p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$

b $4(\frac{1}{2})^3(\frac{1}{2}) = \frac{1}{4}$

2 **a** $(p+q)^5 = p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$

b **i** $5(\frac{1}{2})^4(\frac{1}{2}) = \frac{5}{32}$ **ii** $10(\frac{1}{2})^2(\frac{1}{2})^3 = \frac{5}{16}$

iii $\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) = \frac{1}{32}$

3 **a** $\left(\frac{2}{3}+\frac{1}{3}\right)^4 = \left(\frac{2}{3}\right)^4 + 4\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2$
$+4\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4$

b **i** $\left(\frac{2}{3}\right)^4 = \frac{16}{81}$ **ii** $6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 = \frac{8}{27}$ **iii** $\frac{8}{9}$

4 **a** $\left(\frac{3}{4}+\frac{1}{4}\right)^5 = \left(\frac{3}{4}\right)^5 + 5\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^1 + 10\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2$
$+10\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3 + 5\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5$

b **i** $10\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 = \frac{135}{512}$ **ii** $\frac{53}{512}$ **iii** $\frac{47}{128}$

5 **a** ≈ 0.154 **b** ≈ 0.973 **6** **a** ≈ 0.0305 **b** ≈ 0.265

7 $\approx 0.000\,864$ **8** ≈ 0.0341 **9** 4 dice

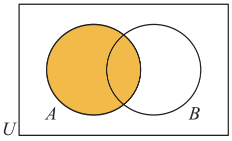
EXERCISE 15I.1

1 **a** $A = \{1, 2, 3, 6\}$, $B = \{2, 4, 6, 8, 10\}$

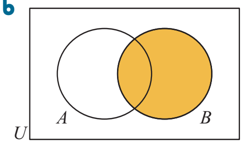
b **i** $n(A) = 4$ **ii** $A \cup B = \{1, 2, 3, 4, 6, 8, 10\}$

iii $A \cap B = \{2, 6\}$

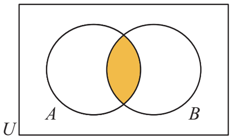
2 **a**



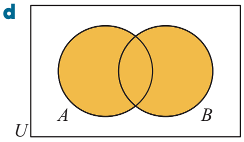
b



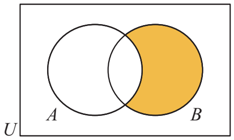
c



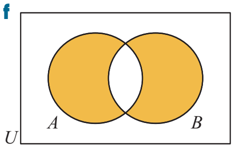
d



e



f



3 **a** 29 **b** 17 **c** 26 **d** 5

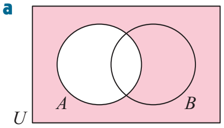
4 **a** 65 **b** 9 **c** 4 **d** 52

5 **a** $\frac{19}{40}$ **b** $\frac{1}{2}$ **c** $\frac{4}{5}$ **d** $\frac{5}{8}$ **e** $\frac{13}{40}$ **f** $\frac{7}{20}$

6 **a** $\frac{19}{25}$ **b** $\frac{13}{25}$ **c** $\frac{6}{25}$ **d** $\frac{7}{19}$

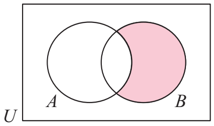
7 **a** $\frac{7}{15}$ **b** $\frac{1}{15}$ **c** $\frac{2}{15}$ **d** $\frac{6}{7}$

8 **a**



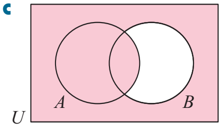
A' is shaded.

b



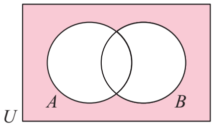
$A' \cap B$ is shaded.

c



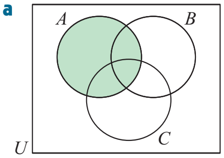
$A \cup B'$ is shaded.

d

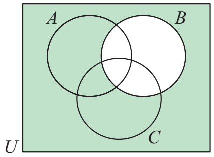


$A' \cap B'$ is shaded.

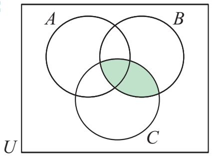
9 **a**



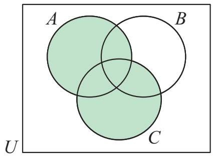
b



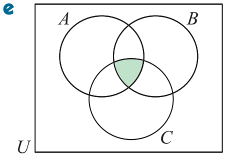
c



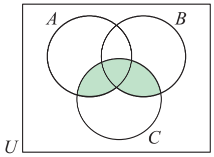
d



e



f



EXERCISE 15I.2

1 For each of these draw **two** diagrams, shade the first with the LHS set and the second with the RHS set.

2 **a** $A = \{7, 14, 21, 28, 35, \ldots\ldots, 98\}$
$B = \{5, 10, 15, 20, 25, \ldots\ldots, 95\}$
i $n(A) = 14$ **ii** $n(B) = 19$ **iii** 2 **iv** 31

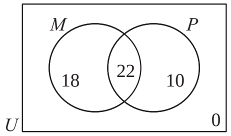
3 **a** **i** $\dfrac{b+c}{a+b+c+d}$ **ii** $\dfrac{b}{a+b+c+d}$

iii $\dfrac{a+b+c}{a+b+c+d}$ **iv** $\dfrac{a+b+c}{a+b+c+d}$

b P(A or B) = P(A) + P(B) − P(A and B)

EXERCISE 15J

1 **a**



22 study both

b **i** $\frac{9}{25}$ **ii** $\frac{11}{20}$

2 **a** $\frac{3}{8}$ **b** $\frac{7}{20}$ **c** $\frac{1}{5}$ **d** $\frac{15}{23}$

3 **a** $\frac{14}{25}$ **b** $\frac{4}{5}$ **c** $\frac{1}{5}$ **d** $\frac{5}{23}$ **e** $\frac{9}{14}$ **4** $\frac{5}{6}$

5 **a** $\frac{13}{20}$ **b** $\frac{7}{20}$ **c** $\frac{11}{50}$ **d** $\frac{7}{25}$ **e** $\frac{4}{7}$ **f** $\frac{1}{4}$

6 **a** $\frac{3}{5}$ **b** $\frac{2}{3}$ **7** **a** 0.46 **b** $\frac{14}{23}$ **8** $\frac{70}{163}$

9 **a** 0.45 **b** 0.75 **c** 0.65

10 **a** 0.0484 **b** 0.3926 **11** $\frac{2}{3}$

EXERCISE 15K

1 $\text{P}(R \cap S) = 0.2$ and $\text{P}(R) \times \text{P}(S) = 0.2$
$\therefore$ are independent events

2 **a** $\frac{7}{30}$ **b** $\frac{7}{12}$ **c** $\frac{7}{10}$ No, as $\text{P}(A \cap B) \neq \text{P}(A) \times \text{P}(B)$

3 **a** 0.35 **b** 0.85 **c** 0.15 **d** 0.15 **e** 0.5

4 $\frac{14}{15}$ **5** **a** $\frac{91}{216}$ **b** 26

6 **Hint:** Show $\text{P}(A' \cap B') = \text{P}(A')\,\text{P}(B')$
using a Venn diagram and $\text{P}(A \cap B)$

7 0.9

8 **a** **i** $\frac{13}{20}$ **ii** $\frac{7}{10}$ **b** No, as $\text{P}(C \cap D) \neq \text{P}(C)\,\text{P}(D)$

REVIEW SET 15A

1 ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA

a $\frac{1}{2}$ **b** $\frac{1}{3}$

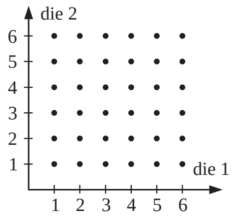
2 **a** $\frac{3}{8}$ **b** $\frac{1}{8}$ **c** $\frac{5}{8}$ **3** **a** $\frac{2}{5}$ **b** $\frac{13}{15}$ **c** $\frac{4}{15}$

4 **a** 0 **b** 0.45 **c** 0.8

5 **a** Two events are independent if the occurrence of each event does not influence the occurrence of the other. For A and B independent, $\text{P}(A) \times \text{P}(B) = \text{P}(A \text{ and } B)$

b Two events A and B are disjoint if they have no common outcomes. $\text{P}(A \text{ or } B) = \text{P}(A) + \text{P}(B)$

6



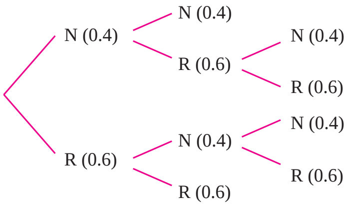
a $\frac{2}{9}$

b $\frac{5}{12}$

7 **a** $\frac{1}{4}$ **b** $\frac{37}{40}$ **c** $\frac{2}{5}$ **8** $\frac{5}{9}$

REVIEW SET 15B

1 P(N wins)
$= \frac{44}{125}$
$= 0.352$

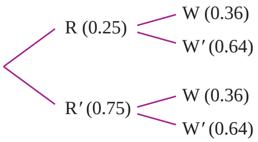


2 **a** $\frac{4}{500} \times \frac{3}{499} \times \frac{2}{498} \approx 0.000\,000\,193$

b $1 - \frac{496}{500} \times \frac{495}{499} \times \frac{494}{498} \approx 0.023\,86$

3 **a** ≈ 0.259 **b** ≈ 0.703

4



a 0.09
b 0.52

5 $1 - 0.9 \times 0.8 \times 0.7 = 0.496$

6 **a** $\left(\frac{3}{5}+\frac{2}{5}\right)^4 = \left(\frac{3}{5}\right)^4 + 4\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right) + 6\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^2$
$+4\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4$

b **i** $\frac{216}{625}$ **ii** $\frac{328}{625}$

7 **a**

	Female	*Male*	*Total*
smoker	20	40	60
non-smoker	70	70	140
total	90	110	200

b **i** $\frac{7}{20}$ **ii** $\frac{1}{2}$ **c** ≈ 0.121

REVIEW SET 15C

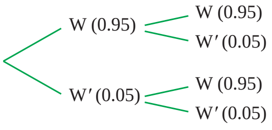
1 BBBB, BBBG, BBGB, BGBB, GBBB, BBGG, BGBG, BGGB, GGBB, GBBG, GBGB, BGGG, GBGG, GGBG, GGGB, GGGG.
P(2 children of each sex) $= \frac{3}{8}$

2 **a** $\frac{5}{33}$ **b** $\frac{19}{66}$ **c** $\frac{5}{11}$ **d** $\frac{16}{33}$

3 **a** $\frac{3}{25}$ **b** $\frac{24}{25}$ **c** $\frac{11}{12}$

4 **a** $\frac{5}{8}$ **b** $\frac{1}{4}$

5



0.9975

6 **a** $\frac{31}{70}$ **b** $\frac{21}{31}$

7 **a** $\left(\frac{4}{5}+\frac{1}{5}\right)^5 = \left(\frac{4}{5}\right)^5 + 5\left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right) + 10\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right)^2$
$+10\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^3 + 5\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)^4 + \left(\frac{1}{5}\right)^5$

b **i** ≈ 0.0205 **ii** ≈ 0.205

EXERCISE 16A

1 **a** 7 **b** 7 **c** 11 **d** 16 **e** 0 **f** $\frac{3}{2}$ **g** 5 **h** -2

2 **a** -3 **b** 6 **c** -8 **d** $\frac{1}{2}$ **e** $\frac{1}{2}$ **f** 5

3 **a** 0 **b** 3 **c** $-\frac{2}{3}$ **d** -1 **e** 1 **f** 1

EXERCISE 16B

1 **a** vertical asymptote $x = -3$, horizontal asymptote $y = 3$
as $x \to -3^-$, $f(x) \to \infty$ as $x \to \infty$, $f(x) \to 3^-$
as $x \to -3^+$, $f(x) \to -\infty$ as $x \to -\infty$, $f(x) \to 3^+$

b horizontal asymptote $y = 1$. as $x \to \infty$, $y \to 1^-$
as $x \to -\infty$, $y \to 1^-$

c horizontal asymptote $y = 0$. as $x \to \infty$, $f(x) \to 0^+$
as $x \to -\infty$, $f(x) \to 0^-$

d vertical asymptotes $x=-2$, $x=1$,
horizontal asymptote $y=0$.
as $x\to-2^-$, $f(x)\to-\infty$ as $x\to\infty$, $f(x)\to0^+$
as $x\to-2^+$, $f(x)\to\infty$ as $x\to-\infty$, $f(x)\to0^-$
as $x\to1^-$, $f(x)\to\infty$
as $x\to1^+$, $f(x)\to-\infty$

EXERCISE 16D.1

1 **a** **i** 0.6 units^2 **ii** 0.4 units^2 **b** 0.5 units^2

2 **a** 0.737 units^2 **b** 0.653 units^2

EXERCISE 16D.2

1

n	A_L	A_U
10	2.185 0	2.485 0
25	2.273 6	2.393 6
50	2.303 4	2.363 4
100	2.318 4	2.348 4
500	2.330 3	2.336 3

converges to $\frac{7}{3}$

2 **a** **i**

n	A_L	A_U
5	0.160 00	0.360 00
10	0.202 50	0.302 50
50	0.240 10	0.260 10
100	0.245 03	0.255 03
500	0.249 00	0.251 00
1000	0.249 50	0.250 50
10 000	0.249 95	0.250 05

ii

n	A_L	A_U
5	0.400 00	0.600 00
10	0.450 00	0.550 00
50	0.490 00	0.510 00
100	0.495 00	0.505 00
500	0.499 00	0.501 00
1000	0.499 50	0.500 50
10 000	0.499 95	0.500 05

iii

n	A_L	A_U
5	0.549 74	0.749 74
10	0.610 51	0.710 51
50	0.656 10	0.676 10
100	0.661 46	0.671 46
500	0.665 65	0.667 65
1000	0.666 16	0.667 16
10 000	0.666 62	0.666 72

iv

n	A_L	A_U
5	0.618 67	0.818 67
10	0.687 40	0.787 40
50	0.738 51	0.758 51
100	0.744 41	0.754 41
500	0.748 93	0.750 93
1000	0.749 47	0.750 47
10 000	0.749 95	0.750 05

b **i** $\frac{1}{4}$ **ii** $\frac{1}{2}$ **iii** $\frac{2}{3}$ **iv** $\frac{3}{4}$ **c** area $=\dfrac{1}{a+1}$

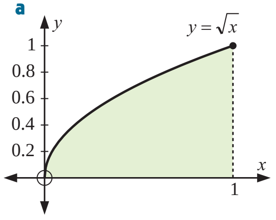
3 **a**

n	Rational bounds for π
10	$2.9045 < \pi < 3.3045$
50	$3.0983 < \pi < 3.1783$
100	$3.1204 < \pi < 3.1604$
200	$3.1312 < \pi < 3.1512$
1000	$3.1396 < \pi < 3.1436$
10 000	$3.1414 < \pi < 3.1418$

b $n = 10\,000$

EXERCISE 16D.3

1 **a**

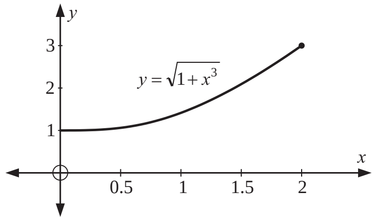


b

n	A_L	A_U
5	0.5497	0.7497
10	0.6105	0.7105
50	0.6561	0.6761
100	0.6615	0.6715
500	0.6656	0.6676

c $\int_0^1 \sqrt{x}\,dx \approx 0.67$

2 **a**



b

n	A_L	A_U
50	3.2016	3.2816
100	3.2214	3.2614
500	3.2373	3.2453

c $\int_0^2 \sqrt{1+x^3}\,dx \approx 3.24$

3 **a** 18 **b** 4.5 **c** 2π

REVIEW SET 16

1 **a** -4 **b** $\frac{1}{4}$ **c** 8 **d** $-\frac{1}{2}$

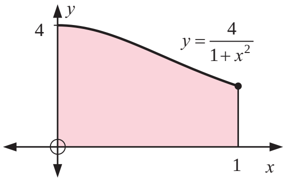
2 **a** horizontal asymptote $y=-3$
as $x\to\infty$, $y\to\infty$ as $x\to-\infty$, $y\to-3^+$

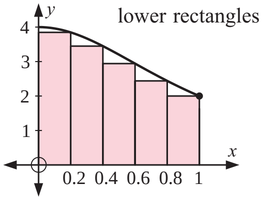
b no asymptotes

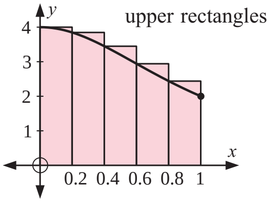
c vertical asymptote $x=0$, horizontal asymptote $y=0$
as $x\to0^+$, $f(x)\to-\infty$ as $x\to\infty$, $f(x)\to0^+$

d vertical asymptote $x=\frac{3}{2}$ as $x\to\frac{3}{2}^+$, $y\to-\infty$

3 **a**





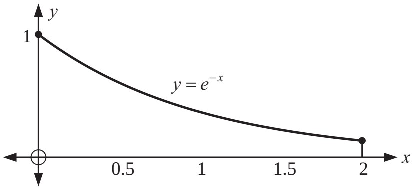


b

n	A_L	A_U
5	2.9349	3.3349
50	3.1215	3.1615
100	3.1316	3.1516
500	3.1396	3.1436

c $\int_0^1 \frac{4}{1+x^2}\,dx \approx 3.1416$

4 **a**



b $A_U \approx 0.977$ units^2, $A_L \approx 0.761$ units^2
c $A_U \approx 0.8733$ units^2, $A_L \approx 0.8560$ units^2

5 **a** 2π **b** 4

6 **a** $A = \frac{17}{4}$, $B = \frac{25}{4}$ **b** $\int_0^2 (4 - x^2)\,dx \approx \frac{21}{4}$

EXERCISE 17A.1

1 **a** 3 **b** 0 **2** **a** 4 **b** -1 **3** $f(2) = 3$, $f'(2) = 1$

EXERCISE 17A.2

1 **a** 1 **b** 0 **c** $3x^2$ **d** $4x^3$ **2** $f'(x) = nx^{n-1}$

3 **a** 2 **b** -1 **c** $2x - 3$
d $4x + 1$ **e** $-2x + 5$ **f** $3x^2 - 4x$

EXERCISE 17B

1 **a** 3 **b** -12 **c** 9 **d** 10 **2** **a** 12 **b** 108

EXERCISE 17C

1 **a** $3x^2$ **b** $6x^2$ **c** $14x$
d $\frac{3}{\sqrt{x}}$ **e** $\frac{1}{\sqrt[3]{x^2}}$ **f** $2x + 1$
g $-4x$ **h** $2x + 3$ **i** $2x^3 - 12x$
j $\frac{6}{x^2}$ **k** $-\frac{2}{x^2} + \frac{6}{x^3}$ **l** $2x - \frac{5}{x^2}$
m $2x + \frac{3}{x^2}$ **n** $-\frac{1}{2x\sqrt{x}}$ **o** $8x - 4$
p $3x^2 + 12x + 12$

2 **a** $7.5x^2 - 2.8x$ **b** $2\pi x$ **c** $-\frac{2}{5x^3}$
d 100 **e** 10 **f** $12\pi x^2$

3 **a** 6 **b** $\frac{3\sqrt{x}}{2}$ **c** $2x - 10$
d $2 - 9x^2$ **e** $2x - 1$ **f** $-\frac{2}{x^3} + \frac{3}{\sqrt{x}}$
g $4 + \frac{1}{4x^2}$ **h** $6x^2 - 6x - 5$

4 **a** 4 **b** $-\frac{16}{729}$ **c** -7 **d** $\frac{13}{4}$ **e** $\frac{1}{8}$ **f** -11

5 $b = 3$, $c = -4$

6 **a** $\frac{2}{\sqrt{x}} + 1$ **b** $\frac{1}{3\sqrt[3]{x^2}}$ **c** $\frac{1}{x\sqrt{x}}$
d $2 - \frac{1}{2\sqrt{x}}$ **e** $-\frac{2}{x\sqrt{x}}$ **f** $6x - \frac{3}{2}\sqrt{x}$
g $\frac{-25}{2x^3\sqrt{x}}$ **h** $2 + \frac{9}{2x^2\sqrt{x}}$

7 **a** $\frac{dy}{dx} = 4 + \frac{3}{x^2}$, $\frac{dy}{dx}$ is the gradient function of $y = 4x - \frac{3}{x}$ from which the gradient at any point can be found.
b $\frac{dS}{dt} = 4t + 4$ ms^{-1}, $\frac{dS}{dt}$ is the instantaneous rate of change in position at the time t, or the velocity function.
c $\frac{dC}{dx} = 3 + 0.004x$ \$ per toaster, $\frac{dC}{dx}$ is the instantaneous rate of change in cost as the number of toasters changes.

EXERCISE 17D.1

1 **a** $g(f(x)) = (2x+7)^2$ **b** $g(f(x)) = 2x^2 + 7$
c $g(f(x)) = \sqrt{3 - 4x}$ **d** $g(f(x)) = 3 - 4\sqrt{x}$
e $g(f(x)) = \frac{2}{x^2 + 3}$ **f** $g(f(x)) = \frac{4}{x^2} + 3$

2 **a** $g(x) = x^3$, $f(x) = 3x + 10$
b $g(x) = \frac{1}{x}$, $f(x) = 2x + 4$
c $g(x) = \sqrt{x}$, $f(x) = x^2 - 3x$
d $g(x) = \frac{10}{x^3}$, $f(x) = 3x - x^2$

EXERCISE 17D.2

1 **a** u^{-2}, $u = 2x - 1$ **b** $u^{\frac{1}{2}}$, $u = x^2 - 3x$
c $2u^{-\frac{1}{2}}$, $u = 2 - x^2$ **d** $u^{\frac{1}{3}}$, $u = x^3 - x^2$
e $4u^{-3}$, $u = 3 - x$ **f** $10u^{-1}$, $u = x^2 - 3$

2 **a** $8(4x-5)$ **b** $2(5-2x)^{-2}$ **c** $\frac{1}{2}(3x-x^2)^{-\frac{1}{2}} \times (3-2x)$
d $-12(1-3x)^3$ **e** $-18(5-x)^2$
f $\frac{1}{3}(2x^3 - x^2)^{-\frac{2}{3}} \times (6x^2 - 2x)$ **g** $-60(5x-4)^{-3}$
h $-4(3x - x^2)^{-2} \times (3 - 2x)$ **i** $6(x^2 - \frac{2}{x})^2 \times (2x + \frac{2}{x^2})$

3 **a** $-\frac{1}{\sqrt{3}}$ **b** -18 **c** -8 **d** -4 **e** $-\frac{3}{32}$ **f** 0

4 **a** $\frac{dy}{dx} = 3x^2$, $\frac{dx}{dy} = \frac{1}{3}y^{-\frac{2}{3}}$ **Hint**: Substitute $y = x^3$
b $\frac{dy}{dx} \times \frac{dx}{dy} = \frac{dy}{dy} = 1$

EXERCISE 17E

1 **a** $2x(2x-1) + 2x^2$ **b** $4(2x+1)^3 + 24x(2x+1)^2$
c $2x(3-x)^{\frac{1}{2}} - \frac{1}{2}x^2(3-x)^{-\frac{1}{2}}$
d $\frac{1}{2}x^{-\frac{1}{2}}(x-3)^2 + 2\sqrt{x}(x-3)$
e $10x(3x^2-1)^2 + 60x^3(3x^2-1)$
f $\frac{1}{2}x^{-\frac{1}{2}}(x-x^2)^3 + 3\sqrt{x}(x-x^2)^2(1-2x)$

2 **a** -48 **b** $406\frac{1}{4}$ **c** $\frac{13}{3}$ **d** $\frac{11}{2}$ **3** $x = 3$ or $\frac{3}{5}$

EXERCISE 17F

1 **a** $\frac{7}{(2-x)^2}$ **b** $\frac{2x(2x+1) - 2x^2}{(2x+1)^2}$
c $\frac{(x^2-3) - 2x^2}{(x^2-3)^2}$ **d** $\frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x) + 2\sqrt{x}}{(1-2x)^2}$
e $\frac{2x(3x-x^2) - (x^2-3)(3-2x)}{(3x-x^2)^2}$
f $\frac{(1-3x)^{\frac{1}{2}} + \frac{3}{2}x(1-3x)^{-\frac{1}{2}}}{1-3x}$

2 **a** 1 **b** 1 **c** $-\frac{7}{324}$ **d** $-\frac{28}{27}$

3 **b** **i** never $\{\frac{dy}{dx}$ is undefined at $x = -1\}$
ii $x \leqslant 0$ and $x = 1$

4 **b** **i** $x = -2 \pm \sqrt{11}$ **ii** $x = -2$

c $\dfrac{dy}{dx}$ is zero when the tangent to the function is horizontal (gradient 0), at its turning points or points of horizontal inflection. $\dfrac{dy}{dx}$ is undefined at vertical asymptotes.

EXERCISE 17G

1 **a** $y=-7x+11$ **b** $4y=x+8$ **c** $y=-2x-2$
d $y=-2x+6$ **e** $y=-5x-9$ **f** $y=-5x-1$

2 **a** $6y=-x+57$ **b** $7y=-x+26$ **c** $3y=x+11$
d $x+6y=43$

3 **a** $y=21$ and $y=-6$ **b** $(\frac{1}{2},\ 2\sqrt{2})$
c $k=-5$ **d** $y=-3x+1$

4 **a** $a=-4,\ b=7$ **b** $a=2,\ b=4$

5 **a** $3y=x+5$ **b** $9y=x+4$
c $16y=x-3$ **d** $y=-4$

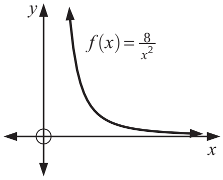
6 **a** $y=2x-\frac{7}{4}$ **b** $y=-27x-\frac{242}{3}$
c $57y=-4x+1042$ **d** $2y=x+1$

7 $a=4,\ b=3$

8 **a** $(-4,\ -64)$ **b** $(4,\ -31)$
c does not meet the curve again

9 **a** $y=(2a-1)x-a^2+9$; $y=5x$, contact at $(3,\ 15)$
$y=-7x$, contact at $(-3,\ 21)$
b $y=0,\ \ y=27x+54$
c $y=0,\ \ y=-\sqrt{14}x+4\sqrt{14}$

10 **a**



b $16x+a^3y=24a$
c $A(\frac{3}{2}a,\ 0)$, $B\left(0,\ \dfrac{24}{a^2}\right)$
d Area $=\dfrac{18}{|a|}$ units^2,
area → 0 as $|a|\to\infty$

EXERCISE 17H

1 **a** 6 **b** $12x-6$ **c** $\dfrac{3}{2x^{\frac{5}{2}}}$
d $\dfrac{12-6x}{x^4}$ **e** $24-48x$ **f** $\dfrac{20}{(2x-1)^3}$

2 **a** $-6x$ **b** $2-\dfrac{30}{x^4}$ **c** $-\frac{9}{4}x^{-\frac{5}{2}}$
d $\dfrac{8}{x^3}$ **e** $6(x^2-3x)(5x^2-15x+9)$
f $2+\dfrac{2}{(1-x)^3}$

3 **a** $x=1$ **b** $x=0,\ \pm\sqrt{6}$

4

x	-1	0	1
$f(x)$	$-$	0	$+$
$f'(x)$	$+$	$-$	$+$
$f''(x)$	$-$	0	$+$

REVIEW SET 17A

1 **a** -17 **b** -17 **c** -6 **2** $y=4x+2$

3 **a** $6x-4x^3$ **b** $1+\dfrac{1}{x^2}$ **4** $2x+2$ **5** $x=1$

6 $a=\frac{5}{2},\ b=-\frac{3}{2}$

7 **a** $f'(x)=8x(x^2+3)^3$
b $g'(x)=\dfrac{\frac{1}{2}x(x+5)^{-\frac{1}{2}}-2(x+5)^{\frac{1}{2}}}{x^3}$

8 **a** $\frac{23}{4}$ **b** $-\frac{1}{8\sqrt{2}}$ **9** $a=-14,\ b=21$

10 $x=-\frac{1}{2},\ \frac{3}{2}$ **11** $a=64$ **12** $P(0,\ 7.5)$, $Q(3,\ 0)$

REVIEW SET 17B

1 **a** $\dfrac{dy}{dx}=5+3x^{-2}$ **b** $\dfrac{dy}{dx}=4(3x^2+x)^3(6x+1)$
c $\dfrac{dy}{dx}=2x(1-x^2)^3-6x(1-x^2)^2(x^2+1)$

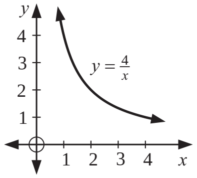
2 $y=7,\ \ y=-25$ **3** $\frac{3267}{152}$ units^2 **4** $(-2,\ -25)$

5 $a=\frac{1}{2}$ **6** $(-2,\ 19)$ and $(1,\ -2)$

7 **a** $-2(5-4x)^{-\frac{1}{2}}$ **b** $-4(5-4x)^{-\frac{3}{2}}$ **8** $5y=x-11$

10 $g(x)=-2x^2+6x+3$

11 **a**



b $y=-\dfrac{4}{k^2}x+\dfrac{8}{k}$
c $A(2k,\ 0)$ $B\left(0,\ \dfrac{8}{k}\right)$
d Area = 8 units^2
e $k=2$

REVIEW SET 17C

1 **a** $\dfrac{dy}{dx}=3x^2(1-x^2)^{\frac{1}{2}}-x^4(1-x^2)^{-\frac{1}{2}}$
b $\dfrac{dy}{dx}=\dfrac{(2x-3)(x+1)^{\frac{1}{2}}-\frac{1}{2}(x^2-3x)(x+1)^{-\frac{1}{2}}}{x+1}$

2 $y=16x-\frac{127}{2}$

3 **a** $-\dfrac{2}{x\sqrt{x}}-3$ **b** $4\left(x-\dfrac{1}{x}\right)^3\left(1+\dfrac{1}{x^2}\right)$
c $\frac{1}{2}(x^2-3x)^{-\frac{1}{2}}(2x-3)$

4 $f(3)=2,\ \ f'(3)=-1$

5 **a** $f'(x)=\dfrac{3(x+3)^2\sqrt{x}-\frac{1}{2}x^{-\frac{1}{2}}(x+3)^3}{x}$
b $f'(x)=4x^3\sqrt{x^2+3}+x^5(x^2+3)^{-\frac{1}{2}}$

6 $a=-1,\ \ b=2$

7 $a=2$ and the tangent is $y=3x-1$ which meets the curve again at $(-4,\ -13)$

8 $BC=\frac{8\sqrt{10}}{3}$ **Hint:** The normal is $y=-3x+8$.

9 **a** $\dfrac{d^2y}{dx^2}=36x^2-\dfrac{4}{x^3}$ **b** $\dfrac{d^2y}{dx^2}=6x+\frac{3}{4}x^{-\frac{5}{2}}$

10 $a=9,\ \ b=2,\ \ f''(-2)=-18$ **11** $4y=3x+5$

EXERCISE 18A

1 **a** \$118 000 **b** $\dfrac{dP}{dt}=4t-12$ \$1000 per year
c $\dfrac{dP}{dt}$ is the rate of change in profit with time
d **i** $t\leqslant 3$ years **ii** $t>3$ years
e minimum profit is \$100 000 when $t=3$

f $\left.\dfrac{dP}{dt}\right|_{t=4} = 4$ Profit is increasing at \$4000 per year after 4 years.

$\left.\dfrac{dP}{dt}\right|_{t=10} = 28$ Profit is increasing at \$28 000 per year after 10 years.

$\left.\dfrac{dP}{dt}\right|_{t=25} = 88$ Profit is increasing at \$88 000 per year after 25 years.

2 **a** 19 000 m^3 per minute **b** 18 000 m^3 per minute

3 **a** 1.2 m

b $s'(t) = 28.1 - 9.8t$ represents the instantaneous velocity of the ball

c $t = 2.87$ s. The ball has stopped and reached its maximum height.

d 41.5 m

e **i** 28.1 m s^{-1} **ii** 8.5 m s^{-1} **iii** 20.9 m s^{-1}

$s'(t) \geqslant 0$ when the ball is travelling upwards.

$s'(t) \leqslant 0$ when the ball is travelling downwards.

f 5.78 s

g $\dfrac{d^2s}{dt^2}$ is the rate of change of $\dfrac{ds}{dt}$, or the instantaneous acceleration.

4 **b** 69.6 m s^{-1}

EXERCISE 18B

1 **a** **i** $Q = 100$ **ii** $Q = 50$ **iii** $Q = 0$

b **i** decr. 1 unit per year **ii** decr. $\frac{1}{\sqrt{2}}$ units per year

2 **a** 0.5 m

b $t = 4$: 9.17 m, $t = 8$: 12.5 m, $t = 12$: 14.3 m

c $t = 0$: 3.9 m year^{-1}, $t = 5$: 0.975 m year^{-1}, $t = 10$: 0.433 m year^{-1}

d as $\dfrac{dH}{dt} = \dfrac{97.5}{(t+5)^2} > 0$, for all $t \geqslant 0$, the tree is always growing, and $\dfrac{dH}{dt} \to 0$ as t increases

3 **a** **i** €4500 **ii** €4000

b **i** decr. of €210.22 per km h^{-1}

ii incr. of €11.31 per km h^{-1}

c $\dfrac{dC}{dv} = 0$ at $v = \sqrt[3]{500\,000} \approx 79.4$ km h^{-1}

4 **a** The near part of the lake is 2 km from the sea, the furthest part is 3 km.

b $\dfrac{dy}{dx} = \frac{3}{10}x^2 - x + \frac{3}{5}$.

$\left.\dfrac{dy}{dx}\right|_{x=\frac{1}{2}} = 0.175$, height of hill is increasing as gradient is positive.

$\left.\dfrac{dy}{dx}\right|_{x=1\frac{1}{2}} = -0.225$, height of hill is decreasing as gradient is negative.

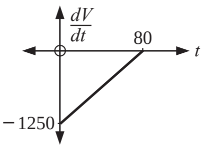
$\therefore$ top of the hill is between $x = \frac{1}{2}$ and $x = 1\frac{1}{2}$.

c 2.55 km from the sea, 63.1 m deep

5 **a** $\dfrac{dV}{dt} = -1250\left(1 - \dfrac{t}{80}\right)$

b at $t = 0$ when the tap was first opened

c $\dfrac{d^2V}{dt^2} = \dfrac{125}{8}$



This shows that the rate of change of V is constantly increasing, so the outflow is decreasing at a constant rate.

6 **a** When $\dfrac{dP}{dt} = 0$, the population is not changing over time, so it is stable.

b 4000 fish **c** 8000 fish

7 **a** $C'(x) = 0.0009x^2 + 0.04x + 4$ dollars per pair

b $C'(220) = \$56.36$ per pair. This estimates the additional cost of making one more pair of jeans if 220 pairs are currently being made.

c \$56.58 This is the actual increase in cost to make an extra pair of jeans (221 rather than 220).

d $C''(x) = 0.0018x + 0.04$.

$C''(x) = 0$ when $x = -22.2$. This is where the rate of change is a minimum, however it is out of the bounds of the model (you cannot make < 0 jeans!).

EXERCISE 18C.1

1 **a** 7 m s^{-1} **b** $(h+5)$ m s^{-1} **c** 5 $\text{m s}^{-1} = s'(1)$

d average velocity $= (2t + h + 3)$ m s^{-1},

$\lim\limits_{h \to 0} (2t + h + 3) = s'(t) \to 2t + 3$ as $h \to 0$

2 **a** -14 cm s^{-1} **b** $(-8 - 2h)$ cm s^{-1}

c -8 $\text{cm s}^{-1} = s'(2)$ $\therefore$ velocity $= -8$ cm s^{-1} at $t = 2$

d $-4t = s'(t) = v(t)$

3 **a** $\frac{2}{3}$ cm s^{-2} **b** $\left(\dfrac{2}{\sqrt{1+h}+1}\right)$ cm s^{-2}

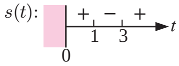
c 1 $\text{cm s}^{-2} = v'(1)$

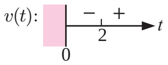
d $\dfrac{1}{\sqrt{t}}$ $\text{cm s}^{-2} = v'(t)$, the instantaneous accn. at time t

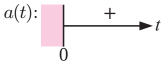
4 **a** velocity at $t = 4$ **b** acceleration at $t = 4$

EXERCISE 18C.2

1 **a** $v(t) = 2t - 4$, $a(t) = 2$





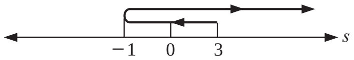


b The object is initially 3 cm to the right of the origin and is moving to the left at 4 cm s^{-1}. It is accelerating at 2 cm s^{-2} to the right.

c The object is instantaneously stationary, 1 cm to the left of the origin and is accelerating to the right at 2 cm s^{-2}.

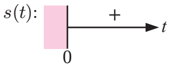
d At $t = 2$, $s(2) = 1$ cm to the left of the origin.

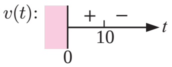
e

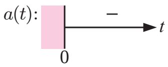


f $0 \leqslant t \leqslant 2$

2 **a** $v(t) = 98 - 9.8t$, $a(t) = -9.8$







b $s(0) = 0$ m above the ground, $v(0) = 98$ m s^{-1} skyward

c $t = 5$ Stone is 367.5 m above the ground and moving skyward at 49 m s^{-1}. Its speed is decreasing.

$t = 12$ Stone is 470.4 m above the ground and moving groundward at 19.6 m s^{-1}. Its speed is increasing.

d 490 m **e** 20 seconds

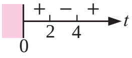
3 **a** $v(t) = 12 - 6t^2$, $a(t) = -12t$

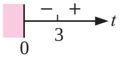
b $s(0) = -1$, $v(0) = 12$, $a(0) = 0$

Particle started 1 cm to the left of the origin and was travelling to the right at a constant speed of 12 cm s^{-1}.

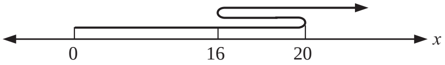
c $t = \sqrt{2}$, $s(\sqrt{2}) = 8\sqrt{2} - 1$ **d** **i** $t \geqslant \sqrt{2}$ **ii** never

4 **a** $v(t) = 3t^2 - 18t + 24$ $a(t) = 6t - 18$





b $x(2) = 20$, $x(4) = 16$



c **i** $0 \leqslant t \leqslant 2$ and $3 \leqslant t \leqslant 4$ **ii** $0 \leqslant t \leqslant 3$ **d** 28 m

5 **Hint:** $s'(t) = v(t)$ and $s''(t) = a(t) = g$

Show that $a = \frac{1}{2}g$, $b = v(0)$, $c = 0$.

EXERCISE 18D.1

1 **a** **i** $x \geqslant 0$ **ii** never

b **i** never **ii** $-2 < x \leqslant 3$

c **i** $x \leqslant 2$ **ii** $x \geqslant 2$

d **i** all real x **ii** never

e **i** $1 \leqslant x \leqslant 5$ **ii** $x \leqslant 1$, $x \geqslant 5$

f **i** $2 \leqslant x < 4$, $x > 4$ **ii** $x < 0$, $0 < x \leqslant 2$

2 **a** increasing for $x \geqslant 0$, decreasing for $x \leqslant 0$

b decreasing for all x

c increasing for $x \geqslant -\frac{3}{4}$, decreasing for $x \leqslant -\frac{3}{4}$

d increasing for $x \geqslant 0$, never decreasing

e decreasing for $x > 0$, never increasing

f incr. for $x \leqslant 0$ and $x \geqslant 4$, decr. for $0 \leqslant x \leqslant 4$

g increasing for $-\sqrt{\frac{2}{3}} \leqslant x \leqslant \sqrt{\frac{2}{3}}$,
decreasing for $x \leqslant -\sqrt{\frac{2}{3}}$, $x \geqslant \sqrt{\frac{2}{3}}$

h decr. for $x \leqslant -\frac{1}{2}$, $x \geqslant 3$, incr. for $-\frac{1}{2} \leqslant x \leqslant 3$

i increasing for $x \geqslant 0$, decreasing for $x \leqslant 0$

j increasing for $x \geqslant -\frac{3}{2} + \frac{\sqrt{5}}{2}$ and $x \leqslant -\frac{3}{2} - \frac{\sqrt{5}}{2}$
decreasing for $-\frac{3}{2} - \frac{\sqrt{5}}{2} \leqslant x \leqslant -\frac{3}{2} + \frac{\sqrt{5}}{2}$

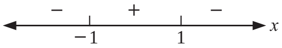
k increasing for $x \leqslant 2 - \sqrt{3}$, $x \geqslant 2 + \sqrt{3}$
decreasing for $2 - \sqrt{3} \leqslant x \leqslant 2 + \sqrt{3}$

l increasing for $x \geqslant 1$, decreasing for $0 \leqslant x \leqslant 1$

m increasing for $-1 \leqslant x \leqslant 1$, $x \geqslant 2$
decreasing for $x \leqslant -1$, $1 \leqslant x \leqslant 2$

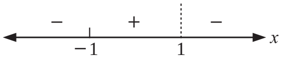
n increasing for $1 - \sqrt{2} \leqslant x \leqslant 1$, $x \geqslant 1 + \sqrt{2}$
decreasing for $x \leqslant 1 - \sqrt{2}$, $1 \leqslant x \leqslant 1 + \sqrt{2}$

3 **a**



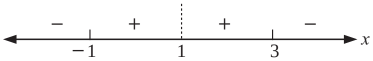
b increasing for $-1 \leqslant x \leqslant 1$
decreasing for $x \leqslant -1$, $x \geqslant 1$

4 **a**



b increasing for $-1 \leqslant x < 1$
decreasing for $x \leqslant -1$, $x > 1$

5 **a**



b increasing for $-1 \leqslant x < 1$, $1 < x \leqslant 3$
decreasing for $x \leqslant -1$, $x \geqslant 3$

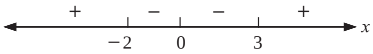
6 **a** increasing for $x \geqslant \sqrt{3}$ and $x \leqslant -\sqrt{3}$
decreasing for $-\sqrt{3} \leqslant x < -1$, $-1 < x < 1$,
$1 < x \leqslant \sqrt{3}$

b increasing for $x \geqslant 2$, decreasing for $x < 1$, $1 < x \leqslant 2$

EXERCISE 18D.2

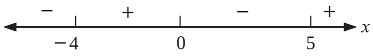
1 **a** A - local max B - horiz. inflection C - local min.

b



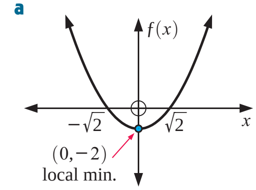
c **i** $x \leqslant -2$, $x \geqslant 3$ **ii** $-2 \leqslant x \leqslant 3$

d

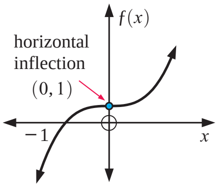


e For **b** we have intervals where the function is increasing (+) or decreasing (−). For **d** we have intervals where the function is above (+) and below (−) the x-axis.

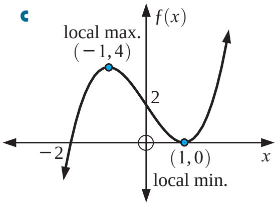
2 **a**



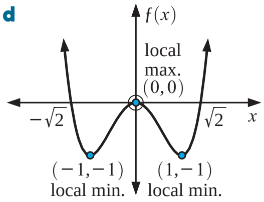
b



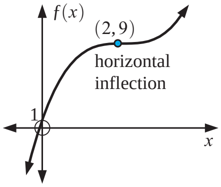
c



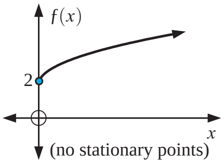
d



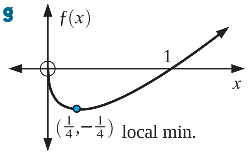
e



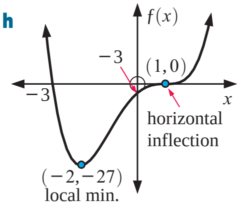
f



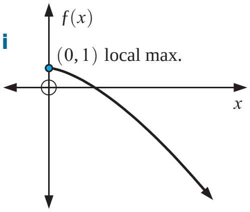
g



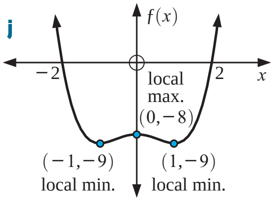
h



i



j



3 $x = -\dfrac{b}{2a}$, local min if $a > 0$, local max if $a < 0$

4 $a = 9$

5 **a** $a = -12$, $b = -13$

b $(-2, 3)$ local max. $(2, -29)$ local min

6 $P(x) = -9x^3 - 9x^2 + 9x + 2$

7 **a** greatest value is 63 when $x = 5$,
least value is -18 when $x = 2$

b greatest value is 4 when $x = 3$ and $x = 0$,
least value is -16 when $x = -2$.

8 Maximum hourly cost is \$680.95 when 150 hinges are made per hour. Minimum hourly cost is \$529.80 when 104 hinges are made per hour.

EXERCISE 18E.1

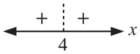
1 **a** $f(x) = 3 - \dfrac{5}{x+1}$ H.A. $y = 3$, V.A. $x = -1$

b $f(x) = \frac{1}{2} - \dfrac{7}{2(2x-1)}$ H.A. $y = \frac{1}{2}$, V.A. $x = \frac{1}{2}$

c $f(x) = -2 + \dfrac{2}{x-1}$ H.A. $y = -2$, V.A. $x = 1$.

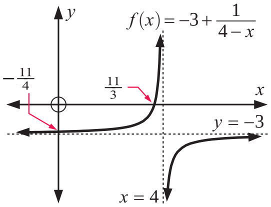
2 **a** **i** H.A. $y = -3$, V.A. $x = 4$

ii $f'(x) = (4-x)^{-2}$



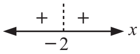
iii x-int. is $\frac{11}{3}$, y-int. is $-\frac{11}{4}$.

iv



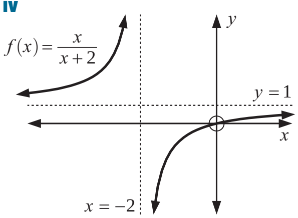
b **i** H.A. $y = 1$, V.A. $x = -2$

ii $f'(x) = \dfrac{2}{(x+2)^2}$



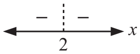
iii x-int. is 0, y-int. is 0.

iv



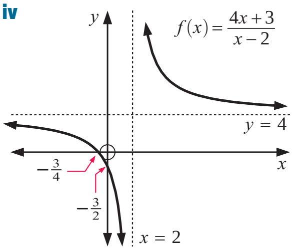
c **i** H.A. $y = 4$, V.A. $x = 2$

ii $f'(x) = \dfrac{-11}{(x-2)^2}$



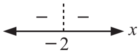
iii x-int. is $-\frac{3}{4}$, y-int. is $-\frac{3}{2}$.

iv



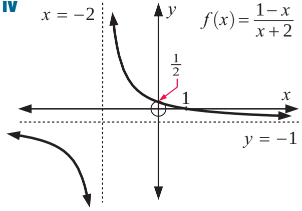
d **i** H.A. $y = -1$, V.A. $x = -2$

ii $f'(x) = \dfrac{-3}{(x+2)^2}$



iii x-int. is 1, y-int. is $\frac{1}{2}$.

iv



EXERCISE 18E.2

1 **a** H.A. $y = 0$, V.A. $x = 2$ and $x = -2$

b H.A. $y = 0$, V.A. $x = -2$

c H.A. $y = 0$, no V.A

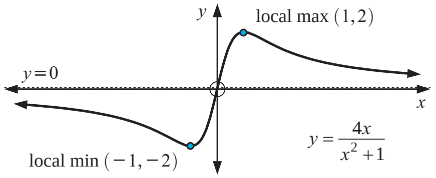
2 **a** **i** H.A. $y = 0$

ii $f'(x) = \dfrac{-4(x+1)(x-1)}{(x^2+1)^2}$

$(1, 2)$ is a local max. $(-1, -2)$ is a local min.

iii x-intercept is 0, y-intercept is 0

iv

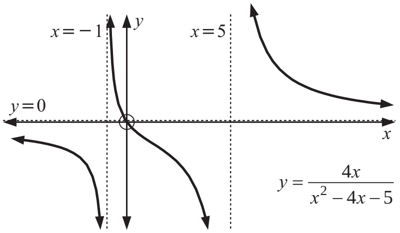


b **i** H.A. $y = 0$, V.A.s $x = 5$ and $x = -1$

ii $f'(x) = \dfrac{-4(x^2+5)}{(x-5)^2(x+1)^2}$, no stationary points

iii x-intercept is 0, y-intercept is 0

iv

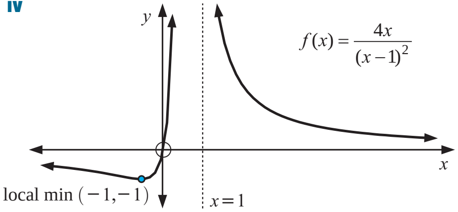


c **i** H.A. $y = 0$, V.A. $x = 1$

ii $f'(x) = \dfrac{-4(x+1)}{(x-1)^3}$, $(-1, -1)$ is a local minimum

iii x-intercept is 0, y-intercept is 0

iv

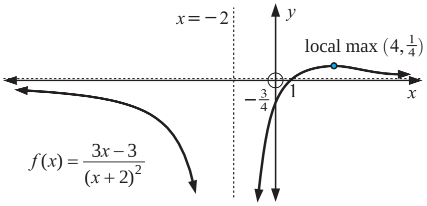


d **i** H.A. $y = 0$, V.A. $x = -2$

ii $f'(x) = \dfrac{-3(x-4)}{(x+2)^3}$, $(4, \frac{1}{4})$ is a local maximum

iii x-intercept is 1, y-intercept is $-\frac{3}{4}$

iv



EXERCISE 18E.3

1 **a** V.A. $x = 1$ and $x = -1$, H.A. $y = 2$

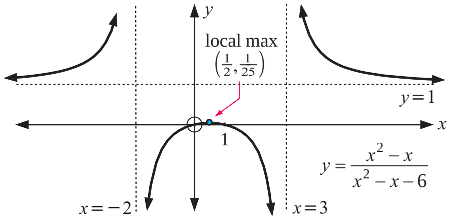
b no V.A.s, H.A. $y = -1$ **c** V.A. $x = -2$, H.A. $y = 3$

2 **a** **i** H.A. $y = 1$, V.A.s $x = 3$ and $x = -2$

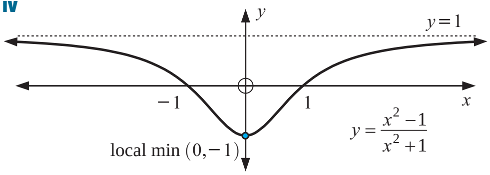
ii $\left(\frac{1}{2}, \frac{1}{25}\right)$ is a local maximum

iii x-intercepts are 0 and 1, y-intercept is 0

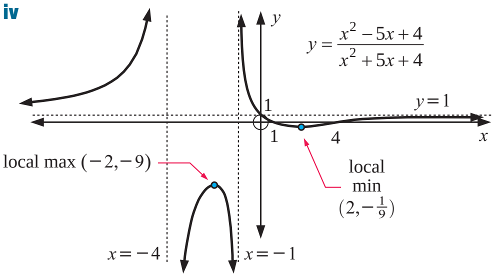
iv



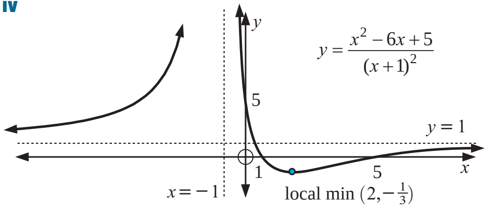
b **i** H.A. $y = 1$, no V.A.
ii $(0,\ -1)$ is a local minimum
iii x-intercepts are 1 and -1, y-intercept is -1
iv



c **i** H.A. $y = 1$, V.A.s $x = -4$ and $x = -1$
ii $(2,\ -\frac{1}{9})$ is a local min., $(-2,\ -9)$ is a local max.
iii x-intercepts are 4 and 1, y-intercept is 1
iv



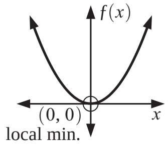
d **i** H.A. $y = 1$, V.A. $x = -1$
ii $(2,\ -\frac{1}{3})$ is a local minimum
iii x-intercepts are 5 and 1, y-intercept is 5
iv



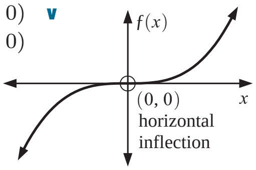
EXERCISE 18F.1

1 **a** no inflection **b** horizontal inflection at $(0,\ 2)$
c non-horizontal inflection at $(2,\ 3)$
d horizontal inflection at $(-2,\ -3)$
e horizontal inflection at $(0,\ 2)$
non-horizontal inflection at $(-\frac{4}{3},\ \frac{310}{27})$ **f** no inflection

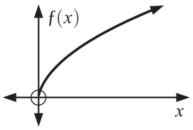
2 **a** **i** local minimum at $(0,\ 0)$ **v**
ii no points of inflection
iii decreasing for $x \leqslant 0$, increasing for $x \geqslant 0$
iv function is concave up for all x



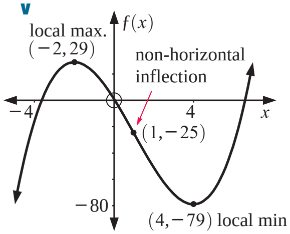
b **i** horizontal inflection at $(0,\ 0)$ **v**
ii horizontal inflection at $(0,\ 0)$
iii increasing for all real x
iv concave down for $x \leqslant 0$, concave up for $x \geqslant 0$



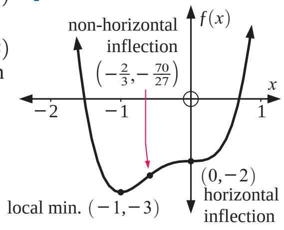
c **i** $f'(x) \neq 0$, no stationary points **v**
ii no points of inflection
iii increasing for $x \geqslant 0$, never decreasing
iv concave down for $x \geqslant 0$, never concave up



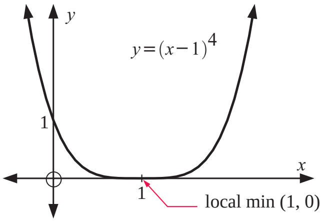
d **i** local max. at $(-2,\ 29)$ local min at $(4,\ -79)$ **v**
ii non-horizontal inflection at $(1,\ -25)$
iii increasing for $x \leqslant -2$, $x \geqslant 4$ decreasing for $-2 \leqslant x \leqslant 4$
iv concave down for $x \leqslant 1$, concave up for $x \geqslant 1$



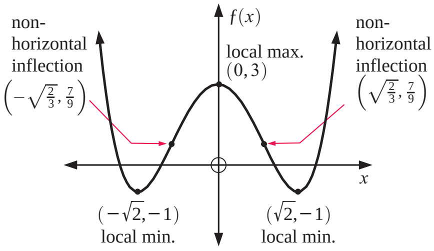
e **i** horiz. inflect. at $(0,\ -2)$ local min. at $(-1,\ -3)$ **v**
ii horiz. inflect. at $(0,\ -2)$ non-horizontal inflection at $(-\frac{2}{3},\ -\frac{70}{27})$
iii increasing for $x \geqslant -1$, decreasing for $x \leqslant -1$
iv concave down for $-\frac{2}{3} \leqslant x \leqslant 0$ concave up for $x \leqslant -\frac{2}{3}$, $x \geqslant 0$



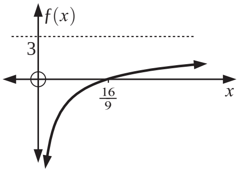
f **i** local min. at $(1,\ 0)$ **v**
ii no points of inflection
iii increasing for $x \geqslant 1$, decreasing for $x \leqslant 1$
iv concave up for all x



g **i** local minimum at $(-\sqrt{2},\ -1)$ and $(\sqrt{2},\ -1)$, local maximum at $(0,\ 3)$
ii non-horizontal inflection at $(\sqrt{\frac{2}{3}},\ \frac{7}{9})$
non-horizontal inflection at $(-\sqrt{\frac{2}{3}},\ \frac{7}{9})$
iii increasing for $-\sqrt{2} \leqslant x \leqslant 0$, $x \geqslant \sqrt{2}$
decreasing for $x \leqslant -\sqrt{2}$, $0 \leqslant x \leqslant \sqrt{2}$
iv concave down for $-\sqrt{\frac{2}{3}} \leqslant x \leqslant \sqrt{\frac{2}{3}}$
concave up for $x \leqslant -\sqrt{\frac{2}{3}}$, $x \geqslant \sqrt{\frac{2}{3}}$
v

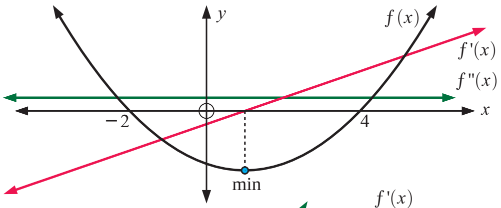


h **i** no stationary points **v**
ii no inflections
iii increasing for $x > 0$, never decreasing
iv concave down for $x > 0$, never concave up

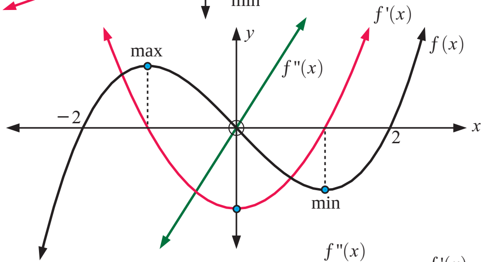


EXERCISE 18F.2

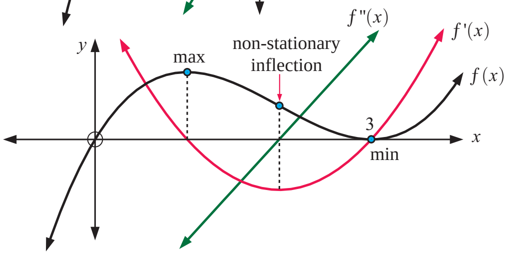
1 **a**



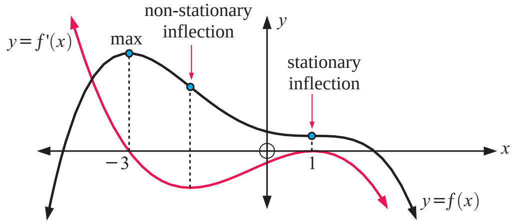
b



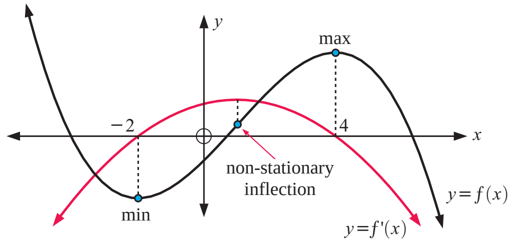
c



2 **a**



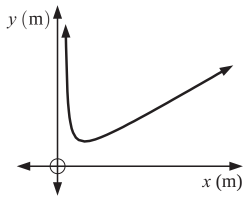
b



EXERCISE 18G

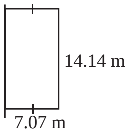
1 50 fittings **2** 250 items **3** 10 blankets **4** 25 km h^{-1}

5 **b**



c $L_{\min} \approx 28.28$ m, $x \approx 7.07$ m

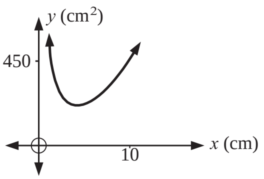
d



6 **a** $2x$ cm **b** $V = 200 = 2x \times x \times h$

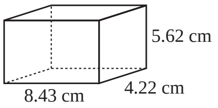
c **Hint**: Show $h = \dfrac{100}{x^2}$ and substitute into the surface area equation.

d



e $SA_{\min} \approx 213$ cm^2, $x \approx 4.22$ cm

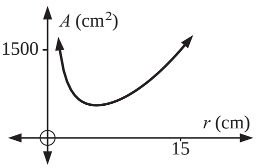
f



7 **a** recall that $V_{\text{cylinder}} = \pi r^2 h$ and that $1 \text{ L} = 1000 \text{ cm}^3$

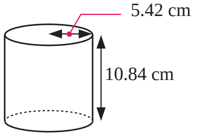
b recall that $SA_{\text{cylinder}} = 2\pi r^2 + 2\pi r h$

c



d $A \approx 554$ cm^2, $r \approx 5.42$ cm

e



8 **b** 6 cm × 6 cm

9 **a** $0 \leqslant x \leqslant 63.7$ **c** $x \approx 63.7$, $l = 0$ (circular)

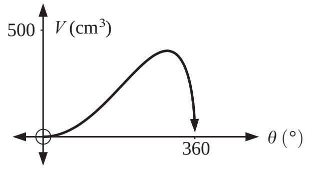
10 **a** **Hint**: Show that $\text{AC} = \frac{\theta}{360} \times 2\pi \times 10$

b **Hint**: Show that $2\pi r = \text{AC}$

c **Hint**: Use the result from **b** and Pythagoras' theorem.

d $V = \frac{1}{3}\pi \left(\frac{\theta}{36}\right)^2 \sqrt{100 - \left(\frac{\theta}{36}\right)^2}$

e



f $\theta \approx 294^o$

11 **a** For $x < 0$ or $x > 6$, X is not on [AC].

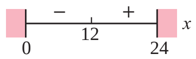
c $x \approx 2.67$ km This is the distance from A to X which minimises the time taken to get from B to C.

12 3.33 km **13** $r \approx 31.7$ cm, $h \approx 31.7$ cm

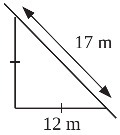
14 4 m from the 40 cp globe

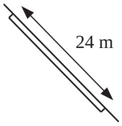
15 **a** $D(x) = \sqrt{x^2 + (24 - x)^2}$

b $\dfrac{d[D(x)]^2}{dx} = 4x - 48$



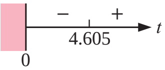
c Smallest $D(x) \approx 17.0$ Largest $D(x) = 24$, which is not an acceptable solution as can be seen in the diagram.





16 **a** **Hint**: Use the cosine rule.

b 3553 km^2



c 5:36 pm

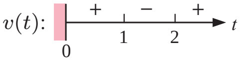
17 **a** $\text{QR} = \left(\dfrac{2+x}{x}\right)$ m

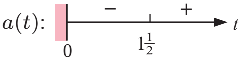
c **Hint**: All solutions < 0 can be discarded as $x \geqslant 0$.

d 416 cm

REVIEW SET 18A

1 **a** $v(t) = (6t^2 - 18t + 12)\ \text{cm s}^{-1}$ $a(t) = (12t-18)\ \text{cm s}^{-2}$



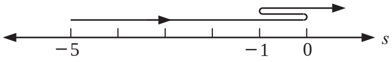


b $s(0) = 5$ cm to left of origin
$v(0) = 12\ \text{cm s}^{-1}$ towards origin
$a(0) = -18\ \text{cm s}^{-2}$ (reducing speed)

c At $t = 2$, particle is 1 cm to the left of the origin, is stationary and is accelerating towards the origin.

d $t = 1,\ s = 0$ and $t = 2,\ s = -1$

e



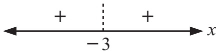
f Speed is increasing for $1 \leqslant t \leqslant 1\frac{1}{2}$ and $t \geqslant 2$.

2 **b** $k = 9$

3 6 cm from each end

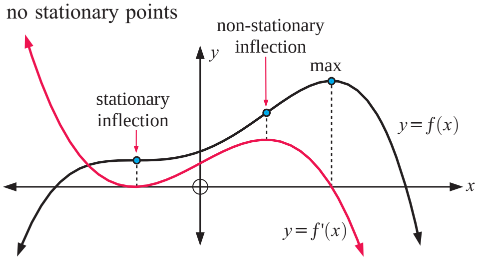
4 **a** $x = -3$ **b** x-int. $\frac{2}{3}$, y-int. $-\frac{2}{3}$

c $f'(x) = \dfrac{11}{(x+3)^2}$



d no stationary points

5

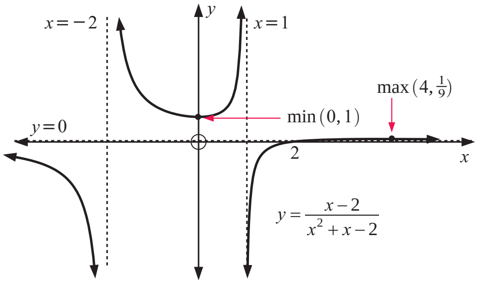


6 **a** V.A.s $x = -2,\ x = 1$, H.A. $y = 0$

b local min. $(0, 1)$, local max. $(4, \frac{1}{9})$

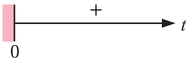
c x-intercept 2, y-intercept 1

d

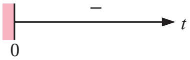


e $p < 0,\ 0 < p < \frac{1}{9},\ p > 1$

7 **a** $v(t) = 2 + \dfrac{4}{t^2}$



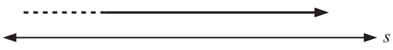
$a(t) = -\dfrac{8}{t^3}$



b The particle is 2 cm to the left of O, moving right at $6\ \text{cm s}^{-1}$, and slowing down.

c The particle never changes direction

d



e **i** never **ii** never

8 $x = \dfrac{k}{2}\left(1 - \frac{1}{\sqrt{3}}\right)$

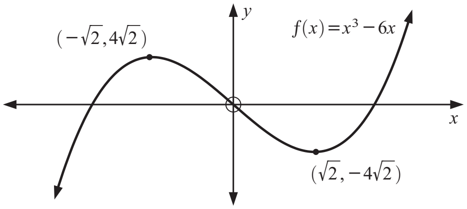
REVIEW SET 18B

1 **b** $\dfrac{d[A(x)]^2}{dx} = 5000x - 4x^3$
Area is a maximum when $x \approx 35.4$, $A = 1250\ \text{m}^2$.

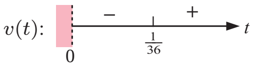
2 **a** $a = -6$

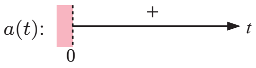
b local max. $(-\sqrt{2}, 4\sqrt{2})$, local min. $(\sqrt{2}, -4\sqrt{2})$

c



3 **a** $v(t) = 3 - \dfrac{1}{2\sqrt{t}}$ $a(t) = \dfrac{1}{4t\sqrt{t}}$





b $x(0) = 0$, $v(0)$ is undefined, $a(0)$ is undefined

c Particle is 24 cm to the right of the origin and is travelling to the right at $2.83\ \text{cm s}^{-1}$. Its speed is increasing.

d Changes direction at $t = \frac{1}{36}$, 0.083 cm to the left of the origin.

e Particle's speed is decreasing for $0 < t \leqslant \frac{1}{36}$.

4 **a** **i** \$535 **ii** \$1385.79

b **i** $-\$0.27$ per km h^{-1} **ii** \$2.33 per km h^{-1}

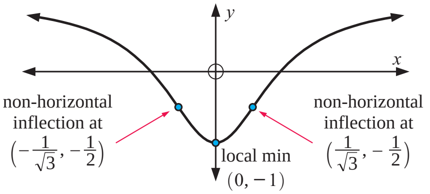
c 51.3 km h^{-1}

5 **a** y-int. at $y = -1$. x-int. at $x = 1$, $x = -1$.

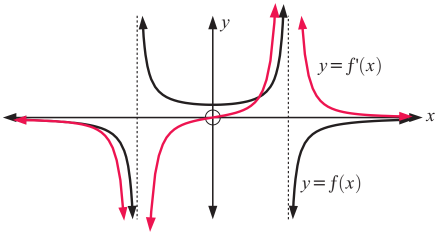
b $x^2 + 1 > 0$ for all real x (i.e., denominator is never 0)

c local minimum at $(0, -1)$

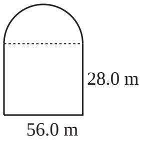
e



6



7 **b** $A = 200x - 2x^2 - \frac{1}{2}\pi x^2$ **c**



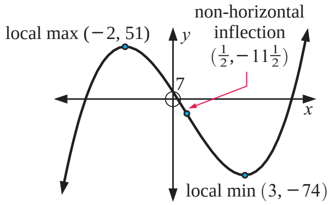
REVIEW SET 18C

1 **a** local maximum at $(-2, 51)$, local minimum at $(3, -74)$
non-horizontal inflection at $(\frac{1}{2}, -11.5)$

b increasing for $x \leqslant -2,\ x \geqslant 3$
decreasing for $-2 \leqslant x \leqslant 3$

c concave down for $x \leqslant \frac{1}{2}$, concave up for $x \geqslant \frac{1}{2}$

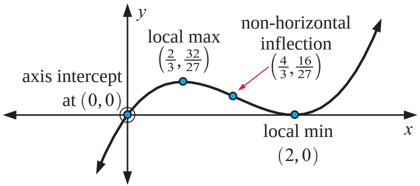
d



2 **a** y-intercept at $y = 0$, x-intercept at $x = 0$ and $x = 2$

b local maximum at $\left(\frac{2}{3}, \frac{32}{27}\right)$, local minimum at $(2, 0)$, non-horizontal inflection at $\left(\frac{4}{3}, \frac{16}{27}\right)$

c



3 **a** $y = \dfrac{1}{x^2},\ x > 0$ **c** base is 1.26 m square, height 0.630 m

4 **a** $v(t) = 15 + \dfrac{120}{(t-1)^3}$ cm s^{-1}, $a(t) = \dfrac{-360}{(t-1)^4}$ cm s^{-2}

b At $t = 3$, particle is 30 cm to the right of the origin, moving to the right at 30 cm s^{-1} and decelerating at 22.5 cm s^{-2}.

c $0 \leqslant t < 1$

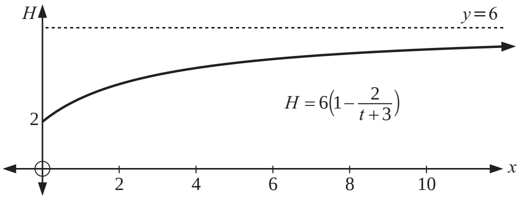
5 **a** 2 m **b** $H(3) = 4$ m, $H(6) = 4\frac{2}{3}$ m, $H(9) = 5$ m

c $H'(0) = \frac{4}{3}$ m year^{-1}, $H'(3) = \frac{1}{3}$ m year^{-1}
$H'(6) = \frac{4}{27}$ m year^{-1}, $H'(9) = \frac{1}{12}$ m year^{-1}

d $H'(t) = \dfrac{12}{(t+3)^2} > 0$ for all $t \geqslant 0$

$\therefore$ the height of the tree is always increasing.

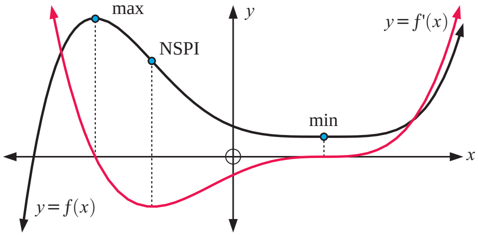
e



6 **a** **Hint**: Use Pythagoras to find h as a function of x and then substitute into the equation for the volume of a cylinder.

b radius ≈ 4.08 cm, height ≈ 5.77 cm

7



EXERCISE 19A

1 **a** $4e^{4x}$ **b** e^x **c** $-2e^{-2x}$ **d** $\frac{1}{2}e^{\frac{x}{2}}$

e $-e^{-\frac{x}{2}}$ **f** $2e^{-x}$ **g** $2e^{\frac{x}{2}} + 3e^{-x}$

h $\dfrac{e^x - e^{-x}}{2}$ **i** $-2xe^{-x^2}$ **j** $e^{\frac{1}{x}} \times \dfrac{-1}{x^2}$

k $20e^{2x}$ **l** $40e^{-2x}$ **m** $2e^{2x+1}$

n $\frac{1}{4}e^{\frac{x}{4}}$ **o** $-4xe^{1-2x^2}$ **p** $-0.02e^{-0.02x}$

2 **a** $e^x + xe^x$ **b** $3x^2e^{-x} - x^3e^{-x}$ **c** $\dfrac{xe^x - e^x}{x^2}$

d $\dfrac{1-x}{e^x}$ **e** $2xe^{3x} + 3x^2e^{3x}$ **f** $\dfrac{xe^x - \frac{1}{2}e^x}{x\sqrt{x}}$

g $\frac{1}{2}x^{-\frac{1}{2}}e^{-x} - x^{\frac{1}{2}}e^{-x}$ **h** $\dfrac{e^x + 2 + 2e^{-x}}{(e^{-x}+1)^2}$

3 **a** $4e^x(e^x+2)^3$ **b** $\dfrac{-e^{-x}}{(1-e^{-x})^2}$

c $\dfrac{e^{2x}}{\sqrt{e^{2x}+10}}$ **d** $\dfrac{6e^{3x}}{(1-e^{3x})^3}$

e $-\dfrac{e^{-x}}{2}\left(1-e^{-x}\right)^{-\frac{3}{2}}$ **f** $\dfrac{1 - 2e^{-x} + xe^{-x}}{\sqrt{1-2e^{-x}}}$

5 **Hint:** Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ and substitute into the equation.

6 $k = -9$

7 **a** local maximum at $(1, e^{-1})$

b local max. at $(-2, 4e^{-2})$, local min. at $(0, 0)$

c local minimum at $(1, e)$ **d** local maximum at $(-1, e)$

EXERCISE 19B

1 **a** 2 **b** $\frac{1}{2}$ **c** -1 **d** $-\frac{1}{2}$ **e** 3 **f** 9 **g** $\frac{1}{5}$ **h** $\frac{1}{4}$

2 **a** $e^{\ln 2}$ **b** $e^{\ln 10}$ **c** $e^{\ln a}$ **d** $e^{x \ln a}$

3 **a** $x = \ln 2$ **b** no real solutions **c** no real solutions

d $x = \ln 2$ **e** $x = 0$ **f** $x = \ln 2$ or $\ln 3$ **g** $x = 0$

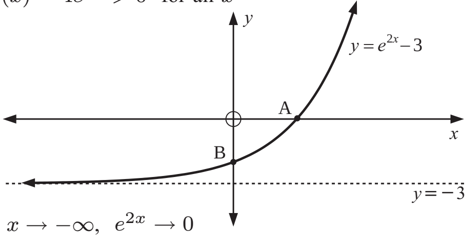
h $x = \ln 4$ **i** $x = \ln\left(\frac{3+\sqrt{5}}{2}\right)$ or $\ln\left(\frac{3-\sqrt{5}}{2}\right)$

4 **a** $(\ln 3, 3)$ **b** $(\ln 2, 5)$ **c** $(0, 2)$ and $(\ln 5, -2)$

5 **a** $\mathrm{A}\left(\frac{\ln 3}{2}, 0\right)$, $\mathrm{B}(0, -2)$ **b** $f'(x) = 2e^{2x} > 0$ for all x

c $f''(x) = 4e^{2x} > 0$ for all x

d



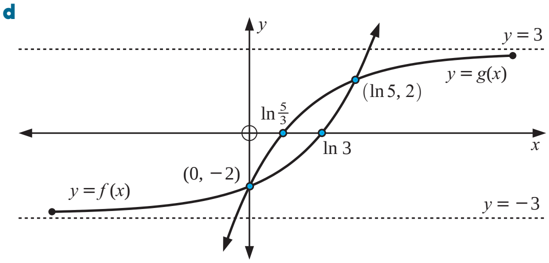
e as $x \to -\infty$, $e^{2x} \to 0$

6 **a** $f(x)$: x-int. at $x = \ln 3$, y-int. at $y = -2$
$g(x)$: x-int. at $x = \ln\left(\frac{5}{3}\right)$, y-int. at $y = -2$

b $f(x)$: as $x \to \infty$, $f(x) \to \infty$
as $x \to -\infty$, $f(x) \to -3$ (above)
$g(x)$: as $x \to \infty$, $g(x) \to 3$ (below)
as $x \to -\infty$, $g(x) \to -\infty$

c intersect at $(0, -2)$ and $(\ln 5, 2)$

d

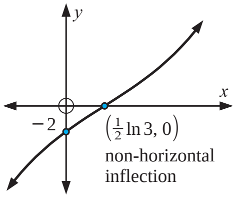


7 **a** $P(\frac{1}{2}\ln 3,\ 0)$, $Q(0,\ -2)$

b $\frac{dy}{dx} = e^x + 3e^{-x}$
> 0 for all x

c y is concave down below the x-axis and concave up above the x-axis

d



EXERCISE 19C

1 **a** $\frac{1}{x}$ **b** $\frac{2}{2x+1}$ **c** $\frac{1-2x}{x-x^2}$

d $-\frac{2}{x}$ **e** $2x\ln x + x$ **f** $\frac{1-\ln x}{2x^2}$

g $e^x\ln x + \frac{e^x}{x}$ **h** $\frac{2\ln x}{x}$ **i** $\frac{1}{2x\sqrt{\ln x}}$

j $\frac{e^{-x}}{x} - e^{-x}\ln x$ **k** $\frac{\ln(2x)}{2\sqrt{x}} + \frac{1}{\sqrt{x}}$ **l** $\frac{\ln x - 2}{\sqrt{x}(\ln x)^2}$

m $\frac{4}{1-x}$ **n** $\ln(x^2+1) + \frac{2x^2}{x^2+1}$

2 **a** $\ln 5$ **b** $\frac{3}{x}$ **c** $\frac{4x^3+1}{x^4+x}$ **d** $\frac{1}{x-2}$

e $\frac{6}{2x+1}[\ln(2x+1)]^2$ **f** $\frac{1-\ln(4x)}{x^2}$

g $-\frac{1}{x}$ **h** $\frac{1}{x\ln x}$ **i** $\frac{-1}{x(\ln x)^2}$

3 **a** $\frac{-1}{1-2x}$ **b** $\frac{-2}{2x+3}$ **c** $1 + \frac{1}{2x}$

d $\frac{1}{x} - \frac{1}{2(2-x)}$ **e** $\frac{1}{x+3} - \frac{1}{x-1}$ **f** $\frac{2}{x} + \frac{1}{3-x}$

g $\frac{9}{3x-4}$ **h** $\frac{1}{x} + \frac{2x}{x^2+1}$ **i** $\frac{2x+2}{x^2+2x} - \frac{1}{x-5}$

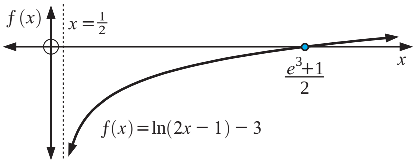
4 **a** $\frac{dy}{dx} = 2^x\ln 2$

5 **a** $x = \frac{e^3+1}{2} \approx 10.5$ **b** no, $\therefore$ there is no y-int.

c gradient $= 2$ **d** $x > \frac{1}{2}$

e $f''(x) = \frac{-4}{(2x-1)^2} < 0$ for all $x > \frac{1}{2}$, so $f(x)$ is concave down

f



6 **a** $x > 0$

7 **Hint:** Show that as $x \to 0$, $f(x) \to -\infty$,
and as $x \to \infty$, $f(x) \to 0$.

8 **Hint:** Show that $f(x) \geqslant 1$ for all $x > 0$.

EXERCISE 19D

1 $y = -\frac{1}{e}x + \frac{2}{e}$ **2** $3y = -x + 3\ln 3 - 1$

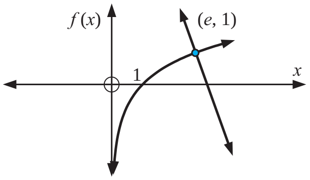
3 A is $\left(\frac{2}{3},\ 0\right)$, B is $(0,\ -2e)$ **4** $y = -\frac{2}{e^2}x + \frac{2}{e^4} - 1$

5 $y = e^a x + e^a(1-a)$ so $y = ex$ is the tangent to $y = e^x$ from the origin

6 **a** $x > 0$

b $f'(x) > 0$ for all $x > 0$, so $f(x)$ is always increasing. Its gradient is always positive. $f''(x) < 0$ for all $x > 0$, so $f(x)$ is concave down for all $x > 0$.

c



normal has equation
$f(x) = -ex + 1 + e^2$

7 $\approx 63.43^o$

8 **a** $k = \frac{1}{50}\ln 2 \approx 0.0139$

b **i** 20 grams **ii** 14.3 grams **iii** 1.95 grams

c 9 days and 6 minutes (216 hours)

d **i** -0.0693 g h^{-1} **ii** -2.64×10^{-7} g h^{-1}

e **Hint:** You should find $\frac{dW}{dt} = -\frac{1}{50}\ln 2 \times 20e^{-\frac{1}{50}\ln 2t}$

9 **a** $k = \frac{1}{15}\ln\left(\frac{19}{3}\right) \approx 0.123$ **b** 100^oC

d **i** decreasing by 11.7^oC min^{-1}

ii decreasing by 3.42^oC min^{-1}

iii decreasing by 0.998^oC min^{-1}

10 **a** 43.9 cm **b** 10.4 years

c **i** growing by 5.45 cm per year

ii growing by 1.88 cm per year

11 **a** $A = 0$ **b** $k = \frac{\ln 2}{3}$ (≈ 0.231)

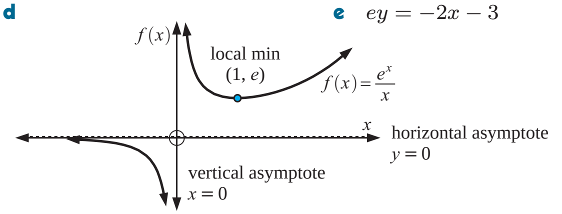
c 0.728 litres of alcohol produced per hour

12 **a** $f(x)$ does not have any x or y-intercepts

b as $x \to \infty$, $f(x) \to \infty$
as $x \to -\infty$, $f(x) \to 0$ (below)

c local minimum at $(1,\ e)$

d



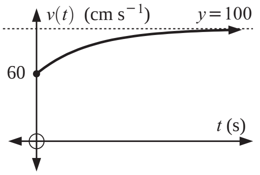
e $ey = -2x - 3$

13 **a** $v(t) = 100 - 40e^{-\frac{t}{5}}$ cm s^{-1}, $a(t) = 8e^{-\frac{t}{5}}$ cm s^{-2}

b $s(0) = 200$ cm on positive side of origin
$v(0) = 60$ cm s^{-1}, $a(0) = 8$ cm s^{-2}

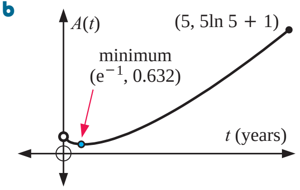
c as $t \to \infty$, $v(t) \to 100$ cm s^{-1} (below)

d



e after 3.47 s

14 **a** at 4.41 months old **b**



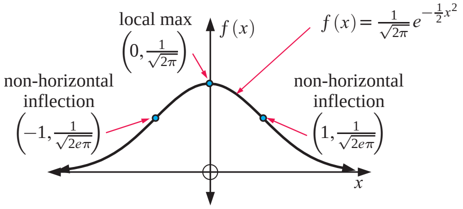
15 **a** There is a local maximum at $\left(0, \frac{1}{\sqrt{2\pi}}\right)$.

$f(x)$ is incr. for all $x \leqslant 0$ and decr. for all $x \geqslant 0$.

b Inflections at $\left(-1, \frac{1}{\sqrt{2e\pi}}\right)$ and $\left(1, \frac{1}{\sqrt{2e\pi}}\right)$

c as $x \to \infty$, $f(x) \to 0$ (positive)
as $x \to -\infty$, $f(x) \to 0$ (positive)

d



16 20 kettles **17** C $\frac{1}{\sqrt{2}}, e^{(-\frac{1}{2})}$ **18** 266 or 267 torches

19 **a** **Hint:** They must have the same y-coordinate at $x = b$ and the same gradient.

c $a = \frac{1}{2e}$ **d** $y = e^{-\frac{1}{2}}x - \frac{1}{2}$

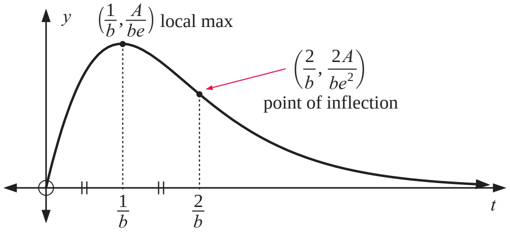
20 after 13.8 weeks

21 **a** t-intercept 0, y-intercept 0

b local maximum at $\left(\frac{1}{b}, \frac{A}{be}\right)$

c non-stationary inflection at $\left(\frac{2}{b}, \frac{2A}{be^2}\right)$

d



e After 40 minutes

REVIEW SET 19A

1 **a** $3x^2e^{x^3+2}$ **b** $\frac{1}{x+3} - \frac{2}{x}$

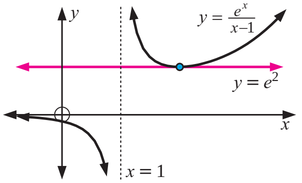
2 $y = \frac{e}{2}x + \frac{1}{e} - \frac{e}{2}$ **3** **a** P$(\ln 2, 4)$

4 **a** y-intercept at $y = -1$, no x-intercept

b $f(x)$ is defined for all $x \neq 1$

c $f'(x) \leqslant 0$ for $x < 1$ and $1 < x \leqslant 2$ and $f'(x) > 0$ for $x \geqslant 2$.
$f''(x) > 0$ for $x > 1$, $f''(x) < 0$ for $x < 1$.
The function is decreasing for all defined values of $x \leqslant 2$, and increasing for all $x \geqslant 2$. The curve is concave down for $x < 1$ and concave up for $x > 1$.

d



e tangent is $y = e^2$

5 Tangent is $y = \ln 3$, so it never cuts the x-axis.

6 $p = 1$, $q = -8$ **7** **a** $x = \ln\left(\frac{2}{3}\right)$ or 0 **b** $x = e^2$

REVIEW SET 19B

1 **a** 60 cm **b** **i** 4.24 years **ii** 201 years

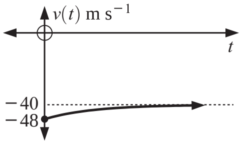
c **i** 16 cm per year **ii** 1.95 cm per year

2 **a** $v(t) = -8e^{-\frac{t}{10}} - 40$ m s^{-1}
$a(t) = \frac{4}{5}e^{-\frac{t}{10}}$ m s^{-2} $\{t \geqslant 0\}$

b $s(0) = 80$ m,
$v(0) = -48$ m s^{-1},
$a(0) = 0.8$ m s^{-2}

c as $t \to \infty$,
$v(t) \to -40$ m s^{-1} (below)

d



e $t \approx 6.93$ seconds

3 100 or 101 shirts, $938.63 profit

4 **a** $20 000 **b** $146.53 per year

5 197 or 198 clocks per day

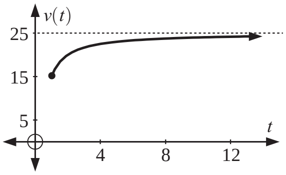
6 **a** $v(t) = 25 - \frac{10}{t}$ cm min^{-1}, $a(t) = \frac{10}{t^2}$ cm min^{-2}

b $s(e) = 25e - 10$ cm, $v(e) = 25 - \frac{}{e}$ cm min^{-1},

$a(e) = \frac{10}{2}$ cm min^{-2}

c As $t \to \infty$, $v(t) \to 25$ cm min^{-1} from below

d



e $t = 2$ min

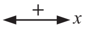
REVIEW SET 19C

1 **a** $\frac{3x^2 - 3}{x^3 - 3x}$ **b** $\frac{e^x(x-2)}{x^3}$

2 $(0, \ln 4 - 1)$ **3** **a** $x = \ln 3$ **b** $x = \ln 4$ or $\ln 3$

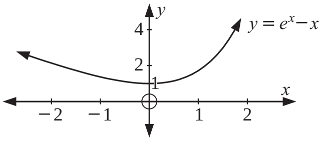
4 **a** local minimum at $(0, 1)$ **b** As $x \to \infty$, $f(x) \to \infty$

c $f''(x) = e^x$,



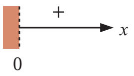
thus $f(x)$ is concave up for all x.

d

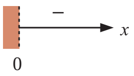


5 **a** $f'(x) = \frac{e^x}{e^x + 3}$ **b** $f'(x) = \frac{2(x-1)}{x(x+2)}$

6 **a** $x > 0$ **b** Sign diag of $f'(x)$

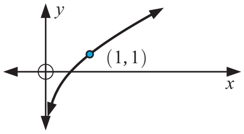


Sign diag of $f''(x)$



$f(x)$ is increasing for all $x > 0$ and is concave downwards for all $x > 0$.

c



d normal is $x + 2y = 3$

7 $A\left(\frac{1}{2}, \frac{1}{e}\right)$

EXERCISE 20A

1 **a** $2\cos(2x)$ **b** $\cos x - \sin x$
c $-3\sin(3x) - \cos x$ **d** $\cos(x+1)$ **e** $2\sin(3-2x)$
f $\dfrac{5}{\cos^2(5x)}$ **g** $\frac{1}{2}\cos(\frac{x}{2}) + 3\sin x$
h $\dfrac{3\pi}{\cos^2(\pi x)}$ **i** $4\cos x + 2\sin(2x)$

2 **a** $2x - \sin x$ **b** $\dfrac{1}{\cos^2 x} - 3\cos x$
c $e^x \cos x - e^x \sin x$ **d** $-e^{-x}\sin x + e^{-x}\cos x$
e $\dfrac{\cos x}{\sin x}$ **f** $2e^{2x}\tan x + \dfrac{e^{2x}}{\cos^2 x}$
g $3\cos(3x)$ **h** $-\frac{1}{2}\sin\left(\frac{x}{2}\right)$ **i** $\dfrac{6}{\cos^2(2x)}$
j $\cos x - x\sin x$ **k** $\dfrac{x\cos x - \sin x}{x^2}$ **l** $\tan x + \dfrac{x}{\cos^2 x}$

3 **a** $2x\cos(x^2)$ **b** $-\dfrac{1}{2\sqrt{x}}\sin(\sqrt{x})$ **c** $-\dfrac{\sin x}{2\sqrt{\cos x}}$
d $2\sin x\cos x$ **e** $-3\sin x\cos^2 x$
f $-\sin x\sin(2x) + 2\cos x\cos(2x)$ **g** $\sin x\sin(\cos x)$
h $-12\sin(4x)\cos^2(4x)$ **i** $-\dfrac{\cos x}{\sin^2 x}$
j $\dfrac{2\sin(2x)}{\cos^2(2x)}$ **k** $-\dfrac{8\cos(2x)}{\sin^3(2x)}$ **l** $\dfrac{-12}{\cos^2(\frac{x}{2})\tan^4(\frac{x}{2})}$

4 **b** $f''(x) = 3\sin x\cos 2x + 6\cos x\sin 2x$

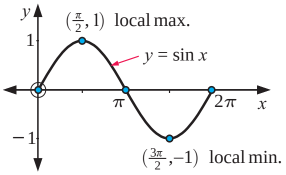
6 **a** $y = x$ **b** $y = x$ **c** $2x - y = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$ **d** $x = \frac{\pi}{4}$

7 **a** rising **b** rising at 2.73 m per hour

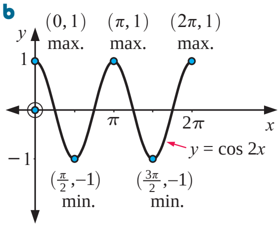
8 **a** $-34\,000\pi$ units per second **b** $V'(t) = 0$

9 **b** **i** 0 **ii** 1 **iii** ≈ 1.11

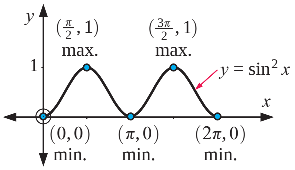
10 **a**



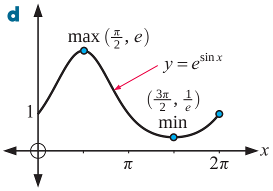
b



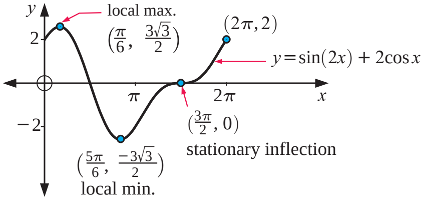
c



d



e



11 **a** $x(0) = -1$ cm $v(0) = 0$ cm s^{-1} $a(0) = 2$ cm s^{-2}
b At $t = \frac{\pi}{4}$ seconds, the particle is $(\sqrt{2} - 1)$ cm left of the origin, moving right at $\sqrt{2}$ cm s^{-1}, with increasing speed.
c changes direction when $t = \pi$, $x(\pi) = 3$ cm
d increasing for $0 \leqslant t \leqslant \frac{\pi}{2}$ and $\pi \leqslant t \leqslant \frac{3\pi}{2}$

EXERCISE 20B

1 $\approx 109.5^o$ **2** **c** $\theta = 30^o$
3 1 hour 34 min 53 s when $\theta \approx 36.9^o$ **4** **c** $4\sqrt{2}$ m
5 9.87 m

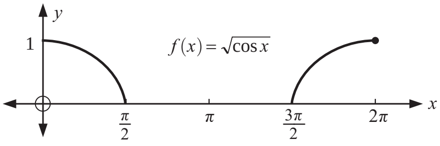
REVIEW SET 20

1 **a** $5\cos(5x)\ln(x) + \dfrac{\sin(5x)}{x}$
b $\cos x\cos(2x) - 2\sin x\sin(2x)$
c $-2e^{-2x}\tan x + \dfrac{e^{-2x}}{\cos^2 x}$ **d** $10 - 10\cos(10x)$
e $\tan x$ **f** $5\cos(5x)\ln(2x) + \dfrac{\sin(5x)}{x}$

3 **a** $f'(x) = 3\cos x + 8\sin(2x)$,
$f''(x) = -3\sin x + 16\cos(2x)$
b $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}\cos(4x) - 4x^{\frac{1}{2}}\sin(4x)$,
$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}\cos(4x) - 4x^{-\frac{1}{2}}\sin(4x)$
$-16x^{\frac{1}{2}}\cos(4x)$

4 **a** $x(0) = 3$ cm, $x'(0) = 2$ cm s^{-1}, $x''(0) = 0$ cm s^{-2}
b $t = \frac{\pi}{4}$ s and $\frac{3\pi}{4}$ s **c** 4 cm

5 **a** for $0 \leqslant x \leqslant \frac{\pi}{2}$ and $\frac{3\pi}{2} \leqslant x \leqslant 2\pi$
b increasing for $\frac{3\pi}{2} \leqslant x \leqslant 2\pi$, decreasing for $0 \leqslant x \leqslant \frac{\pi}{2}$
c



6 **a** $v(0) = 0$ cm s^{-1}, $v(\frac{1}{2}) = -\pi$ cm s^{-1}, $v(1) = 0$ cm s^{-1}, $v\left(\frac{3}{2}\right) = \pi$ cm s^{-1} $v(2) = 0$ cm s^{-1}
b $0 \leqslant t \leqslant 1$, $2 \leqslant t \leqslant 3$, $4 \leqslant t \leqslant 5$, etc.
So, for $2n \leqslant t \leqslant 2n+1$, $n \in \{0, 1, 2, 3, ...\}$

7 **b** $\frac{1}{\sqrt{2}}$ m above the floor

8 **a** $2x + 3y = \frac{2\pi}{3} + 2\sqrt{3}$ **b** $\sqrt{2}y - 4x = 1 - 2\pi$

9 **a** $f(x) = -5\sin 4x$ **b** $x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$

EXERCISE 21A

1 **a** **i** $\dfrac{x^2}{2}$ **ii** $\dfrac{x^3}{3}$ **iii** $\dfrac{x^6}{6}$ **iv** $-\dfrac{1}{x}$ **v** $-\dfrac{1}{3x^3}$
vi $\frac{3}{4}x^{\frac{4}{3}}$ **vii** $2\sqrt{x}$
b the antiderivative of x^n is $\dfrac{x^{n+1}}{n+1}$.

2 **a** **i** $\frac{1}{2}e^{2x}$ **ii** $\frac{1}{5}e^{5x}$ **iii** $2e^{\frac{1}{2}x}$ **iv** $100e^{0.01x}$
v $\frac{1}{\pi}e^{\pi x}$ **vi** $3e^{\frac{x}{3}}$
b the antiderivative of e^{kx} is $\dfrac{1}{k}e^{kx}$

3 **a** $\dfrac{d}{dx}(x^3+x^2)=3x^2+2x$

$\therefore$ antiderivative of $6x^2+4x=2x^3+2x^2$

b $\dfrac{d}{dx}(e^{3x+1})=3e^{3x+1}$

$\therefore$ antiderivative of $e^{3x+1}=\frac{1}{3}e^{3x+1}$

c $\dfrac{d}{dx}(x\sqrt{x})=\frac{3}{2}\sqrt{x}$

$\therefore$ antiderivative of $\sqrt{x}=\frac{2}{3}x\sqrt{x}$

d $\dfrac{d}{dx}(2x+1)^4=8(2x+1)^3$

$\therefore$ antiderivative of $(2x+1)^3=\frac{1}{8}(2x+1)^4$

EXERCISE 21B

2 **a** $\frac{1}{4}$ units^2 **b** $3\frac{3}{4}$ units^2 **c** $24\frac{2}{3}$ units^2 **d** $\frac{4\sqrt{2}}{3}$ units^2

e ≈ 3.48 units^2 **f** 2 units^2 **g** ≈ 3.96 units^2

3 **a** 4.06 units^2 **b** 2.41 units^2 **c** 2.58 units^2

4 **c** **i** $\int_0^1(-x^2)dx=-\frac{1}{3}$, the area between $y=-x^2$ and the x-axis from $x=0$ to $x=1$ is $\frac{1}{3}$ units^2.

ii $\int_0^1(x^2-x)dx=-\frac{1}{6}$, the area between $y=x^2-x$ and the x-axis from $x=0$ to $x=1$ is $\frac{1}{6}$ units^2.

iii $\int_{-2}^0 3x dx=-6$, the area between $y=3x$ and the x-axis from $x=-2$ to $x=0$ is 6 units^2

d $-\pi$

EXERCISE 21C.1

1 $\dfrac{dy}{dx}=7x^6;\quad \int x^6\,dx=\frac{1}{7}x^7+c$

2 $\dfrac{dy}{dx}=3x^2+2x;\quad \int(3x^2+2x)\,dx=x^3+x^2+c$

3 $\dfrac{dy}{dx}=2e^{2x+1};\quad \int e^{2x+1}\,dx=\frac{1}{2}e^{2x+1}+c$

4 $\dfrac{dy}{dx}=8(2x+1)^3;\quad \int(2x+1)^3\,dx=\frac{1}{8}(2x+1)^4+c$

5 $\dfrac{dy}{dx}=\frac{3}{2}\sqrt{x};\quad \int\sqrt{x}\,dx=\frac{2}{3}x\sqrt{x}+c$

6 $\dfrac{dy}{dx}=-\dfrac{1}{2x\sqrt{x}};\quad \displaystyle\int\frac{1}{x\sqrt{x}}\,dx=-\frac{2}{\sqrt{x}}+c$

7 $\dfrac{dy}{dx}=-2\sin 2x;\quad \int\sin 2x dx=-\frac{1}{2}\cos 2x+c$

8 $\dfrac{dy}{dx}=-5\cos(1-5x);\quad \int\cos(1-5x)dx=-\frac{1}{5}\sin(1-5x)+c$

10 $\dfrac{dy}{dx}=\dfrac{-2}{\sqrt{1-4x}};\quad \displaystyle\int\frac{1}{\sqrt{1-4x}}\,dx=-\frac{1}{2}\sqrt{1-4x}+c$

11 $2\ln(5-3x+x^2)+c\quad$ {since $5-3x+x^2$ is >0}

EXERCISE 21C.2

1 **a** $\dfrac{x^5}{5}-\dfrac{x^3}{3}-\dfrac{x^2}{2}+2x+c$ **b** $\frac{2}{3}x^{\frac{3}{2}}+e^x+c$

c $3e^x-\ln x+c,\ x>0$ **d** $\frac{2}{5}x^{\frac{5}{2}}-2\ln x+c,\ x>0$

e $-2x^{-\frac{1}{2}}+4\ln x+c,\ x>0$

f $\frac{1}{8}x^4-\frac{1}{5}x^5+\frac{3}{4}x^{\frac{4}{3}}+c$ **g** $\frac{1}{3}x^3+3\ln x+c,\ x>0$

h $\frac{1}{2}\ln x+\frac{1}{3}x^3-e^x+c,\ x>0$

i $5e^x+\frac{1}{12}x^4-4\ln x+c,\ x>0$

2 **a** $-3\cos x-2x+c$ **b** $2x^2-2\sin x+c$

c $-\cos x-2\sin x+e^x+c$ **d** $\frac{2}{7}x^3\sqrt{x}+10\cos x+c$

e $\frac{1}{9}x^3-\frac{1}{6}x^2+\sin x+c$ **f** $\cos x+\frac{4}{3}x\sqrt{x}+c$

3 **a** $\frac{1}{3}x^3+\frac{3}{2}x^2-2x+c$ **b** $\frac{2}{3}x^{\frac{3}{2}}-2x^{\frac{1}{2}}+c$

c $2e^x+\dfrac{1}{x}+c$ **d** $-2x^{-\frac{1}{2}}-8x^{\frac{1}{2}}+c$

e $\frac{4}{3}x^3+2x^2+x+c$ **f** $\frac{1}{2}x^2+x-3\ln x+c,\ x>0$

g $\frac{4}{3}x^{\frac{3}{2}}-2x^{\frac{1}{2}}+c$ **h** $2x^{\frac{1}{2}}+8x^{-\frac{1}{2}}-\frac{20}{3}x^{-\frac{3}{2}}+c$

i $\frac{1}{4}x^4+x^3+\frac{3}{2}x^2+x+c$

4 **a** $\frac{2}{3}x^{\frac{3}{2}}+\frac{1}{2}\sin x+c$ **b** $2e^t+4\cos t+c$

c $3\sin t-\ln t+c,\ t>0$

5 **a** $y=6x+c$ **b** $y=\frac{4}{3}x^3+c$

c $y=\frac{10}{3}x\sqrt{x}-\frac{1}{3}x^3+c$ **d** $y=-\dfrac{1}{x}+c$

e $y=2e^x-5x+c$ **f** $y=x^4+x^3+c$

6 **a** $y=x-2x^2+\frac{4}{3}x^3+c$ **b** $y=\frac{2}{3}x^{\frac{3}{2}}-4\sqrt{x}+c$

c $y=x+2\ln x+\dfrac{5}{x}+c,\ x>0$

7 **a** $f(x)=\frac{1}{4}x^4-\frac{10}{3}x\sqrt{x}+3x+c$

b $f(x)=\frac{4}{3}x^{\frac{3}{2}}-\frac{12}{5}x^{\frac{5}{2}}+c$

c $f(x)=3e^x-4\ln x+c,\ x>0$

8 **a** $f(x)=x^2-x+3$ **b** $f(x)=x^3+x^2-7$

c $f(x)=e^x+2\sqrt{x}-1-e$ **d** $f(x)=\frac{1}{2}x^2-4\sqrt{x}+\frac{11}{2}$

9 **a** $f(x)=\dfrac{x^3}{3}-4\sin x+3$

b $f(x)=2\sin x+3\cos x-2\sqrt{2}$

10 **a** $f(x)=\frac{1}{3}x^3+\frac{1}{2}x^2+x+\frac{1}{3}$

b $f(x)=4x^{\frac{5}{2}}+4x^{\frac{3}{2}}-4x+5$

c $f(x)=-\cos x-x+4$ **d** $f(x)=\frac{1}{3}x^3-\frac{16}{3}x+5$

EXERCISE 21D

1 **a** $\frac{1}{8}(2x+5)^4+c$ **b** $\dfrac{1}{2(3-2x)}+c$ **c** $\dfrac{-2}{3(2x-1)^3}+c$

d $\frac{1}{32}(4x-3)^8+c$ **e** $\frac{2}{9}(3x-4)^{\frac{3}{2}}+c$ **f** $-4\sqrt{1-5x}+c$

g $-\frac{3}{5}(1-x)^5+c$ **h** $-2\sqrt{3-4x}+c$

2 **a** $-\frac{1}{3}\cos(3x)+c$ **b** $-\frac{1}{2}\sin(-4x)+x+c$

c $6\sin\left(\frac{x}{2}\right)+c$ **d** $-\frac{3}{2}\cos(2x)+e^{-x}+c$

e $-\cos\left(2x+\frac{\pi}{6}\right)+c$ **f** $3\sin\left(\frac{\pi}{4}-x\right)+c$

g $\frac{1}{2}\sin(2x)-\frac{1}{2}\cos(2x)+c$

h $-\frac{2}{3}\cos(3x)+\frac{5}{4}\sin(4x)+c$

i $\frac{1}{16}\sin(8x)+3\cos x+c$

3 **a** $y=\frac{1}{3}(2x-7)^{\frac{3}{2}}+2$ **b** $(-8,-19)$

4 **a** $\frac{1}{2}x+\frac{1}{4}\sin(2x)+c$ **b** $\frac{1}{2}x-\frac{1}{4}\sin(2x)+c$

c $\frac{3}{2}x+\frac{1}{8}\sin(4x)+c$ **d** $\frac{5}{2}x+\frac{1}{12}\sin(6x)+c$

e $\frac{1}{4}x+\frac{1}{32}\sin(8x)+c$ **f** $\frac{3}{2}x+2\sin x+\frac{1}{4}\sin(2x)+c$

5 **a** $\frac{1}{2}(2x-1)^3+c$ **b** $\frac{1}{5}x^5-\frac{1}{2}x^4+\frac{1}{3}x^3+c$

c $-\frac{1}{12}(1-3x)^4+c$ **d** $x-\frac{2}{3}x^3+\frac{1}{5}x^5+c$

e $-\frac{8}{3}(5-x)^{\frac{3}{2}}+c$ **f** $\frac{1}{7}x^7+\frac{3}{5}x^5+x^3+x+c$

6 **a** $2e^x+\frac{5}{2}e^{2x}+c$ **b** $\frac{3}{5}e^{5x-2}+c$
c $-\frac{1}{3}e^{7-3x}+c$ **d** $\frac{1}{2}\ln(2x-1)+c,\ x>\frac{1}{2}$
e $-\frac{5}{3}\ln(1-3x)+c,\ x<\frac{1}{3}$
f $-e^{-x}-2\ln(2x+1)+c,\ x>-\frac{1}{2}$
g $\frac{1}{2}e^{2x}+2x-\frac{1}{2}e^{-2x}+c$ **h** $-\frac{1}{2}e^{-2x}-4e^{-x}+4x+c$
i $\frac{1}{2}x^2+5\ln(1-x)+c,\ x<1$

7 **a** $y=x-2e^x+\frac{1}{2}e^{2x}+c$
b $y=x-x^2+3\ln(x+2)+c,\ x>-2$
c $y=-\frac{1}{2}e^{-2x}+2\ln(2x-1)+c,\ x>\frac{1}{2}$

8 Both are correct. Recall that:

$$\frac{d}{dx}(\ln(\mathrm{A}x))=\frac{d}{dx}(\ln \mathrm{A}+\ln x)=\frac{1}{x},\quad \mathrm{A},\ x>0$$

9 $p=-\frac{1}{4},\ f(x)=\frac{1}{2}\cos(\frac{1}{2}x)+\frac{1}{2}$

11 **a** $f(x)=-e^{-2x}+4$
b $f(x)=x^2+2\ln(1-x)+2-2\ln 2,\ x<1$
c $f(x)=\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{8}e^{-4x}+\frac{1}{8}e^{-4}-\frac{2}{3}$

12 $x-\frac{1}{2}\cos(2x)+c$ **13** $\frac{1}{4}\sin 2x+2\sin x+\frac{3}{2}x+c$

EXERCISE 21E.1

1 **a** $\frac{1}{4}$ **b** $\frac{2}{3}$ **c** $e-1$ (≈ 1.72) **d** $\frac{1}{2}$
e $1\frac{1}{2}$ **f** $6\frac{2}{3}$ **g** $\ln 3$ (≈ 1.10) **h** $\frac{1}{2}$
i ≈ 1.52 **j** 2 **k** $e-1$ (≈ 1.72) **l** $\frac{1}{3}$
m $\frac{\pi}{8}+\frac{1}{4}$ **n** $\frac{\pi}{4}$

2 **a** ≈ 1.30 **b** ≈ 1.49 **c** ≈ -0.189

EXERCISE 21E.2

1 **a** $\int_1^4 \sqrt{x}\,dx\approx 4.67,\ \int_1^4(-\sqrt{x})\,dx\approx -4.67$
b $\int_0^1 x^7\,dx=\frac{1}{8},\ \int_0^1(-x^7)dx=-\frac{1}{8}$

2 **a** $\frac{1}{3}$ **b** $\frac{7}{3}$ **c** $\frac{8}{3}$ **d** 1 **3** **a** -4 **b** 6.25 **c** 2.25
4 **a** $\frac{1}{3}$ **b** $\frac{2}{3}$ **c** 1 **5** **a** 6.5 **b** -9 **c** 0 **d** -2.5
6 **a** 2π **b** -4 **c** $\frac{\pi}{2}$ **d** $\frac{5\pi}{2}-4$
7 **a** $\int_2^7 f(x)\,dx$ **b** $\int_1^9 g(x)\,dx$ **8** **a** -5 **b** 4
9 **a** 4 **b** 0 **c** -8 **d** $k=-\frac{7}{4}$ **10** 0

REVIEW SET 21A

1 **a** $8\sqrt{x}+c$ **b** $-\frac{3}{2}\ln(1-2x)+c,\ x<\frac{1}{2}$
c $-\frac{1}{4}\cos(4x-5)+c$ **d** $-\frac{1}{3}e^{4-3x}+c$

2 **a** $12\frac{4}{9}$ **b** $\sqrt{2}$

3 $\frac{dy}{dx}=\frac{x}{\sqrt{x^2-4}};\ \int$ is $\sqrt{x^2-4}+c$

4 $b=\frac{\pi}{4},\ \frac{3\pi}{4}$

5 **a** $2x-2\sin x+c$ **b** $\frac{9x}{2}-4\sin x+\frac{1}{4}\sin(2x)+c$

6 $\frac{d}{dx}(3x^2+x)^3=3(3x^2+x)^2(6x+1)$
$\int(3x^2+x)^2(6x+1)\,dx=\frac{1}{3}(3x^2+x)^3+c$

7 **a** 6 **b** 3 **8** $a=\ln\sqrt{2}$ **9** $f\left(\frac{\pi}{2}\right)=3-\frac{\pi}{2}$
10 $\frac{\pi}{12}-\frac{1}{4}$

REVIEW SET 21B

1 **a** $y=\frac{1}{5}x^5-\frac{2}{3}x^3+x+c$ **b** $y=400x+40e^{-\frac{x}{2}}+c$

2 **a** 3.528 **b** 2.963 **3** $\frac{2(\ln x)}{x},\ \frac{1}{2}(\ln x)^2+c$
4 $f(x)=3x^3+5x^2+6x-1$ **5** **a** $1.236\,17$ **b** $1.952\,49$
6 **a** $f(x)=\frac{1}{4}x^4+\frac{1}{3}x^3-\frac{10}{3}x+3$ **b** $3x+26y=84$
7 **a** $e^{3x}+6e^{2x}+12e^x+8$ **b** $\frac{1}{3}e^3+3e^2+12e-7\frac{1}{3}$

REVIEW SET 21C

1 **a** $-2e^{-x}-\ln x+3x+c,\ x>0$
b $\frac{1}{2}x^2-2x+\ln x+c,\ x>0$
c $9x+3e^{2x-1}+\frac{1}{4}e^{4x-2}+c$

2 $f(x)=\frac{1}{3}x^3-\frac{3}{2}x^2+2x+2\frac{1}{6}$ **3** $\frac{2}{3}(\sqrt{5}-\sqrt{2})$
4 $\frac{\pi}{6}+\frac{\sqrt{3}}{4}$ **5** $e^{-\pi}$

6 if $n\neq-1,\ \frac{1}{2(n+1)}(2x+3)^{n+1}+c$
if $n=-1,\ \frac{1}{2}\ln(2x+3)+c,\ x>-\frac{3}{2}$

7 $a=\frac{1}{3},\ f'(x)=2\sqrt{x}+\frac{1}{3\sqrt{x}}$ is never 0 as $\sqrt{x}\geqslant 0$ for all x
$\therefore\ f'(x)>0$ for all x

8 $a=0$ or ± 3

EXERCISE 22A.1

1 **a** 30 units^2 **b** $\frac{9}{2}$ units^2 **c** $\frac{27}{2}$ units^2 **d** 2 units^2

2 **a** $\frac{1}{3}$ units^2 **b** 2 units^2 **c** $63\frac{3}{4}$ units^2 **d** $e-1$ units^2
e $20\frac{5}{6}$ units^2 **f** 18 units^2 **g** $\ln 4$ units^2 **h** $\ln 3$ units^2
i $4\frac{1}{2}$ units^2 **j** $2e-\frac{2}{e}$ units^2

3 $\frac{2}{3}$ units^2

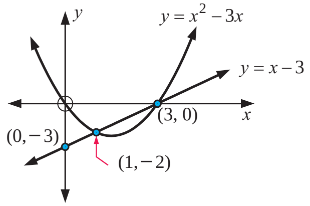
4 **a** 2.55 units^2 **b** 0.699 units^2 **c** 1.06 units^2

EXERCISE 22A.2

1 **a** $4\frac{1}{2}$ units^2 **b** $1+e^{-2}$ units^2 **c** $1\frac{5}{27}$ units^2 **d** 2 units^2
e $2\frac{1}{4}$ units^2 **f** $\frac{\pi}{2}-1$ units^2 **g** $\frac{\pi}{2}$ units^2

2 $10\frac{2}{3}$ units^2

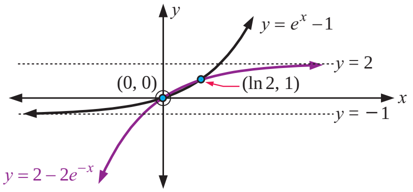
3 **a, b**



c $1\frac{1}{3}$ units^2

4 $\frac{1}{3}$ units^2

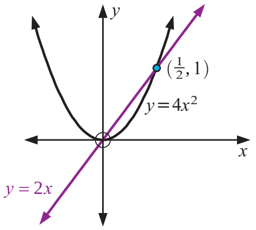
5 **a, b**



c enclosed area $=3\ln 2-2$ (≈ 0.0794) units^2

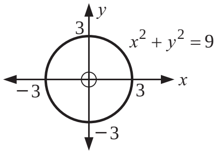
6 $\frac{1}{2}$ units^2

7



enclosed area $= \frac{1}{12}$ units^2

8 **a**



b $\frac{9\pi}{4}$ units^2 (≈ 7.07 units^2)

9 **a** $40\frac{1}{2}$ units^2 **b** 8 units^2 **c** 8 units^2

10 **a** C_1 is $y = \sin 2x$, C_2 is $y = \sin x$

b $A(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$ **c** $2\frac{1}{2}$ units^2

11 **a** $\int_3^5 f(x)\,dx = -$ (area between $x = 3$ and $x = 5$)

b $\int_1^3 f(x)\,dx - \int_3^5 f(x)\,dx + \int_5^7 f(x)\,dx$

12 **a** C_1 is $y = \cos^2 x$, C_2 is $y = \cos(2x)$

b $A(0, 1)$ $B(\frac{\pi}{4}, 0)$ $C(\frac{\pi}{2}, 0)$ $D(\frac{3\pi}{4}, 0)$ $E(\pi, 1)$

13 **a** 2.88 units^2 **b** 4.97 units^2 **14** $k \approx 1.7377$

15 $b \approx 1.3104$ **16** $a = \sqrt{3}$

EXERCISE 22B.1

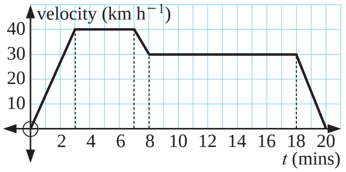
1 110 m

2 **a** **i** travelling forwards

ii travelling backwards (opposite direction)

b 16 km **c** 8 km from starting point (on positive side)

3 **a**



b 9.75 km

EXERCISE 22B.2

1 **a** $\frac{1}{2}$ cm **b** 0 cm **2** **a** $5\frac{1}{6}$ cm **b** $1\frac{1}{2}$ cm left

3 **a** 41 units **b** 34 units **4** **b** 2 units

5 **a** 40 m s^{-1}

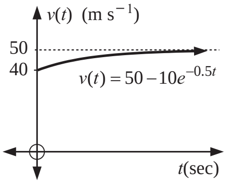
b 47.8 m s^{-1}

c 1.39 seconds

d as $t \to \infty$, $v(t) \to 50$

e $a(t) = 5e^{-0.5t}$ and as $e^x > 0$ for all x, $a(t) > 0$ for all t.

f



g 134.5 m

6 900 m

7 **a** Show that $v(t) = 100 - 80e^{-\frac{1}{20}t}$ m s^{-1} and as $t \to \infty$, $v(t) \to 100$ m s^{-1}

b 370.4 m

EXERCISE 22C

1 €4250

2 **a** $P(x) = 15x - 0.015x^2 - 650$ dollars

b maximum profit is \$3100, when 500 plates are made

c $46 \leqslant x \leqslant 954$ plates (you can't produce part of a plate)

3 14 400 calories **4** 76.3° C

EXERCISE 22D.1

1 **a** 36π units^3 **b** 8π units^3 **c** $\frac{127\pi}{7}$ units^3

d $\frac{255\pi}{4}$ units^3 **e** $\frac{992\pi}{5}$ units^3 **f** $\frac{250\pi}{3}$ units^3

g $\frac{\pi}{2}$ units^3 **h** $\frac{40\pi}{3}$ units^3

2 **a** 18.6 units^3 **b** 30.2 units^3

3 **a** 186π units^3 **b** $\frac{146\pi}{5}$ units^3 **c** $\frac{\pi}{2}(e^8 - 1)$ units^3

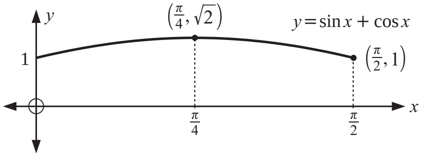
4 **a** 63π units^3 **b** ≈ 198 cm^3

5 **a** a cone of base radius r and height h

b $y = -\left(\frac{r}{h}\right)x + r$ **c** $V = \frac{1}{3}\pi r^2 h$

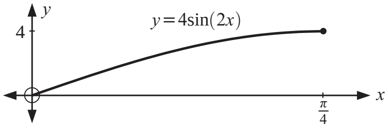
6 **a** a sphere of radius r **7** **a** $\frac{\pi^2}{4}$ units^3 **b** $\frac{\pi^2}{8}$ units^3

8 **a**



b $\pi\left(\frac{\pi}{4} + \frac{1}{2}\right)$ units^3

9 **a**



b $2\pi^2$ units^3

EXERCISE 22D.2

1 **a** $A(-1, 3)$, $B(1, 3)$ **b** $\frac{136\pi}{15}$ units^3

2 **a** $A(2, e)$ **b** $\pi(e^2 + 1)$ units^3

3 **a** $A(1, 1)$ **b** $\frac{11\pi}{6}$ units^3

4 $\frac{162\pi}{5}$ units^3 **5** **a** $A(5, 1)$ **b** $\frac{9\pi}{2}$ units^3

REVIEW SET 22A

1 **a** $2 + \pi$ **b** -2 **c** π

2 $A = \int_a^b [f(x) - g(x)]dx + \int_b^c [g(x) - f(x)]\,dx$
$+ \int_c^d [f(x) - g(x)]\,dx$

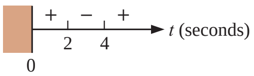
3 no, $\int_1^3 f(x)\,dx = -$ (area from $x = 1$ to $x = 3$)

4 $k = \sqrt[3]{16}$

5 **Hint:** Show that the areas represented by the integrals can be arranged to form a $1 \times e$ unit rectangle.

6 4.5 units^2

7 **a** $v(t)$:



b The particle moves in the positive direction initially, then at $t = 2$, $6\frac{2}{3}$ m from its starting point, it changes direction. It changes direction again at $t = 4$, $5\frac{1}{3}$ m from its starting point, and at $t = 5$, it is $6\frac{2}{3}$ m from its starting point.

c $6\frac{2}{3}$ m **d** $9\frac{1}{3}$ m

8 $(3 - \ln 4)$ units^2 **9** $\frac{128\pi}{5}$ units^3

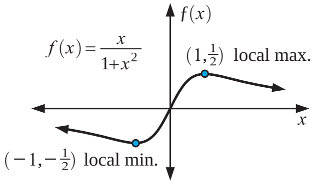
REVIEW SET 22B

1 29.6 cm

2 **a** local maximum at $(1, \frac{1}{2})$, local minimum at $(-1, -\frac{1}{2})$

b as $x \to \infty$, $f(x) \to 0$ (above)
as $x \to -\infty$, $f(x) \to 0$ (below)

c



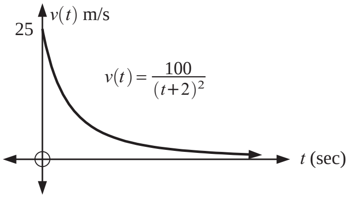
d 0.805 units^2

3 2.35 m

4 **a** $v(0) = 25$ m s^{-1}, $v(3) = 4$ m s^{-1}

b as $t \to \infty$, $v(t) \to 0$

c



d 3 seconds **e** $a(t) = \frac{-200}{(t+2)^3}$, $t \geqslant 0$ **f** $k = \frac{1}{5}$

5 **a** 0 and -0.7292 **b** 0.2009 units^2

6 £408 **7** $m = \frac{\pi}{3}$

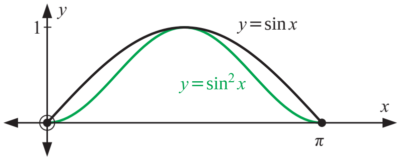
8 **a** $a \approx 0.8767$ **b** ≈ 0.1357 units^2

9 **a** $\pi\left(\frac{3\pi}{32} - \frac{1}{8\sqrt{2}}\right)$ units^3 **b** ≈ 124 units^3

REVIEW SET 22C

1 269 cm

2 **a**



b $\left(1 - \frac{\pi}{4}\right)$ units^2

3 $a = \ln 3$, $b = \ln 5$ **4** $\left(\frac{\pi^2}{2} - 2\right)$ units^2

5 $\left(2 - \frac{\pi}{2}\right)$ units^2 **6** $k = 1\frac{1}{3}$

7 **a** 312π units^3 **b** 402π units^3 **c** $\frac{\pi^2}{2}$ units^3

d $\pi\left(\frac{3\pi - 8}{4}\right)$ units^3

8 $\frac{\pi}{2}$ units^3 **10** **a** $\frac{128\pi}{3}$ units^3

EXERCISE 23A

1 **a** continuous **b** discrete **c** continuous **d** continuous
e discrete **f** discrete **g** continuous **h** continuous

2 **a** **i** height of water in the rain gauge **ii** $0 \leqslant x \leqslant 200$ mm
iii continuous

b **i** stopping distance **ii** $0 \leqslant x \leqslant 50$ m **iii** continuous

c **i** number of switches until failure **ii** any integer $\geqslant 1$
iii discrete

3 **a** $0 \leqslant X \leqslant 4$

b

YYYY	YYYN	YYNN	NNNY	NNNN
	YYNY	YNYN	NNYN	
	YNYY	YNNY	NYNN	
	NYYY	NNYY	YNNN	
		NYNY		
		NYYN		
$(X = 4)$	$(X = 3)$	$(X = 2)$	$(X = 1)$	$(X = 0)$

c **i** $X = 2$ **ii** $X = 2$, 3 or 4

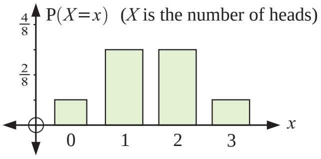
4 **a** $X = 0, 1, 2, 3$

b

HHH	HHT	TTH	TTT
	HTH	THT	
	THH	HTT	
$(X = 3)$	$(X = 2)$	$(X = 1)$	$(X = 0)$

c $P(X = 3) = \frac{1}{8}$
$P(X = 2) = \frac{3}{8}$
$P(X = 1) = \frac{3}{8}$
$P(X = 0) = \frac{1}{8}$

d



EXERCISE 23B

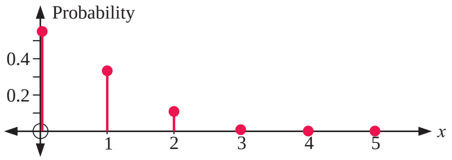
1 **a** $k = 0.2$ **b** $k = \frac{1}{7}$

2 **a** $P(2) = 0.1088$

b $a = 0.5488$, the probability that Jason does not hit a home run in a game.

c $P(1) + P(2) + P(3) + P(4) + P(5) = 0.4512$ and is the probability that Jason will hit one or more home runs in a game.

d



3 **a** $\sum P(x_i) > 1$ **b** $P(5) < 0$ which is not possible

4 **a** The random variable represents the number of hits that Sally has in each game.

b $k = 0.23$

c **i** $P(X \geqslant 2) = 0.79$ **ii** $P(1 \leqslant X \leqslant 3) = 0.83$

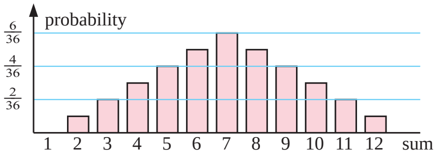
5 **a**

	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
roll 1	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
		1	2	3	4	5	6

roll 2

b $P(0)=0,\ P(1)=0,\ P(2)=\frac{1}{36},\ P(3)=\frac{2}{36},$
$P(4)=\frac{3}{36},\ P(5)=\frac{4}{36},\ P(6)=\frac{5}{36},\ P(7)=\frac{6}{36},$
$P(8)=\frac{5}{36},\ P(9)=\frac{4}{36},\ P(10)=\frac{3}{36},\ P(11)=\frac{2}{36},$
$P(12)=\frac{1}{36}$

c



6 **a** $k=\frac{1}{12}$ **b** $k=\frac{12}{25}$

7 **a** $P(0)=0.1975k;\ \ P(1)=0.0988k;\ \ P(2)=0.0494k;$
$P(3)=0.0247k;\ P(4)=0.0123k$
b $k=\frac{81}{31}$ (≈ 2.61), $\mathrm{P}(X\geqslant 2)=0.226$

8 **a** $P(0)=0.665$ **b** $\mathrm{P}(X\geqslant 1)=0.335$

9 **a**

x	0	1	2
$\mathrm{P}(X=x)$	$\frac{3}{28}$	$\frac{15}{28}$	$\frac{10}{28}$

b

x	0	1	2	3
$\mathrm{P}(X=x)$	$\frac{1}{56}$	$\frac{15}{56}$	$\frac{30}{56}$	$\frac{10}{56}$

10 **a**

		Die 2					
		1	2	3	4	5	6
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
Die 1	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

b $\frac{1}{6}$
d $\frac{15}{26}$

c

d	2	3	4	5	6	7	8	9	10	11	12
$\mathrm{P}(D=d)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

11 **a**

		Die 2					
		1	2	3	4	5	6
	1	0	1	2	3	4	5
	2	1	0	1	2	3	4
Die 1	3	2	1	0	1	2	3
	4	3	2	1	0	1	2
	5	4	3	2	1	0	1
	6	5	4	3	2	1	0

b

N	0	1	2	3	4	5
$\mathrm{P}(N=n)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

c $\frac{1}{6}$ **d** $\frac{2}{5}$

EXERCISE 23C

1 102 days **2** **a** $\frac{1}{8}$ **b** 25 **3** 30 times

4 \$1.50 **5** 15 days **6** 27

7 **a** **i** 0.55 **ii** 0.29 **iii** 0.16
b **i** 4125 **ii** 2175 **iii** 1200

8 **a** €3.50 **b** −€0.50, no **c** **i** $k=3.5$ **ii** $k>3.5$

9 **a** **i** $\frac{1}{6}$ **ii** $\frac{1}{3}$ **iii** $\frac{1}{2}$
b **i** \$1.33 **ii** \$0.50 **iii** \$3.50
c lose 50 cents **d** lose \$50

10 **a** £2.75 **b** £3.75 **11** **a** $k=0.03$ **b** $\mu=0.74$

12 $\mu=2.5$

13 **a**

m	1	2	3	4	5	6
$\mathrm{P}(M=m)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

b $\mu\approx 4.47$

14 **a** $\mathrm{P}(X\leqslant 3)=\frac{1}{12},\ \mathrm{P}(4\leqslant X\leqslant 6)=\frac{1}{3},$
$\mathrm{P}(7\leqslant X\leqslant 9)=\frac{5}{12},\ \mathrm{P}(X\geqslant 10)=\frac{1}{6}$
c $a=5$ **d** organisers would lose \$1.17 per game
e \$2807

EXERCISE 23D

1 **a** The binomial distribution applies, as tossing a coin has two possible outcomes (H or T) and each toss is independent of every other toss.
b The binomial distribution applies, as this is equivalent to tossing one coin 100 times.
c The binomial distribution applies as we can draw out a red or a blue marble with the same chances each time.
d The binomial distribution does not apply as the result of each draw is dependent upon the results of previous draws.
e The binomial distribution does not apply, assuming that ten bolts are drawn without replacement. We do not have a repetition of independent trials.

2 **a** ≈ 0.268 **b** ≈ 0.800 **c** ≈ 0.200

3 **a** ≈ 0.476 **b** ≈ 0.840 **c** ≈ 0.160 **d** ≈ 0.996

4 **a** ≈ 0.231 **b** ≈ 0.723 **c** 1.25 apples

5 **a** ≈ 0.0280 **b** $\approx 0.002\,46$ **c** ≈ 0.131 **d** ≈ 0.710

6 **a** ≈ 0.998 **b** ≈ 0.807 **c** 105 students

7 **a** **i** ≈ 0.290 **ii** ≈ 0.885 **b** 18.8

REVIEW SET 23A

1 **a** $a=\frac{5}{9}$ **b** $\frac{4}{9}$ **2** 4.8 defectives

3 **a** $k=0.05$ **b** $\mu=1.7$

4 **a** **i** $\frac{1}{10}$ **ii** $\frac{3}{5}$ **iii** $\frac{3}{10}$ **b** $1\frac{1}{5}$

5 **a** £7 **b** No, she would lose £1 per game in the long run.

REVIEW SET 23B

1 **a** $k=\frac{8}{5}$ **b** 0.975 **c** 2.55

2 **a** 0.302 **b** 0.298 **c** 0.561

3 6.43 surgeries

4 **a** 0.849 **b** 2.56×10^{-6} **c** 0.991
d 0.000 246

5 **a** 42 donations **b** 0.334

6 **a** **i** 0.100 **ii** 0.114 **b** 3.41 games

REVIEW SET 23C

1 **a** $k=\frac{12}{11}$ **b** $k=\frac{1}{2}$

2 **a**

x	0	1	2	3	4
$\mathrm{P}(X=x)$	0.0625	0.25	0.375	0.25	0.0625

b $\mu=2$

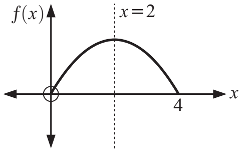
3 480 **4** **a** 0.259 **b** 0.337 **c** 0.922

5 **a** **i** $\frac{2}{5}$ **ii** $\frac{1}{10}$ **iii** $\frac{1}{10}$ **b** \$2.70

EXERCISE 24A

1 **b** $\frac{2}{3}$

2 **a** $a = -\frac{3}{32}$ **b**



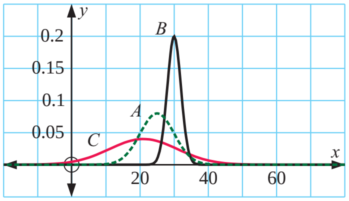
c $\frac{5}{32}$ **d** 2

3 **a** $k = -\frac{4}{375}$ **b** $3\frac{1}{3}$ **c** 0.64 **4** $a = \frac{5}{32}$, $k = 2$

5 **a** $k = \frac{3}{16}$ **b** 2.4 hours **c** 0.369 **d** $a = \sqrt[3]{16}$

EXERCISE 24B.1

1



2 **a,b** The mean volume (or diameter) is likely to occur most often with variations around the mean occurring symmetrically as a result of random variations in the production process.

3 **a** 0.683 **b** 0.477

4 **a** 84.1% **b** 2.28% **c** **i** 2.15% **ii** 95.4%
d **i** 97.7% **ii** 2.28%

5 **a** 0.954 **b** 2.83

6 **a** $\mu = 176$ g, $\sigma = 24$ g **b** 81.9%

7 **a** **i** 34.1% **ii** 47.7%
b **i** 0.136 **ii** 0.159 **iii** 0.0228 **iv** 0.841
c $k = 178$

8 **a** $\mu = 155$ cm $\sigma = 12$ cm **b** 0.84

9 **a** ≈ 41 days **b** ≈ 254 days **c** ≈ 213 days

EXERCISE 24B.2

1 **a** 0.341 **b** 0.383 **c** 0.106

2 **a** 0.341 **b** 0.264 **c** 0.212
d 0.945 **e** 0.579 **f** 0.383

3 **a** $a \approx 21.4$ **b** $a \approx 21.8$ **c** $a \approx 2.82$

EXERCISE 24C.1

1 **a** 0.885 **b** 0.195 **c** 0.302 **d** 0.947 **e** 0.431

2 **a** 0.201 **b** 0.524 **c** 0.809 **d** 0.249 **e** 0.249

3 **a** 0.383 **b** 0.950

4 **a** $a = 1.645$ **b** $a = -1.282$

5 **a** Physics 0.463, Chemistry 0.431, Maths 0.990, German 0.521, Biology 0.820
b Maths, Biology, German, Physics, Chemistry

6 65.6%

EXERCISE 24C.2

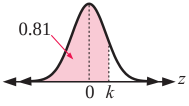
1 **a** 0.159 **b** 0.309 **c** 0.335

2 **a** 0.348 **b** 0.324 **c** 0.685

3 **a** 0.585 **b** 0.805 **c** 0.528

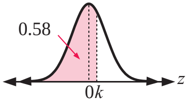
EXERCISE 24D

1 **a**



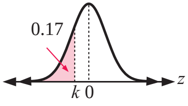
$k \approx 0.878$

b



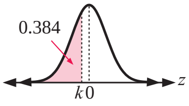
$k \approx 0.202$

c



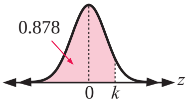
$k \approx -0.954$

2 **a**



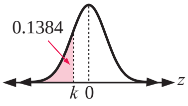
$k \approx -0.295$

b



$k \approx 1.17$

c



$k \approx -1.09$

3 **b** **i** $k \approx 0.303$ **ii** $k \approx 1.04$

4 **a** $k \approx 79.1$ **b** $k \approx 31.3$

EXERCISE 24E

1 0.378 **2** **a** 90.4% **b** 4.78% **3** 83

4 **a** 0.003 33 **b** 61.5% **c** 23 eels

5 $\mu \approx 23.6$, $\sigma \approx 24.3$

6 **a** $\mu = 52.4$, $\sigma = 21.6$ **b** 54.4% **7** 112

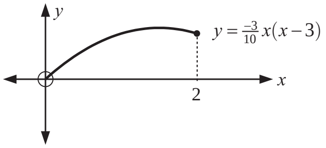
8 0.193 m **9** **a** $\mu = 2.00$, $\sigma = 0.0305$ **b** 0.736

REVIEW SET 24A

1 **a** **i** 2.28% **ii** 84% **b** 0.341

2 **a** $a = 6.3$ grams **b** $b \approx 32.3$ grams

3 **a** $a = -\frac{3}{10}$ **b**



c 1.2
d $\frac{13}{20}$

4 $k \approx 0.524$ **5** 29.5 m **6** **a** 0.136 **b** 0.341

REVIEW SET 24B

1 **a** **i** 81.9% **ii** 84.1% **b** 0.477 **c** $x \approx 61.9$

2 $\mu \approx 31.2$

3 **a** $\mu = 29$, $\sigma \approx 10.7$ **b** **i** 0.713 **ii** 0.250

4 **a** 1438 students **b** 71 marks

5 **a** 0.260 **b** 29.3 weeks

6 **a** $\mu = 61.218$, $\sigma \approx 22.559$ **b** ≈ 0.244

REVIEW SET 24C

1 **a** 0.364 **b** 0.356 **c** $k \approx 18.2$

2 $\sigma \approx 0.501$ mL **3** 0.207 **4** $\mu \approx 80.0$ cm **5** 0.0708

6 0.403

EXERCISE 25A

1 **a** $r = 3$ **b** 2×3^{19} **2** **Hint:** $u_1 = \ln 2 = d$

3 **a** b^2x **b** $2\ln b + x$ **c** $x^* = \dfrac{2\ln b}{b^2 - 1}$

4 **a** $(b, 2)$ **b** y-intercept is $2 - 2b^2$, x-intercepts are $b \pm 1$
c **i** $b = -2$ **ii** $b < -2$ **iii** $b = \dfrac{1 \pm \sqrt{17}}{4}$

5 **a** $x^3 - 6x^2 + 12x - 8$ **b** 29

6 **a** 1 **b** 3 **c** $\{x \mid x \leqslant \frac{1}{2}, x \in \mathbb{R}\}$ **d** $\{y \mid y \geqslant 0, y \in \mathbb{R}\}$

7 **a** a **b** $-b$ **c** a **d** $\dfrac{a}{\sqrt{1-a^2}}$

8 **a** $x = 0, \pi, 2\pi$ **b** $x = \frac{\pi}{3}, \frac{5\pi}{3}$

9 **a**

Constant	a	b	c	d	e	h
Sign	> 0	< 0	> 0	< 0	> 0	$= 0$

Constant	Δ of $f(x)$	Δ of $g(x)$
Sign	< 0	> 0

10 **a** $x = -5$ **b** $a = -\frac{1}{2}$

11 **a** $\mathbf{A} = \frac{1}{3}(\mathbf{B}^{-1} - \mathbf{I})$ **b** $\mathbf{A} = \begin{pmatrix} -2 & -1\frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}$ **c** $\frac{4}{9}$

12 **a** **i** $p = 1$ **ii** $\sqrt{6p^2 + 60}$

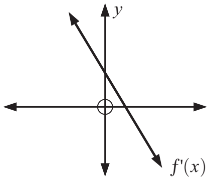
13 **a** $\overrightarrow{\text{BA}} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ $\overrightarrow{\text{BC}} = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$ **b** both are $\sqrt{11}$ units

c a rhombus **d** $\frac{1}{11}$ **e** $\frac{\sqrt{120}}{11}$ **f** $2\sqrt{30}$ units^2

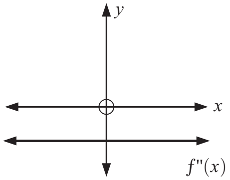
14 **a** g **b** **i** $m - a$ **ii** $\dfrac{j + k - c - d}{2}$

15 **a** 35, 6.4 **b** 19.5, 3.2 **c** 57.5, 9.6

17 **a**

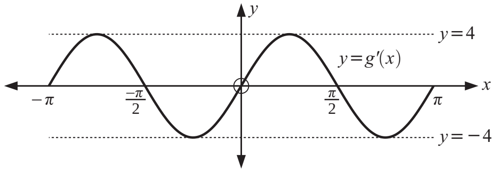


b



18 **a** $4\sin(2x)$

b

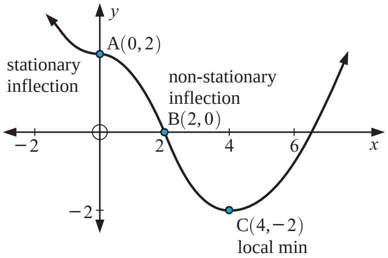


c $x = -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$ $\therefore$ 5 solutions

d M is at $(-\pi, 0)$, $(0, 0)$ or $(\pi, 0)$

19 **a** $\text{P}(A \cup B) = x + 0.57$ **b** $x = 0.16$

20



21 **b** $\left(\frac{1}{2}, \frac{1}{2}\right)$ **c** **i** $x > 0$ **ii** $x < \frac{1}{2}$

22 **a** $v(t) = k - 8e^{2t}$ m s^{-1} **b** $k = 72$

23 **a** $x = 3$ **b** $x = \sqrt{7}$ **c** $x = \dfrac{5 - \ln 8}{2}$ **d** $x = 3$

24 **a** 1 m s^{-1}, the initial velocity

b 0, uniform (constant) velocity

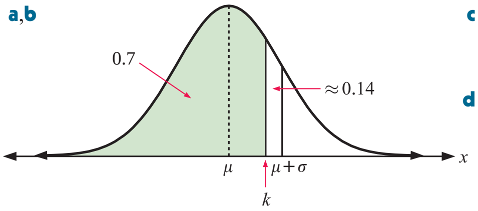
c 4, 4 m displacement on $1 \leqslant t \leqslant 3$

25 **a** -8 **b** $k = \frac{1}{2}$ **26** **b** $\frac{\pi}{4} - \frac{1}{2}$ **27** $a = 0.3$, $b = 0.2$

28 **a** $r = \dfrac{1}{e^2}$ **b** e^{-199} **c** $\dfrac{e^3}{e^2 - 1}$

29 **a** $x = 3$ **b** $x = 2$

30 **a,b**



c **i** 0.3

ii 0.2

iii 0.541

d 0.1

31 **a** 70%

b **i** $m \approx 27.5$ **ii** $n \approx 35$ **iii** $p \approx 42.5$ **iv** $q = 100$

32 **a** $p = 10\sqrt{3}$ **b** $x + \sqrt{3}y = 40$

33 **a** $v(t) = t - \frac{3}{2}\sin\left(2t + \frac{\pi}{2}\right) + 6\frac{1}{2}$ cm s^{-1} **b** $\dfrac{\pi + 26}{4}$ cm s^{-1}

34 **b** $\frac{1}{3}\ln\left(\frac{7}{2}\right)$ **35** **a** $-\frac{2}{\sqrt{21}}$ **b** $-\frac{4\sqrt{21}}{25}$

36 **a** $\frac{5}{16}\sqrt{2}$ **b** $40 + 20\sqrt{2}$

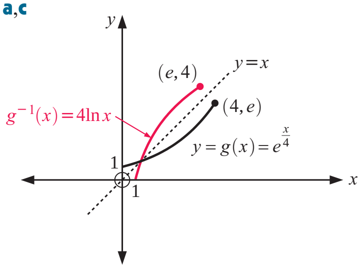
37 **a** $\begin{pmatrix} -2 \\ 2 \\ 6 \end{pmatrix}$ **b** $\sqrt{11}$ units s^{-1}

c $x = 3 - 2t, \ y = 1 + 2t, \ z = -2 + 6t, \ t \geqslant 0$

38 **a** $\mathbf{PQ} = \begin{pmatrix} -2 & 3b + 3 \\ 8 - 4a & -3a - 4b \end{pmatrix}$

b $a = 2, \ b = -1, \ k = -2$ **c** $\mathbf{P}^{-1} = -\frac{1}{2}\mathbf{Q}$

39 **a,c**



b $\{y \mid 1 \leqslant y \leqslant e\}$

d $\{x \mid 1 \leqslant x \leqslant e\}$, $\{y \mid 0 \leqslant y \leqslant 4\}$

e $g^{-1}(x) = 4\ln x$

40 **c** $\text{SR} = 5\sqrt{3}$ cm

d perimeter $= 15 + 5\sqrt{3}$ cm, area $= \frac{25}{2}\sqrt{3}$ cm^2

41 **a** **i** $-\frac{1}{2}x + 3$ **ii** $x + 2y = 20$ **iii** A(12, 4)

b **i** $\int_2^6 (-\frac{1}{4}x^2 + 3x + 4)\,dx$ **ii** $46\frac{2}{3}$ units^2

iii $\pi \int_2^6 (-\frac{1}{4}x^2 + 3x + 4)^2\,dx$

42 **a** **i** $r = -3$ **ii** -4×3^{13}

b **i** $x = 4$ or -1

ii $S = 8$ when $x = 4$; when $x = -1$, S does not exist

c **i** -55 **ii** -2300

43 **a** **i** $\begin{pmatrix} -1 \\ -3 \\ -7 \end{pmatrix}$ **ii** $\frac{1}{\sqrt{59}}\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}$

b no **c** $a = \frac{1}{5}$

d $\overrightarrow{\text{OM}} = \frac{1}{2}(5\mathbf{i} + \mathbf{j} - 9\mathbf{k})$

e $\mathbf{r}_1 = \frac{1}{2}\begin{pmatrix} 5 \\ 1 \\ -9 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}, \ t \in \mathbb{R}$

f **i** $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \neq k\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ for some $k \in \mathbb{R}$

ii $m = -45\frac{1}{2}$ **iii** $\text{P}(-30\frac{1}{2}, -21\frac{1}{2}, 6\frac{1}{2})$

44 **a** **i** $\begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$ **ii** $\begin{pmatrix} 8 & 7 \\ 0 & 1 \end{pmatrix}$ **iii** $\begin{pmatrix} 16 & 15 \\ 0 & 1 \end{pmatrix}$

c $\begin{pmatrix} 1024 & 1023 \\ 0 & 1 \end{pmatrix}$

d **i** $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix} + \ldots\ldots + \begin{pmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{pmatrix}$

iii $p = 2 + 4 + 8 + 16 + \ldots\ldots + 2^n$
{use sum of a geometric series}
$q = 2^{n+1} - 2 - n$

iv $\begin{pmatrix} 2^{15} - 2 & 2^{15} - 16 \\ 0 & 14 \end{pmatrix}$

45 **a** **ii** $x = 6$ **iii** 16

b **i** $y = 12 - x$ **ii** $y^2 = x^2 + 64 - 16x\cos\theta$

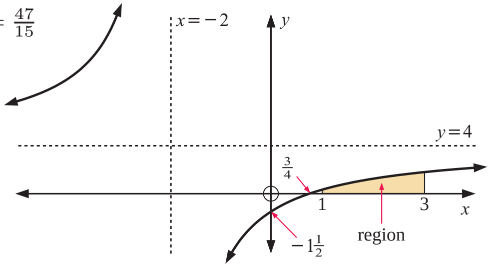
e $8\sqrt{5}$ units^2 when $x = y = 6$ **f** isosceles

46 **a** $f^{-1}(x) = \frac{x+3}{4}$, $g^{-1}(x) = x - 2$

b $4x - 11$

c $x = \frac{47}{15}$

d **i,iv**



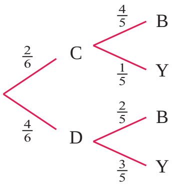
ii $A = 4$, $B = -11$ **iii** $12 - 11\ln 4$

47 **a** The probabilities do not add to 1.

b $a + b = 0.3$, $0 \leqslant a \leqslant 0.3$, $0 \leqslant b \leqslant 0.3$

c **i** 0.16 **ii** 0.84

48 **a**



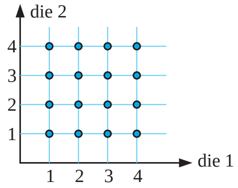
b $\frac{2}{5}$

c $\frac{7}{15}$

d $\frac{1}{2}$

e \$7.40

49 **a**



b $X = 2, 3, 4, 5, 6, 7, 8$

c **i** $\frac{3}{16}$ **ii** $\frac{5}{8}$ **iii** $\frac{3}{10}$ **d** $d = 8\frac{1}{3}$

50 **a** 0 cm s^{-2}, $\left(\frac{3\pi}{2} - 1\right) \text{ cm s}^{-2}$

b $v(t) = \frac{3}{2}t^2 + \cos t + 2 \text{ cm s}^{-1}$

c $\left(\frac{\pi^3}{16} + \pi + 1\right)$ cm, which is positive as $\pi > 3$.

d the integral is the displacement in the first $\frac{\pi}{2}$ seconds.

51 **a** $a = 7$, $b = \frac{\pi}{8}$, $c = 1$, $d = 10$

b **i** $A'(7, 28)$ **ii** $y = 14\sin\frac{\pi}{8}(t - 3) + 14$

c a vertical stretch (x-axis invariant) of factor $\frac{1}{2}$, followed by a translation of $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

52 **a** $(2^x + 4)(2^x - 5)$ **b** $x = \log_2 5$

c **i** $x = \dfrac{1}{p}$ **ii** $x = \dfrac{1}{3p + 1}$

53 **b** $2a - b$ when $x = \frac{3\pi}{4}, \frac{7\pi}{4}$.

c Max TP's: $(0, a)$, (π, a), $(2\pi, a)$
Min TP's: $(\frac{\pi}{2}, b - a)$, $(\frac{3\pi}{2}, b - a)$

54 **c** $S(x)$ **d** $\dfrac{1}{[C(x)]^2}$

55 **a** $P'(t) = \dfrac{30\,000\, e^{-\frac{t}{4}}}{(1 + 2e^{-\frac{t}{4}})^2}$ and use the fact that $e^{-\frac{t}{4}}$ is never negative.

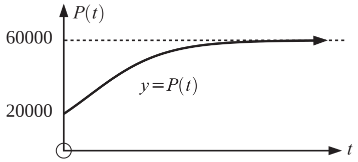
b $P(t)$ is increasing for all $t \geqslant 0$

c $P''(t) = \dfrac{7500e^{-\frac{t}{4}}(2e^{-\frac{t}{4}} - 1)}{(1 + 2e^{-\frac{t}{4}})^3}$

d 3750 per year when $t = 4\ln 2$ years

e $P(t) \to 60\,000$ and $P(0) = 20\,000$

f



56 **b** $\frac{1}{2}\sqrt{7}$ units when P is at $\left(-\frac{\sqrt{3}}{\sqrt{2}}, \frac{3}{2}\right)$ or at $\left(\frac{\sqrt{3}}{\sqrt{2}}, \frac{3}{2}\right)$.

57 **a** $\{y \mid -1 \leqslant y \leqslant 1\}$ **b** 2 solutions **c** $-3\sin x\cos^2 x$

d π units^3

58 **a** $25\sin\alpha$ cm^2 **b** $\left(\frac{25\pi}{2} - 25\sin\alpha\right)$ cm^2

c $A_{\max} = \frac{25\pi}{2}$ cm^2 when $\alpha = 0$ or π
$A_{\min} = 25(\frac{\pi}{2} - 1)$ cm^2 when $\alpha = \frac{\pi}{2}$.

59 **a** **i** $h = 4$ **ii** $k = 18$ **iii** $a = -2$ **b** $18\frac{2}{3}$ units^2

60 **a** **i** $x = 1$ **ii** $x = \sqrt[5]{7}$ **b** $x = 0$ or 1

61 **a** **ii** $\theta = \frac{\pi}{3}$ **b** $\cos x = \frac{1 - \sqrt{3}}{2}$

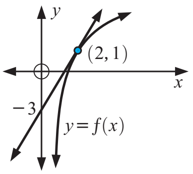
62 $u_1 = 2$, $u_n = 3n^2 - 3n + 3$, $n > 1$

63 $x = -\frac{3\pi}{2}$ or $\frac{\pi}{2}$ **64** **a** $-e^2$ **b** $e^2 - 3$

65 $(0, -1, -1)$

66 **a** $y = 2x - 3$ **b**

d $\frac{3}{2} < x < 2$



67 $x = 2$, $y = \frac{1}{8}$ or $x = 64$, $y = 4$ **69** $a = \frac{1}{2}$

70 $P(A \cup B) = 1$ or $P(A \cap B) = 0$

71 $\theta = -\frac{11\pi}{12}, -\frac{7\pi}{12}, \frac{\pi}{12}, \frac{5\pi}{12}$

72 **a** $\{x \mid x < 0 \text{ or } x > 2\}$ **b** $\dfrac{1}{x} + \dfrac{1}{x - 2}$

c $4x - 3y = 12 - 3\ln 3$

73 **a** $\frac{24}{49}$ **b** $\frac{16}{25}$ **74** **a** ≈ 0.34 **b** $\sigma \approx 5$

75 $a = \frac{3}{5}$ **76** $\dfrac{8}{x}$ **77** $9b = 2a^2$

78 **a** (0, 4), a translation of $\begin{pmatrix}2\\1\end{pmatrix}$

b (0, 6), a translation of $\begin{pmatrix}2\\0\end{pmatrix}$ followed by a vertical stretch of factor 2 (with x-axis invariant)

c $(\frac{1}{2},\ 3)$, a horizontal stretch of factor $\frac{1}{2}$, followed by a translation of $\begin{pmatrix}\frac{3}{2}\\0\end{pmatrix}$

d $(-2,\ \frac{1}{3})$ **e** $(3,\ -2)$, a reflection in $y = x$

79 $y = 4\sin\left(\frac{\pi}{2}x\right) - 1$ **80** **a** $x = 0$ **b** $x = 0.2$ or 0.3

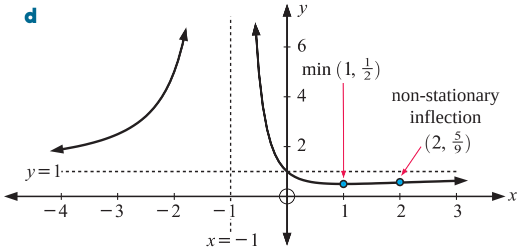
81 $a = -2$, $b = 3$, $\mathbf{A}^{11} = \begin{pmatrix}-2 & -1\\3 & 2\end{pmatrix}$

82 **a** VA is $x = -1$, HA is $y = 1$

b $f'(x) = \dfrac{2(x-1)}{(x+1)^3}$; local min $(1,\ \frac{1}{2})$

c $f''(x) = \dfrac{-4(x-2)}{(x+1)^4}$; inflection $\left(2,\ \frac{5}{9}\right)$

d



83 **a** $\dfrac{3x}{x-2}$ **b** $\dfrac{2x+1}{x-1}$

84 **a** **i** $\frac{13}{21}$ **ii** $\frac{11}{21}$ **b** $\frac{2}{3}$ **85** $x = \dfrac{2}{a^2-1}$

86 **a** $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$ **b** 1

c $32x^5 + 80x^3 + 80x + \dfrac{40}{x} + \dfrac{10}{x^3} + \dfrac{1}{x^5}$

87 **a** $a^2 - 2$ **b** $a^3 - 3a$

88 **a** **i** A(4, 0), B(−4, 0) **ii** C(0, 2), D(0, −2)

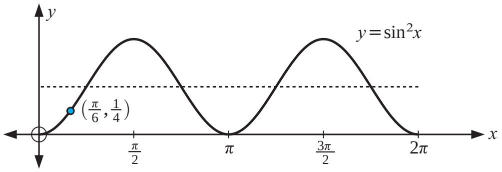
b $y = \sqrt{4 - \dfrac{x^2}{4}}$ **c** area $= 4\displaystyle\int_0^4 \sqrt{4 - \frac{x^2}{4}}\,dx$

d volume $= \frac{64\pi}{3}$ units^3

89 **a**

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$f(0)$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0

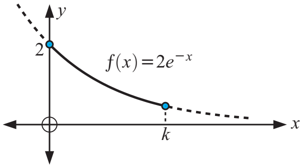
b



c when $x = \frac{\pi}{6}$, $y = \frac{1}{4}$ ✓ **d** $\{y \mid 0 \leqslant y \leqslant 1\}$

e $\frac{\pi}{2}$ units^2 **f** $x - y = \frac{\pi}{4} - \frac{1}{2}$

90 **a** $f(0) = 2$ **b**



c $k = \ln 2$

d $2 - \dfrac{2}{\sqrt{e}}$

91 **a** $f'(x) = 1 - x^{-2}$, $x = 1$ **b** A(1, 2) **c** .... is at least 2

d **i** no solutions **ii** one solution **iii** two solutions

92 **a** $\mathbf{r} = \begin{pmatrix}2\\0\\-3\end{pmatrix} + t\begin{pmatrix}1\\-1\\2\end{pmatrix}$, $t \in \mathbb{R}$

b $x = 2 + t$, $y = -t$, $z = -3 + 2t$, $t \in \mathbb{R}$

c it represents any point on the line **d** $\begin{pmatrix}t+3\\-t-3\\2t-8\end{pmatrix}$

e $6t - 10$ **f** $t = \frac{5}{3}$ **g** $\left(\frac{11}{3},\ -\frac{5}{3},\ \frac{1}{3}\right)$

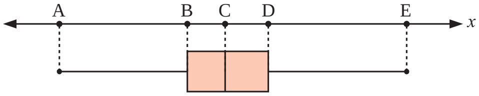
93 **a** **i** A is the minimum value of X **ii** B is Q_1

iii C is the median **iv** D is Q_3

v E is the maximum value of X

b **i** the range **ii** the IQR **c** **i** 0.5 **ii** 0.75

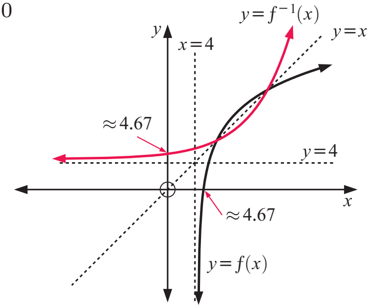
d



EXERCISE 25B

1 $n = 30$

2 **a,c**



b $x \approx 4.82$ **d** $x + 5y = 15$

3 $-\frac{84}{125}$

4 **a** $m = -2$, $n = 4$ **b** $k = 7$ **c** vertex is (2, 5)

d Domain of f is $\{x \mid x \in \mathbb{R}\}$
Range of f is $\{y \mid y \geqslant 3\}$
Domain of g is $\{x \mid x \in \mathbb{R}\}$
Range of g is $\{y \mid y \geqslant 5\}$

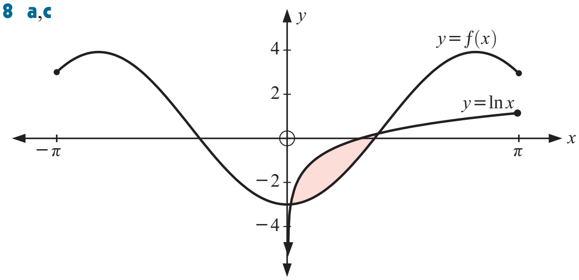
5 **a** 1950 **b** 10 500

6 **a** $x = \pm 1$ **b** $x \approx \pm 0.671$ **c** $x = -0.2$

d $x = -\frac{1}{2}$ or $\frac{2}{5}$

7 **a** $r \approx 35.4$ cm **b** ≈ 1530 cm^2 **c** 59.4 cm

8 **a,c**



b $x \approx \pm 1.68$

d **i** $A \approx \int_{0.0501}^{1.245} (\ln x - x\sin x + 3\cos x)\,dx$

ii $A \approx 1.37$ units^2

9 **a** $\mathbf{A}^{-1} = \begin{pmatrix} -2 & -3 & -2 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{7}{10} & -\frac{9}{10} & -\frac{1}{2} \end{pmatrix}$ **b** $x = -10$, $y = -2.5$, $z = -2.5$

10 **a** $m = 4$, $n = \frac{\pi}{4}$, $p = 1$, $r = 8$

b **i** ≈ 5.17 **ii** $t = 2\frac{1}{3}$

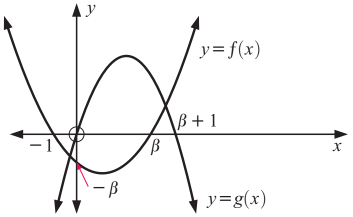
11 **a** $\approx 30.9^o$

b **ii** No, as equations have inconsistent solutions

c $(-8, 2\frac{1}{2}, -\frac{1}{2})$ **d** $a = -5\frac{1}{2}$

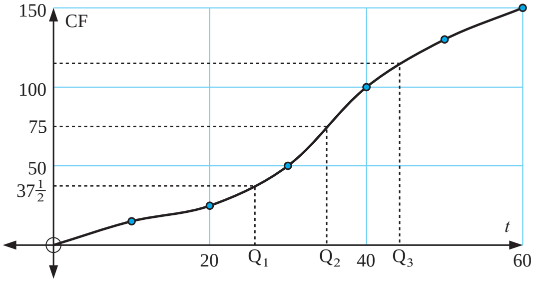
12 **a** x-intercepts are -1 and β, y-intercept is $-\beta$

b,c



13 **a**

Time	*f*	*Cumulative freq.*
$0 < t \leqslant 10$	15	15
$10 < t \leqslant 20$	10	25
$20 < t \leqslant 30$	25	50
$30 < t \leqslant 40$	50	100
$40 < t \leqslant 50$	30	130
$50 < t \leqslant 60$	20	150



b **i** ≈ 35 min **ii** ≈ 19 **iii** ≈ 0.5

14 ≈ 0.0548

15 **a** ≈ 0.470

b No, it is $\binom{5}{3}(0.86)^3(0.14)^2$ where $\binom{5}{3} = 10$.

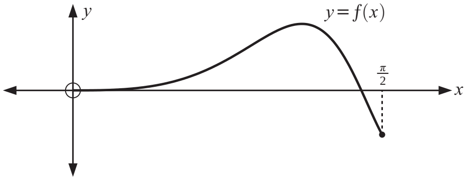
16 $\frac{1}{2}$

17 **a** z-score for 100 m ≈ -1.86 **b** the 100 m

z-score for 200 m ≈ -1.70

18 **a** $0, \approx 1.46$

b



c **i** ≈ 1.64 **ii** $y \approx 1.64x - 0.820$ **d** P(0.903, 0.671)

19 **a** **i** ≈ 0.672 **ii** ≈ 0.705

b Method is ok. Although not strictly binomial, the binomial distribution is very close in this case.

20 **a** $-12e^{1-4x}$ **b** $-\frac{3}{4}e^{1-4x} + c$ **c** ≈ 2.04

21 **a** A(1, 0), B(π, 0) **b** C(2.128, 0.641)

c (1.101, 0.086) **d** a non-stationary inflection

22 **a** 186 months **b** 371 months

23 **a** $+80x^4 + 80x^2 + 32$

b $\frac{1}{11}x^{11} + \frac{10}{9}x^9 + \frac{40}{7}x^7 + 16x^5 + \frac{80}{3}x^3 + 32x + c$

24 **a** $f(x) = -4(x-1)^2 + 4$

b **i** $A \approx \int_{0.106}^{1.89}[f(x) - g(x)]dx$ **ii** ≈ 4.77 units^2

25 **b** $\mathrm{P}(B) = 0.6$, $\mathrm{P}(A) = 0.2$

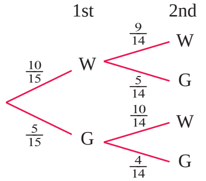
26 **a** $\overline{x} = 70.5$ **b** 76 kg **c** $s \approx 15.1$

d about 1.92 standard deviations above the mean.

27 **a** ≈ 0.0355 **b** ≈ 0.974

28 **a** **i** ≈ 0.544 **ii** ≈ 0.456 **b** **i** $(0.97)^n$ **ii** $n = 12$

29 **a** **i**



ii $\frac{11}{21}$

b $n = 6$

30 $\mu \approx 679$ kg, $\sigma \approx 173$ kg **31** **a** ≈ 2000 m **b** 350^o

32 **a** $z = 0.1$ **b** $y = 0.1$ **c** $x = 0.7$

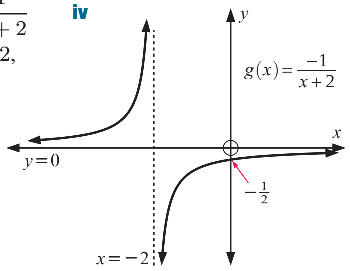
33 **a** $m = -1$, $n = 2$

b **i** $g(x) = -\dfrac{1}{x+2}$

ii VA is $x = -2$, HA is $y = 0$

iii $-\frac{1}{2}$

iv



34 **a** **i** $(-1, 3)$ **ii** $(39, 23)$ **b** ≈ 44.7 m

d no

35 $k \approx -0.969$

36 **a** A(0, 1), B$\left(2, \dfrac{1}{e^4}\right)$ **b** ≈ 0.882 units^2

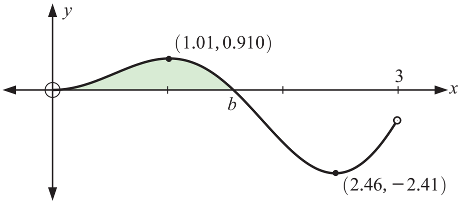
37 **a** 0.8 **b** **i** ≈ 0.0881 **ii** ≈ 0.967

38 **a** $\mu \approx 40.4$ **b** ≈ 0.0117 **c** $a \approx 55.8$

39 **a** $\begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix}$ **b** show $|\mathbf{A}| \neq 0$

d $a = -\frac{1}{8}$, $b = \frac{5}{8}$

40 **a**



b Range is $\{y \mid -2.41 \leqslant y \leqslant 0.91\}$ **c** $b = \frac{\pi}{2}$

d $A \approx 0.785$ **e** ≈ 1.721 units^3

41 **b** $3x + e^{\frac{3\pi}{2}}y = 1 + \frac{3\pi}{2}$ **c** ≈ 0.0847 units^2

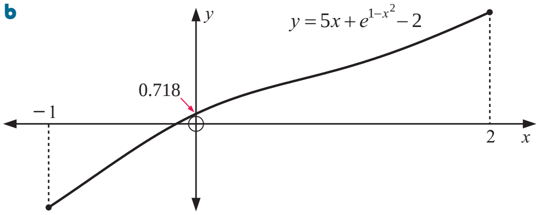
42 **b** $x = 0,\ \ x = \frac{\pi}{2}$ are VAs **c** 2 when $x = \frac{\pi}{4}$
d ≈ 2.046

43 **a** **i** ≈ 0.0362 **ii** ≈ 0.610 **iii** ≈ 0.566 **b** $k \approx 74.4$
c $a \approx 81.0,\ \ b \approx 101.6$ **d** **i** ≈ 0.506 **ii** ≈ 0.168

44 **a** $\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$, $\sqrt{38}$ units **b** D(4, 3, 2) **c** F(7, −4, −2)
d $33,\ \frac{11}{3\sqrt{38}}$ **e** $3\sqrt{221}$ units^2

45 **a** $t = \frac{1}{4}$ **b** 2.675 **c** the mean of the Y distribution

46 **a** $e - 2 \approx 0.718$
b



c ≈ -0.134 **d** 3

47 **a** $a \approx 17.2,\ \ b \approx 30.0$ **b** **i** ≈ 0.514 **ii** ≈ 0.538

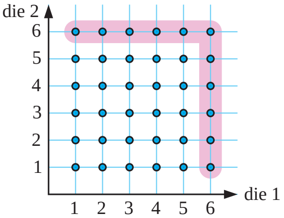
48 **a** 2.59 days **b** **i** ≈ 0.279 **ii** ≈ 0.799

49 **a** A(−3, 4, −2) **b** Yes, at (4, −3, 5) **c** $\approx 75.0^o$

50 **a** A(1, 0), B(2, 0), C(0, 2) **b** $y = 0$ is a HA
d local max. at $x \approx -0.618$, local min. at $x \approx 1.618$
f $x \approx 2.05$ **g** area ≈ 0.959 units^2

51 **a** −1 **b** $x \approx 1.857$ at P, ≈ 4.536 at Q
c $f'(x) = x^2e^{-x}(3 - x)$, B(3, 0.344)
d $3 - \sqrt{3}$ at A and $3 + \sqrt{3}$ at C **e** ≈ 0.595 units^2

52 **a**



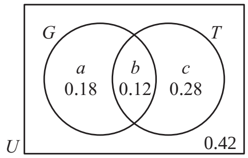
b $\frac{11}{36}$ **c** **i** ≈ 0.227 **ii** ≈ 0.635

53 **b** $a = -3\frac{1}{2},\ b = 5$ **c** D(10, −11, 11)

54 **a** DB ≈ 4.09 m, BC ≈ 9.86 m
b $\widehat{\text{ABE}} \approx 68.2^o$, $\widehat{\text{DBC}} \approx 57.5^o$ **c** ≈ 17.0 m^2
d ≈ 10.9 m

55 **a** $a = -1,\ \ b = 2$ **b** y-intercept is $-2\frac{1}{2}$
c $\frac{-1-\sqrt{21}}{2}$ and $\frac{-1+\sqrt{21}}{2}$ **d** $\text{D}\left(-\frac{1}{2},\ -2\frac{1}{3}\right)$
e **i** $A = -\displaystyle\int_{\frac{\sqrt{21}-1}{2}}^{k} \left(-1 + \frac{3}{x^2 + x - 2}\right) dx$
ii ≈ 0.558 units^2

56 **a** **i**



ii 0.12
iii are independent

b **iii** $b \approx 0.104,\ \ a \approx 0.124$ **iv** ≈ 0.228

57 **b** $9a + 3b + c = 14,\ \ -4a + 2b - c = 1$
c $\begin{pmatrix} 1 & 1 & 1 \\ 9 & 3 & 1 \\ -4 & 2 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -4 \\ 14 \\ 1 \end{pmatrix}$
d $a = 2,\ \ b = 1,\ \ c = -7$ **e** $p = 2,\ \ q = 3$

58 **b** **i** $\approx 0.927^c$ **ii** $\approx 0.644^c$
c **i** ≈ 2.16 cm^2 **ii** ≈ 29.3 cm^2

59 **a** $x = 3$ **b** $x = \frac{\ln 2}{\ln 3}$ (or $\log_3 2$)

60 **a** ≈ 1.48 units **b** ≈ 3.82 units

61 **a** $2p^2 - p^4$ **b** $p \approx 0.541$

62 **a** $\frac{3}{5}$ **b** 5 cm or 2.2 cm **c** AB = 5 cm is not possible

63 ≈ 6.40 cm **64** ≈ 0.114 **65** ≈ 0.842

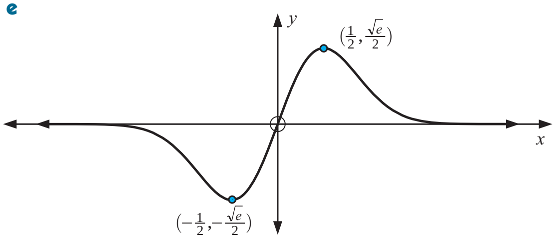
66 **a** $a = 13,\ \ b = 12,\ \ c = \frac{\pi}{30},\ \ d = 15$ **b** ≈ 24.9 m

67 $\mathbf{AB} = \mathbf{I},\ \ a = 2,\ \ b = -1,\ \ c = 3$

68 **a** $x = \frac{\pi}{2}$ **b** $f''(x) = e^{\sin^2 x}(2 - 4\sin^4 x),\ \ \sin^2 x = \frac{1}{\sqrt{2}}$
c $\approx (0.999,\ e^{\frac{1}{\sqrt{2}}}),\ \ \approx (2.14,\ e^{\frac{1}{\sqrt{2}}})$

69 $31\frac{1}{7}$ or $46\frac{6}{7}$

70 **a** $f'(x) = e^{1-2x^2}(1 - 4x^2),\ \ f''(x) = e^{1-2x^2}(16x^3 - 12x)$
b local min at $\left(-\frac{1}{2},\ \frac{\sqrt{e}}{2}\right)$, local max at $\left(\frac{1}{2},\ \frac{\sqrt{e}}{2}\right)$
c $x = 0$ or $\pm\frac{\sqrt{3}}{2}$
d as $x \to \infty,\ \ f(x) \to 0$ (from above)
as $x \to -\infty,\ \ f(x) \to 0$ (from below)
e



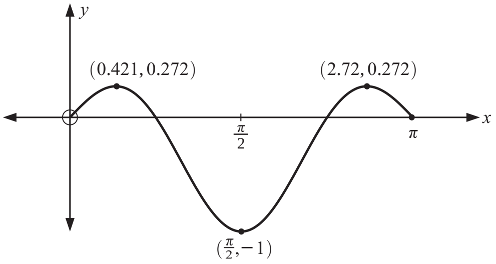
71 **a** $\overline{x} \approx 16.0$ **b** $s \approx 2.48$

72 **c** $\theta \approx 1.02,\ 2.59,\ 4.16,\ 5.73$

73 **a** no solutions exist **b** $x \approx 3.82$

74 **a** $k = 2$ **b** $\mu = 3.2$ **c** $\frac{47}{50}$

75 **a** $f'(x) = 6\cos^3 x - 5\cos x$
c local max. at (0.421, 0.272), (2.72, 0.272),
local min. at $\left(\frac{\pi}{2},\ -1\right)$
d



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