Homogenous differential equations
and substitution methods

A homogenous differential equation is a differential equation of the form $\frac{dy}{dx}=f\left(\frac{y}{x}\right)$.

For example, $\frac{dy}{dx}=\left(\frac{x^2-y^2}{xy}\right)$ is a homogenous first order differential equation because it can be written as $\frac{dy}{dx}=\left(\frac{1-\left(\frac{y}{x}\right)^2}{\frac{y}{x}}\right)$. The method that may be used to solve the problem is to substitute. Using the product rule, $\frac{dy}{dx}=x\frac{dv}{dx}+v$. These homogenous equations can then be transformed into separable equations of the form $\frac{dv}{dx}=\frac{f(v)-v}{x}$.

For example,

Find the general solution of the homogenous differential equation $\frac{dy}{dx}=\left(\frac{x^2-y^2}{xy}\right)$.

$$\frac{dy}{dx}=\left(\frac{x^2-y^2}{xy}\right)\Rightarrow\frac{dy}{dx}=\left(\frac{1-\left(\frac{y}{x}\right)^2}{\frac{y}{x}}\right)$$

$$x\frac{dv}{dx}+v=\frac{1-v^2}{v}\Rightarrow\frac{dv}{dx}=\frac{1}{x}\cdot\frac{1-2v^2}{v}$$

$$\frac{v}{1-2v^2}dv=\frac{1}{x}dx\Rightarrow-\frac{1}{4}\int\frac{-4v}{1-2v^2}dv=\int\frac{1}{x}dx$$

$$-\frac{1}{4}\ln\left|1-2v^2\right|=\ln\left|x\right|+c$$

$$\ln\left|1-2v^2\right|=\ln\left|x^{-4}\right|+c\Rightarrow$$

$$\left|\frac{x^2-2y^2}{x^2}\right|=A\left|x^{-4}\right|$$

EXAMPLE: Show that the substitution $v = \frac{y}{x}$ transforms the homogenous equation $\frac{dy}{dx} = \frac{2x - y}{x}$ into a first order linear equation and a separable variables differential equation. Hence solve the equation using two different methods.

$$\frac{dy}{dx} = \frac{2x - y}{x} \Rightarrow \frac{dy}{dx} = 2 - \frac{y}{x}$$

Let $y = vx$ so $\frac{dy}{dx} = 1 \cdot v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = 2 - \frac{y}{x} = 2 - v$$

$$\frac{dv}{dx} = \frac{2 - 2v}{x} \therefore \text{ separable equation}$$

$$\frac{dv}{dx}x + v = 2 - v \Rightarrow \frac{dv}{dx} + \frac{2}{x}v = \frac{2}{x} \therefore \text{ linear equation}$$

Method 1

$$\frac{dv}{dx} = \frac{2 - 2v}{x}$$

$$-\frac{1}{2}\int \frac{1}{v - 1}dv = \int \frac{1}{x}dx$$

$$-\frac{1}{2}\ln|v - 1| = \ln|x| + c \Rightarrow -\frac{1}{2}\ln\left|\frac{y}{x} - 1\right| = \ln|x| + c$$

$$\ln\left|\frac{y}{x} - 1\right|^{-\frac{1}{2}} = \ln|x| + c \Rightarrow \left|\frac{y - x}{x}\right|^{-\frac{1}{2}} = A|x|$$

$$\Rightarrow \frac{y - x}{x} = \frac{\pm A}{x^2} \Rightarrow y = x + \frac{k}{x}$$

Method 2

$$\frac{dv}{dx} + \frac{2}{x}v = \frac{2}{x} \qquad \text{Integrating factor } e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$x^2\frac{dv}{dx} + 2xv = 2x$$

$$\Rightarrow \frac{d}{dx}\left(x^2 v\right) = 2x$$

$$x^2 v = x^2 + c \Rightarrow xy = x^2 + c \Rightarrow y = x + \frac{c}{x}$$

- For some equations, we can use more than one method to solve them;
- The form of the solutions may be very different but we can verify that each of them is indeed a solution either by differentiation or by manipulating one solution to obtain the same form as the solution obtained when the equation is solved using a different method.