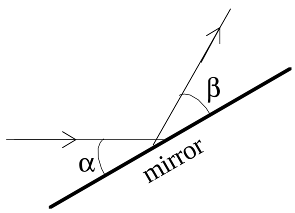
9.6 PROPERTIES OF THE CONIC SECTIONS

This section presents some of the interesting and important properties of the conic sections that can be proven using calculus. It begins with their reflection properties and considers a few ways these properties are used today. Then we derive the polar coordinate form of the conic sections and use that form to examine one of the reasons conic sections are still extensively used: the paths of planets, satellites, comets, baseballs, and even subatomic particles are often conic sections. The section ends with a specialized examination of elliptical orbits. To understand and describe the motions of the universe, at telescopic and microscopic levels, we need conic sections!

Reflections on the Conic Sections



incidence angle α
= reflection angle β
Fig. 1

This discussion of reflection assumes that the angle of incidence of a light ray or billiard ball is equal to the angle of reflection of the ray or ball. The assumption is valid for light rays and mirrors (Fig. 1) but is not completely valid for balls: the spin of the ball before it hits the wall may make the reflection angle smaller than, greater than, or equal to the incidence angle.

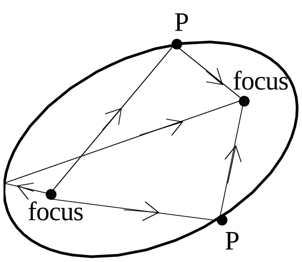


Fig. 2

> **Reflection Property of an Ellipse**
>
> An elliptical mirror reflects light from one focus to the other focus (Fig. 2) and all of the light rays take the same amount of time to be reflected to the other focus.

Outline of a proof: We can assume that the ellipse is oriented so its equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{(Fig. 3) and } a > b > 0.$$

Then the foci are at the points $F_1 = (-c, 0)$ and $F_2 = (c, 0)$ with $c = \sqrt{a^2 - b^2}$.

To show that the light rays from one focus are always reflected to the other focus, we need to show that angle α, the angle between the ray from F_1 and the tangent line to the ellipse, is equal to angle β, the angle between the tangent to the ellipse and the ray to F_2 . The most direct way to show that α = β is to start by calculating the slopes m_1 = slope of the line from F_1 to P,

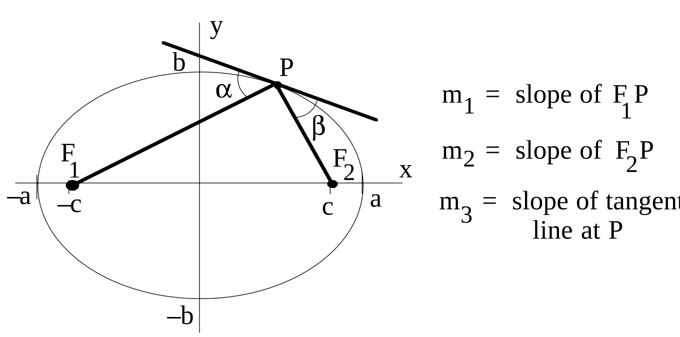


Fig. 3

m_2 = the slope of the line from P to F_2, and m_3 = the slope of the tangent line:

$$m_1 = \frac{y}{x+c} \;,\quad m_2 = \frac{y}{x-c}\text{, and, by implicit differentiation, } m_3 = \frac{-x}{y}\,\frac{b^2}{a^2}\;.$$

Then, from section 0.2, we know that $\tan(\alpha) = \dfrac{m_3 - m_1}{1 + m_1 m_3}$ and $\tan(\beta) = \dfrac{m_2 - m_3}{1 + m_2 m_3}$ so we just need to evaluate $\tan(\alpha)$ and $\tan(\beta)$ and show that they are equal. Since the process is algebraically tedious and not very enlightening, and it has been relegated to an Appendix after the problem set.

Since straight paths from one focus to the ellipse and back to the other focus all have the same length (the definition of an ellipse), all of the light rays from the one focus take the same amount of time to reach the other focus. If a small stone is dropped into an elliptical pool at one focus (Fig. 4), then the waves radiate in all directions, reflect off the sides of the pool to the other focus and create a splash there because they all arrive at the same time.

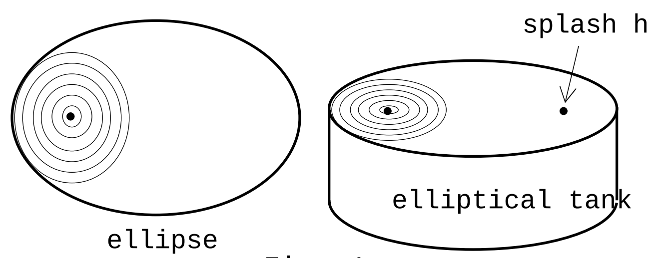


Fig. 4

Similarly, if a room is in the shape of an (half) ellipsoid of revolution (Fig. 5), then the sound waves from a whisper at one focus will bounce off the walls and all arrive at the same time at the other focus where an eavesdropper can hear the conversation.

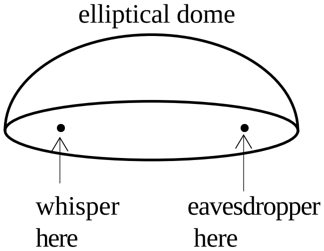
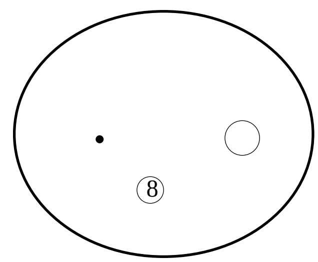


Fig. 5

Practice 1: What simple directions ensure that a ball shot shot from anywhere on an elliptical pool table (Fig. 6) will bounce off one wall and go into the single hole located at a focus of the ellipse?



elliptical pool table
Fig. 6

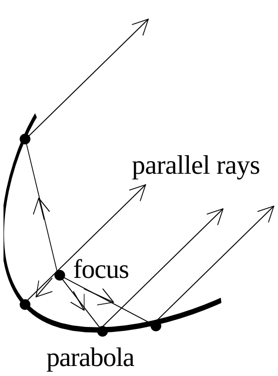


Fig. 7

Reflection Property of a Parabola

A parabolic mirror reflects light from the focus in a line parallel to the axis of the parabola (Fig. 7). In reverse, incoming light rays parallel to the axis are reflected to the focus.

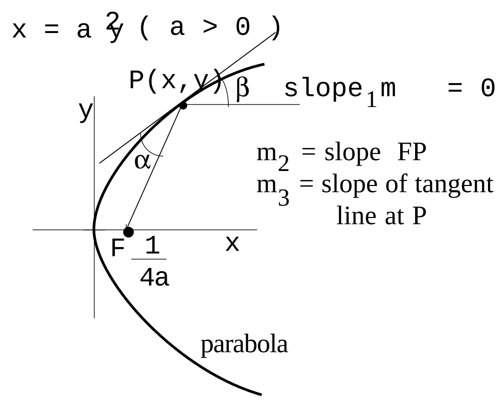


Fig. 8

Outline of a proof: If the parabolic mirror is given by $x = ay^2$ (Fig. 8), then its focus is at $F = (\frac{1}{4a}, 0)$ and the parabola is symmetric with respect to the x–axis. To prove the reflection property of the parabola, we need to show that $\alpha = \beta$ or, since α and β are both acute angles, that $\tan(\alpha) = \tan(\beta)$.
From Fig. 8 we calculate that

$$m_1 = 0, \quad m_2 = \frac{y}{x - \frac{1}{4a}} = \frac{4ay}{4ax - 1} = \frac{4ay}{4a^2y^2 - 1},$$

and, by implicit differentiation, $m_3 = \frac{1}{2ay}$.

Since $m_1 = 0$, we know that $\tan(\alpha) = \frac{m_3 - m_1}{1 + m_1 m_3} = m_3 = \frac{1}{2ay}$. An elementary but tedious algebraic argument shows that $\tan(\beta) = \frac{m_2 - m_3}{1 + m_2 m_3}$ simplifies to the same value as $\tan(\alpha)$ so $\tan(\alpha) = \tan(\beta)$ and $\alpha = \beta$.

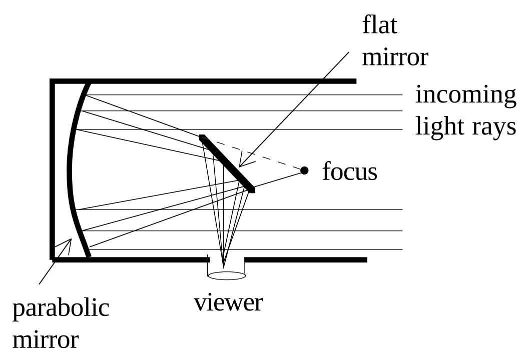


Fig. 9: Reflecting telescope

Because of this reflection property, the parabola is used in a variety of instruments and devices. Mirrors in reflecting telescopes are parabolic (Fig. 9) so that the dim incoming (parallel) light rays from distant stars are all reflected to an eyepiece at the mirror's focus for viewing. Similarly, radio telescopes use a parabolic surface to collect weak signals. A well known scientific supply company sells an 18 inch diameter parabolic reflector "ideal for a broad range of applications including solar furnaces, solar energy collectors, and parabolic and directional microphones." For outgoing light, flashlights and automobile headlights use (almost) parabolic mirrors so a light source set at the focus of the mirror creates a tight beam of light (Fig. 10).

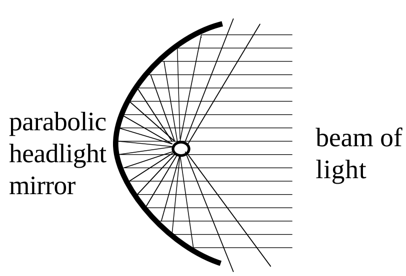


Fig. 10

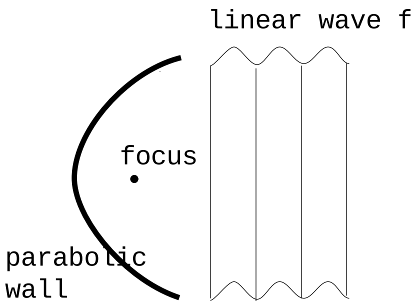


Fig. 11

Not only are incoming parallel rays reflected to the same point, but they reach that point at the same time. More precisely, if two objects start at the same distance from the y–axis and travel parallel to the x–axis, they both travel the same distance to reach the focus. If they are traveling at the same speed, they reach the focus together. An incoming linear wave front (Fig. 11) is reflected by a parabolic wall to create a splash at the focus. A small stone dropped into a wave tank at the focus of a parabola creates a linear outgoing wave. This "same distance" property of the reflection is something you can prove.

Practice 2: An object starts at the point (p,q), travels to the left until it encounters the parabola $x = ay^2$ (Fig. 12) and then goes straight to the focus at $(\frac{1}{4a}, 0)$. Show that the total distance traveled, L_1 plus L_2, equals $p + \frac{1}{4a}$ so the total distance is the same for all values of q. (Assume that $p > aq^2$ so the starting point is to the left of the parabola.)

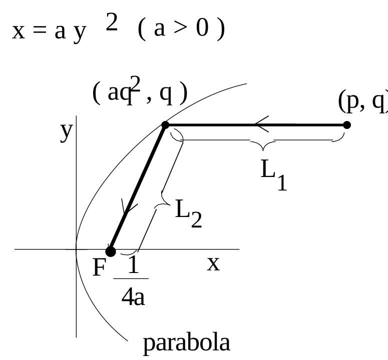


Fig. 12

The hyperbola also has a reflection property, but it is less useful than those for ellipses and parabolas.

> **Reflection Property of an Hyperbola**
>
> An hyperbolic mirror reflects light aimed at one focus to the other focus (Fig. 13).

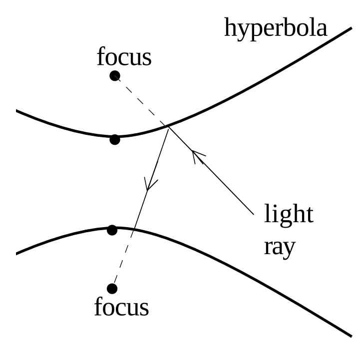


Fig. 13

Polar Coordinate Forms for the Conic Sections

In the rectangular coordinate system, the graph of the general quadratic equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is always a conic section, and the value of the discriminant $B^2 - 4AC$ tells us which type. In the polar coordinate system, an even simpler function describes all of the conic section shapes, and a single parameter in that function tells us the shape of the graph.

For $e \ge 0$, the polar coordinate graphs of $r = \dfrac{k}{1 \pm e\bullet\cos(\theta)}$ and $r = \dfrac{k}{1 \pm e\bullet\sin(\theta)}$

are conic sections with one focus at the origin.

If $e < 1$, the graph is an ellipse. (If $e = 0$, the graph is a circle.)

If $e = 1$, the graph is a parabola.

If $e > 1$, the graph is an hyperbola.

The number e is called the **eccentricity** of the conic section.

Fig. 14 shows graphs of $r = \dfrac{1}{1 + e\bullet\cos(\theta)}$ for several values of e.

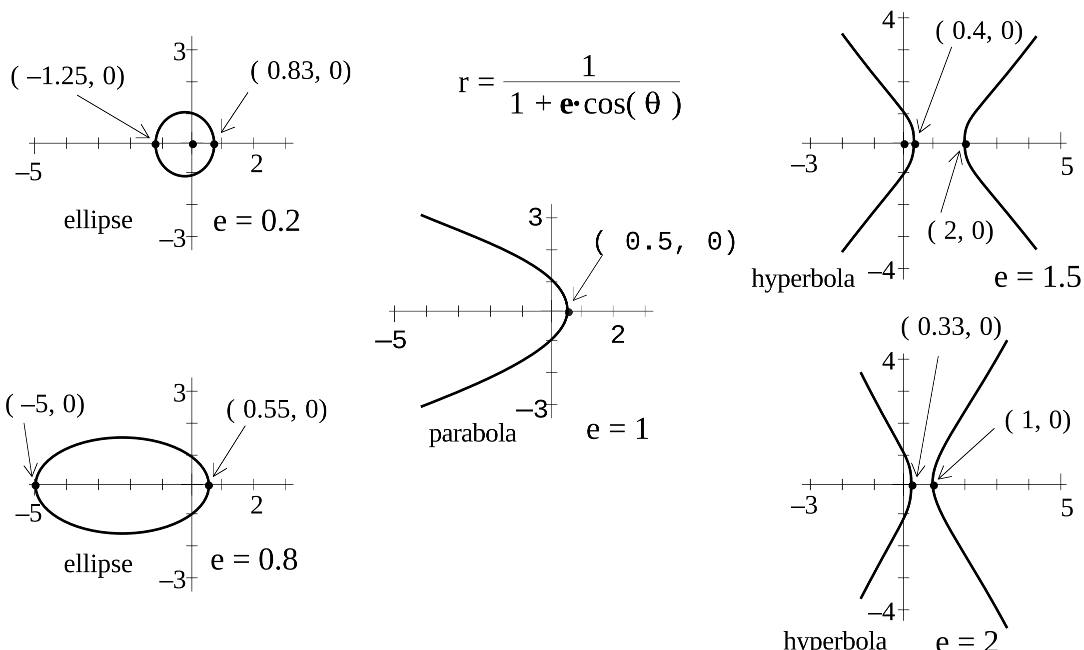


Fig. 14

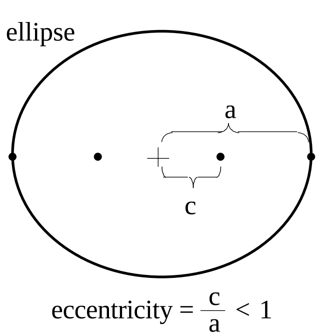


Fig. 15

For an ellipse, the eccentricity = $\dfrac{\text{dist(center, focus)}}{\text{dist(center, vertex)}}$ (Fig. 15) . If the eccentricity of an ellipse is close to zero, then the ellipse is "almost" a circle. If the eccentricity of an ellipse is close to 1, the ellipse is rather "narrow."

For a hyperbola, the eccentricity = $\dfrac{\text{dist(center, focus)}}{\text{dist(center, vertex)}}$ (Fig. 16) . If the eccentricity of a hyperbola is close to 1, then the hyperbola is "narrow." If the eccentricity of a hyperbola is very large, the hyperbola "opens wide."

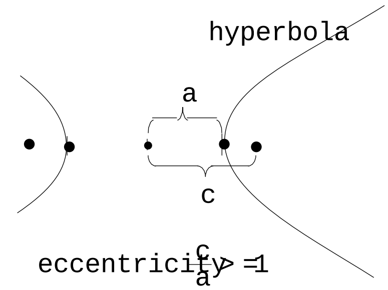


Fig. 16

Proof: The proof uses a strategy common in mathematics: move the problem into a system we know more about. In this case we move the problem from the polar coordinate system to the rectangular coordinate system, put the resulting equation into the form of a general quadratic equation, and then use the discriminant to determine the shape of the graph.

If $r = \dfrac{k}{1 + e \cdot \cos(\theta)}$, then $r + e \cdot r \cdot \cos(\theta) = k$. Replacing r with $\sqrt{x^2 + y^2}$ and $r \cdot \cos(\theta)$ with x, we get $\sqrt{x^2 + y^2} + e \cdot x = k$ and $\sqrt{x^2 + y^2} = k - e \cdot x$.

Squaring each side and collecting all of the terms on the right gives the equivalent general quadratic equation

$$(1 - e^2)x^2 + y^2 + 2kex - k = 0 \text{ so } A = 1 - e^2, B = 0, \text{ and } C = 1.$$

The discriminant of this general quadratic equation is $B^2 - 4AC = 0 - 4(1 - e^2)(1) = 4(e^2 - 1)$ so

if $e < 1$, then $B^2 - 4AC < 0$ and the graph is an ellipse,
if $e = 1$, then $B^2 - 4AC = 0$ and the graph is an parabola, and
if $e > 1$, then $B^2 - 4AC > 0$ and the graph is an hyperbola.

The graph of $r = \dfrac{k}{1 + e \cdot \cos(\theta)}$ is a conic section, and the value of the eccentricity tells which shape the graph has. We will not prove that one focus of the conic section is at the origin, but it's true.

Practice 3: Graph $r = \dfrac{k}{1 + (0.8) \cdot \cos(\theta)}$ for $k = 0.5, 1, 2$, and 3. What effect does the value of k have on the graph?

Subtracting a constant α from θ rotates a polar coordinate graph counterclockwise about the origin by an angle of α, but does not change the shape of the graph, so the graphs of

$r = \frac{k}{1 + e \cdot \cos(\theta - \alpha)}$ are all conic sections whose shapes depend on the size of the parameter e.

In particular, the polar graphs of $r = \frac{1}{1 - e \cdot \cos(\theta)}$, $r = \frac{1}{1 + e \cdot \sin(\theta)}$, and $r = \frac{1}{1 - e \cdot \sin(\theta)}$

are conic sections, rotations about the origin of the graphs of $r = \frac{1}{1 + e \cdot \cos(\theta)}$, since

$-\cos(\theta) = \cos(\theta - \pi)$, $\sin(\theta) = \cos(\theta - \pi/2)$, and $-\sin(\theta) = \cos(\theta - -\pi/2)$. Fig. 17 shows several of these graphs for e = 0.8 .

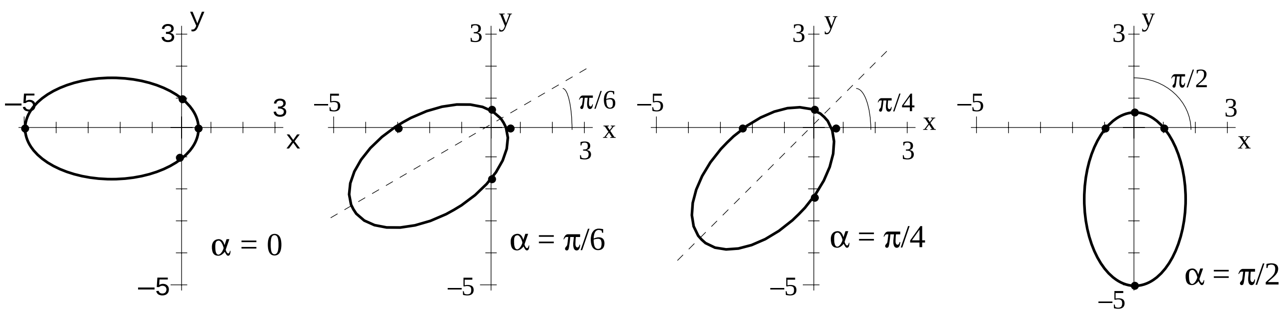


Fig. 17: $r = \frac{1}{1 + 0.8 \cdot \cos(\theta - \alpha)}$

The Path of Every Object in the Universe is a Conic Section (not really)

Rather than engage in endless philosophical discussion about how the planets ought to move, Tycho Brahe (1546–1601) had a better idea about how to find out how they actually do move — collect data! Even before the invention of the telescope, he built an observatory, and with the aid of devices like protractors he carefully cataloged the positions of the planets for 20 years. Just before his death, he passed this accumulated data to Johannes Kepler to edit and publish. From these remarkable data, Kepler deduced his three laws of planetary motion, the first of which says **each planet moves in an elliptical orbit with the sun at one focus** (Fig. 18). From Kepler's laws, Newton was able to deduce that a force, gravity, held the planets in orbits and that the force varied inversely as the square of the distance between the planet and the sun. In section ?.? we will assume that the force of gravity varies inversely with the square of the distance and show that the position of one object (e.g., a planet) with respect to another (e.g., the sun) is given by the the polar coordinate formula

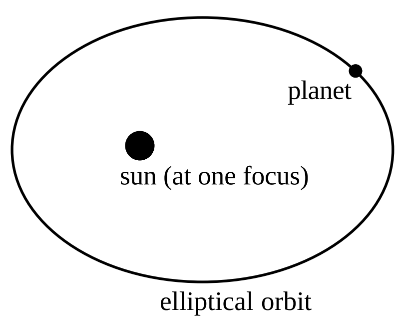


Fig. 18

$$r = \frac{h}{1 + (h-1)\cdot\cos(\theta)} \qquad \text{where} \qquad h = \frac{r_o \cdot v_o^2}{GM}$$

r_o = initial distance (m), v_o = initial velocity (m/s)

G = universal gravition constant ($6.7 \times 10^{-11} \frac{\text{N m}^2}{\text{kg}^2}$)

M = mass of the sun or "other object" (kg) .

You should recognize the pattern of this formula as the pattern for the conic sections with eccentricity

$$e = | h - 1 | = \left| \frac{r_o \cdot v_o^2}{GM} - 1 \right| .$$

In a "two body" universe, all motion paths are conic sections, and the shape of the conic section is determined by the value of $r_o \cdot v_o^2$. If the object is far away or moving very rapidly or both, then $r_o \cdot v_o^2 > 2GM$, the eccentricity is greater than one, and the path is an hyperbola. If the objects are relatively close and/or moving slowly, then $r_o \cdot v_o^2 < 2GM$ (the situation with each planet and the sun), the eccentricity is less than one and the path is an ellipse. If $r_o \cdot v_o^2 = 2GM$, then the eccentricity is 1 and the path is a parabola. If $r_o \cdot v_o^2 = GM$, then the eccentricity is 0 and the path is a circle. It is rare to encounter values of r_o and v_o so $r_o \cdot v_o^2$ exactly equals GM or 2GM.

The universe obviously contains more than two bodies and the paths of most objects are not conic sections, but there are still important situations in which the force on an object is due almost entirely to the gravitational attraction between it and one other body. For example, scientists and engineers use the position formula to determine the orbital position and velocity needed to put a satellite into an orbit with the desired eccentricity, and the position formula was used to help calculate how close Voyager 2 should come to Jupiter and Saturn so the gravity of those planets could be used to change the path of Voyager 2 to a hyperbola and send it on to other planets (Fig. 19). The conic sections even appear at less grand scales. In a vacuum, the path of a thrown baseball (or bat) is a parabola, unless it is thrown hard enough to achieve an elliptical orbit. And at the subatomic scale, Rutherford (1871 – 1937) discovered that alpha particles shot toward the nucleus of an atom are repelled away from the nucleus along hyperbolic paths (Fig. 20). Conic sections are everywhere.

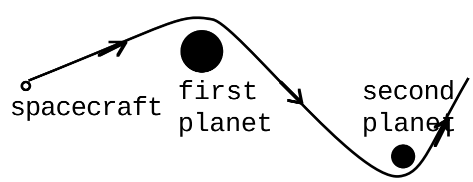


Fig. 19

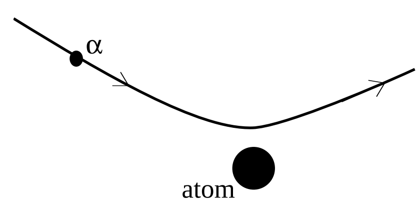


Fig. 20

Elliptical Orbits (Optional)

When a planet orbits a sun, the orbit is an ellipse, and we can use information about ellipses to calculate information about these orbits.

The position of a planet in elliptical orbit around a sun is given by the polar equation

$$r = \frac{k}{1 + e \cdot \cos(\theta)}$$

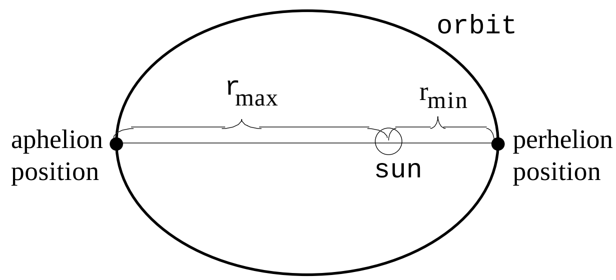


Fig. 21

for some value of the eccentricity $e < 1$ (Fig. 21). The planet is closest (at perhelion) to the sun when $\theta = 0$ and this minimum distance is $r_{min} = \frac{k}{1 + e}$. The greatest distance (at aphelion) occurs when $\theta = \pi$ and that distance is $r_{max} = \frac{k}{1 - e}$. If the width of the ellipse (technically, the length of the major axis) is 2a, then

$$2a = r_{min} + r_{max} = \frac{k}{1 + e} + \frac{k}{1 - e} = \frac{2k}{1 - e^2}$$

so $k = a(1 - e^2)$ and the position of the planet is given by

$$r = \frac{a(1 - e^2)}{1 + e \cdot \cos(\theta)} \quad \text{with } r_{min} = \frac{a(1 - e^2)}{1 + e} = a(1 - e) \text{ and } r_{max} = \frac{a(1 - e^2)}{1 - e} = a(1 + e).$$

Example 4: We want to put a satellite in an elliptical orbit around the earth (radius ≈ 6360 km) so the maximum height of the satellite is 20,000 km and the minimum height is 10,000 km (Fig. 22). Find the eccentricity of the orbit and give a polar formula for its position.

Solution: r_{max} = maximum height plus the radius of the earth = 26,360 km and r_{min} = minimum height plus the radius of the earth = 16,360 km so $a(1 + e) = 26,360$ and $a(1 - e) = 16,360$. Dividing these last two quantities, we have

$$\frac{r_{max}}{r_{min}} = \frac{26360}{16360} = \frac{a(1 + e)}{a(1 - e)} = \frac{1 + e}{1 - e} \quad \text{and } \mathbf{e} = \frac{10000}{42720} \approx \mathbf{0.234} .$$

Using $r_{min} = a(1 - e) = a(1 - 0.234) = 16360$ we have $a \approx 21358$.

Finally, $$r = \frac{a(1 - e^2)}{1 + e \cdot \cos(\theta)} = \frac{\mathbf{20189}}{\mathbf{1 + (0.234) \cdot \cos(\theta)}} .$$

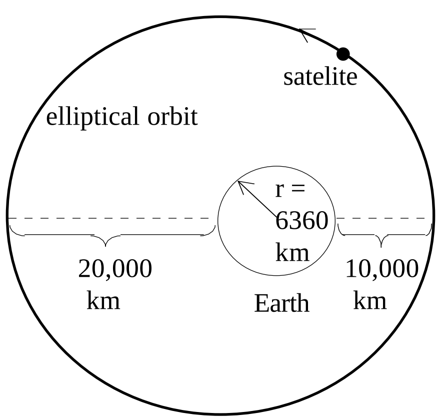


Fig. 22

Practice 4: Pluto's orbit has an eccentricity of 0.2481 and its semimajor axis is 5,909 million kilometers. Find the minimum and maximum distance of Pluto from the sun during one orbit. The orbit of Neptune has an eccentricity of 0.0082 with a semimajor axis of 4,500 million kilometers. Is Neptune ever farther from the sun than Pluto?

PROBLEMS

Reflection properties

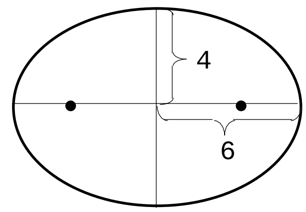


Fig. 23

1. For the ellipse in Fig. 23, how far does a ball travel as it moves from one focus to the ellipse and on to the othe focus?

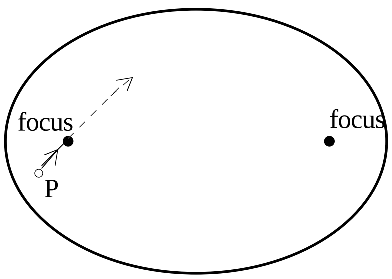


Fig. 24

2. In Fig. 24 a ball rolls from point P over the the focus at A and keeps rolling. Sketch the path of the ball for the first 5 bounces it makes off of the ellipse. What does the path of the ball look like after "a long time?"

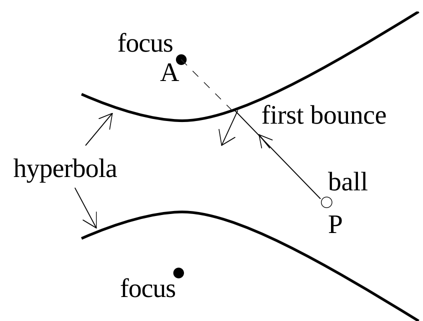


Fig. 25

3. In Fig. 25 a ball rolls from point P toward the the focus at A and bounces off of the hyperbola. Sketch the path of the ball for the first 5 bounces it makes off of the hyperbola. What does the path of the ball look like after "a long time?"

4. An explosion is set off at one focus inside a very strong ellipsoidal shell. What might happen to a piece of graphite located at the other focus?

5. A straight wave front is approaching a parabolic jetty. Why wouldn't you want your boat to be at the focus of the parabola?

6. The members of a marching band are grouped near point A (Fig. 26), the focus of a parabola. At a signal from the director, the band members each march (at the same speed) in different directions toward the parabola, immediately turn and then march due west. What shape will the formation have after all of the marchers have made their turns?

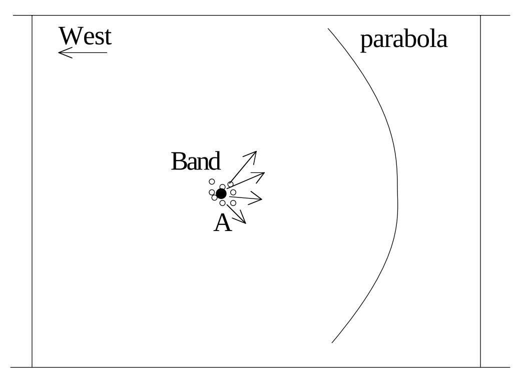


Fig. 26

7. A reflecting telescope is built with a parabolic mirror and a hyperbolic mirror (Fig. 27) so F_1 is the focus of the parabola and F_1 and F_2 are the foci of the hyperbola.

   Trace the paths of the incoming parallel light rays a, b, and c as they rlect off both mirrors. Why is the eyepiece located at F_2 ?

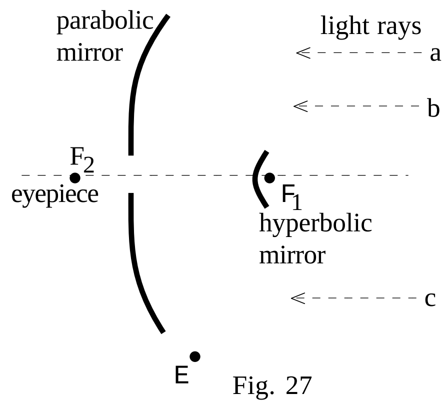


Fig. 27

8. Make a slight design change in the telescope in Fig. 27 so the eyepiece can be located at the point E to the side of the parabolic mirror.

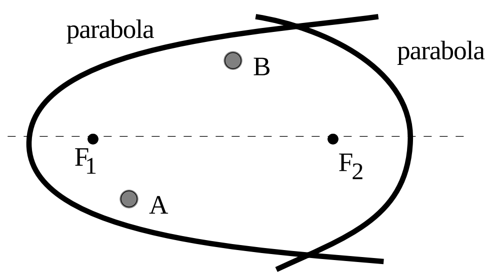


Fig. 28

9. The billiards table in Fig. 28 consists of parts of two parabolas: the right parabola has focus F_1 and the left parabola has focus F_2 . (a) Determine a strategy for shooting the balls located at A and B to make a two–cushion (i.e., two bounce) shot into the hole located at F_2 . (b) Are there any places on the table where your strategy in part (a) does not work?

10. The billiards table in Fig. 29 consists of parts of two ellipses: the shot ellipse has foci F_1 and F_2 and the tall ellipse has foci F_2 and F_3 . (a)

    (a) Determine a strategy for shooting the balls located at A and B to make a two–cushion (i.e., two bounce) shot into the hole located at F_3 .

    (b) Are there any places on the table where your strategy in part (a) does not work?

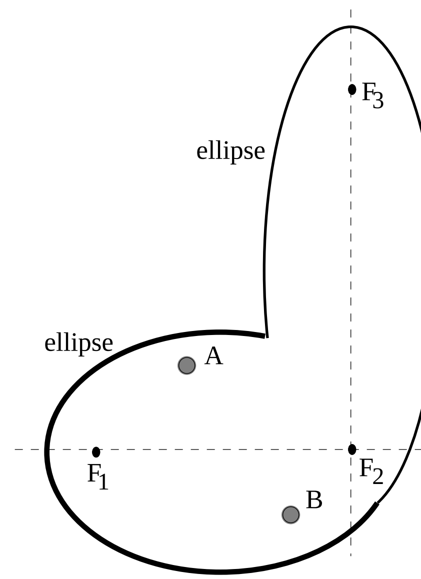


Fig. 29

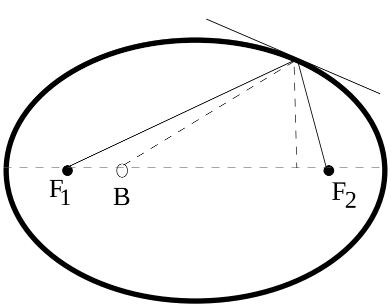


Fig. 30

11. Use Fig. 30 to help explain geometrically why a ball located on the major axis of an ellipse between the two foci is always reflected back to a point on the major axis between the two foci.

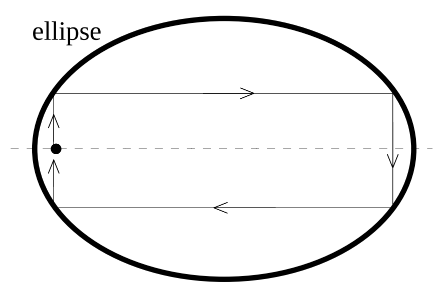


Fig. 31

12. Is a rectangular reflection path (Fig. 31) possible for the ellipse

    (a) $\frac{x^2}{25} + \frac{y^2}{16} = 1$? (b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$?

Polar forms

In problems 13–18, determine the eccentricity of each conic section, identify the shape, and determine where it crosses the x and y axes.

13. $r = \dfrac{11}{3 + 5\bullet\cos(\theta)}$ 14. $r = \dfrac{8}{7 + 3\bullet\sin(\theta + \pi/6)}$ 15. $r = \dfrac{1}{2 + 2\bullet\sin(\theta - \pi/3)}$

16. $r = \dfrac{-4}{3 - 3\bullet\cos(\theta)}$ 17. $r = \dfrac{17}{7 - 5\bullet\cos(\theta + 3\pi)}$ 18. $r = \dfrac{3}{4 - 2\bullet\sin(\theta + \pi/11)}$

In problems 19–24, sketch each ellipse and determine the coordinates of the foci. (Reminder: In the standard polar coordinate form used here, one focus is always at the origin.)

19. $r = \dfrac{6}{2 + \cos(\theta)}$ 20. $r = \dfrac{6}{2 + \sin(\theta)}$ 21. $r = \dfrac{12}{3 - \sin(\theta)}$

22. $r = \dfrac{12}{3 - \cos(\theta)}$ 23. $r = \dfrac{3}{2 + \sin(\theta - \pi/4)}$ 24. $r = \dfrac{3}{2 + \cos(\theta + \pi/4)}$

In problems 25 and 26, represent the length and area of each ellipse as definite integrals and use Simpson's rule with n = 100 to approximate the values of the integrals.

25. $r = \dfrac{1}{1 + 0.5\bullet\cos(\theta)}$ 26. $r = \dfrac{1}{1 + 0.9\bullet\cos(\theta)}$

Conic section paths:

Problems 27–30 refer to the two objects in Fig. 32. Determine the shape of the path of object B. (Object A has mass 10^{19} kg, r = 10^5 m)

27. v = 17.6 m/s 28. v = 115 m/s

29. v = 120 m/s

30. Determine a velcity for B so that the path is a parabola.

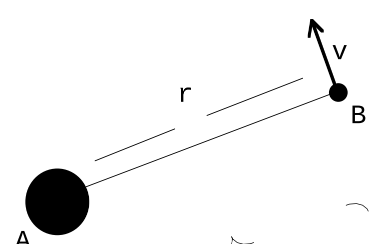


Fig. 32

31. An object at a distance r from the center of a planet of mass M (Fig. 33) has velocity v. Determine conditions on v as a function of r, M, and G so the resulting path is (a) circular, (b) elliptical, (c) parabolic, and (d) hyperbolic.

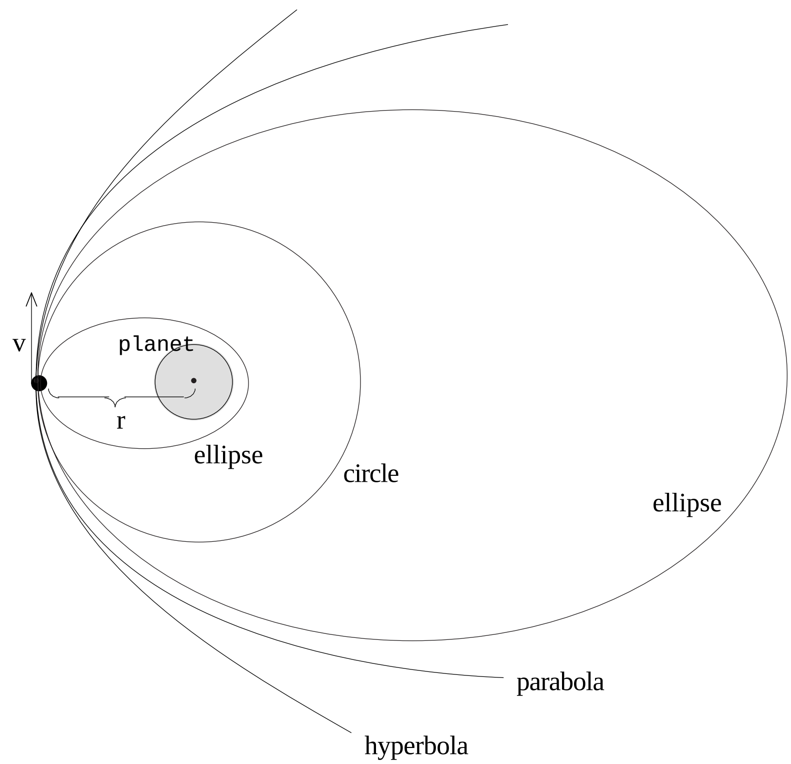


Fig. 33

Problems 32–34 refer to Earth and moon orbits: use

$G = 6.7\text{x}10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2$,
$M_{Earth} = 5.98\text{x}10^{24}$ kg,
$r_{Earth} = 6.360\text{x}10^{6}$ m,
$M_{moon} = 7\text{x}10^{22}$ kg , and
$r_{moon} = 1.738\text{x}10^{6}$.

32. At the Earth's surface, determine a velocity so that the resulting path is circular. This is the minimum velocity needed of achieve "orbit," a very low orbit.

33. At the Earth's surface, determine a velocity so that the resulting path is parabolic. This is the minimum velocity needed to escape from orbit, and is called the "escape velocity".

34. At the moon's surface, determine the minimum orbital velocity and the (minimum) escape velocity.

Elliptical orbits (optional)

35. We want to put a satellite into orbit around Earth so the maximum altitude of the satellite is 1000 km and the minimum altitude is 800 km. Find the eccentricity of this orbit and give a polar coordinate formula for its position.

36. The Earth follows an elliptical orbit around the sun, and this ellipse has a semimajor axis of $149.6\text{x}10^{6}$ km and an eccentricity of 0.017. (a) Determine the maximum and minimum distances of the Earth from the sun. (b) How far apart are the two foci of this ellipse?

37. Determine the altitude needed for an Earth satellite to make one orbit on a circular path every 24 hours ("geosyncronous"). (Since the orbit is circular and the satellite makes one orbit every 24 hours, you can determine the velocity v (in m/s) as a function of the distance r from the center of the Earth. Since the orbit is circular, $e = 0$ and $r\cdot v^2 = GM$.)

Section 9.6 PRACTICE Answers

Practice 1: "Shoot the ball toward the left focus." Since the table is an ellipse, the ball will roll over the focus, hit the wall, and be reflected into the hole at the other focus.

Practice 2: L_1 = distance from (p, q) to $(a \cdot q^2, q)$ = $p - a \cdot q^2$.

L_2 = distance from $(a \cdot q^2, q)$ to $(\frac{1}{4a}, 0)$

$$= \sqrt{(a \cdot q^2 - \frac{1}{4a})^2 + (q-0)^2} = \sqrt{a^2q^4 - 2aq^2(\frac{1}{4a}) + \frac{1}{16a^2} + q^2}$$

$$= \sqrt{a^2q^4 - \frac{1}{2}q^2 + \frac{1}{16a^2} + q^2}$$

$$= \sqrt{a^2q^4 + \frac{1}{2}q^2 + \frac{1}{16a^2}} = \sqrt{(aq^2 + \frac{1}{4a})^2} = aq^2 + \frac{1}{4a} .$$

Therefore, $L_1 + L_2 = (p - a \cdot q^2) + (aq^2 + \frac{1}{4a}) = p + \frac{1}{4a}$.

Practice 3: The graphs of $r = \dfrac{k}{1 + (0.8) \cdot \cos(\theta)}$ for k = 0.5, 1, and 2 are shown in Fig. 34.

Each graph is an ellipse with eccentricity 0.8 and one focus at the origin. The value of k determines the size of the ellipse. The larger the magnitude of k, the larger the ellipse.

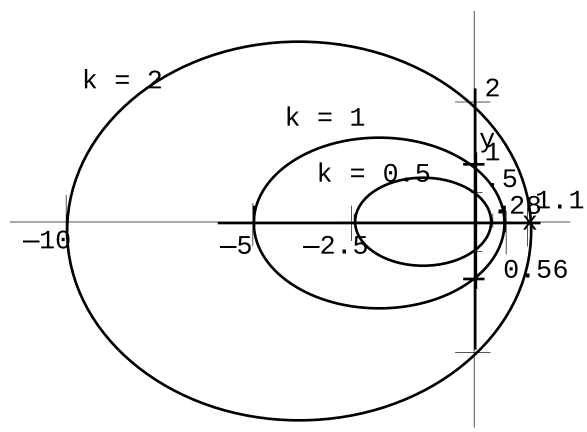


Fig. 34

Practice 4: For Pluto: e = 0.2481 and semimajor axis length = 5,909 km (a = 5,909 km).

r_{min} = a(1 – e) = 5909(0.7519) ≈ 4443 km.

r_{max} = a(1 + e) = 5909(1.2481) ≈ 7375 km.

For Neptune: e = 0.0082 and a = 4,500 km.

r_{min} = a(1 – e) = 4500(0.9918) ≈ 4463 km, a distance closer than Pluto at its closest!

r_{max} = a(1 + e) = 4500(1.0082) ≈ 4537 km.

In fact, Neptune is the farthest planet from the sun between January 1979 and March 1999.

Appendix: Reflection Property of the Ellipse

Let P = (x,y) be on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci at (–c, 0) and (c, 0) for $c = \sqrt{a^2 - b^2}$.

Then the slopes (Fig. 40) are m_1 = slope from P to (–c,0) = $\frac{y}{x+c}$, m_2 = slope from P to (c,0) = $\frac{y}{x-c}$
,

and m_3 = slope of tangent line to ellipse at (x,y) = $\frac{-x}{y}\frac{b^2}{a^2}$ (by implicit differentiation) .

We know that $\tan(\alpha) = \frac{m_3 - m_1}{1 + m_1 m_3}$ and $\tan(\beta) = \frac{m_2 - m_3}{1 + m_2 m_3}$, and we want to show that $\alpha = \beta$ or, equivalently, that $\tan(\alpha) = \tan(\beta)$.

$$\tan(\alpha) = \frac{m_3 - m_1}{1 + m_1 m_3} = \frac{\frac{-x}{y}\frac{b^2}{a^2} - \frac{y}{x+c}}{1 + \frac{y}{x+c}\frac{-x}{y}\frac{b^2}{a^2}} \quad \text{multiply top \& bottom by } ya^2(x+c)$$

$$= \frac{-xb^2(x+c) - y^2a^2}{ya^2(x+c) - xyb^2} = \frac{-x^2b^2 - xb^2c - y^2a^2}{xya^2 + ya^2c - xyb^2}$$

$$= \frac{-xb^2c - (x^2b^2 + y^2a^2)}{xy(a^2 - b^2) + ya^2c} \qquad x^2b^2 + y^2a^2 = a^2b^2 \text{ and } a^2 - b^2 = c^2$$

$$= \frac{-xb^2c - a^2b^2}{xyc^2 + ya^2c} = \frac{-b^2}{yc}\frac{xc + a^2}{xc + a^2} = \frac{-b^2}{yc} .$$

Similarly,

$$\tan(\beta) = \frac{m_2 - m_3}{1 + m_2 m_3} = \frac{\frac{y}{x-c} - \frac{-x}{y}\frac{b^2}{a^2}}{1 + \frac{y}{x-c}\frac{-x}{y}\frac{b^2}{a^2}} \quad \text{multiply top \& bottom by } ya^2(x-c)$$

$$= \frac{y^2a^2 + xb^2(x-c)}{ya^2(x-c) - xyb^2} = \frac{x^2b^2 - xb^2c + y^2a^2}{xya^2 - ya^2c - xyb^2}$$

$$= \frac{-xb^2c + (x^2b^2 + y^2a^2)}{xy(a^2 - b^2) - ya^2c} \qquad (\, x^2b^2 + y^2a^2 = a^2b^2 \text{ and } a^2 - b^2 = c^2\,)$$

$$= \frac{-xb^2c + a^2b^2}{xyc^2 - ya^2c} = \frac{-b^2}{yc}\frac{xc - a^2}{xc - a^2} = \frac{-b^2}{yc} = \tan(\alpha). \quad \text{(Yes!)}$$

Reflection Property of the Parabola $x = ay^2$ with focus $(\frac{1}{4a}, 0)$

The slopes of the line segments in Fig. 41 are m_1 = slope of "incoming" ray = 0,

m_2 = slope from P to focus = $\dfrac{y}{x - \frac{1}{4a}} = \dfrac{4ay}{4ax - 1} = \dfrac{4ay}{4a^2y^2 - 1}$, and, by implicit differentiation,

m_3 = slope of the tangent line at (x,y) = $\dfrac{1}{2ay}$.

We know that $\tan(\alpha) = \dfrac{m_3 - m_1}{1 + m_1 m_3}$ and $\tan(\beta) = \dfrac{m_2 - m_3}{1 + m_2 m_3}$, and we want to show that $\alpha = \beta$

or, equivalently, that $\tan(\alpha) = \tan(\beta)$.

$$\tan(\alpha) = \frac{m_3 - m_1}{1 + m_1 m_3} = m_3 = \frac{1}{2ay}$$

$$\tan(\beta) = \frac{m_2 - m_3}{1 + m_2 m_3} = \frac{\dfrac{4ay}{4a^2y^2 - 1} - \dfrac{1}{2ay}}{1 + \dfrac{4ay}{4a^2y^2 - 1}\,\dfrac{1}{2ay}}$$ multiply top & bottom by $(4a^2y^2 - 1)(2ay)$

$$= \frac{8a^2y^2 - 4a^2y^2 + 1}{(4a^2y^2 - 1)(2ay) + 4ay} = \frac{4a^2y^2 + 1}{8a^3y^3 - 2ay + 4ay}$$

$$= \frac{4a^2y^2 + 1}{8a^3y^3 + 2ay} = \frac{4a^2y^2 + 1}{2ay(4a^2y^2 + 1)} = \frac{1}{2ay} = \tan(\alpha) .$$ (Yes!)

There are other, more geometric ways to prove this result.