Calculus: A Modern Approach

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Contents

P-1: Functions

The Function Concept

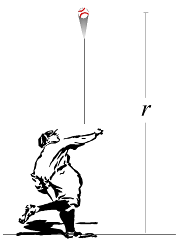
Intuitively, a function is any process that produces only one output for each input. For example, the output on a calculator screen is a function of the input from the keyboard, since pressing a number such as "9" once on the calculator keyboard produces only the number "9" on the screen.

In calculus, most functions have numerical inputs and outputs. Moreover, functions in calculus are often developed as *models* of real-world processes that produce only one output for each input.

Functions as Models
Functions model processes which produce only one output for each input

These function models may be presented either numerically, graphically, or symbolically.

For example, suppose a ball is thrown into the air with an initial speed of 88 feet per second (about 60 m.p.h.) from an initial height of 7 feet. Then the height r of the ball is a function of the time t that has elapsed since the ball was released.



P1-1: Height r at time t.

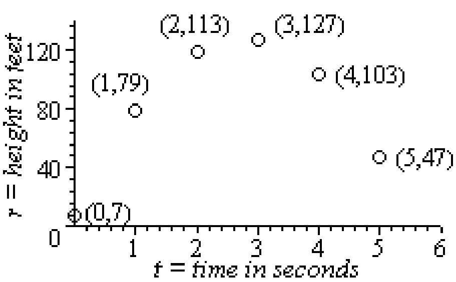
We can generate a *numerical representation* of the ball's height as function of time by measuring the height at one second intervals for 5 seconds. Doing so results in the following data set

t in seconds	0	1	2	3	4	5
r in feet	7	79	113	127	103	47

(0.1)

That is, the ball has a height of 79 feet after 1 second, a height of 113 feet after 2 seconds, and so on.

The data set for a function can then be plotted as points in the plane. The result is known as a *graphical representation* of the function. For example, plotting the points from the data set (0.1) as points in the plane results in a graphical representation of the ball's height r as a function of time t.



P1-2: Data set plotted as points in the plane

Finally, functions can also be written *symbolically* in the form

$$f\,(input) = \text{“formula to be applied to the } input\text{”}$$

In doing so, we often use a letter such as x to denote the *input variable.* The output from that process is denoted by $f\,(x)$, which is pronounced “eff of x.” Or we may denote the input variable by t and then denote the output by $r\,(t)$, which is “r of t.” Graphical and numerical representations of a function can subsequently be generated from a symbolic representation.

EXAMPLE 1 If a ball is thrown upward from an initial height of 7 feet with an initial velocity of 88 feet per second, then the height $r\,(t)$ of the ball at time t is given by

$$r\,(t) = 7 + 88t - 16t^2 \tag{0.2}$$

Use (1.2) to generate the numerical and graphical representations of the data set representing heights measured at half-second intervals for 5.5 seconds.

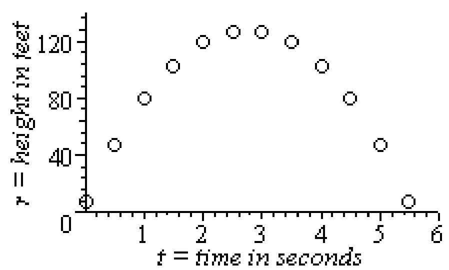
Solution: The data set (0.1) represents heights measured at one second intervals, so we need only supplement it by computing $r\,(0.5)$, $r\,(1.5)$, $r\,(2.5)$, $r\,(3.5)$, $r\,(4.5)$, and $r\,(5.5)$. To compute $r\,(0.5)$, we let $t = 0.5$ in (1.2) to obtain

$$r\,(0.5) = 7 + 88 \cdot (0.5) - 16 \cdot (0.5)^2 = 47$$

Likewise, we compute the remaining heights to obtain the data set

t	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5
r	7	47	79	103	113	127	127	113	103	79	47	7

A plot of the points in the data set then yields a graphical representation of $r\,(t)$:



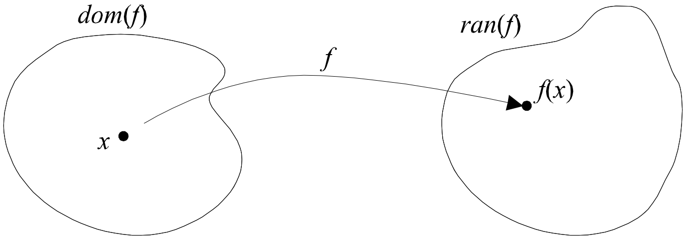
P1-3: A Graphical Representation of $r\,(t)$

Check your Reading *What is the height of the ball after 2.5 seconds? Which representation of* $r\,(t)$ *did you use to determine your answer?*

Domain and Range

Let’s translate the function concept into a mathematical definition. To begin with, the set of inputs for which a function f is defined is called the *domain of the function*, which is denoted $dom\,(f)$, and the set of output values produced by a function f is known as the *range of the function*, which is denoted $ran\,(f)$. The mathematical definition of function then follows:

Definition P.1: A function f is a rule which transforms each input in a set $dom\,(f)$ into only one output in a set $ran\,(f)$.



P1-4: Only one $f\,(x)$ in $ran\,(f)$ for each x in $dom\,(f)$

By convention, if no domain is stated for a function, then the domain is assumed to be the *largest* set of inputs for which the function is defined and produces real number outputs. We call this domain the *natural domain* of the function.

Domains and ranges are often expressed in *interval notation.* For example, $[a, b)$ is the set of points x on the x-axis for which $a \leq x < b$.

P1-5: The interval $[a, b)$

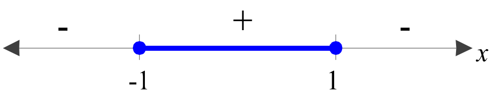
To illustrate, $[0, 1]$ denotes all the real numbers between 0 and 1, inclusive.

EXAMPLE 2 Find the domain and range of $f\,(x) = \sqrt{1 - x^2}$

Solution: In order for the function to be defined, the expression under the square root cannot be negative. That is, we require that $1 - x^2 \geq 0$. Since $1 - x^2 = (1 - x)\,(1 + x)$, the expression $1 - x^2$ can change signs only at 1 and -1. However, if $x = -2$, then $1 - x^2$ is

$$1 - (-2)^2 = -3 < 0$$

Thus, $1 - x^2$ must be negative for $x < -1$. Similarly, if we test points in $[-1, 1,]$ and $[1, \infty)$, we find that



P1-6: $1 - x^2 \geq 0$ if $-1 \leq x \leq 1$

As a result, the domain of $f\,(x) = \sqrt{1 - x^2}$ is

$$dom\,(f) = [-1, 1]$$

By convention, $\sqrt{u} \geq 0$ for u a non-negative real number. Since $1 - x^2 \leq 1$ for all x in $dom\,(f)$ and since the square root cannot be negative, the range of f is

$$ran\,(f) = [0, 1]$$

In this textbook we work with domains of functions much more frequently than we work with ranges of functions.

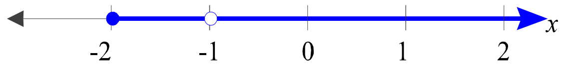
EXAMPLE 3 Find the domain of the function

$$f(x) = \frac{x}{1 - \sqrt{x+2}}$$

Solution: The expression $x+2$ under the radical cannot be negative, and $x+2 \geq 0$ implies that $x \geq -2$. In addition, $\sqrt{x+2}$ cannot be 1 since otherwise we would have division by 0. But $\sqrt{x+2} \neq 1$ implies that $x+2 \neq 1$, which is the same as $x \neq -1$. Thus, the domain of f is the set of inputs $x \geq -2$, but not including -1. In interval notation, we have

$$dom(f) = [-2, -1) \cup (-1, \infty)$$

where $\cup$ represents the *union* of the two intervals.



P1-7: $[-2, -1) \cup (-1, \infty)$

Check your Reading *What is the domain of* $f(x) = \sqrt{x-1}$?

Operations with Functions

We can use the fact that the outputs from functions are real numbers to define operations on functions. If f and g are functions, then their sum $f+g$ is defined

$$(f+g)(x) = f(x) + g(x) \tag{0.3}$$

for every x that belongs to both $dom(f)$ and $dom(g)$. Likewise, for every x that belongs to both $dom(f)$ and $dom(g)$, we define $f-g$, fg, and f/g to be

$$(f-g)(x) = f(x) - g(x) \tag{0.4}$$

$$(fg)(x) = f(x)\,g(x) \tag{0.5}$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad (\text{ where } g(x) \neq 0 \text{ }) \tag{0.6}$$

A consequence of (0.5) is that if k is a constant, then

$$(kf)(x) = kf(x)$$

For example, if $f(x) = x^2$, then $(3f)(x) = 3f(x) = 3x^2$.

EXAMPLE 4 Find $f+g$, $f-g$, fg, and f/g when

$$f(x) = x^2 + 2, \qquad g(x) = x^4 - 2x$$

Solution: The definitions (0.3), (0.4), (0.5), and (0.6) imply that

$$(f+g)(x) = f(x) + g(x) = x^2 + 2 + x^4 - 2x = x^4 + x^2 - 2x + 2$$

$$(f-g)(x) = f(x) - g(x) = \left(x^2+2\right) - \left(x^4 - 2x\right) = -x^4 + x^2 + 2x + 2$$

$$(fg)(x) = f(x)\,g(x) = \left(x^2+2\right)\left(x^4-2x\right) = x^6 - 2x^3 + 2x^4 - 4x$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2+2}{x^4-2x}$$

We can also define an operation between functions by using the *output from* a function g as the *input to* a function f. The result is called the *composition* of f with g and is denoted $f \circ g$. That is, composition of f with g is defined

$$(f \circ g)(x) = f(g(x))$$

Equivalently, we often interpret $f \circ g$ as

$$(f \circ g)(x) = f(\text{input}) \tag{0.7}$$

where the input is $g(x)$.

EXAMPLE 5 Evaluate $f \circ g$ and $g \circ f$ when $f(x) = x^2 + x$ and $g(x) = 2x + 3$

Solution: The equation (0.7) tells us that $(f \circ g)(x) = f(\text{input})$, where

$$f(\text{input}) = (\text{input})^2 + (\text{input})$$

and where the input is $g(x) = 2x + 3$. Replacing the input by $2x + 3$ thus yields

$$f(2x+3) = (2x+3)^2 + (2x+3)$$

As a result, $f \circ g$ is given by

$$(f \circ g)(x) = (2x+3)^2 + (2x+3) = 4x^2 + 14x + 12$$

In contrast, equation (0.7) tells us that $(g \circ f)(x) = g(\text{input})$, where

$$g(\text{input}) = 2(\text{input}) + 3$$

and where the input is $f(x) = x^2 + x$. Replacing the input by $x^2 + x$ thus yields

$$g\left(x^2 + x\right) = 2\left(x^2 + x\right) + 3$$

As a result, $g \circ f$ is given by

$$(g \circ f)(x) = 2\left(x^2 + x\right) + 3 = 2x^2 + 2x + 3$$

Check your Reading *Is $f \circ g$ the same as $g \circ f$ in example 5?*

Polynomial Functions

Functions formed solely by addition, subtraction, and multiplication of real numbers and an input variable are known as *polynomial functions.* In particular, we have the following definition.

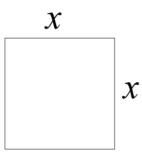
Definition 1.2: If $a_0, a_1, \ldots, a_n$ are numbers with $a_n \neq 0$, then the function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

is a *polynomial function of degree n.*

The numbers $a_0, \ldots, a_n$ are known as the *coefficients* of the polynomial function $f(x)$. Polynomials often arise in modeling simple input-output processes.

EXAMPLE 6 Write the area A of a square as a function of the length x of one of its sides, and then find its domain and range:



P1-8: Area is a function of x

Solution: Since x^2 is the area of a square of length x, we can write the function relationship symbolically as

$$A(x) = x^2$$

Polynomial functions are added by summing coefficients of like terms. For example, if $f(x) = 2x^2 + 1$ and $g(x) = 3x^2 + 2x + 3$, then

$$\begin{aligned} f(x) + g(x) &= 2x^2 + 1 + 3x^2 + 2x + 3 \\ &= 5x^2 + 2x + 4 \end{aligned}$$

However, the product of two polynomials is computed with the distributive law, such as in

$$\begin{aligned} f(x)\,g(x) &= \left(2x^2 + 1\right)\left(3x^2 + 2x + 3\right) \\ &= 2x^2\left(3x^2 + 2x + 3\right) + 1\left(3x^2 + 2x + 3\right) \\ &= 6x^4 + 4x^3 + 9x^2 + 2x + 3 \end{aligned}$$

When entering functions into a graphing calculator, an applet, or a computer algebra system, it is important to remember the conventions for the order of operations:

> **Order of Operations:**
> Powers take precedence over multiplication and division. Multiplication and division take precedence over addition and subtraction.

Grouping takes predence over all other operations. For example, in most computer algebra systems, the code fragment

$$x/4 + x$$

represents the expression $\frac{x}{4} + x$. To enter the expression

$$\frac{x}{4 + x}$$

the code fragment must be of the form $x/(4 + x)$ because division takes precedence over addition. Similarly, the expression xy^2 is **not** the same as $(xy)^2$.

Exercises

Evaluate each function $f(x)$ below at the given input for x, and simplify completely.

1. $f(x) = x^2 - 5, \quad x = 3$
2. $f(x) = 3x^2 - 5x + 2, \quad x = 0.5$
3. $f(x) = 2x^3 + 6x, \quad x = 2$
4. $f(x) = 13 - 4x + 3x^2, \quad x = 1.7$
5. $f(x) = \sqrt{x^2 - 5x}, \quad x = 5$
6. $f(x) = \sqrt{2x^2 - \pi x}, \quad x = \pi$
7. $f(x) = x^2 - 5, \quad x = t - 1$
8. $f(x) = 3x^2 - 5x + 2, \quad x = t + 2$
9. $f(x) = 2x^3 + 6x, \quad x = t - 5$
10. $f(x) = \sqrt{x^2 - 5x}, \quad x = 2t - 5$

Find the domains of the following functions:

11. $f(x) = \sqrt{x - 5}$
12. $f(x) = \sqrt{5 - x}$
13. $f(x) = \sqrt{16 - x^2}$
14. $f(x) = \sqrt{x^2 - 9}$
15. $f(x) = \dfrac{1}{x + 4}$
16. $f(x) = \dfrac{1}{2x - 9}$
17. $f(x) = \dfrac{2x + 1}{\sqrt{5 - x}}$
18. $f(x) = \dfrac{2x + 1}{(2x - 9)\sqrt{5 - x}}$

Numerical:[1] *Construct a numerical representation by evaluating the function at the given input values. Then construct a graphical representation by plotting the resulting input-output pairs as points in the xy-plane:*

19.

x	1	2	3	4	5
$f(x) = x^2 - 2x + 1$					

20.

x	-2.5	-1.5	0.5	1.5	2.5
$f(x) = -2x + 1$					

21.

x	1	2	3	4	5
$f(x) = x^2 + 2x - 1$					

22.

x	1.1	1.01	1.001	1.0001
$f(x) = x^3 - x$				

23. Sketch a graphical representation of $f(x) = x^2 + 1$. What does the range of $f(x)$ appear to be?

24. Sketch a graphical representation of $f(x) = x^2 + 2x$. What does the range of $f(x)$ appear to be?

25. Sketch a graphical representation of $f(x) = 3x - 1$. What does the range of $f(x)$ appear to be?

26. Sketch a graphical representation of $f(x) = x^3 - 3x$. What does the range of $f(x)$ appear to be?

In 27-38, assume that $f(x) = x^2 - 1$, $g(x) = 2 - x^3$, and $h(x) = \sqrt{2x + 3}$:

[1] **Numerical** exercises require either the use of a table or the analysis of data.

27.	$(f+g)(x)$	28.	$(g-h)(x)$
29.	$(2f)(x)$	30.	$(2f+g)(x)$
31.	$(fg)(x)$	32.	$(gh)(x)$
33.	$(f/g)(x)$	34.	$(h\circ g)(x)$
35.	$(f\circ h)(x)$	36.	$h\left(t^2+1\right)$
37.	$(h\circ g)(1)$	38.	$((f(f-g))\circ h)(x)$

39. Consider the position function $r(t) = -16t^2 + 48$ which represents the height of an object above the ground in feet at time t in seconds when the object is in free-fall near the surface of the earth.

(a) How high above the ground is the object initially (i.e., at $t = 0$)?
(b) How high above the ground is the object at $t = 1$ second?
(c) At what time t does it strike the ground?(Hint: solve $r(t) = 0$)

40. Consider the position function $r(t) = -16t^2 + 16t$ which represents the height of an object above the ground in feet at time t in seconds.

(a) How high above the ground is the object initially (i.e., at $t = 0$)?
(b) How high above the ground is the object at $t = 1/2$ second?
(c) At what time t does it strike the ground?

41. The position function $r(t) = -16t^2 + 96t$ represents the height in feet above level ground of a golf ball that is struck with the club at time $t = 0$.

(a) **Numerical:** Create a data set for the function by evaluating $r(t)$ at the given input values.

t	0	1	2	3	4	5	6
$r(t)$							

(b) Sketch a graphical representation for the data set.
(c) About how long is the golf ball in the air?
(d) If the ball is travelling a 36 $feet/\sec$ horizontally, how far does it travel horizontally before striking the ground?

42. The golf ball in problem 41 is now placed on level ground on the surface of Mars and struck with the same force. The function $r(t) = -6.1t^2 + 96t$ represents the height in feet of the golf ball.

(a) **Numerical:** Create a data set for the function by evaluating $r(t)$ at the given input values.

t	0	2	6	10	12	14
$r(t)$						

(b) Sketch a graphical representation for the data set.
(c) How long is the golf ball in the air?
(d) If the ball is travelling a 36 $feet/\sec$ horizontally, how far does it travel horizontally before striking the ground?

43. Write the circumference of a circle as a function of its radius, r.

44. Write the circumference of a circle as a function of its diameter, $d = 2r$.

45. Write the area of a circle as a function of its diameter.

46. Write the area of a right isosceles triangle as a function of the length of one of its legs, b.

47. Given the volume of a right pyramid is one-third the area of the base times the height, write the volume of a right-circular pyramid as a function of its base radius r if the height is $h = 3r$.

48. Given the volume of a right pyramid is one-third the area of the base times the height, write the volume of a right- pyramid with a square base as a function of one of the lengths of the side of the base, x, if the height is $h = 1.5x$.

49. Given the volume of the sphere of radius r is $\frac{4}{3}\pi r^3$, write the volume of the sphere as a function of its diameter $d = 2r$.

50. Given the volume of the sphere of radius r is $\frac{4}{3}\pi r^3$, write the volume of the sphere as a function of time t if its radius r, at time t is given by

$$r = \frac{2t}{t+1}$$

Grapher[2]**:** *Use a computer or graphing calculator to evaluate the following expressions at the given input value for x .*

51. $f(x) = \dfrac{x^2 - 5}{3x + x^2}, \quad x = 3$

52. $f(x) = \dfrac{3.0x^2 - 5.2x + 2.9}{13.7x^3 - 40.4x + 12.1}, \quad x = 0.7$

53. $f(x) = \dfrac{x - 1}{(x^2 - 1)(x + 2)}, \quad x = -1.2$

54. $f(x) = x^{3/2} + 2x - \dfrac{x^2 - 5}{3x + x^2}, \quad x = 3$

55. $f(x) = \dfrac{13 - 4x + 3x^2}{6x + 32 - 5/x}, \quad x = 1.7,\ 2.3$

56. $f(x) = \dfrac{\sqrt{2.3x^2 - 5.1x + 1.0}}{3.6x^2 - 5.8x + 4.9}, \quad x = 2,\ 6.3$

[2] **Grapher** exercises require the use of a graphing calculator or computer to produce the graph of a function.

P-2: Graphs of Functions

Graphs of Functions

The graph of a function f is the set of *all* points (x, y) in the xy-plane that satisfy the equation $y = f(x)$. For example, the graph of $f(x) = a_0 + a_1 x + \ldots + a_n x^n$ is the set of all points in the xy-plane that satisfy

$$y = a_0 + a_1 x + \ldots + a_n x^n \tag{0.8}$$

For this reason, curves of the type (0.8) are often called *polynomial curves.*

In addition, we define the graph of $f(x)$ over an interval $[a, b]$ to be the section of $y = f(x)$ for which x is in $[a, b]$. Often the graph of a function $f(x)$ over an interval $[a, b]$ is approximated by connecting individual points in a finite data set by line segments. We will use the term *grapher* to refer to a machine such as a computer or graphing calculator which produces such an approximation.

EXAMPLE 1 Use a grapher to visualize the graph of

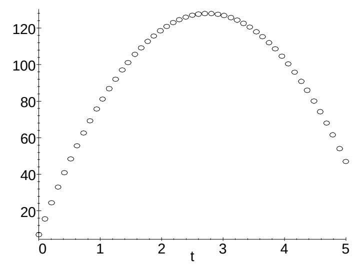
$$f(x) = 7 + 88x - 16x^2$$

over the interval $[0, 5]$.

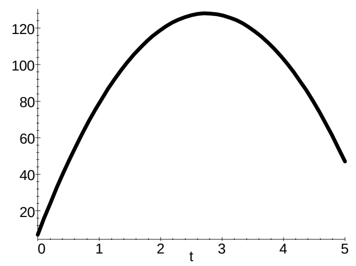
Solution: Our grapher (yours may vary) begins by computing the outputs $f(x)$ at each of the 50 equally spaced inputs x in the following set:

$$\{0, 0.1, 0.2, 0.3, \ldots, 4.7, 4.8, 4.9, 5.0\}$$

The resulting input-output pairs are plotted as points in the xy-plane (see figure P2-1). Joining successive points with line segments then produces an approximation of the curve $y = 7 + 88x - 16x^2$ (see figure P2-2).

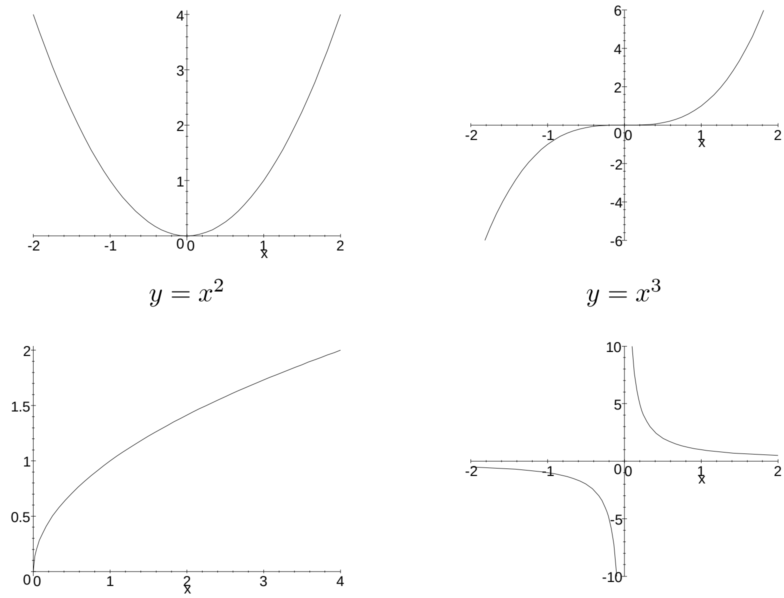


P2-1



P2-2

Graphs of some common functions are shown in figure P2-3:



$y = x^2$ $y = x^3$

$y = \sqrt{x}$ $y = 1/x$

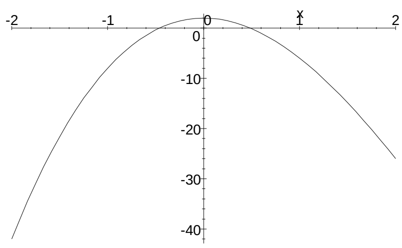
P2-3: Graphs of $f(x) = x^2$, $g(x) = x^3$, $p(x) = \sqrt{x}$, and $q(x) = 1/x$

Graphs of other functions can be obtained with a grapher. However, in doing so, it is important to produce graphs that contain all of the features of a function, features such as the maxima (high points), minima (low points), asymptotes, and symmetry, which is discussed in more detail at the end of this section.

EXAMPLE 2 Use a grapher to sketch the graph of the function

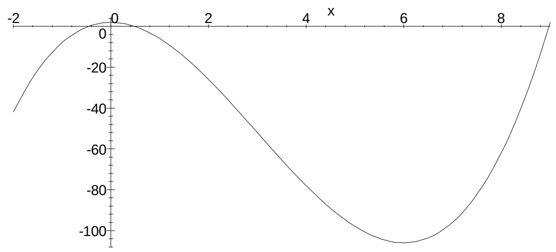
$$f(x) = x^3 - 9x^2 + 2$$

Solution: The graph of $f(x) = x^3 - 9x^2 + 2$ over $[-2, 2]$ is given by



P2-4: $y = x^3 - 9x^2 + 2$ over $[-2, 2]$

However, this is not the whole story. Consider the graph on the bigger interval $[-2, 9]$



P2-5: $y = x^3 - 9x^2 + 2$ over $[-2, 9]$

Clearly, the graph has both a high point and a low point, but this was not revealed by the graph over $[-2, 2]$.

How then do we know when we have identified all the features of a graph? The answer to that questions is one of the reasons we study calculus. In fact, graphing without calculus often produces misleading representations or omit key features.

Check your Reading *Are there any points on $y = x^3 - 9x^2 + 2$ that are higher than the y-intercept?*

Linear Functions

Recall that the equation of a line[3] with slope m which passes through a point (x_1, y_1) is given by $y - y_1 = m(x - x_1)$. Solving for y yields

$$y = y_1 + m(x - x_1) \tag{0.9}$$

Thus, (0.9) is the point-slope equation in the form of a graph of a function.
In particular, $y = y_1 + m(x - x_1)$ is the graph of the function

$$L(x) = y_1 + m(x - x_1) \tag{0.10}$$

Thus, the graph of a first degree polynomial is a straight line, and correspondingly, first degree polynomials are known as *linear functions.*

EXAMPLE 3 Find the linear function whose graph is the line with slope 2 that passes through the point $(1, 3)$.

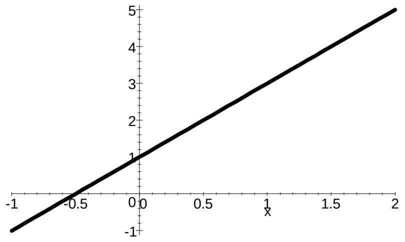
Solution: Since the slope is $m = 2$ and the point is $(x_1, y_1) = (1, 3)$, the equation (0.10) tells us that

$$L(x) = 3 + 2(x - 1) = 2x + 1$$

The graph of the linear function $L(x) = 2x + 1$ is the curve $y = 2x + 1$,

[3] Unless otherwise stated, the word line is understood to mean a straight line.

which is the line with slope 2 that passes through $(1,3)$.



P2-6: $y = 2x + 1$

If we let $x = x_2$ and $y = y_2$ in (0.9), then

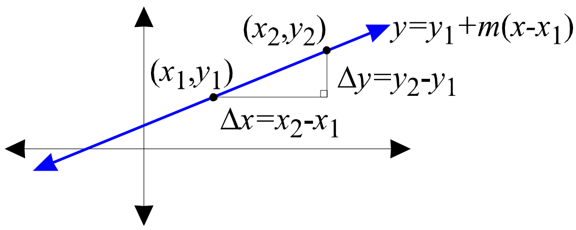
$$y_2 = y_1 + m\,(x_2 - x_1)$$

Solving for m then leads to the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Often, we denote the denominator, or *run*, by $\Delta x = x_2 - x_1$, where Δ is the Greek capital letter delta and denotes a change in a variable. Likewise, the numerator is often denoted $\Delta y = y_2 - y_1$ and is called the *rise*, so that the slope is

$$m = \frac{\Delta y}{\Delta x} = \frac{rise}{run}$$



P2-7: Slope m is ratio of rise Δy to run Δx.

Since (x_1, y_1) and (x_2, y_2) are any two points on the line, it follows that the slope is the same for any two points on the line.

EXAMPLE 4 Find the equation of the line through the points $(-1,4)$ and $(3,2)$.

Solution: The slope of the line through $(-1,4)$ and $(3,2)$ is

$$m = \frac{2-4}{3--1} = \frac{-2}{4} = \frac{-1}{2}$$

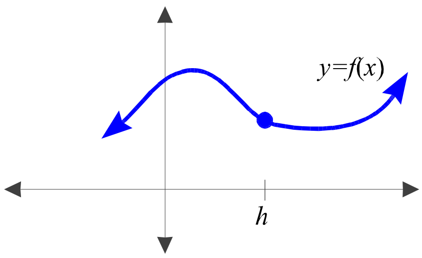
Thus, the linear function with slope $m = -1/2$ and through the point $(-1,4)$ is

$$y = 4 - \frac{1}{2}\,(x - -1) = \frac{7}{2} - \frac{1}{2}x$$

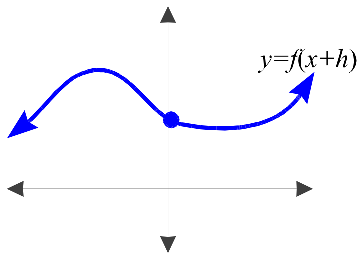
Check your Reading *Would we obtain the same result in example 4 if $(2,3)$ is used as (x_1, y_1)?*

Translations of Graphs

The graph of the function $f(x+h)$ is the graph of the function $f(x)$ shifted h units to the left if $h > 0$ and $|h|$ units to the right if $h < 0$.

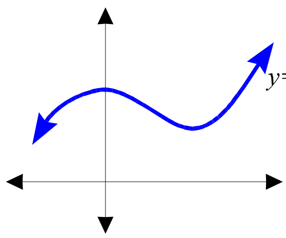


P2-8: Graph of $f(x)$

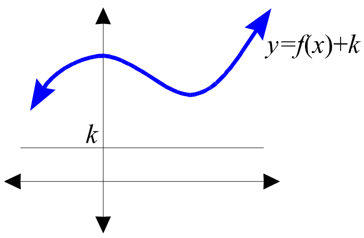


P2-9: Graph of $f(x+h)$ for $h > 0$

In contrast, the graph of $f(x)+k$ is the graph of the function $f(x)$ shifted k units upward if $k > 0$ and $|k|$ units downward if $k < 0$.

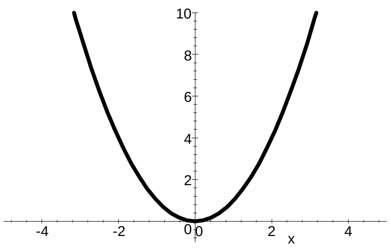


P2-10: Graph of $f(x)$



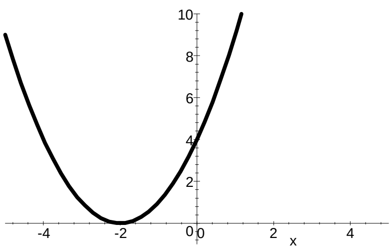
P2-11: Graph of $f(x)+k$ for $k > 0$

EXAMPLE 6 Sketch the graph of $g(x) = (x+2)^2 + 3$ given the graph of $f(x) = x^2$ shown below:



P2-12: Graph of $f(x) = x^2$

Solution: To begin with, let's notice that $f(x+2) = (x+2)^2$. Thus, the graph of $f(x+2)$ is the same as the graph of $f(x) = x^2$ shifted 2 units to the left

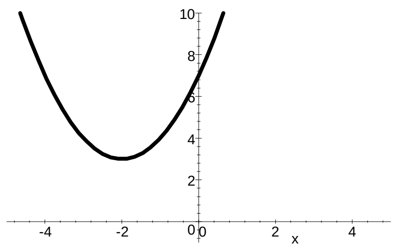


P2-13: Graph of $f(x+2) = (x+2)^2$

Moreover, adding 3 to $f(x+2)$ results in

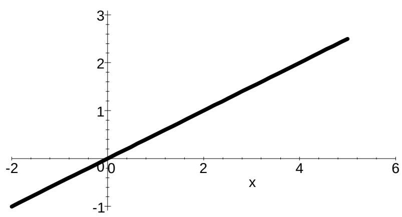
$$f(x+2)+3=(x+2)^2+3$$

Consequently, the graph of $g(x)=(x+2)^2+3$ is the graph of $f(x+2)$ shifted 3 units upward:



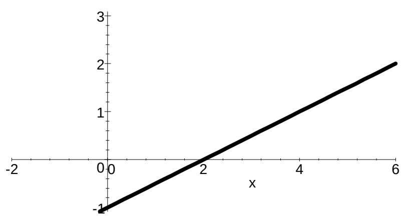
P2-14: Graph of $g(x)=f(x+2)+3$

EXAMPLE 7 Sketch the graph of $g(x)=\frac{1}{2}(x-2)+1$ given the graph of $f(x)=\frac{1}{2}x$ shown below:



P2-15: Graph of $f(x)=\frac{1}{2}x$

Solution: To begin with, let's notice that $f(x-2)=\frac{1}{2}(x-2)$. Thus, the graph of $f(x-2)$ is the same as the graph of $f(x)=\frac{1}{2}$ shifted 2 units to the right

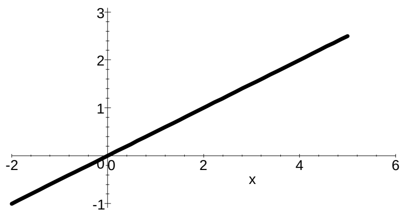


P2-16: Graph of $f(x-2)=\frac{1}{2}(x-2)$

Moreover, adding 1 to $f(x-2)$ results in

$$f(x-2)+1=\frac{1}{2}(x-2)+1$$

Consequently, the graph of $g(x) = \frac{1}{2}(x-2)+1$ is the graph of $f(x-2)$ shifted 1 unit upward:

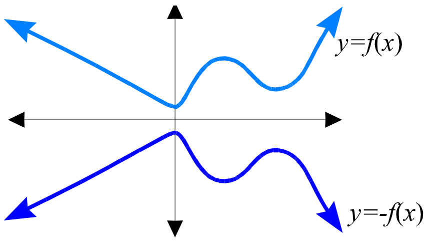


P2-17: Graph of $g(x) = \frac{1}{2}(x-2)+1$

Check your Reading *Why is P2-17 the same as P2-15?*

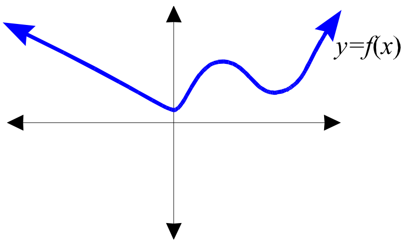
Symmetry

The graph of $-f(x)$ is the same as the graph of $f(x)$ reflected through the x-axis,

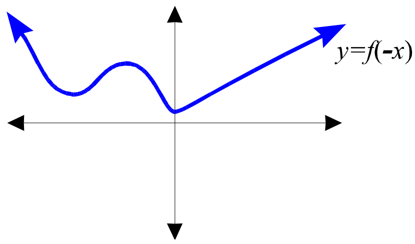


P-19: The graph of $-f(x)$ versus the graph of $f(x)$

and the graph of $f(-x)$ is the same as the graph of $f(x)$ reflected about the y-axis

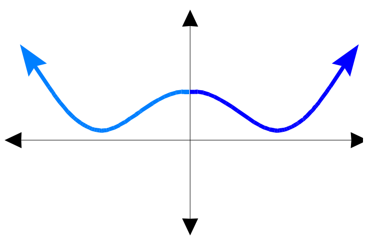


P-20: Graph of $f(x)$

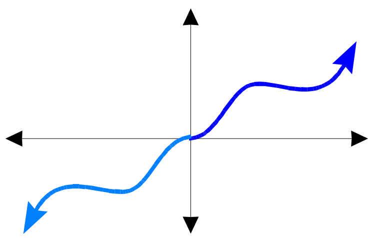


P-21: Graph of $f(-x)$

Functions which satisfy the identity $f(-x) = f(x)$ are said to be *even functions.* Because $f(-x) = f(x)$, the graph of an even function is *symmetric about the y-axis.* Correspondingly, an *odd function* is a function that satisfies $f(-x) = -f(x)$. Thus, the graph of an odd function is *symmetric about the origin.*



P-22: Graph of an even function



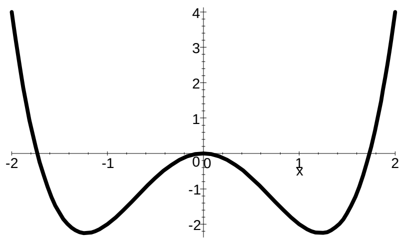
P-23: Graph of an odd function

EXAMPLE 8 Determine if $f(x) = x^4 - 3x^2$ is even, odd, or neither.

Solution: To begin with, let us compute $f(-x)$:

$$f(-x) = (-x)^4 - 3(-x)^2 = x^4 - 3x^2$$

The result is the same as $f(x)$, which is to say that $f(-x) = f(x)$. Thus, $f(x) = x^4 - 3x^2$ is even.



P-24: Graph of $f(x) = x^4 - 3x^2$

Exercises:

Find the equation of the linear function whose graph satisfies the following:

1. Slope $m = 2$ passing through the point $(1, 3)$.

2. Slope $m = -1$ passing through the point $(-2, 1)$.

3. Slope $m = 0.5$ passing through the point $(0.5, -0.75)$

4. Slope $m = 0.2$ passing through the point $(-1.4, 3.3)$

5. Passing through the points $(1, 1)$ and $(-2, 2)$.

6. Passing through the points $(-1, 2)$ and $(2, -3)$.

7. Passing through the points $(3, 5)$ and $(\frac{1}{2}, 4)$.

8. Passing through the origin and the point $(-2, -3)$.

Use a grapher to visualize the graphs of the following functions. In particular, look for features such as where the function reaches its highest or lowest values or if the function has any symmetry.

9.	$f(x) = 16x^2 + 4x - 3$	10.	$f(x) = x^3 + 4x - 1$
11.	$f(x) = 2x^4 + 4x^2 - 1$	12.	$f(x) = x^6 + 4x^2 + 7$
13.	$f(x) = x^3 - 3x$	14.	$f(x) = x^3 - 3x^2$
15.	$f(x) = \cos(\pi x)$	16.	$f(x) = \sin(\pi x)$

Sketch the graph of the function, and then sketch its translation.

17. $f(x) = 3x$ and the translation $f(x+2)$ over $[-4, 4]$

18. $f(x) = 1 - 2x$ and the translation $f(x+1)$ over $[-4, 4]$

19. $f(x) = \frac{1}{2}x^2$ and the translation $f(x) + 1$ over $[-4, 4]$.

20. $f(x) = x^2 - x$ and the translation $f(x - 2)$ over $[-1, 4]$.

21. $f(x) = \frac{1}{x}$ and the translation $f(x + 2) - 1$ over $[-6, 4]$.

22. $f(x) = -x + \frac{7}{2}$ and the translation $f(x - 1) + 2$ over $[-1, 4]$.

Determine if the following functions are even, odd, or neither. Sketch their graph to confirm your answer.

23.	$f(x) = x^3 - x$	24.	$f(x) = 2x^2 + 3$
25.	$f(x) = 4x^5 + 5x$	26.	$f(x) = 2x^4 + 5x^2$
27.	$f(x) = x^3 + 1$	28.	$f(x) = x^5 - 3x^2$
29.	$f(x) = 2x^2 + x - 3$	30.	$f(x) = 2x^5 + 4x^3$

31. Complete the following table of values for $f(x) = 3x^3 - 2x^2$.

x	-1	0	1	2	3
$f(x)$					

Then graph the $(x, f(x))$ pairs to obtain a graphical representation of the function.

32. Complete the following table of values for $f(x) = x^2 - 1$.

x	1.5	2.0	3.5	4.0	4.5
$f(x)$					

Then graph the $(x, f(x))$ pairs to obtain a graphical representation of the function.

33. The height r in feet as a function of time t in seconds of an object thrown upward from the surface of Mars is represented numerically in the table below:

t	0	2 .625	5.25	7.875	10.5
$r(t)$	4	129.97	171.86	129.70	4.475

(a) How high above the surface is the object when it begins its motion?

(b) How high above the surface is the object at $t = 7.875$ seconds after the object began moving?

(c) Graph the (t, r) pairs. Given that r as a function of t must have a graph that is a parabola, why can we conclude that 129.70 feet is the maximum altitude of the object.

34. The function $r(t) = -16t^2 + 4$ describes the height in feet at time t in seconds of an object rolling off of a table. We set $t = 0$ to be the instant the object leaves the table's edge.

(a) **Numerical:** Create a data set for the function by completing the table.

t	0	0.1	0.2	0.3	0.4	0.5
$r(t)$						

(b) How far has the object fallen after 0.1 seconds ?

(c) How long does it take for the object to strike the floor?

35. The population $P(t)$ of a certain small town on January 1 of year t is given in the table below.

t	1990	1992	1994	1996	1998	2000
$P(t)$	1031	1081	1111	1186	1286	1300

Graph the (t, P) pairs to obtain a graphical representation of the function. What is significant about the graph of $P(t)$?

36. A 3.25 inch tall soup can is filled with water and a small hole is punched in the bottom of the can allowing the water to drain out. The height y in inches of the water in the can at time t in minutes since water began draining is modeled by

$$y(t) = 0.08t^2 - 1.08t + 3.25$$

(a) How high is the water in the can one minute after it begins to drain?

(b) How long until the can is empty?

(c) Sketch the graph of the function. For what values of t is the model valid?

37. Suppose that the price p which can be charged per shirt if x number of shirts are to be sold is modeled by the *demand function*

$$p(x) = -0.007x + 32$$

(a) What is the price charged per shirt if we can sell $x = 100$ shirts?

(b) How many shirts must we sell in order to charge \$25.00 per shirt?

(c) Sketch the graph of the demand function (it is linear). Why would we not want to use the model if more than 4571 shirts are to be sold?

38. If $p(x) = -0.007x + 32$ is the price p which can be charged from the sale of x numbers of shirts, then the revenue R from the sale of x shirts is $R(x) = xp(x)$.

(a) What is $R(x)$ for the given price function $p(x)$?

(b) What is the revenue received when 1500 shirts are sold?

(c) Sketch the graph of the demand function. What is significant about its graph?

39. The unit circle has an equation of $x^2 + y^2 = 1$. What function is the upper half of the unit circle the graph of? Is it even or odd?

40. Translate the parabola $y = x^2$ by h units to the right and k units up. What is the resulting curve? What is the equation of the curve and what is significant about the point (h, k)?

41. Suppose that $f(x)$ is even and $g(x)$ is odd. What type of symmetry do the following functions have, if any?

(a)	$h(x) = f(x)\,g(x)$	(b)	$h(x) = f(x) - [g(x)]^2$
(c)	$h(x) = f(x) + g(x)$	(d)	$h(x) = xf(x) - g(x)$

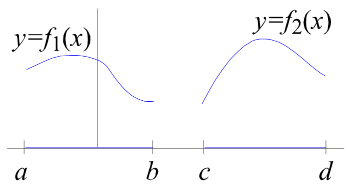
42. Show that if $f(l-x) = f(x)$, then $y = f(x)$ is symmetric about the line $x = l$.

Supplemental Review:

In the next few paragraphs, we look ahead at the topics from precalculus that will be important in the next chapter. To begin with, a function $f(x)$ is *piecewise defined* if there are non-overlapping intervals (a,b) and (c,d) such that

$$f(x) = \begin{cases} f_1(x) & if \quad a < x < b \\ f_2(x) & if \quad c < x < d \end{cases}$$

for functions $f_1(x)$ and $f_2(x)$. That is, if $a < x < b$, then $f(x) = f_1(x)$ and if $c < x < d$, then $f(x) = f_2(x)$. It follows that the graph of $f(x)$ is $y = f_1(x)$ over (a,b) and $y = f_2(x)$ over (c,d).



L1: Graph of $f(x)$

Moreover, the intervals used to define $f(x)$ need not be open.

EXAMPLE 1 Evaluate $f(-1)$, $f(1)$, and $f(2)$, and sketch the graph of the function

$$f(x) = \begin{cases} 2x+3 & if \quad -2 \le x < 1 \\ x^2 - 1 & if \quad 1 \le x \le 3 \end{cases}$$

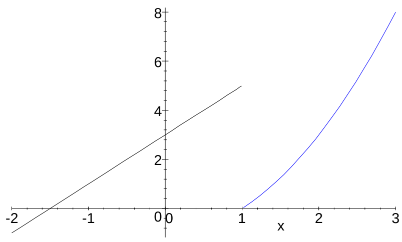
Solution: Since -1 is in $[-2,1)$, we use $2x+3$ in evaluating $f(-1)$:

$$f(-1) = 2(-1) + 3 = 1$$

Since 1 and 2 are both in $[1,3]$, we use $x^2 - 1$ to evaluate $f(1)$ and $f(2)$:

$$f(1) = 1^2 - 1 = 0, \qquad f(2) = 2^2 - 1 = 3$$

Moreover, the graph of $f(x)$ over $[-2,3]$ is the union of the curve $y = 2x+3$ over $[-2,1)$ and the curve $y = x^2 - 1$ over $[1,3]$.



L2: Graph of $f(x)$

More than two intervals may be used to define a piecewise-defined function. For example,

$$s(x) = \begin{cases} 1 & if \quad 0 \leq x < 2 \\ -2 & if \quad 2 \leq x < 3 \\ 1.5 & if \quad 3 \leq x < 5 \end{cases}$$

is a piecewise-defined function using three non-overlapping intervals in its definition. Moreover, when each of the functions used to define $s(x)$ is a constant function, then we say that $s(x)$ is a *simple function.* Simple functions will be used extensively in later chapters.

A *parameter* is a constant whose value is not specified. They are useful for defining families of curves. For example, the family of circles with radius R centered at the origin all have equations of the form

$$x^2 + y^2 = R^2 \tag{0.11}$$

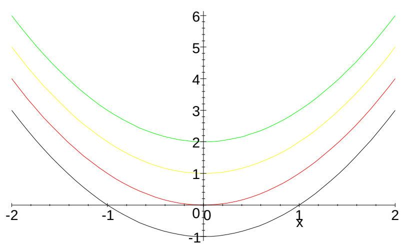
The quantity R in (0.11) is a parameter—that is, it is constant but its value will not be specified until equation (0.11) is actually used. Likewise, parameters can be used to define families of functions.

EXAMPLE 2 Sketch the graphs of the family of functions

$$f(x) = x^2 + k$$

when $k = -1, 0, 1, 2$.

Solution: If $k = -1$, then $f(x) = x^2 - 1$, whose graph has a y-intercept of -1. If $k = 0$, then $f(x) = x^2$, whose graph has a y-intercept of 0. Likewise, $k = 1, 2$ lead to y-intercepts of $1, 2$, respectively:



L3: Graphs of functions $f(x) = x^2 + k$ when $k = -1, 0, 1, 2$

We will also use parameters in chapter 1. For example, we will work with the difference quotient of a function, which is of the form

$$\frac{f(x+h) - f(x)}{h}$$

In particular, we will need to be able to evaluate and simplify a difference quotient.

EXAMPLE 3 Simplify the difference quotient of $f(x) = x^2$.

Solution: To begin with, $f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$, so that

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\left(x^2 + 2xh + h^2\right) - \left(x^2\right)}{h} \\ &= \frac{2xh + h^2}{h} \\ &= \frac{h(2x+h)}{h} \\ &= 2x + h, \ h \neq 0 \end{aligned}$$

Exercises:

Evaluate the piecewise-defined functions at the inputs $x = -1, 0, 1,$ and then sketch their graphs.

1. $f(x) = \begin{cases} x+2 & if \quad x < 0 \\ x^2 - 3 & if \quad x \geq 0 \end{cases}$

2. $f(x) = \begin{cases} x & if \quad x < 0 \\ -x & if \quad x \geq 0 \end{cases}$

3. $f(x) = \begin{cases} x^2 + 3x & if \quad -2 < x < 1 \\ 5 & if \quad 1 \leq x < 3 \end{cases}$

4. $f(x) = \begin{cases} x^2 & if \quad -2 \leq x < 0 \\ x & if \quad 0 \leq x < 2 \end{cases}$

5. $s(x) = \begin{cases} 1 & if \quad -2 \leq x < 0 \\ 3 & if \quad 0 \leq x < 2 \\ -2 & if \quad 2 \leq x < 4 \end{cases}$

6. $s(x) = \begin{cases} 2 & if \quad -2 \leq x < -1 \\ 1 & if \quad -1 \leq x < 1 \\ 5 & if \quad 1 \leq x < 2 \\ 3 & if \quad 2 \leq x < 5 \end{cases}$

Sketch the graphs of the families of functions given below for the indicated values of the parameter.

7. $f(x) = kx + 2, \ k = -1, 0, 1$
8. $f(x) = 0.5x - k, \ k = -1, 0, 1$
9. $f(x) = kx^2, \ k = -1, 1, 2$
10. $f(x) = (x-k)^2, \ k = -1, 0, 1$
11. $f(x) = x^3 + k, \ k = -1, 0, 1$
12. $f(x) = kx^3, \ k = -1, 1, 2$

Evaluate and simplify the difference quotient

$$\frac{f(x+h) - f(x)}{h}$$

for each of the following functions: (Hint: on #18 you will want to rationalize the numerator).

13. $f(x) = 5x - 3$
14. $f(x) = 0.7x + 3.5$
15. $f(x) = 2x^2 - 1$
16. $f(x) = x^2 + 2x$
17. $f(x) = \dfrac{1}{x}$
18. $f(x) = \sqrt{x}$

1. LIMITS, TANGENTS, AND RATES OF CHANGE

We experience our world through our five senses, and yet there is more to our world than our five senses can reveal. We can extend our five senses with tools such as microscopes and telescopes, and yet there are many things that still remain hidden from us. We cannot see electricity. We cannot hear the wind blowing on Mars. We cannot taste, smell or feel the building blocks of matter. Even with the most advanced tools known to man, our five senses simply cannot reveal all there is to know about world around us.

We need a sixth sense, one which will allow us to see inside an atom or hear the pitch of a radio wave. Since the late seventeenth century, that sixth sense has been Calculus. No instrument in the world will allow us to see a planet outside of our solar system, and yet we know of several such planets. Their presence was inferred using Calculus. No extension of our five senses will allow us to explore the twists and turns in our world economy, but with our sixth sense—Calculus—we can make forecasts and predict recessions. We cannot see the wind. But with Calculus, we can describe it, model it and make predictions as to where it is going and where it has been.

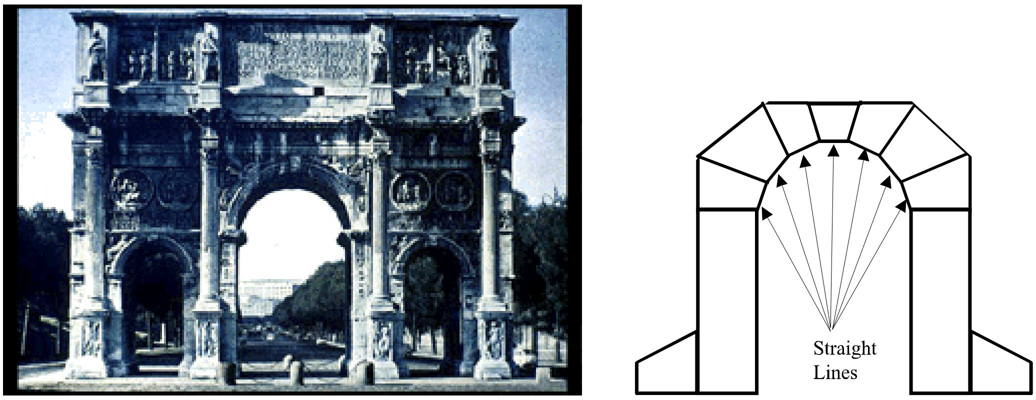
In this book, we learn to use the sixth sense of Calculus to investigate our world. We explore the role of Calculus in modern science and mathematics, and we revisit the discoveries of the past to see how Calculus has grown over the centuries. This chapter begins that exploration by laying the foundation upon which subsequent chapters will be built.

1.1 Tangent Lines

Introduction to Calculus

We have known calculus all of our lives. We use it daily to make sense of our world. When we measure short distances on the earth's surface with straight lines—yardsticks, tape measures, etc.—then we are using calculus. When we ignore the roundness of the earth with phrases like "a straight road" and "a flat field," then again we are using calculus. THE EARTH IS FLAT! ...or so we imagine when the distances being considered are small. This simple idea is the essence of Calculus.

In fact, the concepts underlying calculus have been used throughout history. Since antiquity, architects have used collections of short line segments to imply more complicated curves, such as when bricks and blocks with perfectly straight sides are used to construct semicircular arches.

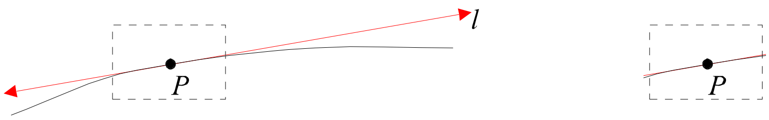


1-1: An arch is often formed by short line segments

That is, since antiquity, people have approximated small sections of curves with straight lines.

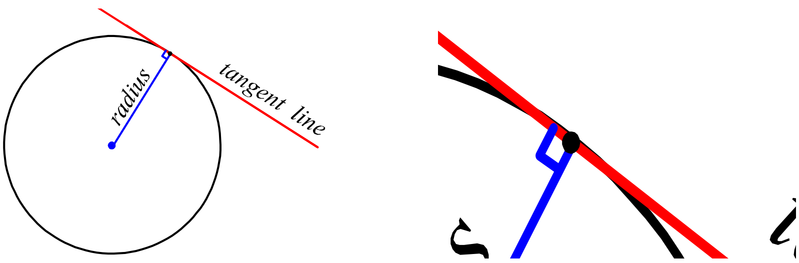
A Fundamental Concept in Calculus
Some curves can be divided into sections in which each section is nearly the same as a segment of a straight line

In calculus, curves are approximated by *tangent lines,* where a line l is tangent to a curve at the *point of tangency* P if the line and the curve are "practically the same" for small sections of the curve containing P.



1-2: The line l is tangent to the curve at point P.

Indeed, the Greek mathematician Euclid based much of his geometry on the fact that a tangent line to a circle is perpendicular to the radius—i.e., to the line through the origin and the point of tangency.



1-3: Tangent Line to a Circle

Notice again that the tangent line to a circle is "practically the same" as a small section of the circle which contains the point of tangency.

Check your Reading *What are some examples of a curve on the earth's surface being considered "practically the same" as a tangent line?*

Tangent Lines to Polynomials

Tangent lines are related to the fact that when h is close to 0, then higher powers like h^2, h^3, and so on are much, much closer to 0. For example, if $h = 0.001$, then

$$h^2 = 0.000001$$

which is 1000 times smaller than h. Likewise, if $h = 0.0001$, then

$$h^3 = 0.000000000001$$

which is much, much closer to 0 than h itself is

> **Neglible Powers of h**
> If h is sufficiently close to 0, then h^2, h^3, h^4, and so on are much, much closer to 0 than h is and thus can often be ignored.

If $f(x)$ is a polynomial, which is a function of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

then this concept can be used to calculate a tangent line to the curve $y = f(x)$ at a given input $x = p$.

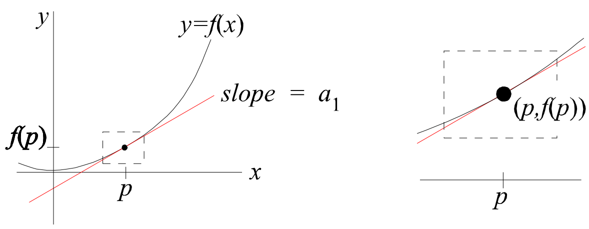
To do so, we let $x = p + h$, where h is assumed to be close to 0. Thus, $f(p+h)$ as a function of h is a polynomial of the form

$$f(p+h) = a_0 + a_1 h \quad + \quad \text{“higher powers of } h\text{”} \qquad (1.1)$$

Since the higher powers are negligible, $y = f(p+h)$ is practically the same as $y = a_0 + a_1 h$. Finally, $x = p + h$ implies that $h = x - p$, so that $y = a_0 + a_1 h$ becomes

$$y = a_0 + a_1 (x - p)$$

Near the point of tangency $(p, f(p))$, the polynomial curve $y = f(x)$ is practically the same as the tangent line $y = a_0 + a_1 (x - p)$.



1-4: $y = f(x)$ is almost a straight line at $(p, f(p))$

Let's look at some examples.

EXAMPLE 1 Find the equation of the tangent line to $y = x^2$ when $p = 1$.

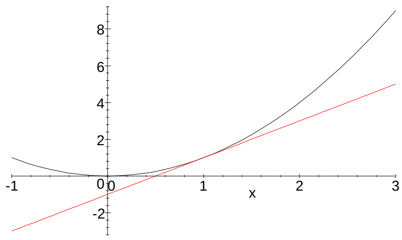
Solution: Since $p = 1$, we let $x = 1 + h$ to obtain $y = (1+h)^2$. Expanding the result leads to

$$y = (1+h)^2 = 1 + 2h + h^2$$

Since h^2 is negligible, the tangent line is $y = 1 + 2h$, which because $h = x - 1$ becomes

$$y = 1 + 2(x - 1) = 2x - 1$$

Thus, $y = 2x - 1$ is the tangent line to $y = x^2$ at $(1, 1)$, as is shown in figure 1-5:



1-5: $y = x^2$ versus $y = 2x - 1$

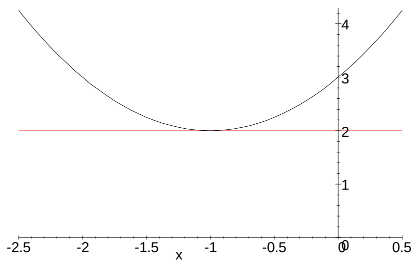
If $a_1 = 0$ in (1.1), then the tangent line is *horizontal*.

EXAMPLE 2 Find the equation of the tangent line to $y = x^2 + 2x + 3$ when $p = -1$.

Solution: Since $p = -1$, we let $x = -1 + h$ to obtain

$$\begin{aligned} y &= (-1 + h)^2 + 2(-1 + h) + 3 \\ &= 1 - 2h + h^2 - 2 + 2h + 3 \\ &= 2 + h^2 \end{aligned}$$

That is, $y = 2 + h^2$, but since h^2 is negligible, the tangent line is simply $y = 2$, which is a horizontal line (slope is 0). The parabola and its tangent line are shown in figure 1-6:



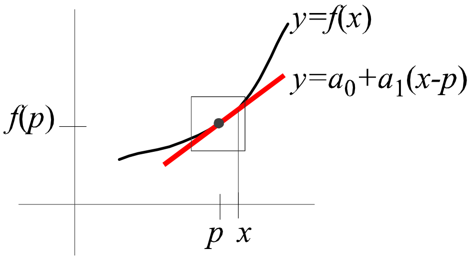
1-6: $y = x^2 + 2x + 3$ versus $y = 2$

Check your Reading *Why might horizontal tangent lines be important? (See figure 1-6 for help)*

In summary, to find the equation of the tangent line to the graph of an n^{th} degree polynomial when $x = p$, we use the following steps:

1. Let $x = p + h$ for h close to 0 and expand to obtain a polynomial in h of the form
$$y = a_0 + a_1 h + a_2 h^2 + \ldots + a_n h^n$$
2. Since h^2, h^3, h^4, and so on are negligible when h is close to 0, the polynomial is nearly the same as
$$y = a_0 + a_1 h$$
for x close to p.
3. Since $x = p + h$ implies that $h = x - p$, the *equation of the tangent line to* $y = f(x)$ *when* $x = p$ is
$$y = a_0 + a_1 (x - p)$$

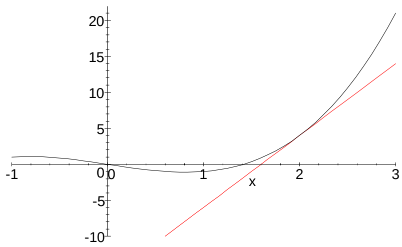
It follows that when x is close to p, which is when h is close to 0, then the tangent line is a good approximation of the curve itself.



1-7: Tangent is a good approximation to curve when x is close to p

EXAMPLE 3 Find the equation of the tangent line to $y = f(x)$ at $p = 2$ when $f(x) = x^3 - 2x$?

Solution: Substituting $x = 2 + h$ and expanding leads to
$$y = (2 + h)^3 - 2(2 + h) = 4 + 10h \quad + \quad 6h^2 + h^3$$
Since $6h^2 + h^3$ is negligible for h close to 0, the curves is practically the same as
$$y = 4 + 10h$$
Since $x = 2 + h$ implies $h = x - 2$, the equation of the tangent line is
$$y = 4 + 10(x - 2) = 10x - 16$$
That is, $y = 10x - 16$ is tangent to $y = x^3 - 2x$ at $(2, 4)$.



1-8: $y = 10x - 16$ is tangent to $y = x^3 - 2x$ at $(2, 4)$

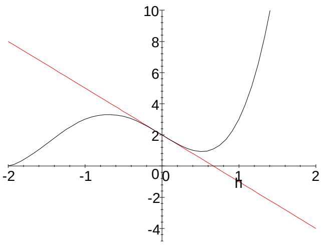
In particular, the tangent line is the line that becomes *ever more indistinguishable* from the curve as we choose shorter and shorter sections containing the point of tangency. A tangent line may even *cross the curve*, just as long as it does so by becoming arbitrarily close to the curve itself.

EXAMPLE 4 Find the equation of the tangent line to $y = x^4 + 3x^3 - 3x + 2$ when $p = 0$. Then graph both the curve and its tangent line over the intervals $[-2, 2]$, $[-1, 1]$, $[-0.5, 0.5]$ and $[-0.2, 0.2]$.

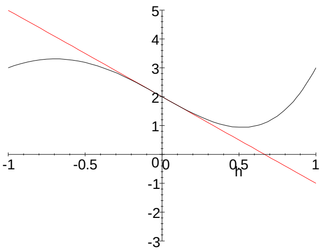
Solution: Since $x = 0 + h$, the curve is of the form

$$y = \overbrace{2 - 3h}^{\text{linear part}} + \overbrace{3h^3 + h^4}^{\text{higher powers of } h}$$

Since $x = h$, the tangent to the curve when $p = 0$ is $y = 2 - 3x$. Graphs of the curve and the line are shown over $[-2, 2]$, $[-1, 1]$, $[-0.5, 0.5]$ and $[-0.2, 0.2]$ in figure 1-8a through 1-8d, respectively.



1-9a



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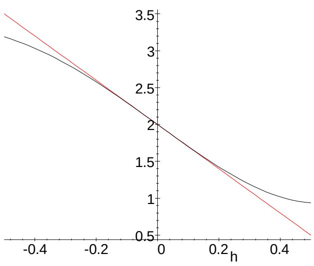

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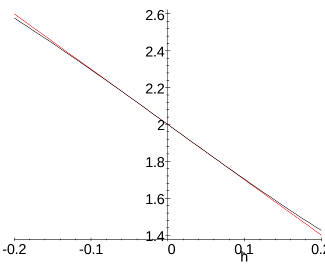


1-9d

As the intervals become shorter and shorter, the curve becomes more and more like the straight line. Thus, the line is tangent to the curve, even though it crosses over the curve itself.

Check your Reading *Can a line be tangent to a curve WITHOUT EVER INTERSECTING THE CURVE?*

Applications of the Tangent Concept

In an xy-coordinate system, a "rise" is a change in the y coordinate and a "run" is a change in the x-coordinate. The slope of a line is a ratio of a "rise" to a "run", which means that the slope of the tangent line a_1 satisfies

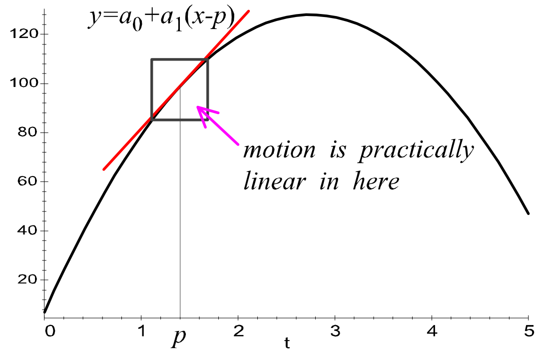
$$a_1 = \frac{rise}{run} = \frac{change\ in\ y}{change\ in\ x}$$

That is, the slope a_1 is the *rate of change* of the tangent line. Since $y = f(x)$ is nearly the same as its tangent line at a point of tangency $(p, f(p))$, the rate of change of the function itself over a short interval $[p, p+h]$ is practically the same as a_1. That is, the *slope of the tangent line* at $x = p$ is *the rate of change of the function at* $x = p$.

For example, if a ball is thrown upward from an initial height of 7 feet with an initial velocity of 88 feet per second, then the height $r(t)$ of the ball at time t is given by

$$r(t) = 7 + 88t - 16t^2 \tag{1.2}$$

Over a short period of time, the motion of the ball is practically a straight line with slope a_1.



1-10: Over short time intervals, motion is practically linear with slope $r'(p)$.

Since the slope is the rate of change of the tangent line, a_1 is the *rate of change* of the ball at p seconds, which is also known as the *velocity* of the ball. That is, the slope of the tangent line tells us about how fast the ball is traveling at time $t = p$ seconds.

EXAMPLE 5 If $r(t) = 7 + 88t - 16t^2$ is the height in feet of a ball at time t in seconds, then how fast is the ball traveling at time $p = 0$ seconds? How fast at $p = 1$ seconds?

Solution: If $p = 0$, then we let $t = 0 + h$ and

$$r(0 + h) = 7 + 88h - 16h^2$$

Since h^2 is negligible, we obtain $y = 7 + 88h$, so that the tangent line is

$$y = 7 + 88(t - 0)$$

which has a slope of 88. Thus, at $p = 0$, the ball has a velocity of 88 feet per second.

If $p = 1$, then we let $t = 1 + h$ and

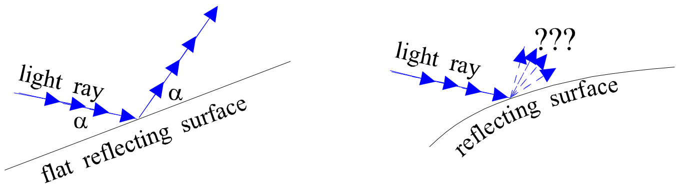
$$r(1+h) = 7 + 88(1+h) - 16(1+h)^2 = 79 + 56h - 16h^2$$

Since h^2 is negligible, we obtain $y = 79 + 56h$, so that the tangent line is

$$y = 79 + 56(t-1)$$

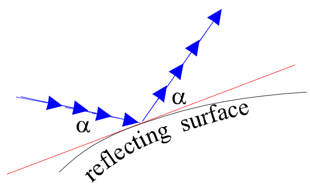
which has a slope of 56. Thus, at $p = 1$ seconds, the ball has a velocity of 56 feet per second.

Rates of change are not the only application of tangent lines. For example, the law of reflection says that if a ray of light reflects off of a flat surface, the angle of incidence is equal to the angle of reflection. But what is the direction of a reflected ray off of a *curved surface?*



1-11: Cross-sections of a flat and a curved surface

To answer that question, we zoom until the curve formed by the cross section of the reflecting surface can be replaced by its tangent line.



1-12: Ray of light reflected off of tangent line

We then compute the angle of reflection off of the flat tangent rather than the curved surface.

Exercises

Find the tangent line to $y = f(x)$ and identify $f'(p)$ for the given value of p. Graph both the curve and the line to verify tangency.

1. $f(x) = 3x^2, \quad p = 1$
2. $f(x) = -x^2, \quad p = 1$
3. $f(x) = x^2 - 1, \quad p = 1$
4. $f(x) = x^2 + 1, \quad p = 2$
5. $f(x) = x + 3x^2, \quad p = 2$
6. $f(x) = x^2 + 3x, \quad p = 1$
7. $f(x) = 3x + 2, \quad p = 1$
8. $f(x) = 3x + 2, \quad p = 2$
9. $f(x) = 1 + 3x - x^2, \quad p = 1$
10. $f(x) = 2 - 3x + x^2, \quad p = 2$
11. $f(x) = x^3 + 3,\ p = -2$
12. $f(x) = x^3 + 3x + 1,\ p = 2$
13. $f(x) = x^3 - 3x,\ p = 1$
14. $f(x) = x(x-1)^2,\ p = 1$

Grapher[1]**:** *Find the tangent line when $p = 0$, and then graph both the curve and the line over the intervals $[-2, 2]$, $[-1, 1]$, $[-0.5, 0.5]$ and $[-0.1, 0.1]$. On which of*

[1] **Grapher** exercises require the use of a graphing calculator or computer to produce the graph of a function.

the intervals would you say that the curve and the line are indistinguishable?

15.	$y = 1 + 3x + x^2$	16.	$y = 2 - 3x + x^2$
17.	$y = x + 3x^2$	18.	$y = (2x+1)^2$
19.	$y = (3+x)(2-x)$	20.	$y = (3+x)(2-x)$
21.	$y = x\left(1+x^2\right)$	22.	$y = 1 + x + x^3$
23.	$y = 1 + x^3 + x^5$	24.	$y = 2x^2 + x + 1$

Each of the following functions represents the height $r(t)$ in feet of an object at time t in seconds. Find the velocity of the object at the given time p by finding the slope of the tangent line to the graph of the curve.

25.	$r(t) = 64 - 16t^2$ *at* $p = 1$ sec	26.	$r(t) = 64 - 16t^2$ *at* $p = 0$ sec
27.	$r(t) = 96t - 16t^2$ *at* $p = 0$ sec	28.	$r(t) = 96t - 16t^2$ *at* $p = 1$ sec
29.	$r(t) = 64 + 4t - 16t^2$ *at* $p = 1$ sec	30.	$r(t) = 32 + 12t - 16t^2$ *at* $p = 1$ sec

31. Grapher: Graph the following lines along with the curve

$$f(x) = \sqrt{4 + x + x^2}$$

on the intervals $[-2, 2]$, $[-1, 1]$, $[-0.5, 0.5]$ *and* $[-0.1, 0.1]$. Which of the lines is tangent to the curve when $p = 0$?

(a) $y = \frac{x}{2} + 2$ (b) $y = \frac{x}{3} + 2$ (c) $y = \frac{x}{4} + 2$

32. Grapher: Graph the following lines along with the curve

$$f(x) = \sqrt{4 + x + x^2}$$

on the intervals $[-2, 2]$, $[-1, 1]$, $[-0.5, 0.5]$ *and* $[-0.1, 0.1]$. Which of the lines is tangent to the curve when $p = 0$?

(a) $y = \frac{x}{2} + 2$ (b) $y = \frac{x}{3} + 2$ (c) $y = \frac{x}{4} + 2$

33. Grapher: In (a)-(c), only one of the lines is tangent to the given curve when $p = 0$. Graph both on the intervals $[-2, 2]$, $[-1, 1]$, $[-0.5, 0.5]$ *and* $[-0.1, 0.1]$. In which is the line tangent to the given curve when $p = 0$? (Be sure to use radians)

(a) $y = \cos(x)$, $y = x$
(b) $y = (1+x)^{10}$, $y = 1 + x$
(c) $y = \sqrt{1+x}$, $y = 1 + x/2$

34. Find the tangent lines to $y = x^2 - 2x + 3$ at $p = 0$, $p = 1$, $p = 2$, and $p = 3$. Then graph **only** the tangent lines on the interval $[-1, 4]$. What information might you infer about $y = x^2 - 2x + 3$ from these 4 tangent lines?

35. Find the tangent lines to the following curves when $p = 0$. The letters k and a are called *parameters* and should be treated as if they have a fixed numerical value. For example, if

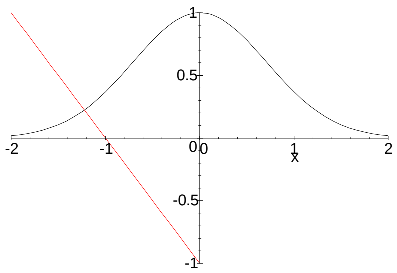
$$y = 1 + ax + x^2$$

then the tangent line when $p = 0$ is $y = 1 + ax$.

(a) $y = (x+a)^2$
(b) $y = (x+a)^3$
(c) $y = \left(1 + \frac{x}{k}\right)^2$
(d) $y = (1-kx)(1+kx)$

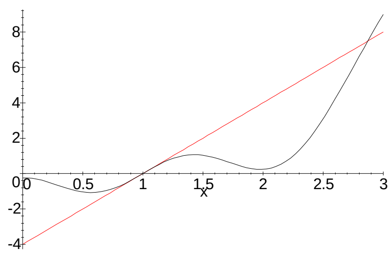
36. Show that in terms of the parameter p, the set of all tangent lines to $y = x^2$ at $x = p$ are given by $y = 2px - p^2$.

37. There are many myths and misunderstandings surrounding tangent lines. One of the most prevalent is that if a line intersects a curve at only one point, then it is a tangent line. The line in figure 1-13 intersects the curve at only one point. Why would we not consider this line to be a tangent line to the curve?



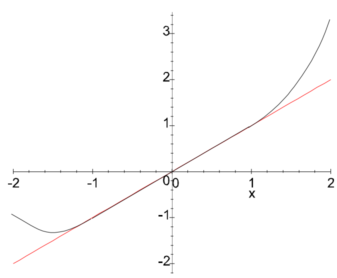
1-13: Is this a tangent line?

38. A related myth is that a tangent line cannot intersect a curve more than once. However, the line in figure 1-14 is tangent to the curve at $x = 1$,and yet it crosses the curve more than once. Explain why we would nonetheless consider it to be a tangent line to the curve at $x = 1$.



1-14: Is this a tangent line?

39. One last myth is that a line must be tangent to a curve at only one point. However, in figure 1-15, the curve in black is tangent to the red line for all x in $[-1, 1]$.



1-15: Is this a tangent line?

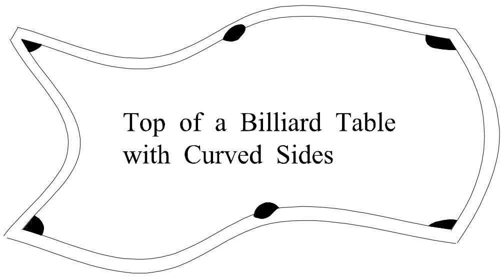
What, then, does it mean for a line to be tangent to a curve, and how do we see that concept illustrated in this example?

40. Computer Algebra System. If you have access to a computer algebra system, use it to find the tangent lines to the given curves at the given point. Then graph both the curve and the tangent line.

(a) $y = (1 - x)(1 - 2x)(1 - 3x)$ when $p = 1$
(b) $y = x(x + 1)(x + 2)(x + 3)$ when $p = 2$
(c) $y = x^2 (x - 1)^{10}$ when $p = 1$
(d) $y = 1 + x(1 + x(1 + x))$ when $p = 3$

41. Write to Learn: The cross section of a satellite dish is a parabola with an equation of $y = x^2 + x + 4$. If a signal received from space travels down the positive y-axis, what will its angle of reflection off of the mirror be. Write a short essay explaining your results and how they were obtained.

42. Write to Learn: Suppose that a billiard table has curved sides that reflect billiard balls so that the angle of incidence is equal to the angle of reflection.



1-16: An Unsual Pool Table

Write a short essay in which you identify and explain *mathematically* which of the six pockets is the "easiest" to hit a ball into (in particular, be sure to use tangent lines in your explanation).

1.2 The Limit Concept

The Limit Process

Unfortunately, we cannot build a mathematical theory on the vague notion that "if h is close enough to 0, then h^2 and higher powers can be ignored." Instead, we base calculus on the more concrete and dynamic idea that "If h gets closer and closer to 0, then h^2 and higher powers become more and more negligible." The concept of h getting closer and closer to 0 is known as the *limit concept* in calculus and is the foundation on which the theory of calculus is built.

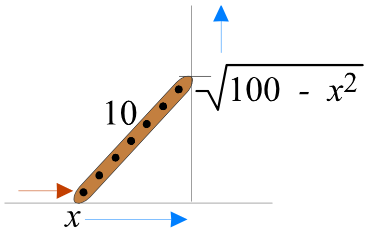
Let's be more specific. The symbol $\lim_{x \to p}$, which is interpreted "the limit as x approaches p," denotes the act of letting x become closer and closer to p. Thus, the equation

$$\lim_{x \to p} f(x) = L \tag{1.3}$$

x approaching p means $x \neq p$.

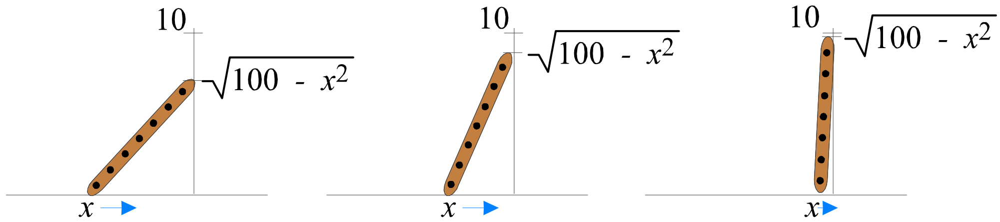
means that "if x approaches p and $x \neq p$, then $f(x)$ approaches L." The requirement $x \neq p$ implies that x approaches p without ever being equal to p.

To illustrate (1.3), let us suppose a 10 foot long ladder is leaning against a wall and that its base is pushed toward the wall. If x denotes the distance from the base of the ladder to the wall, then the Pythagorean theorem says that the height of the end of the ladder touching the wall is $\sqrt{100 - x^2}$.



2-1: Ladder moving up a wall

As the base of the ladder moves to the wall, the other end of the ladder moves up the wall.



2-2: Base of Ladder moving toward wall

Thus, as x approaches 0, the end of the ladder approaches its full height of 10 feet, which we write as

$$\lim_{x \to 0} \sqrt{100 - x^2} = 10 \tag{1.4}$$

Let's verify (1.4) numerically by examining a data set in which the inputs x are approaching 0 *from both sides*.

EXAMPLE 1 Estimate the value of the limit

$$\lim_{x \to 0} \sqrt{100 - x^2}$$

Solution: To do so, we complete a table in which the inputs x are approaching 0. In particular, we choose negative and positive values of x that are becoming successively closer to 0:

x	-0.5	-0.2	-0.1	$\to$	0	$\leftarrow$	0.1	0.2	0.5
$\sqrt{100 - x^2}$				$\to$	???	$\leftarrow$			

That is, we compute $\sqrt{100 - (-0.1)^2}$, $\sqrt{100 - (-0.01)^2}$, and so on. The result is the numerical representation below:

x	-0.5	-0.2	-0.1	$\to$	0	$\leftarrow$	0.1	0.2	0.5
$\sqrt{100 - x^2}$	9.987	9.998	9.9995	$\to$	???	$\leftarrow$	9.9995	9.998	9.987

As the x-values approach 0 from either side, the outputs $\sqrt{100 - x^2}$ seem to be approaching 10. Thus, the table leads us to the estimate

$$\lim_{x \to 0} \sqrt{100 - x^2} = 10$$

Check your Reading *How is the Pythagorean theorem used in figure 3-1?*

Properties of Limits

When limits of functions exist and and are not equal to 0, then limits of arithmetic combinations (sums, differences, products, quotients) follow logically. For example, suppose that

$$\lim_{x \to p} f(x) = L \qquad \textit{and} \qquad \lim_{x \to p} g(x) = K$$

Then for x close to p, the function $f(x)$ is close to L and the function $g(x)$ is close to K, so it follows immediately that $f(x) + g(x)$ is close to the number $L + K$. That is,

$$\lim_{x \to p} [f(x) + g(x)] = L + K$$

Theorem 2.1 further illustrates that arithmetic with non-zero limits is straightforward.

Theorem 2.1: If $\lim_{x \to p} f(x)$ and $\lim_{x \to p} g(x)$ both exist, then

i. $\lim_{x \to p} [f(x) + g(x)] = \lim_{x \to p} f(x) + \lim_{x \to p} g(x)$

ii. $\lim_{x \to p} [f(x) - g(x)] = \lim_{x \to p} f(x) - \lim_{x \to p} g(x)$

iii. $\lim_{x \to p} [kf(x)] = k \lim_{x \to p} f(x)$, where k is a constant

iv. $\lim_{x \to p} [f(x) \cdot g(x)] = \left[\lim_{x \to p} f(x)\right] \cdot \left[\lim_{x \to p} g(x)\right]$

v. $\lim_{x \to p} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to p} f(x)}{\lim_{x \to p} g(x)}$, when $\lim_{x \to p} g(x) \neq 0$

Proofs of some of these properties will be supplied in the next section.

EXAMPLE 2 Suppose that $\lim_{x \to 4} f(x) = 10$ and $\lim_{x \to 4} g(x) = 3$. What is the value of the limit

$$\lim_{x \to 4} \frac{f(x) + 2g(x)}{f(x) - 2g(x)}$$

Solution: Since the denominator does not approach 0, properties (i), (ii), (iii), and (v) imply

$$\lim_{x \to 4} \frac{f(x) + 2g(x)}{f(x) - 2g(x)} = \frac{\lim_{x \to 4} f(x) + 2 \lim_{x \to 4} g(x)}{\lim_{x \to 4} f(x) - 2 \lim_{x \to 4} g(x)}$$

Using the given values of the limits thus implies that

$$\lim_{x \to 4} \frac{f(x) + 2g(x)}{f(x) - 2g(x)} = \frac{10 + 2 \cdot 3}{10 - 2 \cdot 3} = \frac{16}{4} = 4$$

Moreover, suppose that $f(x)$ and $g(x)$ are polynomials. If $g(p) \neq 0$, then it can be shown that

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \frac{f(p)}{g(p)} \tag{1.5}$$

We say that the function $\frac{f(x)}{g(x)}$ is *continuous* at $x = p$ when (1.5) holds, and we say that we are *using continuity* to evaluate a limit when we use (1.5).

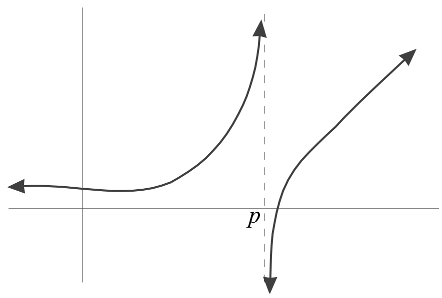
EXAMPLE 3 Use continuity to evaluate the limit

$$\lim_{x \to 1} \frac{x^2 - 3x + 2}{x^2 + 1}$$

Solution: Let $g(x) = x^2 + 1$. Then $g(1) = 2$. Thus, (1.5) applies and

$$\lim_{x \to 1} \frac{x^2 - 3x + 2}{x^2 + 1} = \frac{1^2 - 3 \cdot 1 + 2}{1^2 + 1} = \frac{0}{5} = 0$$

However, when limits of denominators are equal to 0, then limit evaluation becomes more complicated. For example, if $g(p)$ approaches 0 but $f(p)$ approaches a number $L \neq 0$, then the graph of the function $\frac{f(x)}{g(x)}$ has a *vertical asymptote* at $x = p$.



2-3: $x = p$ is a vertical asymptote

If $\frac{f(x)}{g(x)}$ has a vertical asymptote $x = p$, then $\frac{f(x)}{g(x)}$ cannot approach a number L as x approaches p, and consequently, we say that

$$\lim_{x \to p} \frac{f(x)}{g(x)} \text{ does not exist}$$

That is, the quotient is not getting close to any given number.

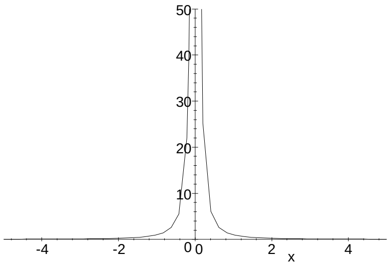
EXAMPLE 4 Discuss the limit

$$\lim_{x \to 0} \frac{1}{x^2}$$

Solution: As x gets closer to 0, the quantity x^2 also gets closer to 0. As a result, the function $\frac{1}{x^2}$ becomes arbitrarily large, as is shown in the numerical representation below:

x	-0.1	-0.01	-0.001	$\rightarrow$	0	$\leftarrow$	0.001	0.01	0.1
$\frac{1}{x^2}$	100	$10,000$	$1,000,000$	$\rightarrow$	???	$\leftarrow$	$1,000,000$	$10,000$	100

Thus, $y = \frac{1}{x^2}$ has a vertical asymptote of $x = 0$



2-4: The graph of $f(x) = \frac{1}{x^2}$ has a vertical asymptote of $x = 0$

Consequently, $\lim\limits_{x\to 0} \frac{1}{x^2}$ does not exist.

Check your Reading *How does the following data set relate to example 3?*

x	0.9	0.99	0.999	$\rightarrow$	1	$\leftarrow$	1.001	1.01	1.1
$\frac{x^2-3x+2}{x^2+1}$	0.061	0.0051	0.0005	$\rightarrow$	???	$\leftarrow$	-0.0005	-0.0049	-0.041

Limits of the Form $\frac{0}{0}$

Limits of the form $\frac{0}{0}$ must be simplified before they can be analyzed.

Here is a situation that is very important in calculus. It is possible that both $f(x)$ and $g(x)$ approach 0 as x approaches p. In that case, we say that the limit

$$\lim_{x\to p} \frac{f(x)}{g(x)}$$

is *of the form* $\frac{0}{0}$. A limit of the form $\frac{0}{0}$ may exist, or it may not. However, it must be simplified before it can be analyzed.

EXAMPLE 5 Show that the following limit is of the form $\frac{0}{0}$ and then evaluate it:

$$\lim_{x\to -2} \frac{x^2-4}{x^2+x-2}$$

Solution: At $x = -2$, the numerator becomes $(-2)^2 - 4 = 0$ and the denominator becomes $(-2)^2 - 2 - 2 = 0$. Thus, the limit is of the form $\frac{0}{0}$ and must be reduced to a non-$\frac{0}{0}$ form. Since x approaching -2 implies that $x \neq -2$, we can factor and cancel to obtain

$$\lim_{x\to -2} \frac{x^2-4}{x^2+x-2} = \lim_{x\to -2} \frac{(x-2)(x+2)}{(x-1)(x+2)} = \lim_{x\to -2} \frac{x-2}{x-1}$$

Since $x - 1 \neq 0$ when $x = -2$, we can now evaluate using continuity:

$$\lim_{x\to -2} \frac{x^2-4}{x^2+x-2} = \lim_{x\to -2} \frac{x-2}{x-1} = \frac{-2-2}{-2-1} = \frac{4}{3}$$

EXAMPLE 6 Evaluate the limit

$$\lim_{x\to -1} \frac{x^3+1}{x^2+2x+1} \tag{1.6}$$

Solution: Substituting $x = -1$ into the numerator and into the denominator shows us that (1.6) is of the form $\frac{0}{0}$. As a result, we must reduce (1.6) to a non-$\frac{0}{0}$ form. Since $x^3 + 1 = (x+1)\left(x^2 - x + 1\right)$, we have

$$\lim_{x \to -1} \frac{x^3 + 1}{x^2 + 2x + 1} = \lim_{x \to -1} \frac{(x+1)\left(x^2 - x + 1\right)}{(x+1)^2} = \lim_{x \to -1} \frac{x^2 - x + 1}{x + 1}$$

The graph of $f(x) = \frac{x^2 - x + 1}{x + 1}$ has a vertical asymptote at $x = -1$, which implies that

$$\lim_{x \to -1} \frac{x^3 + 1}{x^2 + 2x + 1} \text{ does not exist}$$

Check your Reading *Does every limit of the form $\frac{0}{0}$ exist?*

Graphical Estimation of Limits

Finally, limits of the form $\frac{0}{0}$ can also be estimated graphically by *zooming* centered at a given point p to reveal outputs of $f(x)$ as they get closer and closer to p.

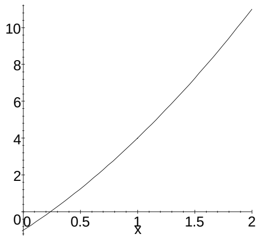
EXAMPLE 7 Use zooming to estimate the limit

$$\lim_{x \to 1} \frac{x^3 + 3x^2 - 5x + 1}{x - 1}$$

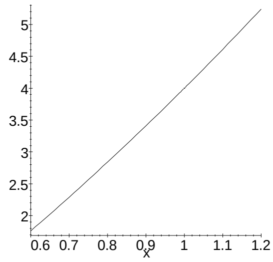
Solution: Notice first that both the denominator and the numerator reduce to 0 when $x = 1$, thus implying that the limit is of the form $\frac{0}{0}$. However, if we define

$$f(x) = \frac{x^3 + 3x^2 - 5x + 1}{x - 1}$$

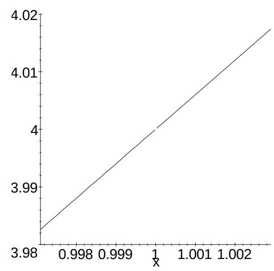
then we can produce a sequence of zooms centered at $x = 1$.



2-5



2-6



2-7

In each plot in the sequence of zooms, the graph of $f(x)$ seems to imply an output of 4 at an input of 1, even though $f(x)$ is not defined at 1. Moreover, the smaller the interval on the x-axis, the closer the range of the y-axis is to being a single output of 4. Thus, we are lead to the estimate

$$\lim_{x \to 1} \frac{x^3 + 3x^2 - 5x + 1}{x - 1} = 4 \qquad (1.7)$$

However, no matter what the graphs seem to imply, the function $f(x)$ in example 7 **does not** produce an output of 4 when the input is 1. Instead, we are seeing that $f(x)$ is **approaching** 4 as x **approaches** 1 with $x \neq 1$, which means that the fact that the graphs are not defined at at point is not relevant.

Exercises:

Given that $\lim_{x\to 5} f(x) = 2$ *and* $\lim_{x\to 5} g(x) = 7$, *use the properties of limits to evaluate the following:*

1. $\lim_{x\to 5} [f(x) + 4g(x)]$
2. $\lim_{x\to 5} [f(x)\, g(x)]$
3. $\lim_{x\to 5} \dfrac{f(x) - g(x)}{f(x) + g(x)}$
4. $\lim_{x\to 5} [f(x)]^3$

Evaluate the following limits. Identify all limits which are of the form $\frac{0}{0}$. *If the limit does not exist, write d.n.e.*

5. $\lim_{x\to 2} \left(x^2 + 3x - 4\right)$
6. $\lim_{x\to 1} \left(x^3 + 3x^5\right)$
7. $\lim_{x\to -1} \dfrac{x^2 + 3x + 2}{x^2 - 1}$
8. $\lim_{x\to 0.5} \dfrac{2x^2 + x - 1}{x - 0.5}$
9. $\lim_{x\to -2} \dfrac{6x^2 + 15x + 6}{x^2 + 5x + 6}$
10. $\lim_{x\to 2} \dfrac{6x^2 + 15x + 6}{x^2 + 5x + 6}$
11. $\lim_{x\to -1} \dfrac{x + 1}{x^2 + 1}$
12. $\lim_{x\to 2} \dfrac{x^2 - 4}{x - 2}$
13. $\lim_{x\to 2} \dfrac{x^2 - 4}{x^2 - 4x + 4}$
14. $\lim_{x\to 7} \dfrac{x^2 - 7x}{x^2 - 14x + 49}$
15. $\lim_{x\to 1} \dfrac{x^5 - 32}{x^2 - 4}$
16. $\lim_{x\to 3} \dfrac{x^4 - 81}{x^3 - 3x^2}$
17. $\lim_{x\to -2} \dfrac{x^4 - 16}{x^4 - 7x^2 + 12}$
18. $\lim_{x\to 5} \dfrac{x^3 - 25x}{4x^2 - 23x + 15}$
19. $\lim_{x\to \sqrt{2}} \dfrac{x^2 - 2}{x - \sqrt{2}}$
20. $\lim_{x\to -\sqrt{2}} \dfrac{x^2 - 2}{x + \sqrt{2}}$
21. $\lim_{x\to \sqrt{2}} \dfrac{x^4 - 4}{x^2 - 2}$
22. $\lim_{x\to 2} \dfrac{x^3 - 8}{x^2 - 4}$

Numerical: *Use a numerical representation of the function to estimate the given* *limit. Then confirm your estimate both analytically and graphically.*

23. $\lim\limits_{x\to 1} \dfrac{x^2 - x}{x^2 - 1}$

24. $\lim\limits_{x\to 1} \dfrac{(x-4)^2}{x^2 - 16}$

25. $\lim\limits_{x\to 1} \dfrac{3x^2 - x - 2}{x^2 - 1}$

26. $\lim\limits_{x\to 1} \dfrac{x - 1}{|x| - 1}$

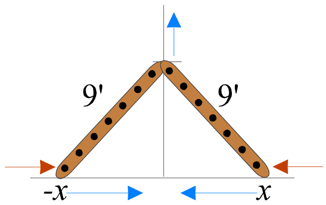
27. $\lim\limits_{x\to 2} \dfrac{x^3 - 8}{x - 2}$

28. $\lim\limits_{x\to 2} \dfrac{x^4 - 16}{x^3 - 8}$

29. $\lim\limits_{x\to 2} \dfrac{3x^2 - 2x - 8}{4x^2 - 5x - 6}$

30. $\lim\limits_{x\to 2} \dfrac{\sqrt{4x} - 2}{x - \sqrt{4x}}$

31. Suppose that two $9'$ long ladders form the sides of an isosceles triangle whose center is moving up the y-axis as the bases move toward the y-axis.



2-8

(a) What is the height $h(x)$ of the vertex on the y-axis as a function of x?

(b) What is the value of the limit

$$\lim_{x\to 0} h(x)$$

and what is the significance of the result?

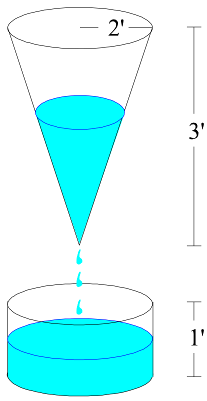
(c) What is the area $A(x)$ of the triangle as a function of x?

(d) What is the value of the limit

$$\lim_{x\to 0} A(x)$$

and what is the significance of the result?

32. Consider the funnel and tank apparatus as shown in figure 2-9.



2-9: A funnel tank system

The funnel is a right circular cone with radius 2 feet and height 3 feet. The tank is a right circular cylinder with radius 2 feet and height 1 foot. The volume of a right circular cone of radius r and height x is $V = \frac{1}{3}\pi r^2 x$ and the volume of a right circular cylinder with radius r and height h is $V = \pi r^2 h$. A fluid is flowing from the funnel into the tank.

(a) Suppose h is the depth of fluid in the tank and x is the depth of the fluid in the funnel. Write h as a function of x. (Volume in tank $= 4\pi -$ Volume in funnel)

(b) Compute and give the limit interpretation of $\lim_{x\to 1} f(x)$.

(c) If the funnel is initially half-full and the tank initially empty find $h = f(x)$, where h is the depth of the fluid in the tank and x is the depth of the fluid in the funnel. (Volume in tank $= 2\pi -$ Volume in funnel)

(d) Compute and give the limit interpretation of $\lim_{x\to 0} f(x)$

33. Explore the following limit numerically and graphically:

$$\lim_{h\to 0} \frac{(h+2)^2 - (h-2)^2}{h}$$

Then evaluate the limit analytically to obtain the same result with all three methods.

34. Explore the following limit numerically and graphically:

$$\lim_{h\to 0} \frac{\sqrt{1+h} - \sqrt{1-h}}{h}$$

Then evaluate the limit analytically to obtain the same result with all three methods.

35. Show that if $f(x) = x^2 + bx + c$ has real roots r_1 and r_2, then

$$\lim_{x\to r_1} \frac{x^2 + bx + c}{x - r_1} = r_1 - r_2$$

36. Show that if $f(x) = (x-r)\, g(x)$ where $g(x)$ is a polynomial and $g(r) \neq 0$, then

$$\lim_{x\to r} \frac{f(x)}{x-r} = g(r)$$

37. **Computer Algebra System:** Use either synthetic division or a computer algebra system to simplify

$$\frac{x^3 + 3x^2 - 5x + 1}{x - 1}$$

and then use the result to evaluate the limit

$$\lim_{x\to 1} \frac{x^3 + 3x^2 - 5x + 1}{x - 1}$$

38. **Computer Algebra System:**Use either synthetic division or a computer algebra system to simplify

$$\frac{2x^3 - 5x^2 + 5x - 6}{x - 2}$$

and then use the result to evaluate the limit

$$\lim_{x\to 2} \frac{2x^3 - 5x^2 + 5x - 6}{x - 2}$$

39. Evaluate the limits

$$\lim_{x\to 2} \frac{x^2-4}{x-2}, \quad \lim_{x\to 2} \frac{x^2+x-6}{x-2}, \quad \text{and} \quad \lim_{x\to 2} \frac{2x^2+x-10}{x-2}$$

How is the last limit related to the other two?

40. Explore the limits

$$\lim_{x\to 2} \frac{x^2-4}{x-2}, \quad \lim_{x\to 2} \frac{x^2+x-6}{x-2}, \quad \text{and} \quad \lim_{x\to 2} \left(x^2+5x+6\right)$$

How is the last limit related to the other two?

41. Write to Learn: Consider that the limits

$$\lim_{x\to -1} \frac{x}{x+1} \qquad \text{and} \qquad \lim_{x\to -1} \frac{1}{x+1}$$

do not exist, but that the limit

$$\lim_{x\to -1} \left(\frac{x}{x+1} + \frac{1}{x+1} \right) = 1$$

Write a short essay using this example to explain why

$$\lim_{x\to p} \left[f(x) + g(x) \right] = L$$

does not necessarily imply that

$$\lim_{x\to p} f(x) + \lim_{x\to p} g(x) = L$$

42. Write to Learn: Consider that

$$\lim_{x\to 1} \frac{x-1}{x^4+2} = 0 \qquad \text{and} \qquad \lim_{x\to 1} \frac{x^2+1}{x^2-1} \text{ does not exist}$$

but that the limit

$$\lim_{x\to 1} \left(\frac{x-1}{x^4+2} \cdot \frac{x^2+1}{x^2-1} \right) = \frac{1}{3}$$

Write a short essay using this example to explain why

$$\lim_{x\to p} \left[f(x)\, g(x) \right] = L$$

does not necessarily imply that

$$\left(\lim_{x\to p} f(x) \right) \left(\lim_{x\to p} g(x) \right) = L$$

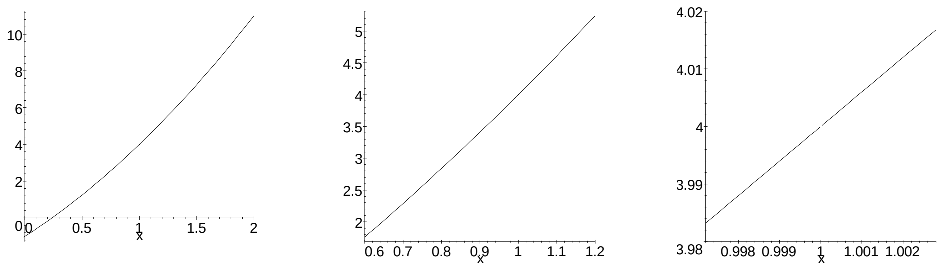
1.3 Definition of the Limit

Target Intervals

As we saw in the last section, limits can be explored numerically, graphically, and analytically. For example, if we let

$$f(x) = \frac{x^3 + 3x^2 - 5x + 1}{x - 1}$$

then we can zoom centered at $x = 1$ to produce a sequence of zooms



3-1: Zooming centered on 1

Because the graph of $f(x)$ seems to imply an output of 4 at an input of 1, we estimate that

$$\lim_{x \to 1} \frac{x^3 + 3x^2 - 5x + 1}{x - 1} = 4 \qquad (1.8)$$

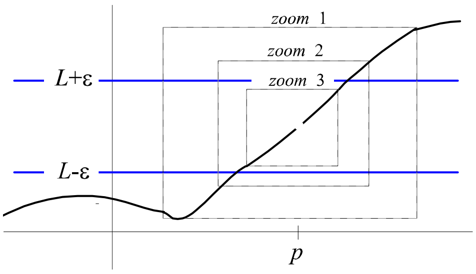
even though $f(x)$ is not defined at $x = 1$ itself.

In order to translate the zooming process into a definition of the limit, we need some new terminology. In particular, any open interval (a, b) containing a point p is said to be a *neighborhood* of p, and the Greek letter ε, which is pronounced "epsilon," will be used here to denote a small positive number.

When we zoom to estimate the limit

$$\lim_{x \to p} f(x) = L \qquad (1.9)$$

we are essentially producing smalller and smaller neighborhoods of (a, b) containing p. Our estimate is the output $f(x)$ for some x in (a, b) with $x \neq p$, and it is a good estimate if it is between $L - \varepsilon$ and $L + \varepsilon$ for some small $\varepsilon > 0$.



3-2: Zoom 3 is a graph where $L - \varepsilon < f(x) < L + \varepsilon$

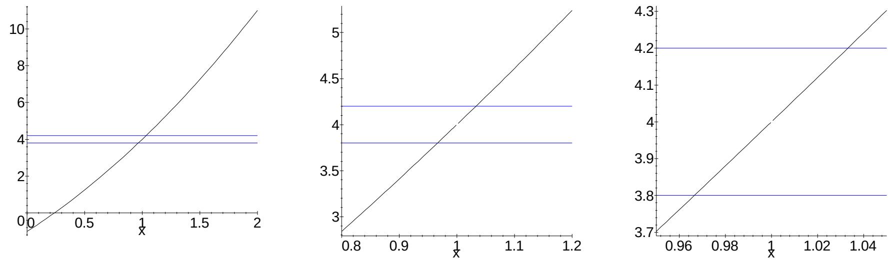
That is, our goal is to find a neighborhood (a, b) of p such that the estimate $f(x)$ for x in (a, b), $x \neq p$ is in the *target interval* $(L - \varepsilon, L + \varepsilon)$, where $\varepsilon > 0$ is a small number.

EXAMPLE 1 Let's explore (1.8) by finding a neighborhood of 1 on which the function

$$f(x) = \frac{x^3 + 3x^2 - 5x + 1}{x - 1}$$

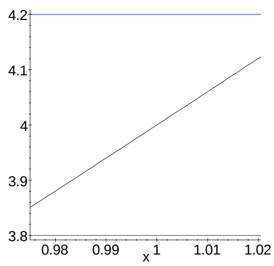
is within $\varepsilon = 0.2$ of $L = 4$ (i.e., $3.8 < f(x) < 4.2$).

Solution: Since $\varepsilon = 0.2$, we must zoom until the graph of $f(x)$ is between the horizontal lines $y = 3.8$ and $y = 4.2$:



3-3: A sequence of zooms centered at $p = 1$

The last graph in the sequence of zooms implies that if x is in $(0.98, 1.02)$, then $f(x)$ is between $y = 3.8$ and $y = 4.2$. This is illustrated more clearly in figure 3-4 below.

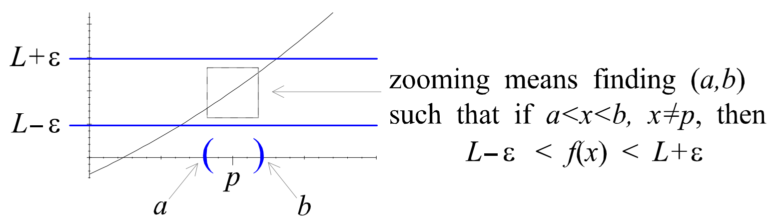


3-4: $f(x)$ is between 3.8 and 4.2

Check your Reading *Would zooming to a domain of* $(0.99, 1.01)$ *also force* $f(x)$ *to be between* $y = 3.8$ *and* $y = 4.2$?

Definition of the Limit

Intuitively, $\lim_{x \to p} f(x) = L$ means that no matter how small $\varepsilon > 0$ is, zooming centered on p will result in an interval (a, b) on which $f(x)$ is between $L - \varepsilon$ and $L + \varepsilon$ for all x in (a, b) such that $x \neq p$.



3-5: Definition of the Limit

Let's translate this intuition into a definition. To begin with, $f(x)$ between $L-\varepsilon$ and $L+\varepsilon$ implies the inequality $L-\varepsilon < f(x) < L+\varepsilon$, which is the same as

$$-\varepsilon < f(x) - L < \varepsilon \tag{1.10}$$

However, (1.10) is equivalent to saying that $|f(x)-L| < \varepsilon$. These two concepts allow us to translate our intuition about limits into the following definition.

Definition 3.1: $\lim_{x\to p} f(x) = L$ means that for all $\varepsilon > 0$, there is a neighborhood (a,b) of p such that

$$|f(x) - L| < \varepsilon$$

for all x in (a,b) such that $x \neq p$.

Moreover, any neighborhood of p contained in (a,b) also leads to $|f(x)-L| < \varepsilon$.

EXAMPLE 2 Find a neighborhood of 1 that forces $f(x)$ in the limit

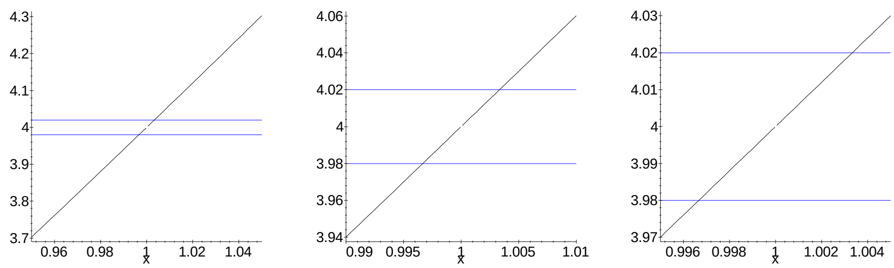
$$\lim_{x\to 1} \frac{x^3+3x^2-5x+1}{x-1} = 4$$

to be within $\varepsilon = 0.02$ of $L = 4$.

Solution: Although the interval $(0.98, 1.02)$ was sufficient for $\varepsilon = 0.2$ (see example 1), it is not sufficient for $\varepsilon = 0.02$. Thus, we must zoom centered at $p = 1$ until the graph of the function

$$f(x) = \frac{x^3+3x^2-5x+1}{x-1}$$

is between the lines $y = 4 - 0.02 = 3.98$ and $y = 4 + 0.02 = 4.02$:



3-6: A sequence of zooms centered at $p = 1$

Since the graph of $f(x)$ over the interval $(0.998, 1.002)$ is between the two horizontal lines, we can conclude that x in $(0.998, 1.002)$ and $x \neq 1$ implies that $f(x)$ is within 0.02 of 4. Moreover, any neighborhood of 1 contained in $(0.998, 1.002)$ also satisfies the definition.

Notice that the limit in example 2 is of the form $\frac{0}{0}$. Thus, the definition of the limit justifies our earlier assertion that when limits of the form $\frac{0}{0}$ exist, they must either be estimated graphically or simplified to a non-$\frac{0}{0}$ form.

EXAMPLE 3 Evaluate the limit

$$\lim_{x\to 2} \frac{x^2-4}{x^2-5x+6}$$

and then find a neighborhood of $x=2$ that forces the function to be within $\varepsilon = 0.03$ of its limit.

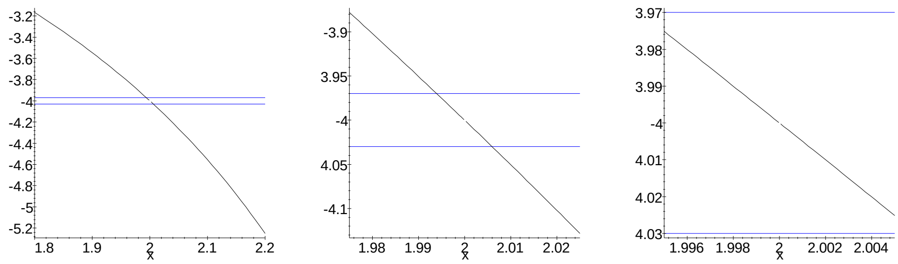
Solution: The limit is of the form $\frac{0}{0}$, which implies that

$$\lim_{x\to 2} \frac{x^2-4}{x^2-5x+6} = \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(x-3)} = \lim_{x\to 2} \frac{x+2}{x-3} = -4$$

Since $\varepsilon = 0.03$, we graph the lines $y = -4 - 0.03 = -4.003$ and $y = -4 + 0.03 = -3.97$ along with the function

$$f(x) = \frac{x^2-4}{x^2-5x+6}$$

Zooming centered on 2 eventually reveals a section of the graph of $f(x)$ between the two lines:



3-7: A sequence of zooms centered at $p=2$

Since the graph of $f(x)$ over the interval $(1.996, 2.004)$ is between the two lines, we can conclude that x in $(1.996, 2.004)$ and $x \neq 2$ implies that $f(x)$ is within 0.03 of -4.

Check your Reading *Will the interval* $(1.998, 2.002)$ *also force* $f(x)$ *to be within* $\varepsilon = 0.05$ *of* 9?

Verifying Limits Analytically

Hypothetically, it is not necessary to use zooming to find neighborhoods of p when

$$\lim_{x\to p} f(x) = L$$

Definition 3.1 implies that for any $\varepsilon > 0$, we need only manipulate the inequality $|f(x) - L| < \varepsilon$ until we have $a < x < b$ for some numbers $a < p < b$.

EXAMPLE 4 Find a neighborhood of 2 that forces $f(x)$ in the limit

$$\lim_{x\to 2} (3x+2) = 8$$

to be within $\varepsilon = 0.03$ of $L = 8$.

Solution: Since $f(x) = 3x + 2$, the inequality $|f(x) - L| < \varepsilon$ is the same as

$$|(3x + 2) - 8| < 0.03$$

which simplifies to $|3x - 6| < 0.03$. As a result,

$$\begin{aligned} -0.03 &< 3x - 6 < 0.03 \\ -0.03 &< 3(x - 2) < 0.03 \\ -0.01 &< x - 2 < 0.01 \\ 1.99 &< x < 2.01 \end{aligned}$$

Thus, $f(x) = 3x + 2$ is within $\varepsilon = 0.03$ of $L = 8$ over the interval $(1.99, 2.01)$.

Moreover, the definition of the limit allows us to actually *verify* that a limit exists by showing that a suitable neighborhood of p exists for every value of $\varepsilon > 0$.

EXAMPLE 5 Verify the limit

$$\lim_{x \to 2} (3x + 2) = 8$$

by specifying a suitable neighborhood of $p = 2$ for each $\varepsilon > 0$.

Solution: Given an $\varepsilon > 0$, we must solve the inequality

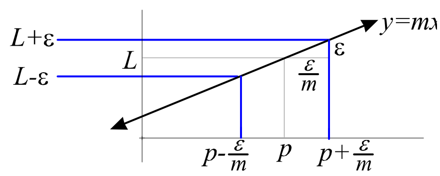
$$|3x + 2 - 8| < \varepsilon$$

The result is

$$\begin{aligned} -\varepsilon &< 3x - 6 < \varepsilon \\ 6 - \varepsilon &< 3x < 6 + \varepsilon \\ 2 - \frac{\varepsilon}{3} &< x < 2 + \frac{\varepsilon}{3} \end{aligned}$$

Thus, given any $\varepsilon > 0$, the interval $\left(2 - \frac{\varepsilon}{3}, 2 + \frac{\varepsilon}{3}\right)$ is a neighborhood of 2 on which $|(3x + 2) - 8| < \varepsilon$.

Limits of linear functions always exist. Indeed, if $f(x) = mx + b$ is a linear function with $m \neq 0$, then a *rise* of $\varepsilon > 0$ must be due to a *run* of $\frac{\varepsilon}{m}$.



3-8: Limit Definition for a Linear Function

Thus, suitable neighborhoods for limits of linear functions can always be selected to be of the form $\left(p - \frac{\varepsilon}{m}, p + \frac{\varepsilon}{m}\right)$ for each $\varepsilon > 0$.

Likewise, it can be shown that if n is an integer, then

$$\lim_{x \to p} x^n = p^n \tag{1.11}$$

for all real numbers $p > 0$. In particular, if $|x^n - p^n| < \varepsilon$, then

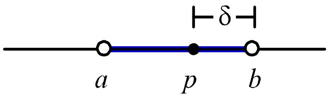
$$\begin{aligned} -\varepsilon &\leq x^n - p^n < \varepsilon \\ p^n - \varepsilon &< x^n < p^n + \varepsilon \\ \sqrt[n]{p^n - \varepsilon} &< x < \sqrt[n]{p^n + \varepsilon} \end{aligned}$$

Thus, a suitable neighborhood $\left(\sqrt[n]{p^n - \varepsilon},\ \sqrt[n]{p^n + \varepsilon}\right)$ exists for each $\varepsilon > 0$.

Check your Reading *Does the limit of a constant function always exist?*

The $\varepsilon - \delta$ Definition of the Limit

Mathematicians often use a slightly different form of definition 3.1, one that gives the limit concept a more numerical flavor. In particular, if x is in (a, b) containing a point p, then we let δ, which is the Greek lowercase "delta," be the smaller of $p - a$ and $b - p$. As a result, if x satisfies $|x - p| < \delta$, then x is in the neighborhood (a, b) of p.



This concept is used to rewrite the definition of the limit in terms of a $\delta > 0$.

Definition 3.2: $\lim_{x \to p} f(x) = L$ also means that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon$$

for all x that satisfy $0 < |x - p| < \delta$.

Computationally, verifying limits is the same as before until the last step, at which time we choose the smaller of $p - a$ and $b - p$.

EXAMPLE 5 Verify the limit

$$\lim_{x \to 2} (3x + 2) = 8$$

by finding a suitable $\delta > 0$ for each $\varepsilon > 0$.

Solution: Given an $\varepsilon > 0$, the inequality $|3x + 2 - 8| < \varepsilon$ leads to

$$\begin{aligned} -\varepsilon &< 3x - 6 < \varepsilon \\ 6 - \varepsilon &< 3x < 6 + \varepsilon \\ 2 - \frac{\varepsilon}{3} &< x < 2 + \frac{\varepsilon}{3} \end{aligned}$$

Thus, if we let $\delta = \frac{\varepsilon}{3}$, then $0 < |x - 2| < \delta$ implies $|(3x + 2) - 8| < \varepsilon$.

The advantage of definition 3.2 is that it leads to limit proofs of theorems which are concise and compact. The key is that if we choose a δ that is the smaller of all the δ's for a given set of limits, then all those limits can be used simultaneously. Let's look at two such theorems whose proofs illustrate this idea, which is italized for emphasis in both cases.

Theorem 3.3: If $\lim_{x \to p} f(x) = L$ and $\lim_{x \to p} g(x) = K$, then

$$\lim_{x \to p} [f(x) + g(x)] = L + K$$

Proof: For all $\varepsilon > 0$, there exists $\delta_1 > 0$ such that if $0 < |x - p| < \delta_1$, then

$$L - \frac{\varepsilon}{2} < f(x) < L + \frac{\varepsilon}{2} \tag{1.12}$$

Likewise, there exists $\delta_2 > 0$ such that if $0 < |x - p| < \delta_2$, then

$$K - \frac{\varepsilon}{2} < g(x) < K + \frac{\varepsilon}{2} \tag{1.13}$$

If we let $\delta > 0$ be the smaller of δ_1 and δ_2, then $0 < |x - p| < \delta$ implies that both (1.12) and (1.13) are true. Combining the two yields

$$\begin{aligned} L - \frac{\varepsilon}{2} + K - \frac{\varepsilon}{2} &< f(x) + g(x) < L + \frac{\varepsilon}{2} + K + \frac{\varepsilon}{2} \\ L + K - \varepsilon &< f(x) + g(x) < L + K + \varepsilon \end{aligned}$$

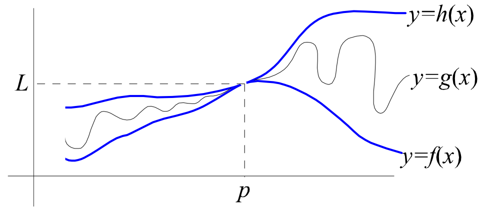
which is the same as

$$|(f(x) + g(x)) - (L + K)| < \varepsilon$$

Thus, for each $\varepsilon > 0$, a suitable $\delta > 0$ exists, so that definition 3.2 implies that

$$\lim_{x \to p} (f(x) + g(x)) = L + K = \lim_{x \to p} f(x) + \lim_{x \to p} g(x)$$

As another example, let us use the definition of the limit to consider the case where the graph of $g(x)$ is "squeezed" to a point at $x = p$ between the graphs of $f(x)$ and $h(x)$:



3-9: $y = g(x)$ is "squeezed" between $y = f(x)$ and $y = h(x)$

In words, this leads to the following:

Theorem 3.4: (Squeeze Theorem) If $f(x) \leq g(x) \leq h(x)$ on some neighborhood (a, b) of p and if

$$\lim_{x \to p} f(x) = L \qquad \textit{and} \qquad \lim_{x \to p} h(x) = L$$

then the limit of $g(x)$ as x approaches p exists and

$$\lim_{x \to p} g(x) = L$$

Proof. **:** For all $\varepsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that if $0 < |x - p| < \delta_1$ and $0 < |x - p| < \delta_2$, respectively, then

$$L - \varepsilon < f(x) < L + \varepsilon \qquad \textit{and} \qquad L - \varepsilon < h(x) < L + \varepsilon$$

respectively. *If δ is chosen to be the smaller of δ_1 and δ_2, then $0 < |x - p| < \delta$ implies that*

$$L - \varepsilon < f(x) \le g(x) \le h(x) < L + \varepsilon$$

which in turn implies that $L - \varepsilon < g(x) < L + \varepsilon$. Thus, $|g(x) - L| < \varepsilon$ for all x that satisfy $0 < |x - p| < \delta$. ■.

Exercises:

Grapher: *Zoom centered at the appropriate input to determine graphically the value of each limit. Be sure to use radians with trigonometric functions.*

1. $\displaystyle\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$
2. $\displaystyle\lim_{x \to 1} \frac{x - 1}{x^2 - 1}$
3. $\displaystyle\lim_{x \to -1} \frac{x^2 + 3x + 2}{x^2 - 1}$
4. $\displaystyle\lim_{x \to 0.5} \frac{2x^2 + x - 1}{x - 0.5}$
5. $\displaystyle\lim_{x \to 2} (3x + 2)$
6. $\displaystyle\lim_{x \to 2} (4x + 2)$
7. $\displaystyle\lim_{x \to 0} (|x| + 1)$
8. $\displaystyle\lim_{x \to 0} \left|x^2 - 1\right|$
9. $\displaystyle\lim_{x \to 1} \frac{x^3 + x^2 - x - 1}{x - 1}$
10. $\displaystyle\lim_{x \to -1} \frac{x^3 + 1}{x + 1}$
11. $\displaystyle\lim_{x \to 0} \frac{\sin(x)}{x}$
12. $\displaystyle\lim_{x \to 0} \frac{\cos(x) - 1}{x}$
13. $\displaystyle\lim_{x \to 0} x \sin\left(\tfrac{1}{x}\right)$
14. $\displaystyle\lim_{x \to \pi} \frac{\sin(x)}{x - \pi}$

Grapher: *Given the following limits, find a neighborhood of p which satisfies the definition of the limit for each of the values $\varepsilon = 0.1$, $\varepsilon = 0.01$, $\varepsilon = 0.001$.*

15. $\displaystyle\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6$
16. $\displaystyle\lim_{x \to 0} \left(x^2 - 1\right) = -1$
17. $\displaystyle\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$
18. $\displaystyle\lim_{x \to -2} \frac{x^3 + 2x^2 + x + 2}{x^2 - 4} = -\frac{5}{4}$
19. $\displaystyle\lim_{x \to 0} \frac{|x| + x^2}{|x|} = 1$
20. $\displaystyle\lim_{x \to 0} \frac{2|x| + x^2}{|x|} = 2$
21. $\displaystyle\lim_{x \to 0} \frac{\sin(4x)}{x} = 4$
22. $\displaystyle\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \frac{1}{2}$

Verify the following limits exist by finding a suitable neighborhood of the given p for each possible $\varepsilon > 0$. (Equivalently, find $\delta > 0$ for an arbitrary $\varepsilon > 0$ that satisfies the limit definition).

23. $\lim_{x \to 2} 3x = 6$

24. $\lim_{x \to 2} (x - 1) = 1$

25. $\lim_{x \to 1} (2x + 1) = 3$

26. $\lim_{x \to 1} (3 - x) = 2$

27. $\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6$

28. $\lim_{x \to 0} \frac{x^2 + x}{x} = 1$

29. $\lim_{x \to 0} \left[(x+1)^2 - (x-1)^2 \right] = 0$

30. $\lim_{x \to 1} \frac{x^2}{x} = 1$

31. Use the definition of the limit to prove that

$$\lim_{x \to p} (3x + 1) = 3p + 1$$

Then use this result to evaluate $\lim_{x \to 1} (3x + 1)$.

32. Use the definition of the limit to prove that

$$\lim_{x \to p} (mx + b) = mp + b$$

Then use this result to evaluate $\lim_{x \to \pi} (\pi x + 1)$.

33. For $f(x) = x^2 + 2x$.

(a) Show that

$$\frac{f(0+h) - f(0)}{h} = \frac{h^2 + 2h}{h}$$

(b) **Numerical:** Complete the table below:.

h	-0.01	-0.001	-0.0001	$\rightarrow$	0	$\leftarrow$	0.0001	0.001	0.01
$\frac{f(0+h)-f(0)}{h}$				$\rightarrow$	?	$\leftarrow$			

(c) Use the table to estimate the value of the limit

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

(d) **Grapher:** Find an neighborhood (a, b) of 0 which satisfies the definition of the limit in (c) for $\varepsilon = 0.01$.

34. For $f(x) = \sqrt{x+3}$.

(a) Show that

$$\frac{f(1+h) - f(1)}{h} = \frac{\sqrt{h+4} - 2}{h}$$

(b) **Numerical:** Complete the table below using the difference quotient.

h	-0.01	-0.001	-0.0001	$\rightarrow$	0	$\leftarrow$	0.0001	0.001	0.01
$\frac{f(1+h)-f(1)}{h}$				$\rightarrow$	?	$\leftarrow$			

(c) Use the table to estimate the value of the limit

$$\lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

(d) **Grapher:** Find an neighborhood (a, b) of 0 which satisfies the definition of the limit in (c) for $\varepsilon = 0.01$.

35. Write to Learn: Write a short essay in which you prove that if $\lim_{x\to p} f(x) = L$ and k is constant, then

$$\lim_{x\to p} [kf(x)] = kL$$

36. Write to Learn: Write a short essay in which you prove that if $\lim_{x\to p} f(x) = L$ and $\lim_{x\to p} g(x) = K$, then

$$\lim_{x\to p} [f(x) - g(x)] = L - K$$

37. Write to Learn: Write a short essay in which you prove that

$$\lim_{x\to p} x^{1/2} = p^{1/2}$$

for all positive numbers p.

38. Write to Learn: Write a short essay in which you prove that if $\lim_{x\to p} f(x) = L$ and $\lim_{x\to p} g(x) = K$ and if $f(x) \le g(x)$ for all x in a neighborhood (a, b) of p, then $L \le K$. (Hint: If $L - \varepsilon < K + \varepsilon$ for all $\varepsilon > 0$, then $L \le K$).

1.4 Horizontal Asymptotes

Limits to Infinity

In many applications of the limit concept, an input variable x does not approach a number p, but it instead becomes larger and larger without bound. In this case, we say that x approaches *infinity*, and we write

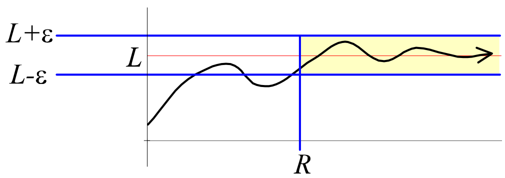
$$\lim_{x\to\infty} f(x) = L \tag{1.14}$$

where ∞ is called the *infinity symbol* and represents the concept of numbers becoming larger and larger without bound. (note: ∞ is not a number, even though we often treat it like one).

Rigorously, (1.14) means that for each $\varepsilon > 0$, there is an interval (R, ∞) for some number R such that

$$L - \varepsilon < f(x) < L + \varepsilon$$

That is, as x increases without bound, $f(x)$ approaches L.



4-1: $|f(x) - L| < \varepsilon$ on (R, ∞)

If we replace the interval (R, ∞) by the inequality $x > R$, then we are led to the following definition.

Definition 4.1a: $\lim_{x\to\infty} f(x) = L$ means that for any $\varepsilon > 0$, there is a number R such that if $x > R$, then

$$|f(x) - L| < \varepsilon$$

The limit as x approaches $-\infty$ of $f(x)$ is similarly defined:

Definition 4.1b: $\lim_{x\to-\infty} f(x) = L$ means that for any $\varepsilon > 0$, there is a number R such that if $x < R$, then $|f(x) - L| < \varepsilon$.

We say that the curve $y = f(x)$ has a *horizontal asymptote* of $y = L$ if either

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L$$

EXAMPLE 1 Use definition 4.1 to show that

$$\lim_{x\to\infty} \frac{1}{x^2} = 0$$

Solution: Suppose we are given a number $\varepsilon > 0$ (ε should be considered small—such as $\varepsilon = 0.01$ or $\varepsilon = 0.00001$, for instance). We must find $R > 0$ such that if $x > R$, then

$$\left|\frac{1}{x^2} - 0\right| < \varepsilon \tag{1.15}$$

However, (1.15) simplifies to

$$\frac{1}{x^2} < \varepsilon \text{ or } x^2 > \frac{1}{\varepsilon}$$

Thus, if $x > \frac{1}{\sqrt{\varepsilon}}$, then (1.15) is true. Thus, given any $\varepsilon > 0$, we let $R = \frac{1}{\sqrt{\varepsilon}}$ and require that $x > R$.

For example, if $\varepsilon = 0.01$ in example 1, then we would let

$$R = \sqrt{\frac{1}{0.01}} = \sqrt{100} = 10$$

As a result, $x > 10$ implies that $\left|\frac{1}{x^2} - 0\right| < 0.01$.

Let's expand example 1 into a much broader result. Consider that if $n > 0$, then x^n becomes arbitrarily large as x approaches ∞. Thus, for any $\varepsilon > 0$, there is a number R such that $|x|^n > \frac{1}{\varepsilon}$ for all $x > R$. It follows that

$$\left|\frac{1}{x^n}\right| < \varepsilon \text{ or } \left|\frac{1}{x^n} - 0\right| < \varepsilon$$

for all $x > R$. That is, $1/x^n$ becomes arbitrarily close to 0 as x approaches ∞, which we write as

$$\lim_{x \to \infty} \frac{1}{x^n} = 0 \quad \text{if} \quad n > 0 \tag{1.16}$$

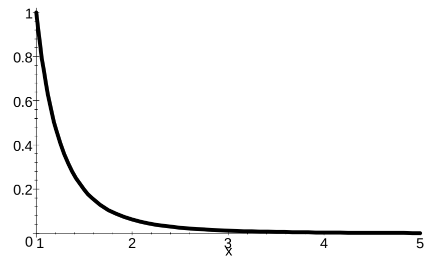
As a result, $y = \dfrac{1}{x^n}$ has a horizontal asymptote of $y = 0$ if $n > 0$.

EXAMPLE 2 What is the horizontal asymptote of $y = \dfrac{1}{x^4}$?

Solution: Equation (1.16) with $n = 4$ is of the form

$$\lim_{x \to \infty} \frac{1}{x^4} = 0$$

As a result, $y = \frac{1}{x^4}$ has a horizontal asymptote of $y = 0$.



4-2: Graph of $y = 1/x^4$

Check your Reading *Does $y = 1/x^3$ have a horizontal asymptote? Explain.*

Limits of Rational Functions

The result (1.16) can be applied to limits of algebraic functions in general. To do so, we first define the *principal term* to be x^n where n denotes the degree of the denominator. We then multiply by

$$\frac{1/x^n}{1/x^n}$$

and use (1.16) to evaluate the resulting limits.

EXAMPLE 3 Evaluate the limit

$$\lim_{x \to \infty} \frac{x^4 + 1}{x^4 + 2x + 3}$$

Solution: The *principal term* is x^4 since 4 is the highest power in the denominator. Multiplying by $\frac{1/x^4}{1/x^4}$ yields

$$\lim_{x \to \infty} \left(\frac{x^4 + 1}{x^4 + 2x + 3} \right) \frac{\frac{1}{x^4}}{\frac{1}{x^4}}$$

which after distributing yields

$$\lim_{x\to\infty}\left(\frac{\frac{x^4}{x^4}+\frac{1}{x^4}}{\frac{x^4}{x^4}+\frac{2x}{x^4}+\frac{3}{x^4}}\right)=\lim_{x\to\infty}\left(\frac{1+\frac{1}{x^4}}{1+\frac{2}{x^3}+\frac{3}{x^4}}\right)$$

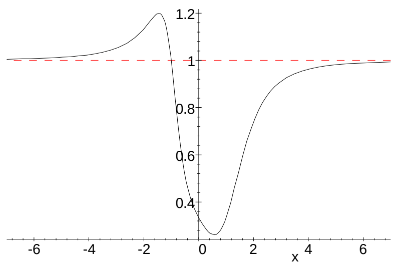
The result is a ratio of limits of the form (1.16), so that

$$\lim_{x\to\infty}\frac{x^4+1}{x^4+2x+3}=\lim_{x\to\infty}\left(\frac{1+\frac{1}{x^4}}{1+\frac{2}{x^3}+\frac{3}{x^4}}\right)=\frac{1+0}{1+0+0}=1$$

It also follows that the graph of the function

$$f(x)=\frac{x^4+1}{x^4+2x+3}$$

has a horizontal asymptote of $y=1$:



4-3: Horizontal Asymptote of $y=1$

Limits in which x approaches $-\infty$ can be converted into limits as x approaches ∞ using the fact that

$$\lim_{x\to-\infty}f(x)=\lim_{x\to\infty}f(-x) \tag{1.17}$$

Thus, limits to either $+\infty$ or $-\infty$ are evaluated in the same way.

EXAMPLE 4 Evaluate the limit

$$\lim_{x\to-\infty}\frac{x^2-1}{1+x}$$

Solution: Using (1.17), we convert this into the limit

$$\lim_{x\to-\infty}\frac{x^2-1}{1+x}=\lim_{x\to\infty}\frac{(-x)^2-1}{1-x}=\lim_{x\to\infty}\frac{x^2-1}{1-x}$$

The principal term is x, so we transform the limit into

$$\lim_{x\to\infty}\frac{x^2-1}{1-x}\,\frac{1/x}{1/x}=\lim_{x\to\infty}\frac{x-\frac{1}{x}}{\frac{1}{x}-1}=\lim_{x\to\infty}\frac{x}{-1}$$

since $\frac{1}{x}$ approaches 0 as x approaches ∞. However, $\frac{x}{-1}$ approaches $-\infty$ as x approaches ∞. We thus write either

$$\lim_{x\to-\infty}\frac{x^2-1}{1+x}=-\infty \quad \textit{or} \quad \lim_{x\to-\infty}\frac{x^2-1}{1+x}\text{ does not exist} \tag{1.18}$$

Check your Reading *Why are both statements true in (1.18)?*

Functions with Two Horizontal Asymptotes

A function may possibly have 2 horizontal asymptotes—one as x approaches $-\infty$ and one as x approaches ∞. For example, if a function is defined by roots of polynomials, then sign changes may be implied by the roots being used.

EXAMPLE 5 Find all the horizontal asymptotes of

$$f(x) = \frac{x+0.5}{\sqrt{x^2+1}}$$

Solution: To begin with, let us evaluate

$$\lim_{x\to\infty} \frac{x+0.5}{\sqrt{x^2+1}}$$

The degree of the denominator is 1, so the principal term is x^1. Moreover, we can assume that $x > 0$, so that $\sqrt{x^2} = x$. Thus, we have

$$\lim_{x\to\infty} \frac{x+0.5}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{x+0.5}{\sqrt{x^2+1}} \frac{1/x}{1/\sqrt{x^2}} = \lim_{x\to\infty} \frac{1+\frac{0.5}{x}}{\sqrt{1+\frac{1}{x^2}}}$$

As x approaches ∞, we obtain

$$\lim_{x\to\infty} \frac{x+0.5}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{1+\frac{0.5}{x}}{\sqrt{1+\frac{1}{x^2}}} = \frac{1+0}{\sqrt{1+0}} = 1$$

and thus, $y = 1$ is a horizontal asymptote of $f(x)$ as x approaches ∞.

For x approaching $-\infty$, we use (1.17):

$$\lim_{x\to-\infty} \frac{x+0.5}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{-x+0.5}{\sqrt{(-x)^2+1}} = \lim_{x\to\infty} \frac{0.5-x}{\sqrt{x^2+1}}$$

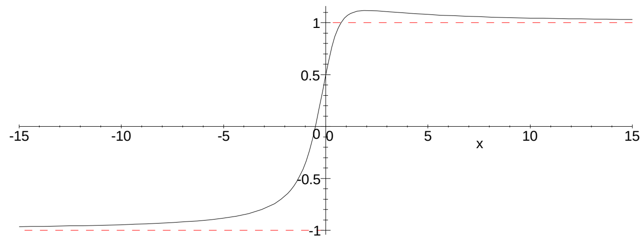
Again, the principal term is x and we can assume that $x > 0$. Thus,

$$\lim_{x\to\infty} \frac{0.5-x}{\sqrt{x^2+1}} \frac{1/x}{1/\sqrt{x^2}} = \lim_{x\to\infty} \frac{\frac{0.5}{x}-1}{\sqrt{1+\frac{1}{x^2}}} = \frac{0-1}{\sqrt{1+0}} = -1$$

As a result, we have

$$\lim_{x\to-\infty} \frac{x+0.5}{\sqrt{x^2+1}} = -1$$

which implies that $f(x)$ has a horizontal asymptote of $y = -1$ as x approaches $-\infty$.



4-4: A function with 2 horizontal asymptotes

The absolute value function can also lead to functions with more than one horizontal asymptote, or possibly even a function with only one horizontal asymptote in only one direction.

EXAMPLE 6 Find the horizontal asymptotes of $f(x) = |x| + x + 1$.

Solution: If x approaches ∞, then we can assume $x > 0$ and thus,

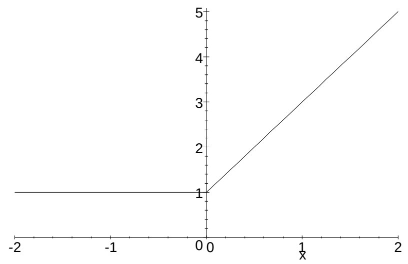
$$\lim_{x\to\infty} (|x| + x + 1) = \lim_{x\to\infty} (2x + 1) = \infty$$

Because $2x + 1$ becomes arbitrarily large as x approaches ∞, the function $f(x) = |x| + x + 1$ does not have a horizontal asymptote as x approaches ∞.

For the limit as x approaches $-\infty$, we use to fact that $|x| = -x$ when $x < 0$, so that

$$\lim_{x\to-\infty} (|x| + x + 1) = \lim_{x\to-\infty} (-x + x + 1) = \lim_{x\to-\infty} (0 + 1) = 1$$

Thus, $f(x) = |x| + x + 1$ has a horizontal asymptote of $y = 1$ as x approaches $-\infty$.



4-5: Graph of $f(x) = |x| + x + 1$

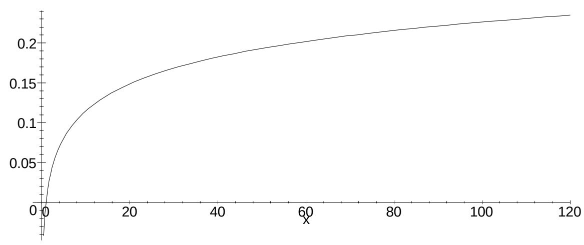
Check your Reading *Can a function have 3 horizontal asymptotes*?

Horizontal Asymptotes and Graphing

Rigorous definitions for limits to ∞ are necessary because even the most sophisticated graphing technology may not be useful as tools for studying horizontal asymptotes. For example, the graph of

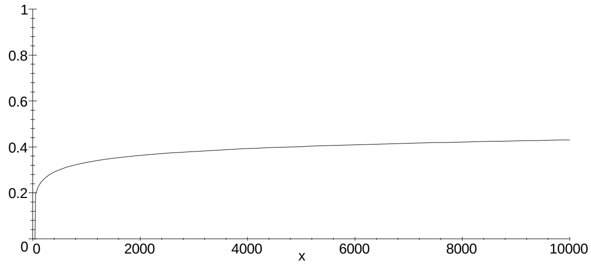
$$f(x) = \frac{x^{0.1} - 1}{x^{0.1} + 1}$$

over the interval $[0, 120]$ does not reveal its horizontal asymptote.



4-6

No horizontal asymptote is observed even for the graph of $f(x)$ over $[0, 10000]$.



4-7

However, it is rather easy to show that

$$\lim_{x \to \infty} \frac{x^{0.1} - 1}{x^{0.1} + 1} = 1$$

which implies that the horizontal asymptote of $y = f(x)$ is $y = 1$.

EXAMPLE 7 Show that

$$\lim_{x \to \infty} \frac{x^{0.1} - 1}{x^{0.1} + 1} = 1$$

Solution: The principal term is $x^{0.1}$. Thus,

$$\lim_{x \to \infty} \frac{x^{0.1} - 1}{x^{0.1} + 1} \frac{1/x^{0.1}}{1/x^{0.1}} = \lim_{x \to \infty} \frac{1 - \frac{1}{x^{0.1}}}{1 + \frac{1}{x^{0.1}}} = \frac{1 - 0}{1 + 0} = 1$$

Exercises:

Evaluate the limit if it exists. If the limit does not exist, explain why not.

1. $\lim_{x \to \infty} \frac{1}{x}$
2. $\lim_{x \to \infty} \frac{1}{x^2}$
3. $\lim_{x \to \infty} \frac{1}{\sqrt{x}}$
4. $\lim_{x \to \infty} \frac{1}{\sqrt[3]{x}}$
5. $\lim_{x \to \infty} 1$
6. $\lim_{x \to \infty} 0$
7. $\lim_{x \to \infty} \frac{1}{x + 1}$
8. $\lim_{x \to \infty} \frac{1}{x + 3}$
9. $\lim_{x \to \infty} \frac{2x + 5}{x - 3}$
10. $\lim_{x \to \infty} \frac{5x - 3}{4x + 2}$
11. $\lim_{x \to \infty} \frac{x^3 + 3x}{x^2 + 2}$
12. $\lim_{x \to \infty} \frac{x^3 - 3x + 5}{x^2 + 2x - 1}$
13. $\lim_{x \to \infty} \frac{x^2 - 4x + 2}{3x^2 - 5}$
14. $\lim_{x \to \infty} \frac{2x^2 - 4x - 2}{1.5x^2 + 4x}$
15. $\lim_{x \to \infty} \left(\frac{1}{x + 1} - \frac{1}{x - 1} \right)$
16. $\lim_{x \to \infty} \left(\frac{1}{x + 1} + \frac{1}{x - 1} \right)$

Find the horizontal asymptote, if one exists, of each of the following functions.

17. $f(x) = \dfrac{1}{x}$

18. $f(x) = \dfrac{1}{\sqrt{x}}$

19. $f(x) = \dfrac{2x^4 - 1}{x^4 - 3x + 2}$

20. $f(x) = \dfrac{x^3 + 3x^2}{x^2 + 1}$

21. $f(x) = \dfrac{x}{x^2 + 1}$

22. $f(x) = \dfrac{x}{x^2 - 1}$

23. $f(x) = \dfrac{(2x^2 + 1)^3}{(2x^3 + 1)^2}$

24. $f(x) = \left(\dfrac{3x + 2}{2x + 3}\right)^2$

Identify all horizontal asymptotes, if any, of the following functions.

25. $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}$

26. $f(x) = \dfrac{2x + 1}{\sqrt{x^2 + 1}}$

27. $f(x) = \dfrac{\sqrt{x^6 + x + 1}}{x^3}$

28. $f(x) = \dfrac{x^3}{\sqrt{x^4 + 1}}$

29. $f(x) = \dfrac{x}{|x| + 1}$

30. $f(x) = \dfrac{x^3}{|x| + 1}$

31. $f(x) = \dfrac{x + |x|}{x + 1}$

32. $f(x) = \dfrac{(|x| + x)^2}{x^2 + 1}$

33. Find the horizontal asymptotes of

$$f(x) = \frac{x + 6}{x^2 + 1}$$

Graph the function and its asymptote over the interval $[-5, 5]$. Does the graph of the function intersect the asymptote at some point in $[-5, 5]$?

34. A linear function $L_\infty(x) = b + mx$ is an *oblique asymptote* of $f(x)$ if

$$\lim_{x \to \infty} [f(x) - L_\infty(x)] = 0 \tag{1.19}$$

Show that the given linear function $L_\infty(x)$ is an oblique asymptote of $f(x)$.(Hint: find a common denominator and combine fractions) Then graph both $L_\infty(x)$ and $f(x)$ over $[-5, 5]$.

(a) $f(x) = \dfrac{x^2 + 1}{x}, \quad L_\infty(x) = x$

(b) $f(x) = \dfrac{2x^2 + 15x + 8}{x + 7}, \quad L_\infty(x) = 2x + 1$

(c) $f(x) = \dfrac{-3x^3 + 2x^2 - 3x + 3}{x^2 + 1}, \quad L_\infty(x) = 2 - 3x$

35. In this exercise, we consider the function

$$f(x) = \frac{\sqrt{x}}{\sqrt{x} + 2}$$

(a) Show that the horizontal asymptote of $f(x)$ is $y = 1$ by evaluating

$$\lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x}+2}$$

(b) Graph $f(x)$ and $y = 1$ over the interval $[0, 10]$. Would you conclude from the graph that $y = 1$ is a horizontal asymptote of $f(x)$?

(c) Graph $f(x)$ and $y = 1$ over the interval $[0, 100]$. Would you conclude from the graph that $y = 1$ is a horizontal asymptote of $f(x)$?

36. In this exercise, we consider the function

$$f(x) = \frac{x^{0.2} - 4x^{0.1} + 3}{x^{0.2} + 3}$$

(a) Show that the horizontal asymptote of $f(x)$ is $y = 1$ by evaluating

$$\lim_{x\to\infty} \frac{x^{0.2} - 4x^{0.1} + 3}{x^{0.2} + 3}$$

(b) Graph $f(x)$ and $y = 1$ over the interval $[0, 1000]$. Would you conclude from the graph that $y = 1$ is a horizontal asymptote of $f(x)$?

(c) Evaluate $f(1\text{E}10)$, where $a\text{E}b$ is scientific notation for $a \times 10^b$. How close is it to 1?

37. Use the definition of the limit to ∞ to show that if

$$\lim_{x\to\infty} f(x) = L \qquad \textit{and} \qquad \lim_{x\to\infty} g(x) = K$$

then

$$\lim_{x\to\infty} [f(x) + g(x)] = L + K$$

38. Use the definition of the limit to ∞ to show that if

$$if \ \lim_{x\to\infty} f(x) = L \qquad and \qquad \lim_{x\to\infty} g(x) = K$$

$$then \ \lim_{x\to\infty} [f(x) - g(x)] = L - K$$

39. A function $f(x)$ is said to be *asymptotically equivalent* to $g(x)$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \neq 0$$

Show that $f(x)$ is asymptotically equivalent to $g(x)$ if and only if $g(x)$ is asymptotically equivalent to $f(x)$.

40. A function $f(x)$ is said to be of order at most $g(x)$ if there exists $M > 0$ such that

$$\frac{f(x)}{g(x)} \leq M$$

when x is sufficiently large. Give an example of a function $f(x)$ that is of at most order $g(x) = 1$ for which

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \ \textit{does not exist}$$

(Hint: Think of the trigonometric functions).

41. **Write to Learn:** Suppose that $a_n \neq 0$ and $b_n \neq 0$ where n is a positive integer. In a short essay, explain why

$$\lim_{x\to\infty} \frac{b_n x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0}{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0} = \frac{b_n}{a_n}$$

42. **Write to Learn:** Suppose that $a_n \neq 0$ and $b_m \neq 0$ where m, n are positive integers. In a short essay, explain why the limit

$$\lim_{x\to\infty} \frac{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0}{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}$$

is equal to 0 when $n > m$ and does not exist when $n < m$.

1.5 Continuity

Continuity at a Point

Sections 1-2 and 1-3 imply that if $f(x)$ is a polynomial, then

$$\lim_{x\to p} f(x) = f(p)$$

That is, if $f(x)$ is a polynomial, then we can evaluate its limit as x approaches p by simply substituting p for x. This idea is very important in mathematics, so important that we state it as a definition.

Definition 5.1: A function $f(x)$ is *continuous* at p if

$$\lim_{x\to p} f(x) = f(p) \tag{1.20}$$

Moreover, continuous functions can be combined arithmetically to create still more continuous functions..

Theorem 5.2: If f and g are continuous at $x = p$ and k is a number, then $f + g$, $f - g$, kf and fg are also continuous at $x = p$. Moreover, if $g(p) \neq 0$, then $\frac{f}{g}$ is also continuous at $x = p$.

In addition, if $\lim_{x\to p} g(x) = L$ and $f(x)$ is continuous at L, then

$$\lim_{x\to p} f[g(x)] = f\left[\lim_{x\to p} g(x)\right] \tag{1.21}$$

which implies that if g is continuous at $x = p$ and f is continuous at $g(p)$, then $f \circ g$ is continuous at p. A special case of (1.21) is given by

$$\lim_{x\to p} [f(x)]^n = \left[\lim_{x\to p} f(x)\right]^n \tag{1.22}$$

when the limit as x approaches p of $f(x)$ exists.

EXAMPLE 1 If $g(x)$ is continuous at $x = 4$ and if $g(4) = 3$, then what is the value of

$$\lim_{x \to 4} \left[x^2 + 2g(x) \right]$$

Solution: Since x^2 is a polynomial, it is continuous at $x = 4$. Thus, theorem 5-2 implies that $x^2 + 2g(x)$ is also continuous at $x = 4$, so that

$$\lim_{x \to 4} \left[x^2 + 2g(x) \right] = 4^2 + 2g(4) = 16 + 2 \cdot 3 = 22$$

EXAMPLE 2 Suppose that $\lim_{x \to 5} g(x) = 2$ and evaluate

$$\lim_{x \to 5} [g(x)]^4$$

Solution: The property (1.22) implies that

$$\lim_{x \to 5} [g(x)]^4 = \left[\lim_{x \to 5} g(x) \right]^4 = [2]^4 = 16$$

Check your Reading *What is* $\lim_{x \to 5} [g(x)]^2$ *given the fact that* $\lim_{x \to 5} g(x) = 2$?

One Sided Limits

To better explore continuity, let us define

$$\lim_{x \to p^+} f(x) = L \tag{1.23}$$

to mean that as x approaches p *with the restriction that* $x > p$, the function $f(x)$ approaches L. Similarly, let us define

$$\lim_{x \to p^-} f(x) = L \tag{1.24}$$

to mean that as x approaches p *with the restriction that* $x < p$, the function $f(x)$ approaches L. We call (1.23) and (1.24) the *limit from the right* and the *limit from the left,* respectively.

It follows immediately from the definition of the limit that

$$\lim_{x \to p} f(x) = L \;\; only\ if \;\; \lim_{x \to p^-} f(x) = L = \lim_{x \to p^+} f(x)$$

That is, the limit exists only if the limits from the right and left are the same. As a result, definition 5.1 says that a function $f(x)$ is continuous at an input value p only if the following hold:

i. $f(p)$ is defined

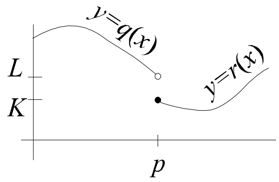
ii. $\lim_{x \to p^-} f(x) = \lim_{x \to p^+} f(x) = L$

iii. L is the same as $f(p)$

For example, suppose that $f(x)$ is the piecewise-defined function

$$f(x) = \begin{cases} q(x) & if \quad x < p \\ r(x) & if \quad x \geq p \end{cases}$$

A typical graph of a piecewise-defined f is shown below:

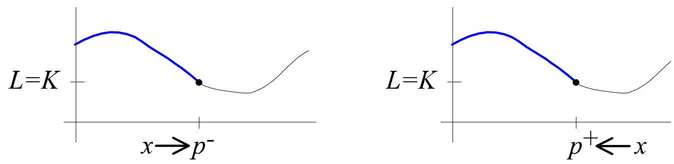


5-1: A piecewise defined function

Then continuity of $f(x)$ at $x = p$ requires that $K = L$, which is equivalent to

$$\lim_{x \to p^-} q(x) = \lim_{x \to p^+} r(x) = f(p)$$

That is, $f(x)$ is continuous at p only if the limit of $f(x)$ exists at p, which occurs only when the limits from the left and right are the same at p.



5-2: Limit from left and right are the same

When $f(x)$ is **not** continuous at $x = p$, as would be the case if $L \neq K$, then $x = p$ is called a *point of discontinuity* of $f(x)$.

EXAMPLE 3 Find a value of k for which

$$f(x) = \begin{cases} 2x + 3 & if \quad x \leq 2 \\ 2x^2 - k & if \quad x > 2 \end{cases} \tag{1.25}$$

is continuous at $x = 2$.

Solution: By definition, $f(2) = 2(2) + 3 = 7$, thus satisfying the first criterion. Next, we notice that the limit from the left is

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x + 3) = 2 \cdot 2 + 3 = 7$$

while the limit from the right is

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \left(2x^2 - k\right) = 2 \cdot (2)^2 - k = 8 - k$$

To achieve continuity, we need

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

which means that we need

$$7 = 8 - k \qquad or \qquad k = 1$$

EXAMPLE 4 Find a value of k for which

$$f(x) = \begin{cases} x^2 - 3x & if \quad x \le -1 \\ 3x^2 + 2k & if \quad x > -1 \end{cases} \tag{1.26}$$

is continuous at $x = -1$.

Solution: By definition, $f(-1) = (-1)^2 - 3(-1) = 4$, thus satisfying the first criterion. Next, we notice that the limit from the left is

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \left(x^2 - 3x\right) = (-1)^2 - 3(-1) = 4$$

while the limit from the right is

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \left(3x^2 + 2k\right) = 3(-1)^2 + 2k = 3 + 2k$$

To achieve continuity, we need

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = f(-1)$$

which means that we need

$$3 + 2k = 4 \qquad or \qquad k = \frac{1}{2}$$

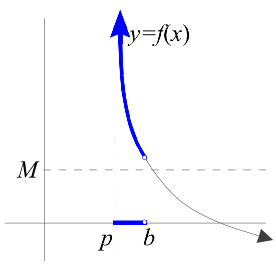
Check your Reading *Does (1.26) with $k = \frac{1}{2}$ satisfy criterion (iii)? Explain.*

Points of Discontinuity

Let us define the equation

$$\lim_{x \to p^+} f(x) = \infty$$

to mean that for all $M > 0$, there exists intervals of the form (p, b) such that if x is in (p, b), then $f(x) > M$.



5-3: x in (p, b) implies that $f(x) > M$

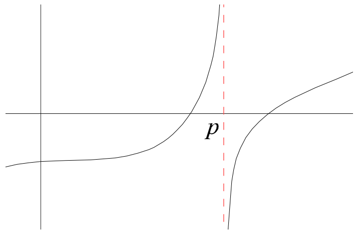
We similarly define the one-sided infinite limits

$$\lim_{x \to p^+} f(x) = -\infty, \qquad \lim_{x \to p^-} f(x) = \infty, \quad and \quad \lim_{x \to p^-} f(x) = -\infty$$

We then define the line $x = p$ to be a *vertical asymptote* of $f(x)$ if any of the following limits hold:

$$\lim_{x \to p^+} f(x) = \infty, \quad \lim_{x \to p^+} f(x) = -\infty, \quad \lim_{x \to p^-} f(x) = \infty, \quad \lim_{x \to p^-} f(x) = -\infty$$

Since ∞ is not a number, a vertical asymptote of $f(x)$ must also be a point of discontinuity of $f(x)$.

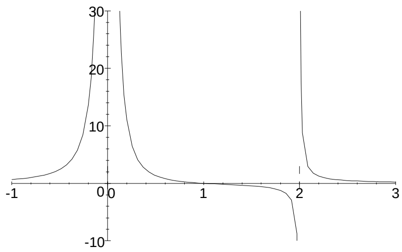


5-4: Vertical Asymptote of $x = p$

EXAMPLE 5 Where are the points of discontinuity of

$$f(x) = \frac{x-1}{x^3 - 2x^2}$$

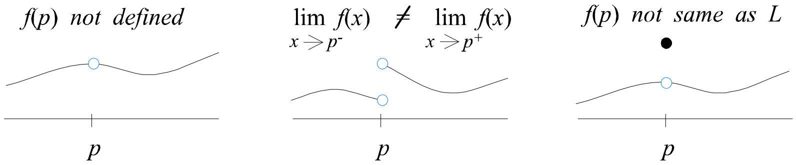
Solution: The function $f(x)$ is the ratio of two polynomials, and polynomials are continuous everywhere. However, $x^3 - 2x^2 = 0$ if $x = 0$ or $x = 2$. Since the numerator is nonzero at both of these inputs, $f(x)$ has vertical asymptotes at $x = 0$ and $x = 2$.



5-5: $f(x)$ has two vertical asymptotes

Correspondingly, $x = 0$ and $x = 2$ are points of discontinuity of $f(x)$.

In addition, a function is **not** continuous at $x = p$ if any of the following occur at $x = p$:



5-6: Criteria for Continuity

That is, a function is not continuous at p if any of the three conditions (i), (ii), or (iii) do not hold at p.

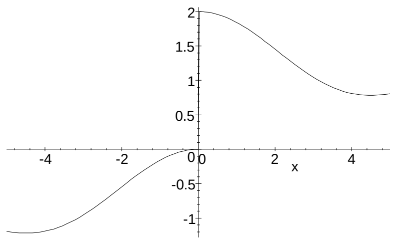
EXAMPLE 6 Is $f(x) = \frac{|x-1|}{x-1}$ continuous at $x = 1$?

Solution: Since $f(1)$ leads to division by 0, *(i)* is not satisfied. Thus, $f(x) = \frac{|x-1|}{x-1}$ is **not** continuous at $x = 0$.

EXAMPLE 7 Determine if the following function is continuous at $x = 0$.

$$f(x) = \begin{cases} \dfrac{|x| + \sin(x)}{x} & if \quad x \neq 0 \\ 2 & if \quad x = 0 \end{cases}$$

Solution: Condition (i) holds since $f(0) = 2$. Let's explore *(ii)* using the graph of $f(x)$:



5-7: Graph of $f(x)$

Clearly, the limits from the left and right are

$$\lim_{x \to 0^+} \frac{|x| + \sin(x)}{x} = 2 \qquad \text{and} \qquad \lim_{x \to 0^-} \frac{|x| + \sin(x)}{x} = 0$$

Since the limits from the left and right are not the same,

$$\lim_{x \to 0} \frac{|x| + \sin(x)}{x} \text{ does not exist}$$

Thus, the function $f(x)$ is **not** continuous at $x = 0$.

Check your Reading *Is* $f(x) = \sin(1/x)$ *continuous at* $x = 0$?

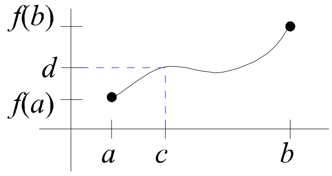
The Intermediate Value Theorem

We say that $f(x)$ is continuous on an interval $[a, b]$ if $f(x)$ is continuous at p for every p in (a, b) and if

$$\lim_{x \to a^+} f(x) = f(a), \qquad \lim_{x \to b^-} f(x) = f(b)$$

The phrase "$f(x)$ is continuous" means that $f(x)$ is continuous at every point.

If $f(x)$ is continuous on an interval $[a, b]$ and if d is a number between $f(a)$ and $f(b)$, then the *intermediate value theorem* says that there is a number c in $[a, b]$ such that $f(c) = d$.



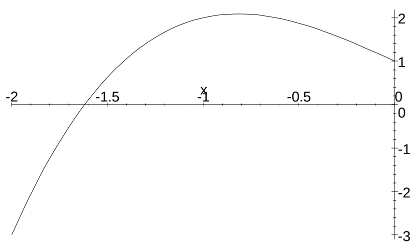
5-8: There must be c in (a, b) such that $f(c) = d$

It is the intermediate value theorem from which we obtain the concept that a continuous function can be "drawn without lifting your pencil."

However, the intermediate value theorem may be more valuable as a tool in numerical calculations. In particular, it can be used to determine if a continuous function has a zero in a given interval.

EXAMPLE 8 Determine if $f(x) = x^3 - 3x + 1$ has a root in the interval $[-2, 0]$.

Solution: Notice that $f(-2) = -1$ while $f(0) = 1$. Since 0 is between -1 and 1, there exists a number c in $[-2, 0]$ such that $f(c) = 0$. That is, $f(x)$ has a root somewhere in $[-2, 0]$:



5-9: Locating roots with the Intermediate Value Theorem

Suppose now for $f(x) = x^3 - 3x + 1$ as in example 8 that we noticed that $f(-1) = 1$. Since $f(-1)$ and $f(-2)$ have different signs, $f(x)$ must have a root at some point in $[-2, -1]$. Indeed, the *Bisection method* in numerical analysis is an algorithm in which we repeatedly compute the midpoint of an interval and then determine which subinterval contains the root of the function.

Exercises:

Identify the point(s), if any, where the function is not continuous. Graph each function on $[-4, 4]$ to verify your identifications. See the users manual for your calculator for instructions on graphing piecewise-defined functions.[2]

1. $f(x) = \dfrac{1}{x-1}$
2. $f(x) = \dfrac{x}{x+1}$
3. $f(x) = (x^2 - 2x + 1)^{1/3}$
4. $f(x) = (x^3 + 1)^{-2}$
5. $f(x) = \left|\dfrac{x^2 - 1}{x^2 + 1}\right|$
6. $f(x) = \left|\dfrac{x^2 + 1}{x^2 - 1}\right|$
7. $f(x) = \dfrac{x^2 + |x|}{|x|}$
8. $f(x) = \dfrac{x^2 + x}{|x|}$
9. $f(x) = \sin(|x|)$
10. $f(x) = \cos(|x|)$

[2] Many graphing programs will not permit the odd root of a negative number unless the variable is first declared to be a real variable. See your manual for details.

11. $f(x) = \frac{\sin(|x|)}{x}$

12. $f(x) = \frac{1 - \cos(x)}{|x|}$

13. $f(x) = \begin{cases} (x-1)^2 & if \quad x < 0 \\ x^2 - 1 & if \quad x \geq 0 \end{cases}$

14. $f(x) = \begin{cases} x^2 & if \quad x < 1 \\ 2x - 1 & if \quad x \geq 1 \end{cases}$

15. $f(x) = \begin{cases} x^2 + 2x & if \quad x < -2 \\ \sin(\pi x) & if \quad x \geq -2 \end{cases}$

16. $f(x) = \begin{cases} x - 1 & if \quad x < 0 \\ \cos(x) & if \quad x \geq 0 \end{cases}$

17. $f(x) = \begin{cases} 3x - 2 & if \quad x < 1 \\ x^3 - 1 & if \quad x \geq 1 \end{cases}$

18. $f(x) = \begin{cases} 2x - 1 & if \quad x < 1 \\ 3 - 2x & if \quad x \geq 1 \end{cases}$

Find a value for k for which the given functions are continuous at the point between the two sections.

19. $f(x) = \begin{cases} 3x - 2 & if \quad x < 1 \\ kx + 2 & if \quad x \geq 1 \end{cases}$

20. $f(x) = \begin{cases} 4x + 1 & if \quad x \leq 2 \\ kx + 3 & if \quad x > 2 \end{cases}$

21. $f(x) = \begin{cases} (kx+1)^2 & if \quad x \neq 1 \\ 4 & if \quad x = 1 \end{cases}$

22. $f(x) = \begin{cases} (kx+1)^2 & if \quad x \neq 1 \\ 4 & if \quad x = 1 \end{cases}$

23. $f(x) = \begin{cases} |x| - x & if \quad x < -3 \\ k & if \quad x = -3 \\ 2|x| & if \quad x > -3 \end{cases}$

24. $f(x) = \begin{cases} x^2 + 2 & if \quad x < 1 \\ k & if \quad x = 1 \\ 2x + 1 & if \quad x > 1 \end{cases}$

Assume that $f(x)$ is continuous at $x = 3$, that $f(3) = 5$, that $f(8) = 2$, and that

$$\lim_{x \to 3} g(x) = 8$$

Evaluate the following limits.

25. $\lim_{x \to 3} \left[x^2 + 3f(x)\right]$

26. $\lim_{x \to 3} \left[\sqrt[3]{g(x)}\right]$

27. $\lim_{x \to 3} [f(x) g(x)]$

28. $\lim_{x \to 3} \frac{f(x)}{g(x)}$

29. $\lim_{x \to 3} \left[g^2(x) - 2f(x)\right]$

30. $\lim_{x \to 3} f[g(x) - 5]$

31. Intermediate Value Theorem: Prove that $f(x) = x^3 + 2x - 4$ has a zero somewhere in $[0, 2]$. Is the zero in $[0, 1]$ or is it in $[1, 2]$?

32. Intermediate Value Theorem: Prove that $f(x) = x^5 - 2x^2 + 4$ has a zero somewhere in $[-2, 0]$. Is the zero in $[-2, -1]$ or is it in $[-1, 0]$?

33. Intermediate Value Theorem: A function $f(x)$ is said to be *even* if

$$f(-x) = f(x)$$

Prove that an even function must have an even number of real roots.

34. Intermediate Value Theorem: Let $f(x)$ be continuous on the real line and suppose that

$$\lim_{x \to -\infty} f(x) = K \qquad and \qquad \lim_{x \to \infty} f(x) = L$$

where $K < L$. Show that if $K < y < L$, there there is a number c such that $f(c) = y$?

35. Grapher: Graph the following function on the interval $[-0.2, 0.2]$:

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$$

Is the function continuous at $x = 0$? Does the function have a vertical asymptote at $x = 0$? Explain.

36. Grapher: Graph the following function on the interval $[-0.2, 0.2]$:

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$$

Is the function continuous at $x = 0$? Does the function have a vertical asymptote at $x = 0$? Explain.

37. Write to Learn: Suppose $f(x) \neq 0$ for all x in (p, q). Explain why $f(x)$ continuous on (p, q) implies that $f(x)$ is either always positive or always negative on (p, q).

38. Write to Learn: In a short essay explain why the definition of the limit implies that if $f(x)$ is continuous at p, then for all $\varepsilon > 0$, there is an interval (a, b) containing p such that

$$|f(x) - f(p)| < \varepsilon$$

for all $x \neq p$ in (a, b).

39. Write to Learn: If f is continuous at a number L, then

$$\lim_{u \to L} f(u) = f(L) \tag{1.27}$$

Suppose that $\lim_{x \to p} g(x) = L$, and in a short essay, explain why letting $u = g(x)$ in (1.27) leads to

$$\lim_{x \to p} f(g(x)) = f(L)$$

40. Write to Learn: Complete the following steps to prove that

$$if \ \lim_{x \to p} f(x) = L \quad and \quad \lim_{x \to p} g(x) = K, \quad then \ \lim_{x \to p} [f(x)g(x)] = KL$$

(a) Prove the following identity

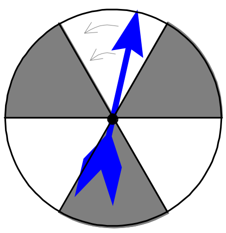
$$f(x)g(x) = \frac{1}{4}(f(x) + g(x))^2 - \frac{1}{4}(f(x) - g(x))^2$$

(b) Use (1.21) to compute the limits

$$\lim_{x \to p} (f(x) + g(x))^2 \qquad and \qquad \lim_{x \to p} (f(x) - g(x))^2$$

(c) Combine the results in (a) and (b) to finish the proof.

41. A spinner is spun counterclockwise as shown in figure 2-9.



5-10

The lines separating the colors are radii drawn every 60^o, and the angle θ is the angle with of the spinner arm with the positive x-axis. The function $f(\theta)$ is defined

$$f(\theta) = \begin{cases} 1 & if \quad \theta \; is \; in \; black \\ 0 & if \quad \theta \; is \; in \; white \end{cases}$$

(a) Sketch the graph of $f(\theta)$? What is its smallest period in degrees?
(b) Compute $\lim_{\theta \to 135^o} f(\theta)$ and $\lim_{\theta \to 700^o} f(\theta)$.
(c) What are the points of discontinuity of $f(\theta)$?
(d) Evaluate the limits

$$\lim_{\theta \to 60^o+} f(\theta) \qquad and \qquad \lim_{\theta \to 60^o-} f(\theta)$$

42. In this exercise, we explore the limit

$$\lim_{x \to 0} \frac{|x|}{x}$$

(a) Explain why $\frac{|x|}{x}$ is equal to 1 when $x > 0$ and $\frac{|x|}{x}$ is equal to -1 when $x < 0$.
(b) Graph $\frac{|x|}{x}$ and then zoom centered on 0. Explain why the range of every graph in the zoom contains the points -1 and 1.
(c) What is the limit from the left and the limit from the right as x approaches 0 of $\frac{|x|}{x}$?
(d) * Given $\varepsilon = 0.01$ and any value of L, explain why the inequality

$$L - 0.01 < \frac{|x|}{x} < L + 0.01$$

does not have a solution.

1.6 Differentiability

Differentiability

Now that we have developed the limit concept, let's use it to place the study of tangent lines on a firm foundation. To begin with, we say that a function $f(x)$ is *differentiable* at an input $x = p$ if its graph has a tangent line at $x = p$. Let's use the limit concept to explore what it means for a function to be differentiable at a given input.

Recall from section 1 that if $f(x)$ is a polynomial, then the equation of a tangent line is calculated with

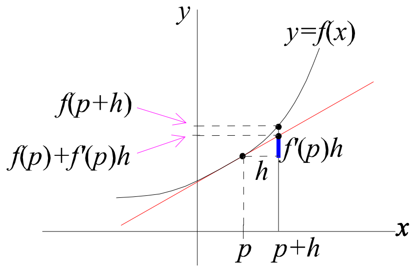
$$f(p+h) = a_0 + a_1 h + \text{ “}higher\ powers\ of\ h\text{”}$$

If $h = 0$, then $f(p+0) = a_0$, so that $a_0 = f(p)$. In contrast, the quantity a_1 is the *slope of the tangent line*, which is called the *derivative* of $f(x)$ at $x = p$ and is denoted $f'(p)$ (“eff prime of p”). Replacing a_0 and a_1 by their new names yields

$$f(p+h) = f(p) + f'(p)\,h \quad + \quad \text{“}higher\ powers\ of\ h\text{”} \qquad (1.28)$$

$f'(p)$ is the slope of the tangent line to $y = f(x)$ at $x = p$.

That is, if h is close to 0, then $f(p+h)$ is practically the same as $f(p) + f'(p)\,h$.

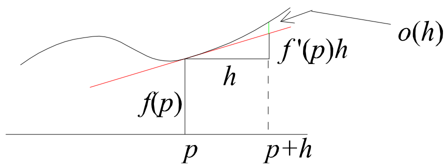


6-1: $f(p+h)$ is nearly the same as $f(p) + f'(p)\,h$

Let us now let $o(h)$, which is pronounced "little oh of h", denote the "higher powers of h, so that (1.28) becomes

$$f(p+h) = f(p) + f'(h)\,h + o(h)$$

That is, $o(h)$ represents the "higher powers" that can be ignored when h is close to 0.



6-2: $o(h)$ represents the negligible "higher powers" of h

Specifically, since $o(h)$ has powers of h higher than 1, the fraction $\frac{o(h)}{h}$ has powers of h greater than 0. Thus, $o(h)$ has the property that

$$\lim_{h\to 0} \frac{o(h)}{h} = 0 \qquad (1.29)$$

This leads us to the following definition:

Definition 6.1: A function $f(x)$ is *differentiable* at an input p if there exists a number $f'(p)$ (called the derivative of $f(x)$ at p) such that

$$f(p+h) = f(p) + f'(p)\;h \quad + \quad o(h) \qquad (1.30)$$

for some $o(h)$ satisfying (1.29).

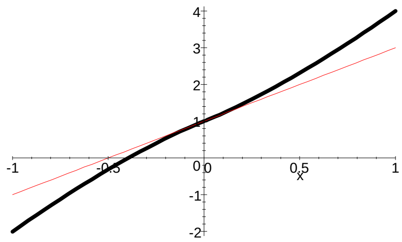
If $f(x)$ is differentiable at each point in an interval (a, b), then we say that $f(x)$ is *differentiable on* (a, b).

EXAMPLE 1 For $f(x) = 1 + 2x + x^{5/3}$, find $f'(0)$ and the equation of the tangent line to $y = f(x)$ at $p = 0$. Also, find $o(h)$ and show that it satisfies (1.29).

Solution: If we let $x = 0 + h$, then we obtain

$$f(0+h) = \overbrace{1 + 2h}^{f'(0)+f'(0)h} + \overbrace{h^{5/3}}^{o(h)}$$

Thus, $f'(0) = 2$ and $y = 1 + 2h$. Since $x = h$, the tangent line to $f(x)$ at $p = 0$ is $y = 1 + 2x$.



6-3: Tangent to $y = 1 + 2x + x^{5/3}$ at the point $(0, 1)$.

Also, $o(h) = h^{5/3}$ and

$$\lim_{h \to 0} \frac{o(h)}{h} = \lim_{h \to 0} \frac{h^{5/3}}{h} = \lim_{h \to 0} h^{2/3} = 0$$

Check your Reading *Is $f(x) = 1 + 2x + x^{5/3}$ a polynomial?*

Derivatives of Polynomials

In principle, we can use definition 6.1 to calculate the derivative of any polynomial. Let's look at a couple of examples.

EXAMPLE 2 For $f(x) = x^3 - 5x$, find $f'(1)$ and the equation of the tangent line to $y = f(x)$ at $p = 1$. Also, find $o(h)$ and show that it satisfies (1.29).

Solution: The problem is essentially the same as those we considered in section 1, but with different notation. We let $x = 1 + h$ and expand $f(1 + h)$:

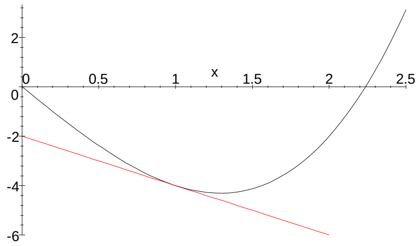
$$f(1 + h) = (1 + h)^3 - 5(1 + h) = -4 - 2h + 3h^2 + h^3$$

Comparing to (1.30) leads us to

$$f(1 + h) = \overbrace{-4 - 2h}^{f'(1) + f'(1)h} + \overbrace{3h^2 + h^3}^{o(h)}$$

Thus, $f'(1) = -2$ and $y = -4 - 2h$, which leads to the tangent line

$$y = -4 - 2(x - 1)$$



6-4: Tangent to $y = x^2 - 4x$ at the point $(1, -3)$.

Also, $o(h) = 3h^2 + h^3$, and

$$\lim_{h\to 0}\frac{o(h)}{h} = \lim_{h\to 0}\frac{3h^2+h^3}{h} = \lim_{h\to 0}\left(3h+h^2\right) = 0$$

Calculating derivatives of higher degree polynomials requires the expansion of $(p+h)^n$. In elementary algebra, it is shown that

$$(p+h)^n = c_0\,p^n + c_1\,p^{n-1}h + c_2\,p^{n-2}h^2 + \ldots + c_n\,h^n \tag{1.31}$$

where the coefficients $c_0, c_1, c_2, \ldots, c_n$ form the n^{th} row of *Pascal's triangle* shown below:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

In a later section we will use Pascal's triangle to develop an important rule for finding derivatives.

EXAMPLE 3 Find $f'(2)$ for $f(x) = x^4$. Also, identify $o(h)$.

Solution: We must first expand $f(2+h) = (2+h)^4$. The $n = 4$ row of Pascal's triangle gives us the coefficients c_0, c_1, c_2, c_3, and c_4 in the expansion of $(2+h)^4$.

$n = 0$	1
$n = 1$	1 1
$n = 2$	1 2 1
$n = 3$	1 3 3 1
$n = 4$	1 **4** **6** **4** 1
$n = 5$	1 5 10 10 5 1
$n = 6$	1 6 15 20 15 6 1
$n = 7$	1 7 21 35 35 21 7 1

Substituting the coefficients from Pascal's triangle into (1.31) yields

$$(2+h)^4 = \mathbf{1}\cdot 2^4 + \mathbf{4}\cdot 2^3\,h + \mathbf{6}\cdot 2^2h^2 + \mathbf{4}\cdot 2h^3 + \mathbf{1}\cdot h^4 \tag{1.32}$$

Simplifying leads us to

$$f(2+h) = 16 + \mathbf{32}h + 24h^2 + 8h^3 + h^4$$

from which we see that $f'(2) = 32$ and $o(h) = 24h^2 + 8h^3 + h^4$.

Check your Reading *Does $o(h) = 24h^2 + 8h^3 + h^4$ satisfy (1.29)?*

Notice that $o(h)$ itself approaches 0 as h approaches 0, because

$$\lim_{h\to 0} o(h) = \lim_{h\to 0}\left(\frac{o(h)}{h}\cdot h\right) = \left(\lim_{h\to 0}\frac{o(h)}{h}\right)\left(\lim_{h\to 0} h\right) = 0$$

Thus, if we apply the limit as h approaches 0 to (1.30), the result is

$$\lim_{h\to 0} f(p+h) = \lim_{h\to 0} f(p) + \lim_{h\to 0} f'(p)\,h + \lim_{h\to 0} o(h)$$

This in turn simplifies to

$$\lim_{h\to 0} f(p+h) = f(p) \tag{1.33}$$

If we let $x = p+h$, then x approaches p as h approaches 0. As a result, (1.33) becomes

$$\lim_{x\to p} f(x) = f(p)$$

which implies that $f(x)$ is continuous at $x = p$.

Theorem 6.2 If $f(x)$ is differentiable at a point p, then it is also continuous at p.

However, a function $f(x)$ can be continuous at p without also being differentiable at p.

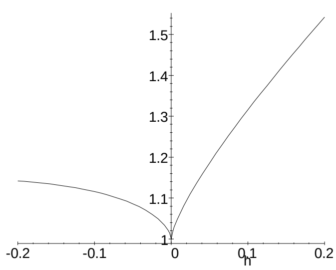
EXAMPLE 4 Explain why the following function is continuous but *not* differentiable at $h = 0$:

$$f(h) = 1 + h + h^{2/3},$$

Solution: Since $\lim_{h\to 0}\left(1 + h + h^{2/3}\right) = 1$, the function is continuous at $h = 0$. But near $h = 0$, the function is of the form

$$f(h) = \text{“linear function”} + \text{“a lower power of } h\text{”}$$

where by a “lower power” we mean an h with an exponent below 1. Lower powers have *larger magnitudes* than h when h is close to 0. Indeed, if $h = 0.001$, then $h^{2/3} = 0.01$, which is 10 times *larger* than $h = 0.01$. Since lower powers of h cannot be ignored, $f'(0)$ does not exist.



6-5: Graph of $f(h) = 1 + h + h^{2/3}$

In fact, the graph of $f(h) = 1 + h + h^{2/3}$ comes to a sharp point at $h = 0$, and thus it cannot be “practically the same” as a straight line at $h = 0$. Equivalently, if we let $o(h) = h^{2/3}$, then $o(h)$ does **not** satisfy (1.29).

We say that $y = 1 + h + h^{2/3}$ has a *cusp* at $h = 0$. Since $y = f(x)$ cannot have a tangent line at a cusp, $f(x)$ is **not differentiable** at an input where a cusp occurs.

EXAMPLE 5 Find the value of k for which

$$f(x) = \begin{cases} x^2 + k & if \quad x \le 1 \\ 6 - 2x & if \quad x > 1 \end{cases}$$

is continuous at $x = 1$. Then determine if it is differentiable at $x = 1$ as well.

Solution: By definition, $f(1) = 1^2 + k = k + 1$, thus satisfying the first criteria. Moreover, the limit from the left is

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \left(x^2 + k\right) = k + 1$$

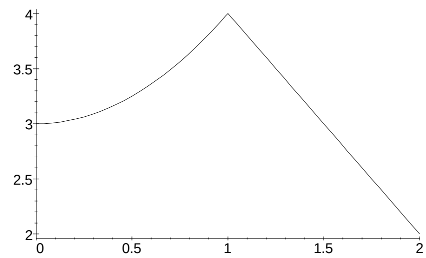
and the limit from the right is

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (6 - 2x) = 6 - 2 = 4$$

To achieve continuity, the limits from the left and right must be the sameand equal to $f(1)$. Thus, $k + 1 = 4$, so that $k = 3$, which leads to

$$f(x) = \begin{cases} x^2 + 3 & if \quad x \le 1 \\ 6 - 2x & if \quad x > 1 \end{cases}$$

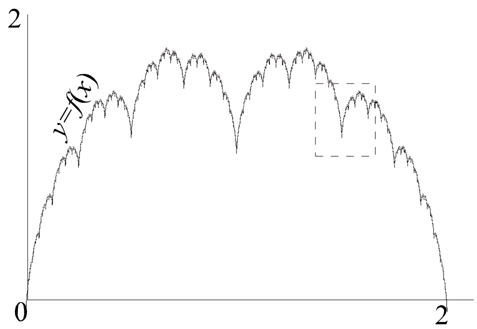
However, the graph of $f(x)$ over $[0, 2]$ reveals a cusp at $x = 1$.



6-6: $f(x)$ has a cusp at $x = 1$

That is, $f(x)$ is continuous at $x = 1$ but is **not** differentiable at $x = 1$.

In fact, there are functions that are continuous at every point but which are **not** differentiable at any point. For example, the function below is a *fractal interpolation function*, which appears to have a cusp at every point on the graph.



6-7: A function which is continuous everywhere but differentiable nowhere

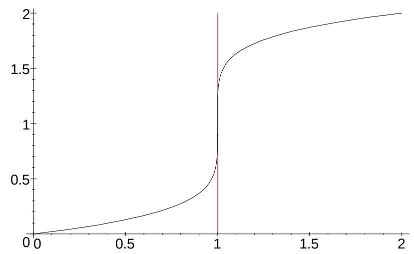
Check your Reading *Is $f(x) = |x|$ differentiable at 0? Explain.*

Additional Points of Nondifferentiability

We saw above that if $f(x)$ has a cusp at $x = p$, then $f'(p)$ does not exist. Other points where $f'(p)$ fails to exist are points where a tangent line is vertical and points of discontinuity.

EXAMPLE 6 Is $f(x) = (x-1)^{1/5} + 1$ differentiable at $x = 1$?

Solution: The graph of $f(x) = (x-1)^{1/5} + 1$ reveals that $f(x)$ has a vertical tangent line when $x = 1$, as is shown below:



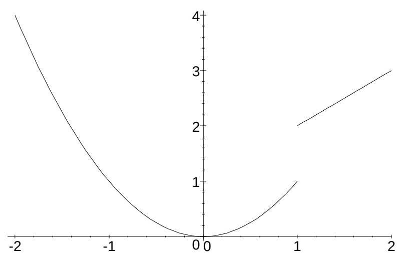
6-8: A vertical tangent line

Since the slope of a vertical line is undefined, $f'(1)$ does not exist.

EXAMPLE 7 Is the following function differentiable at $x = 1$:

$$f(x) = \begin{cases} x^2 & if \quad x < 1 \\ x+1 & if \quad x \geq 1 \end{cases}$$

Solution: Figure 7-9 reveals that $f(x)$ is not continuous at $x = 1$.



6-9

Every zoom centered at 1 will contain both pieces of $f(x)$, thus preventing $f(x)$ from ever being "practically the same" as a straight line. Since $y = f(x)$ does not have a tangent line at $x = 1$, the derivative $f'(1)$ does not exist.

Exercises:

Find $f'(p)$ at the given value of p, and then find the graph of the tangent line to $y = f(x)$ at $(p, f(p))$. Also, identify $o(h)$ and show that it satisfies (1.29).

1. $f(x) = x^2 + 1,\ p = 1$	2. $f(x) = x^3,\ p = 1$
3. $f(x) = x^2 + 2x,\ p = 2$	4. $f(x) = x^2 + 1,\ p = -5$
5. $f(x) = 3x + 2,\ p = -4$	6. $f(x) = \pi,\ p = 0.2736$
7. $f(x) = x^5,\ p = 2$	8. $f(x) = x^7 + 1,\ p = 1$

Grapher: *Graph each function on $[-4, 4]$, and identify the point(s) at which the function is not differentiable. See the users manual for details on graphing piecewise-defined functions.*[3]

9. $f(x) = (x^2 - 2x + 1)^{1/3}$	10. $f(x) = \left\|x^2 - 1\right\|$
11. $f(x) = (x^3 + 1.5)^{-2}$	12. $f(x) = (x^2 + x + \frac{1}{4})^{1/4}$
13. $f(x) = \left\|\frac{x^2 - 1}{x^2 + 1}\right\|$	14. $f(x) = \frac{x^2 + \|x\|}{\|x\|}$
15. $f(x) = \frac{\sin(\|x\|)}{x}$	16. $f(x) = \cos(\|x\|)$
17. $f(x) = \frac{1 - \cos(x)}{\|x\|}$	18. $f(x) = \frac{\sin(\|x\|)}{x - \pi}$

19. $f(x) = \sin(|x|)$

20. $f(x) = \tan(|x|)$

21. $f(x) = x^{1/5}$

22. $f(x) = (x^3 - x)^{1/5}$

23. $f(x) = \begin{cases} x^2 & if \quad x < 1 \\ 2x - 1 & if \quad x \geq 1 \end{cases}$

24. $f(x) = \begin{cases} x^2 + 2x & if \quad x < 1 \\ x^3 + 2 & if \quad x \geq 1 \end{cases}$

25. $f(x) = \begin{cases} 2x - 1 & if \quad x < 1 \\ 3 - 2x & if \quad x \geq 1 \end{cases}$

26. $f(x) = \begin{cases} 3x - 2 & if \quad x < 1 \\ x^3 - 1 & if \quad x \geq 1 \end{cases}$

27. $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & if \quad x \neq 0 \\ 1 & if \quad x = 0 \end{cases}$

28. $f(x) = \begin{cases} \sin(x) + 1 & if \quad x < 0 \\ \cos(x) & if \quad x \geq 0 \end{cases}$

29. Find the number k for which the function

$$f(x) = \begin{cases} 3x - 2 & if \quad x < 1 \\ kx + 2 & if \quad x \geq 1 \end{cases}$$

is continuous at $x = 1$. Then determine if the function is differentiable at that point.

[3] Many graphing programs will not permit the odd root of a negative number unless the variable is first declared to be a real variable. See your manual for details.

30. Find the number k for which the function

$$f(x) = \begin{cases} x^2 + 8x & if \quad x \leq 2 \\ x^3 + k & if \quad x > 2 \end{cases}$$

is continuous at $x = 2$. Then determine if the function is differentiable at that point.

31. Find the number k for which the function

$$f(x) = \begin{cases} (x+3)^3 & if \quad x < -2 \\ x^3 + k & if \quad x \geq -2 \end{cases}$$

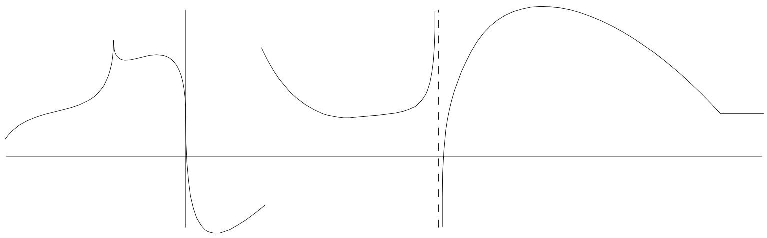
is continuous at $x = -2$. Then determine if the function is differentiable at that point.

32. Find the number k for which the function

$$f(x) = \begin{cases} x^3 - 3x + k & if \quad x < -1 \\ (x+1)^2 & if \quad x \geq -1 \end{cases}$$

is continuous at $x = -1$. Then determine if the function is differentiable at that point.

33. The curve in figure 6-10 is the graph of a function $f(x)$. Determine the points where $f(x)$ is **not** differentiable and explain why $f(x)$ is not differentiable at those points.



6-10: Exercise 33

34. Explain using definition 6.1 why the function $f(x) = x^2 + (x-1)^{2/3}$ is **not** differentiable at $x = 1$.

35. Here is an alternative method for demonstrating that if $f(x)$ is differentiable at $x = p$, then $f(x)$ is also continuous at $x = p$.

(a) Explain why $f(x)$ is practically the same as a linear function $L(x) = mx + b$ for x near p.

(b) Prove that $L(x) = mx + b$ is continuous at p.

(c) Explain why (a) and (b) imply $f(x)$ is continuous at p.

36. Suppose a function $f(x)$ is defined piecewise by

$$f(x) = \begin{cases} g(x) & if \quad x \leq 0 \\ mx + b & if \quad x > 0 \end{cases}$$

and suppose that both $f(x)$ and $g(x)$ are differentiable at $x = 0$. How is m related to the function g?

37. Confirm the following identity assuming $x \neq 0$ and h sufficiently close to 0:

$$\frac{1}{x+h} = \frac{1}{x} - \frac{h}{x^2} + \frac{h^2}{x^2(x+h)}$$

Identify $o(h)$ and show that it satisfies (1.29). What does this imply the derivative of $\frac{1}{x}$ is?

38. Confirm the following identity assuming $x \neq 0$ and h sufficiently small

$$\sqrt{x+h} = \sqrt{x} + \frac{h}{2\sqrt{x}} - \frac{h\left(\sqrt{x+h}-\sqrt{x}\right)}{2\sqrt{x}\left(\sqrt{x+h}+\sqrt{x}\right)}$$

Identify $o(h)$ and show that it satisfies (1.29). What does this imply the derivative of $\sqrt{x}$ is?.

39. Use Pascal's triangle to find $f'(\pi)$ by expanding $f(\pi+h)$ and identifying the coefficient of h in that expansion.

(a) $f(x) = x^6$
(b) $f(x) = x^7$
(c) $f(x) = x^8$
(d) $f(x) = x^9$

40. Based on the pattern observed in exercise 39, what do you expect $f'(\pi)$ to be when $f(x) = x^{10}$? What do you expect $f'(\pi)$ to be when $f(x) = x^{100}$?

41. Computer Algebra System. If you have access to a computer algebra system, use it to find $f'(1)$, $f'(2)$, and $f'(3)$.

(a) $f(x) = (1-x)(1-2x)(1-3x)$
(b) $f(x) = x(x+1)(x+2)(x+3)$
(c) $f(x) = (x-1)^2(x-2)^2(x-3)^2$
(d) $f(x) = 1 + x(1 + x(1+x))$

1.7 Rates of Change

The Limit Definition of the Derivative

One of the most important applications of the derivative is as a measure of the *rate of change* of a function at a given input. In this section, we develop the concept of the derivative as a rate of change by first defining the derivative as a limit of average rates of change.

If $f(x)$ is a differentiable at an input p, then definition 6.1 implies that

$$f(p+h) = f(p) + f'(p)h + o(h)$$

Thus, $f(p+h) - f(p) = f'(p)h + o(h)$, and also

$$\frac{f(p+h) - f(p)}{h} = f'(p) + \frac{o(h)}{h}$$

Also by definition, the limit as h approaches 0 of $\frac{o(h)}{h}$ is 0, and consequently,

$$\begin{aligned} \lim_{h\to 0} f'(p) + \lim_{h\to 0} \frac{o(h)}{h} &= \lim_{h\to 0} \frac{f(p+h) - f(p)}{h} \\ f'(p) + 0 &= \lim_{h\to 0} \frac{f(p+h) - f(p)}{h} \end{aligned}$$

The result is the *limit definition* of the derivative.

Definition 7.1: The derivative $f'(p)$ of a function $f(x)$ at a point p is defined to be

$$f'(p) = \lim_{h\to 0} \frac{f(p+h) - f(p)}{h} \tag{1.34}$$

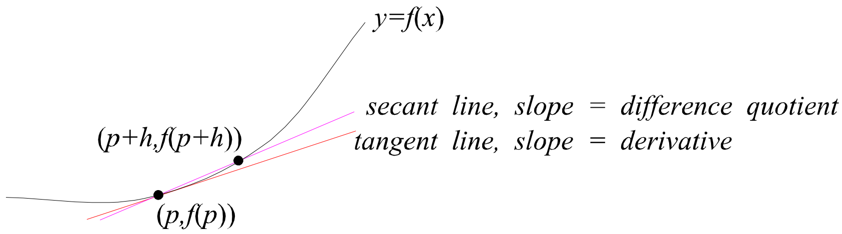
when the limit exists.

Let's develop a geometric interpretation of definition 7.1. Given the points $(p, f(p))$ and $(p+h, f(p+h))$, the change in x is $\Delta x = p + h - p = h$ and the change in y is $\Delta y = f(p+h) - f(p)$. Thus, the slope of the secant line through $(p, f(p))$ and $(p+h, f(p+h))$ is

$$\frac{\Delta y}{\Delta x} = \frac{\text{change in outputs}}{\text{change in inputs}} = \frac{f(p+h) - f(p)}{h} \tag{1.35}$$

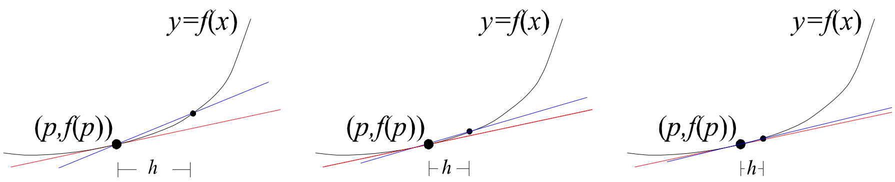
which is known as the *difference quotient* of f over $[p, p+h]$.

Graphically, definition 7.1 implies that the *secant line* through the points $(p, f(p))$ and $(p+h, f(p+h))$ is nearly the same as the tangent line at $(p, f(p))$.



7-1: Secant line approximation of a tangent line.

Indeed, secant line approximations approach the slope of the tangent line as h approaches 0.



7-2: Secant approximation approaches tangent line as h approaches 0

We often rewrite definition 7.1 in the form

$$f'(p) = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} \tag{1.36}$$

to reflect the fact that the derivative is the limit of the slopes of secant line approximations.

EXAMPLE 1 Find $f'(2)$ and the equation of the tangent line to the graph of $f(x) = \frac{1}{x}$ when $p = 2$.

Solution: Since $f(2+h) = \frac{1}{2+h}$, it follows that the change in y is

$$\Delta y = f(2+h) - f(2) = \frac{1}{2+h} - \frac{1}{2}$$

By finding a common denominator, Δy can be simplified to

$$\Delta y = \frac{2}{2(2+h)} - \frac{2+h}{2(2+h)} = \frac{-h}{2(2+h)}$$

Since $\Delta x = h$, the difference quotient is

$$\frac{\Delta y}{\Delta x} = \frac{1}{\Delta x}(\Delta y) = \frac{1}{h}\left(\frac{-h}{2(2+h)}\right) = \frac{-1}{2(2+h)}$$

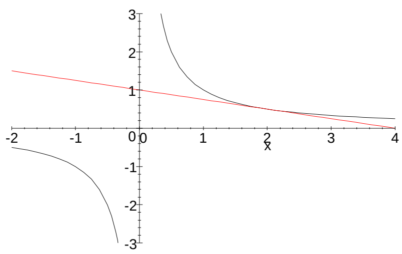
Thus, the derivative at 2 is

$$f'(2) = \lim_{h\to 0} \frac{\Delta y}{\Delta x} = \lim_{h\to 0} \frac{-1}{2(2+h)} = \frac{-1}{4}$$

Since $f(2) = \frac{1}{2}$, the equation of the tangent line is

$$y = y_p + m(x-p) = \frac{1}{2} - \frac{1}{4}(x-2)$$

which simplifies to $y = 1 - \frac{1}{4}x$.



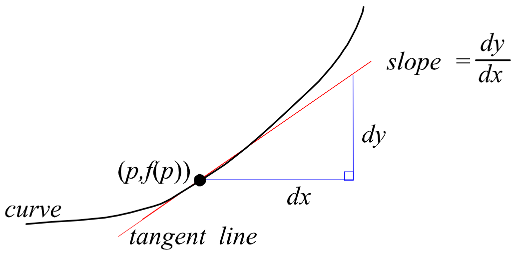
7-3: $y = 1 - \frac{1}{4}x$ is tangent to $y = \frac{1}{x}$ at $\left(2, \frac{1}{2}\right)$

Check your Reading *Is $f(x) = \frac{1}{x}$ a polynomial?*

The Derivative as a Rate of Change

The difference quotient $\frac{\Delta y}{\Delta x}$ is also known as the *average rate of change* of f over $[p, p+h]$. Thus, definition 7.1 says that $f'(p)$ is the limit of average rates of change over arbitrarily short intervals containing p. For this reason, $f'(p)$ is interpreted to be the *instantaneous rate of change* of $f(x)$ at $x = p$. Consequently, the derivative $f'(p)$ tells us *how fast* a process is changing at a given input p.

Often we define the *differential* dy to be the small "rise" along a tangent line due to a small "run" given by the differential dx, so that $\frac{dy}{dx}$ is the slope of the tangent line.



7-4: The derivative is also denoted by $\frac{dy}{dx}$

That is, the derivative can be written in *differential notation* as

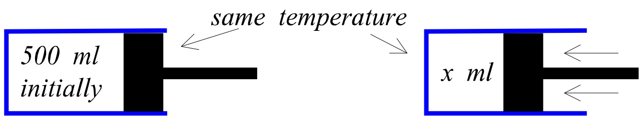
$$\frac{dy}{dx} = f'(p)$$

which further implies that $\frac{dy}{dx}$ should be interpreted to be the *instantaneous* rate of change of y as a function of x at a given input p.

EXAMPLE 2 A cylinder initially contains 500 ml of air at a pressure of 1 atmosphere (atm). If the air in the cylinder is compressed by a piston without changing its temperature, then Boyle's law says that the pressure P is related to the volume x by

$$P = \frac{500}{x}$$

What is $\frac{dP}{dx}$ when $x = 250$ ml? What does it mean?



7-5: Boyles law relates changes in volume to changes in pressure at constant temperature.

Solution: The change in pressure at a volume of 250 ml due to a small change in volume $\Delta x = h$ is given by

$$\Delta P = P(250 + h) - P(250) = \frac{500}{250 + h} - \frac{500}{250}$$

and as a result, the difference quotient is

$$\frac{\Delta P}{\Delta x} = \frac{P(250 + h) - P(250)}{h} = \frac{1}{h}\left(\frac{500}{250 + h} - 2\right)$$

(since division by h is the same as multiplication by $\frac{1}{h}$). Thus, the derivative is

$$P'(250) = \lim_{h \to 0} \frac{1}{h}\left(\frac{500}{250 + h} - 2\right)$$

The limit is of the form $\frac{0}{0}$, so we simplify by combining fractions

$$\begin{aligned} P'(250) &= \lim_{h \to 0} \frac{1}{h}\left(\frac{500}{250 + h} - \frac{2(250 + h)}{250 + h}\right) \\ &= \lim_{h \to 0} \frac{1}{h}\left(\frac{500 - 500 - 2h}{250 + h}\right) \\ &= \lim_{h \to 0} \frac{1}{h}\left(\frac{-2h}{250 + h}\right) \end{aligned}$$

Cancellation leads to a limit that can be evaluated using continuity:

$$P'(250) = \lim_{h \to 0} \frac{-2}{250+h} = \frac{-2}{250+0}$$

Thus, the pressure is changing with respect to volume at a rate of

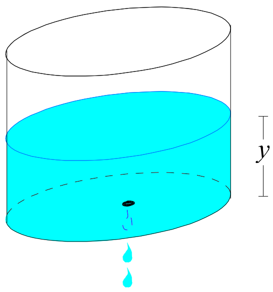
$$\frac{dP}{dx} = \frac{-1}{125} \frac{atm}{ml}$$

when the volume is $x = 250$ ml. We interpret this to mean that for volumes close to 250 ml, the pressure increases at a rate of 1 atm per each 125 ml decrease in volume.

EXAMPLE 3 Water is draining from a small hole in the bottom of a cylindrical can. After t minutes, the height of the water inside the can is

$$y = (-0.3065t + 1.8323)^2$$

where y is in inches. What is $\frac{dy}{dt}$ at $t = 1$ minutes after water begins to drain? How is it interpreted?



7-6: Water leaking from a Tank

Solution: The change in height after 1 minute due to a small change $\Delta t = h$ is given by

$$\begin{aligned} \Delta y &= (-0.3065(1+h) + 1.8323)^2 - (-0.3065 \cdot 1 + 1.8323)^2 \\ &= -0.9353h + 0.93942h^2 \end{aligned}$$

As a result, the difference quotient is

$$\frac{\Delta y}{\Delta t} = \frac{-0.9353h + 0.93942h^2}{h} = -0.9353 + 0.93942h$$

Applying the limit as h approaches 0 results in the desired instantaneous rate of change:

$$\frac{dy}{dt} = \lim_{h \to 0} \frac{\Delta y}{\Delta t} = -0.9353 \frac{in}{\text{min}}$$

That is, the tank is draining at a rate of about 0.9353 inches per minute after 1 minute has elapsed.

Check your Reading *What other method might we have used to find the derivative in example 3?*

Average and Instantaneous Velocity

If $r(t)$ denotes the position of an object at time t, then the derivative $r'(p)$ is often written as $v(p)$ and is called the *instantaneous velocity* of the object at time $t = p$. Moreover, if $r(t)$ is a polynomial, then we can calculate the derivative $r'(p)$ using the definition of differentiability.

EXAMPLE 4 Suppose a $r(t) = 60t - 6.1t^2$ is the height of an object above the surface of Mars, in feet, at time t in seconds. How fast is the projectile traveling at $t = 1$ seconds?

Solution: To find the velocity $r'(1)$, we expand $r(1+h)$:

$$\begin{aligned} r(1+h) &= 60(1+h) - 6.1(1+h)^2 \\ &= 53.9 + 47.8h - 6.1h^2 \end{aligned}$$

Thus, $r'(1) = 47.8$, which means that the object has a velocity after 1 second of

$$v(1) = 47.8\ \frac{ft}{\text{sec}}$$

Rates of change are particularly important in the study of *projectile motion.* In particular, in the late 1500's, Galileo Galilei showed that if an object has a height r_0 and a velocity v_0 at time $t = 0$, then the height of the object at time t is

$$r(t) = r_0 + v_0 t \quad - \quad \frac{1}{2} g t^2 \tag{1.37}$$

where g is the *acceleration due to gravity,* which is about 32 feet per second per second near the earth's surface.

EXAMPLE 5 Suppose a rock is dropped from a building that is 64 feet tall. Neglecting air resistance and supposing that time $t = 0$ corresponds to the moment the rock is released, find the following:

a) The model (1.37) of the rock's height $r(t)$ as a function of t.

b) The time at which the rock strikes the earth.

c) The velocity of the rock when it strikes the earth.

Solution: a) The initial height of the rock is 64 feet, and the initial velocity is $v_0 = 0$ since the rock is not moving prior to being released. Thus, (1.37) implies that at time t, the rock will have a height of

$$r(t) = 64 - 16t^2$$

b) The rock will strike the earth when the rock's height is 0. Setting $r(t) = 0$ yields

$$\begin{aligned} 64 - 16t^2 &= 0 \\ 16(2-t)(2+t) &= 0 \end{aligned}$$

The solutions are $t = 2$ and $t = -2$. The rock is moving only for positive time, so that we conclude that the rock falls for $t = 2$ seconds before it reaches the ground.

c) The velocity when the rock strikes the earth is $v(2)$. To find $v(2)$, we notice that $r(2) = 0$ and that

$$r(2+h) = 64 - 16(2+h)^2 = -64h - 16h^2 \tag{1.38}$$

Since the coefficient of h is -64, we conclude that the velocity with which the rock strikes the earth is

$$v(2) = -64 \, \frac{ft}{\sec}$$

Check your Reading *What is* $\frac{dr}{dt}$ *at* $t = 2$ *when* $r(t) = 64 - 16t^2$*?*

Numerical Representations

It is not always possible to calculate the limit implied in definition 7.1. In such cases, we often use a numerical representation to estimate the desired rate of change.

EXAMPLE 6 Estimate the derivative of $f(x) = \sin(x)$ at $x = \pi$.

Solution: Since $\Delta y = \sin(\pi + h) - \sin(\pi)$ and since $\sin(\pi) = 0$, the difference quotient is

$$\frac{\Delta y}{\Delta x} = \frac{\sin(\pi + h)}{h}$$

Thus, $f'(\pi)$ is given by

$$f'(\pi) = \lim_{h\to 0} \frac{\sin(\pi + h)}{h}$$

which we cannot (yet) evaluate in closed form. Instead, let us use a numerical representation:

h	-0.1	-0.01	$\rightarrow$	0	$\leftarrow$	0.01	0.1
$\frac{\sin(\pi+h)}{h}$	-0.998	-0.99998	$\rightarrow$	???	$\leftarrow$	-0.99998	-0.998

This leads us to estimate that

$$f'(\pi) = \lim_{h\to 0} \frac{\sin(\pi + h)}{h} \approx -1$$

EXAMPLE 7 Suppose that the revenue R in thousands of dollars as a function of the number x of a certain automobile available for sale is

$$R(x) = 2x + 2x^{0.123}$$

How fast is the revenue changing when there are 100 automobiles?

Solution: The difference quotient at $x = 100$ is

$$\frac{R(100+h) - R(100)}{h} = \frac{\left(2(100+h) + 2(100+h)^{0.123}\right) - \left(200 + 2(100)^{0.123}\right)}{h}$$

The derivative—i.e., the rate of change of revenue—is thus given by the limit

$$\frac{dR}{dx} = \lim_{h\to 0} \frac{\left(2(100+h) + 2(100+h)^{0.123}\right) - \left(200 + 2(100)^{0.123}\right)}{h}$$

The limit cannot be evaluated in closed form, so we instead use a numerical representation:

h	-0.1	-0.01	-0.001	$\rightarrow$	0	$\leftarrow$	0.001	0.01	0.1
$\frac{R(100+h)-R(100)}{h}$	2.0043	2.0043	2.0043	$\rightarrow$	???	$\leftarrow$	2.0043	2.0043	2.0043

As a result, the rate of change in the revenue after selling 100 automobiles is

$$\frac{dR}{dx} = \lim_{h\to 0} \frac{R(100+h) - R(100)}{h} \approx \$20,043 \textit{ per car sold}$$

Exercises:

Use definition 7.1 to calculate $f'(p)$, which is the instantaneous rate of change of $f(x)$ at the given input p.

1. $f(x) = 3x^2, \quad p = 1$
2. $f(x) = -x^2, \quad p = 1$
3. $f(x) = x + 3x^2, \quad p = 0$
4. $f(x) = x^2 + 3x, \quad p = 1$
5. $f(x) = 3x + 2, \quad p = 1$
6. $f(x) = 3x + 2, \quad p = 2$
7. $f(x) = x^3, \quad p = 0$
8. $f(x) = x^4, \ p = 0$
9. $f(x) = \dfrac{1}{x+1}, \ p = 2$
10. $f(x) = \dfrac{1}{\sqrt{x+1}}, \quad p = 3$
11. $f(x) = x + x^{-1}, \quad p = 1$
12. $f(x) = x^3 - 3\sqrt{x}, \quad p = 1$
13. $f(x) = x^{1/2}, \quad p = 4$
14. $f(x) = x^{1/3}, \quad p = 8$

Each of the following curves is the model of a distance traveled r in feet at a given time t in seconds. Find the velocity of the object at the given time. Be sure to include units.

15. $r = -16t^2, \ p = 1$ sec
16. $r = -6.1t^2, \ p = 1$ sec
17. $r = 64 - 16t^2, \ p = 1$ sec
18. $r = 64 - 6.1t^2, \ p = 1$ sec
19. $r = 196t - 6.1t^2, \ p = 0$ sec
20. $r = 196t - 16t^2, \ p = 2$ sec
21. $r = 6 + 20t - 6.1t^2, \ p = 0.5$ sec
22. $r = 20 + 68t - 16t^2, \ p = 0.3$ sec
23. $r_0 = 64, \ v_0 = 0, \ p = 2$ sec on the earth
24. $r_0 = 64, \ v_0 = 0, \ p = 1$ sec on Mars ($g = 12.2 \ \frac{ft}{\sec^2}$)

Exercises 25-38 ask you to calculate and interpret rates of change. In doing so, be sure to include the units for the rate of change (such as dollars per shirt, inches per minute, etc.).

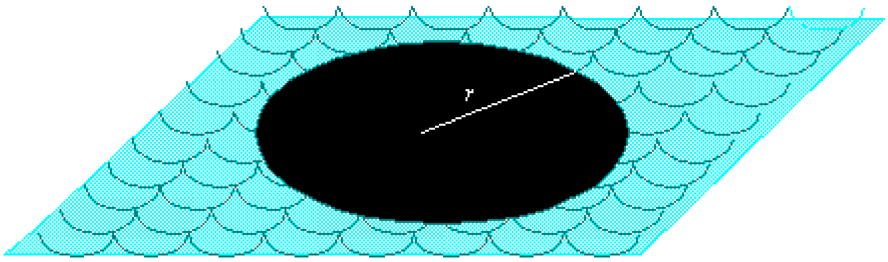
25. Estimate $f'(2\pi)$ when $f(\theta) = \sin(\theta)$ using a numerical representation, where $f(\theta)$ is in meters and θ is in radians.

26. Estimate $f'(0)$ when $f(\theta) = \tan(\theta)$ using a numerical representation, where $f(\theta)$ is in meters and θ is in radians.

27. An oil slick's area is increasing at a constant rate of π square miles per hour has a radius r at time t of

$$r(t) = \sqrt{t}\ miles$$

How fast is the radius increasing after 1 hour?



7-7: An oil slick with radius $r(t) = \sqrt{t}$

28. In the oil slick in exercise 27, suppose instead that the radius r as a function of t is given by

$$r = 5t\ miles$$

How fast is the area of the slick increasing after 1 hour?

29. Use (1.37) to explain why the vertical position of an object thrown straight up with an initial velocity of 12 feet per second from a height of 4 feet above the earth is given by the position function $r = -16t^2 + 12t + 4$. How fast is the ball moving after 1 second?

30. Suppose a ball is rolled off of a level table which is four feet high.

(a) Use (1.37) to explain why the height $r(t)$ in feet of the ball at time t seconds after it leaves the edge of the table is

$$r(t) = 4 - 16t^2$$

(b) What is the average velocity of the ball as it falls from $t = 0.4$ to $t = 0.5$ seconds?

(c) What is the average velocity of the ball as it falls from $t = 0.49$ to $t = 0.5$ seconds?

(d) What is the velocity of the ball when it strikes the ground at $t = 0.5$ seconds?

31. A certain object is dropped from a height of 64 feet *on Mars*. It's height t seconds after being dropped is

$$r(t) = 64 - 6.1t^2$$

Use $r(t)$ to answer the following questions.

(a) What is the average velocity of the object as it falls from time $t = 1$ to time $t = 1.1$? (i.e., $h = 0.1$)

(b) What is the average velocity of the object as it falls from time $t = 1$ to time $t = 1.01$? (i.e., $h = 0.01$).

(c) What is the instantaneous velocity of the object at time $t = 1$?

(d) What is the velocity of the ball when it strikes the ground at $t = 1$ seconds?

32. A rock is dropped from a height of 80 feet *on Mars*. It's height t seconds later is

$$r(t) = 80 - 6.1t^2$$

The same rock dropped on earth from 80 feet would have a height t seconds later of

$$s(t) = 80 - 16t^2$$

(a) How long until the first rock hits the surface of Mars (i.e., until $r(t) = 0$)?

(b) How long until the second rock hits the surface of Earth (i.e., until $s(t) = 0$)?

(c) What is the velocity of the rock when it strikes the surface of Mars?

(d) What is the velocity of the rock when it strikes the surface of Earth?

33. Let $S(t) = 2t^2 + 20t + 1$ be the number of individuals in a large metropolitan area showing cold symptoms at time t in days since the first individual in the area showed cold symptoms. What is the rate of change of the number of people showing cold symptoms initially (i.e., at $t = 0$)?

34. In exercise 33, what is the rate of change of the number of people showing cold symptoms after 20 days?

35. A 3.25 inch tall soup can is filled with water and a small hole is punched in the bottom of the can allowing the water to drain out. The height y in inches of the water in the can at time t in minutes since water began draining is modeled by

$$y = 0.09t^2 - 1.08t + 3.25$$

How fast is the height of the water dropping initially? After 1 minute?

36. The temperature in degrees Celsius of an object is for a short period of time approximated by $T(t) = 2.85t^2 - 0.12t + 31$ where t is the time in minutes since the initial temperature reading. How fast is the temperature changing after 15 minutes?

37. Let $p = -0.007x + 32$ be the price per shirt in dollars when x numbers of shirts are sold and let R denote the revenue from the sale of shirts.

(a) Explain why $R = xp$ and determine the revenue curve by substituting for p.

(b) What is $\frac{dR}{dx}$ when $x = 1000$? What does it mean?

(c) What is the average rate of change of R as x changes from 1000 to 1001. What does it mean? How is it related to (b)?

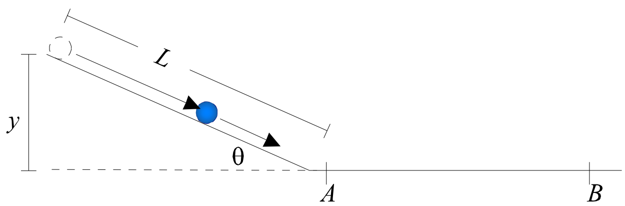
38. Let $P = t^3 - 2t^2 - 30t + 88$ be the population of a suburb of a certain city at time t, where t is the number of years since December 31, 1985.

(a) How fast is the suburb's population changing initially?

(b) How fast is the suburb's population changing after 15 years?

39. **Write to Learn[4]:** On most automobiles, a speedometer reports the ratio of the circumference of the driveshaft to the time required for the driveshaft to make one revolution. Write a short essay which uses the concept of rate of change to explain why the speedometer reading at a given time is a good approximation of the instantaneous speed of the automobile.

40. Galileo used inclined boards to study velocity. In particular, he showed that neglecting friction, a ball released from a point on a board inclined at an angle θ would after t seconds have traveled a distance in feet of $r(t) = 16t^2 \sin\theta$



7-8: Ball is rolling down a plane of length L

(a) What is $r(t)$ when the board is inclined at an angle of $\theta = 30°$?

(b) How long does it take for the ball to roll four feet?

(c) Construct a table of average velocities at the time in (b) with h becoming closer and closer to 0. What is the velocity of the ball after it has rolled four feet?

(d) *Try it out!* Use a stopwatch to time a ball which rolls down a four foot long board inclined at a 30° angle. Does it match the result in (b)?

Self Test

A variety of questions are asked in a variety of ways in the problems below. Answer as many of the questions below as possible before looking at the answers in the back of the book.

1. Answer each statement as true or false.

   (a) A polynomial must have only one output value for each input value

   (b) A function must have only one input value for each output value.

   (c) For $y = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$, the tangent line at an input of 0 is $y = a_1 x + a_0$.

   (d) If $f(h) = a_0 + a_1 h + a_2 h^2 + ... + a_n h^n$ and h is close to 0, then the higher powers of h contribute more to $f(h)$ than do the lower powers of h.

   (e) A tangent line to a curve cannot intersect a curve more than once.

   (f) For a curve given by $y = f(x)$, the slope of the tangent line to the curve at a point $(p, f(p))$ may in some cases differ from $f'(p)$.

   (g) The expression $\frac{\sqrt{1+h}-1}{h}$ is the slope of a secant line to the curve $y = \sqrt{x}$ through $(1, 1)$.

[4] "Write to Learn" exercises are applications in which the solution should be presented with complete sentences and supporting explanations.

(h) The slope of the secant line to a curve gives the average rate of change of the function between two points on the curve, whereas the slope of the tangent line gives the instantaneous rate of change of the function at a point.

(i) $\lim_{h\to 0} g(h)$ is always the same as $g(0)$.

(j) For $y = f(x)$, if x has units in *meters* and y has units in kilograms then $\frac{dy}{dx}$ has units in kilograms/*meter*.

2. Answer each statement as true or false.

(a) From the table

x	0.1	0.01	0.001	0.0001	0.00001
$f(x)$	2.1	2.01	2.001	2.0001	2.00001

one can conclude that $\lim_{x\to 0} f(x) = 2$.

(b) If a function is continuous at a point p, then the function must have a limit as x approaches p.

(c) If a function is continuous at a point p, then the function must have a derivative at p.

(d) If $\lim_{x\to 0} f(x) = 3$, then the curve $y = f(x)$ has a horizontal asymptote of $y = 3$.

(e) If $\lim_{x\to p} f(x) = L$, then $f(x)$ must be continuous at p.

(f) Every polynomial function is continuous everywhere.

(g) Every rational function is continuous everywhere.

(h) If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$ then it follows that $\lim_{x\to c} \frac{f(x)}{g(x)} = 1$.

(i) dy is the change in the y-coordinate along the tangent line to $y = f(x)$ as x changes from p to $p + dx$.

(j) If $f(h) = (2h+3)^2$ then $o(h) = 2h$

3. Which of the following is the equation of the tangent line to $y = 2x^2 - 5x + 3$ when $p = 0$?

(a) $y = 2x^2 - 5x$ (b) $y = 2x^2 + 3$ (c) $y = -5x + 3$ (d) $y = 2x + 3$

4. Which of the following is the *slope* of the tangent line to $y = 1 + x + 2x^2$ at $p = 3$?

(a) $\frac{dy}{dx} = 1$ (b) $\frac{dy}{dx} = 2$ (c) $\frac{dy}{dx} = 3$ (d) $\frac{dy}{dx} = 13$ (e) $\frac{dy}{dx} = 22$ (f) $\frac{dy}{dx} = \frac{\Delta y}{\Delta x}$

5. Which of the following is **not** true of the limit

$$\lim_{h\to 0} \frac{\sqrt{3+h} - \sqrt{3}}{h}$$

(a) It is $f'(1)$ when $f(x) = \sqrt{x+2}$

(b) It is $f'(2)$ when $f(x) = \sqrt{x+1}$

(c) It is $f'(3)$ when $f(x) = \sqrt{x}$

(d) The limit does not exist.

6. If $f(x) = 2 + 5x + 2x^2$ and $p = 2$, then which of the following is true:

   (a) $f(2+h) = 20 + 13h + 2h^2$ and $f'(2) = 20$

   (b) $f(2+h) = 2 + 5h + 2h^2$ and $f'(2) = 5$

   (c) Zooming on the graph of $f(x)$ centered at $p = 2$ produces what appears to be a straight line with a slope of 13.

   (d) Zooming on the graph of $f(x)$ centered at $p = 2$ produces what appears to be a straight line with a slope of 5

7. If $\lim_{x\to c} f(x) = \sqrt{8}/3$ and $\lim_{x\to c} g(x) = \sqrt{2}$, then $\lim_{x\to c}[f(x)g(x)] =$

   (a) $\sqrt{8}/3 + \sqrt{2}$ (b) $\sqrt{8}\sqrt{2}$ (c) $\sqrt{4}/3$ (d) $4/3$

8. $\lim_{x\to\infty} \dfrac{x^2 - k^2}{x - k} =$

   (a) $1/k$ (b) k (c) *does* not exist (d) $2k$ (e) $\frac{0}{0}$

9. How many horizontal asymptotes does the following function have?

$$f(x) = \sqrt{\frac{|x| + 1}{|x| + 2}}$$

   (a) 0 (b) 1 (c) 2 (d) 3

10. Which of the following intervals contains the value of $\lim_{x\to -\infty} \dfrac{\sqrt{x^2+8} - 3}{x + 1}$

   (a) $(-\infty, -2)$ (b) $(-2, 0)$ (c) $(0, 3)$ (d) $(3, \infty)$ (e) does not exist

11. Given that $\lim_{x\to 1} \frac{2x^2-3x+1}{x-1} = 1$, on which of the following neighborhoods of $p = 1$ is the function $f(x) = \frac{2x^2-3x+1}{x-1}$ within $\varepsilon = 0.1$ of 1 but not within $\varepsilon = 0.01$ of 1 for all $x \neq p$ in that neighborhood?

   (a) $(0.94, 1.06)$ (b) $(0.96, 1.04)$ (c) $(0.996, 1.004)$ (d) $(0.999, 1.00$

12. Each of the following functions is not continuous at a point for a different reason. Give the reason for the discontinuity of each one.

   (a) $f(x) = \begin{cases} 2x - 3 & if \quad x < 2 \\ x^2 - 2 & if \quad x \geq 2 \end{cases}$ (b) $f(x) = \begin{cases} 2x - 3 & if \quad x < 2 \\ 2 & if \quad x = 2 \\ \frac{1}{2}x & if \quad x > 2 \end{cases}$

   (c) $f(x) = \dfrac{x^2 - 9}{x + 3}$ (d) $f(x) = \begin{cases} |x| & if \quad x \neq 2 \\ 1 & if \quad x = 2 \end{cases}$

13. What value of k that makes the following function continuous at $x = 1$?

$$f(x) = \begin{cases} 2x - 3, & x < 1 \\ kx^2, & x \geq 1 \end{cases}$$

   Is it differentiable at $x = 1$ for that value of k? Why or why not?

14. Evaluate the following limit, if it exists:

$$\lim_{x \to -3} \sqrt{\frac{x^2 - 9}{x + 3}}$$

15. The height at time t of a ball thrown upward is given by $r(t) = 4 + 10t - 16t^2$

   (a) How far above the ground is the ball at time $t = 0.5$?

   (b) What is the average velocity of the ball as it travels from $t = 0.5$ to $t = 0.51$?

   (c) What is the average velocity of the ball as it travels from $t = 0.5$ to $t = 0.501$?

   (d) What is the ball's instantaneous velocity at $t = 0.5$? Is it falling or rising at $t = 0.5$?

16. Which of the following limits is of the form $\frac{0}{0}$?

   (a) $\lim_{x \to 1} \frac{x^2 - 2x - 1}{x - 1}$ (b) $\lim_{x \to -1} \frac{x^2 - 1}{x + 1}$ (c) $\lim_{x \to 2} \frac{x - 2}{x^2 - 2}$ (d) $\lim_{x \to 0} \frac{x}{x - 1}$

17. What is the value of the limit $\lim_{x \to 1} \frac{(x-2)^2 - 1}{x - 1}$?

   (a) 1 (b) 2 (c) -1 (d) -2

18. Use the limit definition of the derivative to find $f'(2)$ when $f(x) = \sqrt{x + 2}$

19. **Write to Learn:** In 2 or 3 sentences, prove that $\lim_{x \to 2} (2x - 1) = 3$ by finding for each $\varepsilon > 0$ a neighborhood (a, b) of p such that $|f(x) - L| < \varepsilon$ for all x in (a, b) with $x \neq p$. (The endpoints a and b of the interval should be in terms of ε).

20. **Write to Learn:** Write two sentences which explain how we would compute $f'(2)$ when $f(x) = x^2 + 2$.

Looking Ahead:

The laws of exponents and the properties of the trigonometric functions will be important in the next chapter. Thus, in the next few paragraphs we review these important concepts.

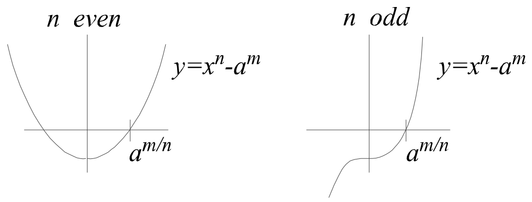
If $a > 0$ and if n is a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}$$

Moreover, if n is a positive integer, then we define

$$a^{-n} = \frac{1}{a^n}$$

If m, n are integers with $n > 0$, then $f(x) = x^n - a^m$ is a polynomial with only one positive real root, which is defined to be $a^{m/n}$.



L1: Definition of $a^{m/n}$

It follows that if p and q are positive integers, then

$$a^p \cdot a^q = \underbrace{a \cdot a \cdot \ldots \cdot a}_{p\ times} \cdot \underbrace{a \cdot a \cdot \ldots \cdot a}_{q\ times} = \underbrace{a \cdot a \cdot \ldots \cdot a}_{p+q\ times} = a^{p+q}$$

For example, $7^2 \cdot 7^3 = (7 \cdot 7) \cdot (7 \cdot 7 \cdot 7) = 7^{2+3}$. Likewise, we can show that

$$(a^p)^q = \underbrace{a^p \cdot a^p \cdot \ldots \cdot a^p}_{q\ times} = a^{\overbrace{p+p+\ldots+p}^{q\ times}} = a^{pq}$$

For example, $\left(5^2\right)^3 = 5^2 \cdot 5^2 \cdot 5^2 = 5^{2+2+2} = 5^{2\cdot3}$. Moreover, $a^n = \frac{1}{a^{-n}}$ implies that $a^n a^{-n} = 1$, and in addition,

$$\frac{a^n}{a^m} = \frac{\overbrace{a \cdot a \cdot \ldots \cdot a}^{n\ times}}{\underbrace{a \cdot a \cdot \ldots \cdot a}_{m\ times}} = a^{n-m}$$

As a result, $\frac{a^n}{a^n} = a^{n-n}$, which implies that $a^0 = 1$.

KEY CONCEPT: Laws of Exponents
If $a > 0$ and m, n are integers, then
(1) $a^m a^n = a^{m+n}$
(2) $(a^n)^m = a^{mn}$
(3) $a^{n-m} = \dfrac{a^n}{a^m}$
(4) $a^0 = 1$

In addition, if $a > 0$ and $b > 0$, then a fifth law of exponents says that

$$(5) \qquad (ab)^n = a^n b^n$$

EXAMPLE 1 Simplify the expression $\dfrac{\left(4x^2\right)^3 y^7}{7xy^2}$ assuming $x > 0$ and $y > 0$.

Solution: To begin with, property (5) implies that

$$\frac{\left(4x^2\right)^3 y^7}{7xy^2} = \frac{4^3 \left(x^2\right)^3 y^7}{7xy^2}$$

Property (2) then implies that

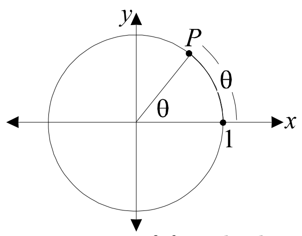
$$\frac{\left(4x^2\right)^3 y^7}{7xy^2} = \frac{4^3\left(x^2\right)^3 y^7}{7xy^2} = \frac{64x^6y^7}{7xy^2}$$

As a result, property (3) implies that

$$\frac{\left(4x^2\right)^3 y^7}{7xy^2} = \frac{4^3\left(x^2\right)^3 y^7}{7xy^2} = \frac{64x^6y^7}{7xy^2} = \frac{64}{7}x^{6-1}y^{7-2} = \frac{64}{7}x^5y^5$$

.

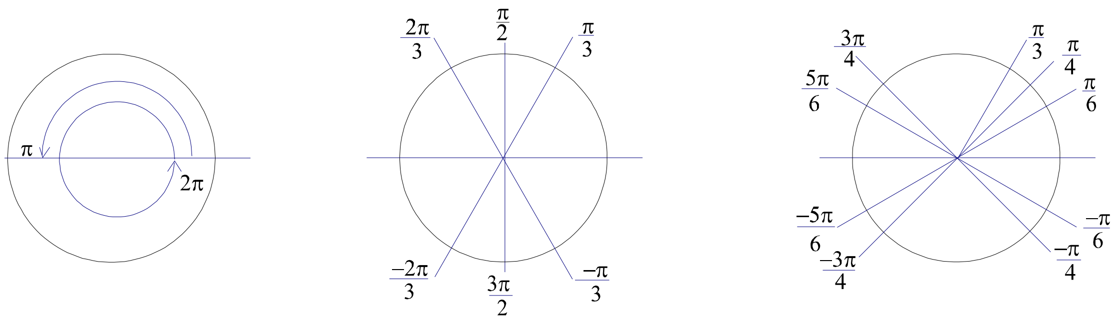
The study of trigonometry begins with *radian measure*. In particular, if P is a point on the unit circle and if θ is the angle between the positive x-axis and the ray from the origin through P, then the *radian measure* of θ is the arclength on the unit circle measured counterclockwise from the x-axis to the point P.



L2: The radian measure of θ is the length of the arc it subtends on the unit circle

(More generally, the radian measure of an angle is the ratio of the arclength of a circle subtended by that angle to the radius of the circle. Thus, radian measure is *dimensionless*, which is to say that it has no units like feet, meters, grams, or pounds).

One *cycle*—that is, one time around the unit circle—has a measure of 2π. Other angles in radian measure are likewise defined as fractions of the circumference of the unit circle.

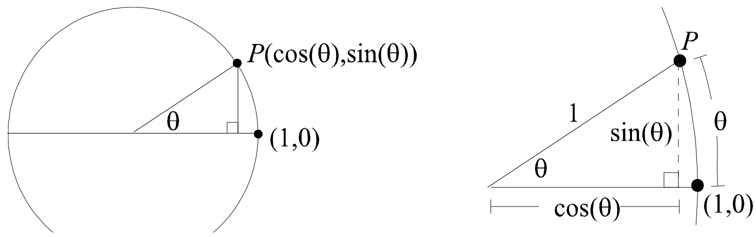


L3: $\theta = \pi$, $\theta = 2\pi$, $\theta = \frac{\pi}{2}$, and $\theta = \frac{3\pi}{2}$

Negative angles correspond to angles measured *clockwise* from the positive x-axis.

If θ is the *radian measure* of the angle formed by the x-axis and the ray from the origin through a point P on the unit circle, then the coordinates of P are

$\big(\cos(\theta), \sin(\theta)\big)$. [5]

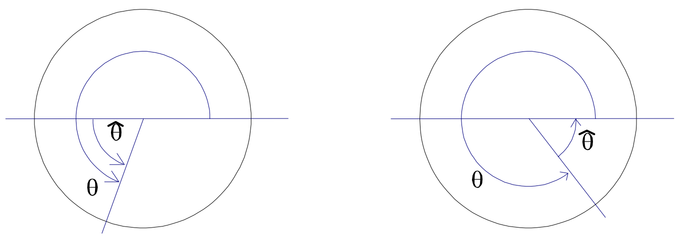


L2: Definition of $\cos(\theta)$ and $\sin(\theta)$

Sines and cosines of some common angles are summarized below:

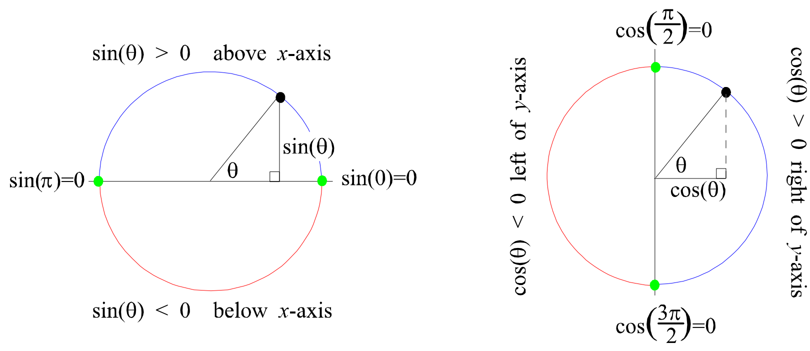
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

The *reference angle* $\hat{\theta}$ of an angle θ is defined to be the smallest positive angle between the termination of θ and the x-axis.



L4: $\hat{\theta}$ is the reference angle θ

The quadrant of an angle θ determines the sign of the sine or cosine of that angle, as shown in the table below:



L5: Signs of Sine and Cosine

The reference angle is then used to determine the magnitude of the sine or cosine.

EXAMPLE 1 Compute $\sin\left(\frac{4\pi}{3}\right)$

Solution: We first recognize that the angle is in the *third* quadrant with a reference angle of $\hat{\theta} = \frac{\pi}{3}$. Since the third quadrant is below the

[5] θ is the Greek letter "theta," and is often used to denote angles.

x-axis, $\sin\left(\frac{4\pi}{3}\right)$ is negative. Combined with the value from the table above, we find that

$$\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

Besides the sine and cosine function, we study the other four trigonometric functions:

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \quad \text{(tangent of } x\text{)} \qquad \cot(x) = \frac{\cos(x)}{\sin(x)} \quad \text{(cotangent of } x\text{)}$$

$$\sec(x) = \frac{1}{\cos(x)} \quad \text{(secant of } x\text{)} \qquad \csc(x) = \frac{1}{\sin(x)} \quad \text{(cosecant of } x\text{)}$$

if let $x = \cos(\theta)$ and $y = \sin(\theta)$, then (**??**) implies that

$$x^2 + y^2 = \cos^2(\theta) + \sin^2(\theta) = 1$$

Exercises

Simplify the following expressions assuming $x > 0$ and $y > 0$.

1. $\left(x^2y\right)^3$
2. $x^2x^3x^5$
3. $\left(4x^2\right)^{3/2}x^4$
4. $(2x)^3(xy)^2$
5. $\dfrac{\left(4x^2y\right)^3}{8x^3y^2}$
6. $\dfrac{\left(\left(2x^2\right)^2y\right)^3}{\left(8x^3\right)^2x^2}$

Evaluate the following

7. $\cos(\pi/6)$
8. $\sin(\pi/6)$
9. $\cos(5\pi/6)$
10. $\sin(5\pi/6)$
11. $\cos(7\pi/6)$
12. $\sin(7\pi/6)$
13. $\cos(\pi/3)$
14. $\sin(3\pi/4)$
15. $\cos(4\pi/3)$
16. $\sin(5\pi/4)$

17. Convert the radian measurement to degrees.

(a) $\frac{\pi}{6}$ (b) $\frac{4\pi}{15}$ (c) $\frac{3\pi}{5}$ (d) $\frac{7\pi}{4}$ (e) 0.3256

18. Convert the degree measurement to radians.

(a) $135°$ (b) $310°$ (c) $215°$ (d) $15°$ (e) $128°$

19. Evaluate the following by first determining the reference angle and quadrant of the angle.

20. In this exercise, we work with the identity $\sin(2x) = 2\sin(x)\cos(x)$

(a) Obtain this identity from (2.43) by letting $a = x$.

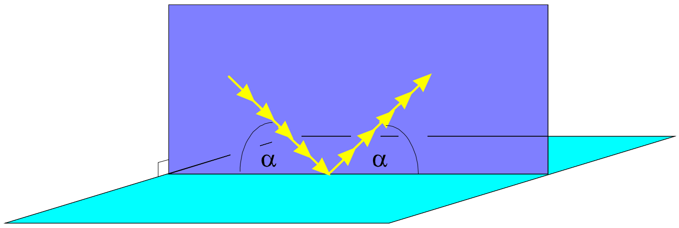
(b) Expand the following and simplify to a function of $\sin(2x)$:

$$(\cos x + \sin x)^2$$

Why do we study tangent lines? Why not simply study approximation in general? Our next step is to show that tangent lines provide information about a curve that cannot be obtained from other types of approximations.

In physics, it is shown that a ray of light reflects off of a flat surface at the same angle it strikes the surface.

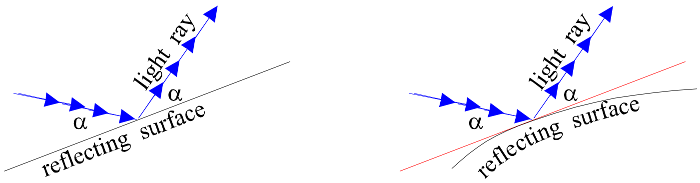
> **Law of Reflection:** If a ray of light is reflected off of a flat surface, then the entire path of the light lies in a plane normal to the surface and the the angle of incidence is equal to the angle of reflection.



NS-1: Law of Reflection

But what if the surface is curved? How do we predict the angle of reflection of a ray of light off of a curved surface?

To answer that, we notice that if the surface has a tangent at the point of reflection, then it is practically the same as the tangent line to the surface at that point. Thus, we may suppose that the light reflects off of the tangent in red instead of off of the surface.

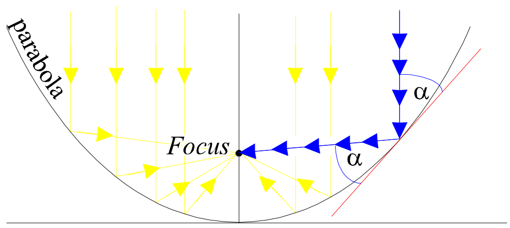


NS-2: Reflection off of a curved surface

We can use tangent lines to study properties of parabolic reflectors, which are used in a variety of applications including satellite dishes and headlight reflectors. For example, let us consider that if the cross-section of a parabolic reflector is given by

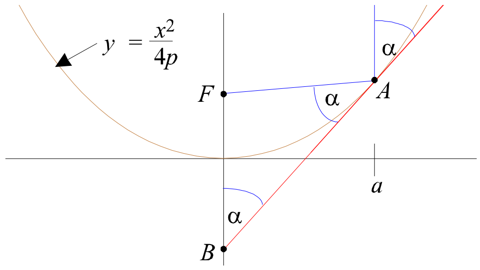
$$y = \frac{x^2}{4p}$$

then any vertical ray which strikes the reflector will be directed to the *focus* of the parabola.



NS-3: Parabolic Reflector Directs Vertical Rays to a common Focus

Let's use this to determine the coordinates of F. To do so, let us assume a vertical ray strikes the reflector at a $A = \left(a, \frac{a^2}{4p}\right)$ and continues to a point F on the y-axis. If B denotes the y-intercept of the tangent line at A, then the law of reflection implies that the triangle AFB is isosceles.



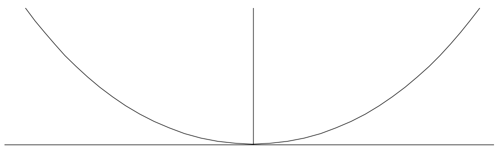
NS-4: ΔABF is isosceles

Moreover, the perpendicular bisector of the line segment $\overline{AB}$ passes through F. Thus, we need only show that the slope of $\overline{AB}$ is $\frac{a}{2p}$ and that the midpoint of $\overline{AB}$ is $\left(\frac{a}{2}, 0\right)$. It then follows that the perpendicular bisector of $\overline{AB}$ is the line with slope $\frac{-2p}{a}$ through the point $\left(\frac{a}{2}, 0\right)$. The y-intercept of this line is the focus, F.

Write to Learn Complete the computation described in the last paragraph to determine the coordinates of F, and then write a short essay with complete sentences which begins with the law of reflection and ends with the demonstration that a parabolic reflector reflects vertical rays toward the focus F.

Write to Learn Locate the focus of the parabola given by $y = 3x - x^2$ by determining where the reflection of the vertical ray $x = 1$ intersects the vertical line $x = 1.5$ which is the axis of symmetry of the parabola . Report your results in a short essay using complete sentences.

Write to Learn Suppose that we choose the point on the parabola such that the segment AF in NS-4 is horizontal. Show that this happens when the tangent line has a slope of 1. Use these ideas to locate the focus of the parabola below.



NS-5

Report your results in a short essay using complete sentences.

Write to Learn Go to the library or explore the internet to learn more about reflection properties of curves. Report your results in an essay using complete sentences and providing references to your sources.

Group Learning Suppose a reflector has a cross-section in the shape of the curve $y = x^4 - 2x^2 + 1$. What are the reflections of the vertical rays through $[-2, 2]$ with increments every 0.25 units (i.e., the rays $x = -2$, $x = -1.75$, $x = -1.5$, and so on)? Have each member of the group determine a set of such reflections, and then plot all the reflections on a common grid. Discuss the results and then present the reflections and the results in either a formal paper or a formal presentation.

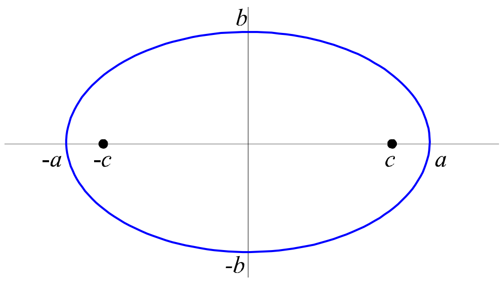
Advanced Contexts

(Advanced contexts are included at the end of each chapter for those students who want something a little more challenging)

Ellipses and hyperbolas also have reflection properties related to their foci. In particular, if $a > b > 0$, then the ellipse

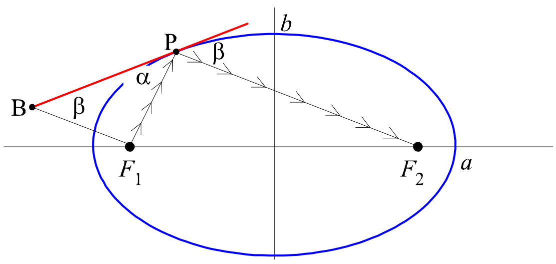
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

has foci at $(c, 0)$ and $(-c, 0)$, where the number c satisfies $b^2 + c^2 = a^2$.



NS-6: Foci at $(-c, 0)$ and $(c, 0)$

If we let F_1 denote the focus at $(-c, 0)$ and let F_2 denote the focus at $(c, 0)$, then a ray emanating from F_1 will be reflected to the Focus at F_2, and vice versa. To prove this fact, we need only show that $\alpha = \beta$ in the figure NS-7, assuming that BF_1 is parallel to PF_2.



NS-7: Need to show that $\alpha = \beta$

To do so, we need only prove that the triangle PF_1B is isosceles.

Exercise 1 *Let (x_1, y_1) denote the coordinates of P. What is the equation of the line through P tangent to the ellipse in figure NS-7?*

Exercise 2 ** What are the coordinates of B in figure NS-7?*

Exercise 3 ** Show that $|BF_1| = |PF_1|$. What does this imply about α and β?*

2. THE DERIVATIVE

In chapter 1, we explored tangent lines, linearizations, and rates of change at a specific point. In the last section of chapter 1, we used the notation $f'(p)$ to denote the *derivative* of a function $f(x)$ at an input of p.

We adopted this notation because the derivative is itself a function. It maps inputs p to rates of change $f'(p)$. Moreover, treating the derivative as a function allows us to develop rules for computing derivatives symbollically and strategies for applying derivatives to problems throughout mathematics and science.

In this chapter, we explore rules for finding derivatives of an important class of functions known as the *elementary functions.* Doing so will allow us to make calculus broad in scope by allowing us to define and explore some of the most important functions encountered in mathematics and science.

2.1 The Derivative Function

The Derivative as a Function

The derivative is itself a function, in that it maps an input p to an output that can be interpreted as either the slope of tangent line to $y = f(x)$ at p or as the rate of change of $f(x)$ at p. Thus, throughout the remainder of this chapter, we concentrate on the process of producing the derivative function, a process known as *differentiation.*

In particular, the derivative as a function of x is denoted by $f'(x)$, which by definition 7.1 satisfies

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Alternatively, when $f(x)$ is differentiable at x, then we can compute $f'(x)$ by identifying it in the expansion of $f(x+h)$:

$$f(x+h) = f(x) + f'(x)\,h \quad + \quad o(h)$$

Moreover, we define the symbol $\frac{d}{dx}$ to mean "the derivative of", so that the equation

$$f'(x) = \frac{d}{dx} f(x)$$

literally means "$f'(x)$ is the derivative of $f(x)$." We call $\frac{d}{dx}$ the *derivative operator* since it represents the act of transforming a given function into its derivative function.

EXAMPLE 1 Find $f'(x)$ for $f(x) = x^2$.

Solution: If $f(x) = x^2$, then $f(x+h) = (x+h)^2$, so that

$$f(x+h) = x^2 + 2x\,h \; + \; h^2$$

Since $2x$ is the coefficient of h, the derivative of $f(x) = x^2$ is $f'(x) = 2x$. We write this in operator notation as

$$\frac{d}{dx}x^2 = 2x$$

EXAMPLE 2 Find $f'(x)$ for $f(x) = x^3$.

Solution: If $f(x) = x^3$, then

$$f(x+h) = (x+h)^3 = x^3 + 3x^2\,h \quad + 3xh^2 + h^3$$

which implies that $f'(x) = 3x^2$. In operator notation, this is written

$$\frac{d}{dx}x^3 = 3x^2$$

In general, if $f(x) = x^n$, then $f(x+h) = (x+h)^n$, which is expanded using Pascal's triangle. Notice, however, that the first two coefficients of the n^{th} row of Pascal's triangle are 1 and n, respectively. :

$$\begin{array}{c|l}
0 & 1 \\
\mathbf{1} & 1 \quad \mathbf{1} \\
\mathbf{2} & 1 \quad \mathbf{2} \quad 1 \\
\mathbf{3} & 1 \quad \mathbf{3} \quad 3 \quad 1 \\
\mathbf{4} & 1 \quad \mathbf{4} \quad 6 \quad 4 \quad 1 \\
\mathbf{5} & 1 \quad \mathbf{5} \quad 10 \quad 10 \quad 5 \quad 1 \\
\mathbf{6} & 1 \quad \mathbf{6} \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
\mathbf{7} & 1 \quad \mathbf{7} \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
\vdots & \quad \nearrow \quad \nearrow \\
n & 1 \quad n
\end{array}$$

As a result, the expansion of $(x+h)^n$ is of the form

$$(x+h)^n = x^n + nx^{n-1}\,h \quad + \quad o(h)$$

which implies the derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$. In operator form, this becomes our first rule for computing derivatives:

The Power Rule: If n is a positive integer, then $\dfrac{d}{dx}x^n = nx^{n-1}$

EXAMPLE 3 Use the power rule to compute $\frac{d}{dx}x^2$ and $\frac{d}{dx}x^3$.

Solution: When $n = 2$ we have

$$\frac{d}{dx}x^2 = 2x^{2-1} = 2x$$

and when $n = 3$, we have $\frac{d}{dx}x^3 = 3x^2$.

Check your Reading *Use the power rule to evaluate* $\frac{d}{dx}x^4$.

Derivatives of Polynomials

If $f(x)$ and $g(x)$ are polynomials and $r(x) = f(x) + g(x)$, then

$$\begin{array}{llll} r(x+h) = & f(x+h) & + & g(x+h) \\ & f(x) + f'(x)h + o_1(h) & + & g(x) + g'(x)h + o_2(h) \end{array}$$

Collecting common terms yields

$$r(x) = f(x) + g(x) + [f'(x) + g'(x)]h + [o_1(h) + o_2(h)]$$

This implies that $r'(x) = f'(x) + g'(x)$. In like fashion, we can obtain each of the following derivative rules:

Theorem 1.2 If $f(x)$ and $g(x)$ are polynomials and k is a constant, then

$$\begin{aligned} \frac{d}{dx}(f(x) + g(x)) &= \frac{d}{dx}f(x) + \frac{d}{dx}g(x) \qquad (2.1) \\ \frac{d}{dx}(f(x) - g(x)) &= \frac{d}{dx}f(x) - \frac{d}{dx}g(x) \\ \frac{d}{dx}(kf(x)) &= k\frac{d}{dx}f(x) \\ \frac{d}{dx}k &= 0 \end{aligned}$$

Theorem 5.2 and the power rule allow us to compute the derivative of a polynomial.

EXAMPLE 4 Find $f'(x)$ for $f(x) = x^3 + 3x + 2$.

Solution: Theorem 5.2 implies that

$$f'(x) = \frac{d}{dx}\left(x^3 + 3x + 2\right) = \left(\frac{d}{dx}x^3\right) + 3\left(\frac{d}{dx}x^1\right) + \left(\frac{d}{dx}2\right)$$

Since 2 is a constant, its rate of change is 0. Thus, the power rule implies that

$$f'(x) = 3x^2 + 3 \cdot 1x^0 + 0 = 3x^2 + 3$$

EXAMPLE 5 Find $f'(x)$ for $f(x) = x^2(x+5)$.

Solution: First, we distribute the x^2 to obtain $f(x) = x^3 + 5x^2$. As a result, the power rule and theorem 5.2 imply that

$$f'(x) = \frac{d}{dx}\left(x^3 + 5x^2\right) = 3x^2 + 10x$$

Moreover, we define the *second derivative* $f''(x)$ to be the derivative of $f'(x)$, we define the *third derivative* $f'''(x)$ to be the derivative of $f''(x)$, and in general, we define the n^{th} derivative $f^{(n)}(x)$ to be the derivative of $f^{(n-1)}(x)$.

EXAMPLE 6 Find f', f'', f''', and $f^{(4)}$ for $f(x) = x^5 + 6x^3 + x$

Solution: The first derivative is

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(x^5 + 6x^3 + x\right) \\ &= \left(\frac{d}{dx}x^5\right) + 6\left(\frac{d}{dx}x^3\right) + \frac{d}{dx}x \\ &= 5x^4 + 18x^2 + 1 \end{aligned}$$

The second derivative $f''(x)$ is then obtained by differentiating $f'(x)$. That is,

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left(5x^4 + 18x^2 + 1\right) \\ &= 20x^3 + 36x \end{aligned}$$

The third derivative $f'''(x)$ is given by

$$f'''(x) = \frac{d}{dx}f''(x) = 60x^2 + 36$$

and the fourth derivative is

$$f^{(4)}(x) = \frac{d}{dx}\left(60x^2 + 36\right) = 120x$$

Check your Reading — *What is the fifth derivative $f^{(5)}(x)$ of $f(x) = x^5 + 6x^3 + x$?*

Velocity and Acceleration

If $r(t)$ denotes the position of an object at time t, then $r'(t)$ is the instantaneous *velocity* of the object and is usually denoted $v(t)$. Moreover, $r''(t) = v'(t)$ is the instantaneous rate of change of the velocity, which is known as the *acceleration,* $a(t)$, of the object at time t. That is, we define

$$a(t) = r''(t)$$

In differential notation, velocity and acceleration are given by

$$v = \frac{dr}{dt} \qquad \text{and} \qquad a(t) = \frac{d^2r}{dt^2}$$

where $\frac{d^2r}{dt^2}$ is the differential notation for the second derivative.

EXAMPLE 7 Find the velocity and acceleration of an object in free fall whose height in feet at time t in seconds is

$$r(t) = 64t - 16t^2$$

Solution: The velocity $v(t)$ is the derivative of $r(t)$, which is

$$v(t) = \frac{dr}{dt} = 64 - 32t$$

The acceleration $a(t)$ is the *second derivative* of $r(t)$, which is

$$a(t) = \frac{d^2r}{dt^2} = \frac{d}{dt}(64 - 32t) = -32$$

EXAMPLE 8 Find the velocity and acceleration of an object whose position at time t is given by $r(t) = t(t-2)^2$.

Solution: First, we expand the square to obtain

$$r(t) = t\left(t^2 - 4t + 4\right)$$

Distributing t then yields $r(t) = t^3 - 4t^2 + 4t$, so that the velocity is

$$v(t) = \frac{dr}{dt} = \frac{d}{dt}\left(t^3 - 4t^2 + 4t\right) = 3t^2 - 8t + 4$$

As a result, the acceleration is given by

$$a(t) = \frac{d^2r}{dt^2} = \frac{d}{dt}\left(3t^2 - 8t + 4\right) = 6t - 8$$

Check your Reading *How is the acceleration $a(t)$ of an object related to its velocity?*

More with Objects in Free-fall

Recall that if an object in free fall has a height r_0 and a velocity v_0 at time $t = 0$, then its height $r(t)$ at time t is given by Galileo's model of projectile motion,

$$r(t) = r_0 + v_0 t - \frac{1}{2}gt^2 \tag{2.2}$$

where g denotes the acceleration due to gravity near that planet's surface. The velocity $v(t) = r'(t)$ is the rate of change of the object at time t, which can be used to study the motion of the projectile.

For example, an object reaches its maximum height when its velocity changes from positive to negative. That is, the time t at what the object reaches its maximum height is the same time at which $v(t) = 0$. Thus, to find the maximum height of an object in free fall, we solve $v(t) = 0$ for t and then compute r at that time.

EXAMPLE 9 Find the model (2.2), the velocity $v(t)$, and the maximum height of a rock which is projected vertically from the earth's surface with a velocity of 64 feet per second (i.e., about 43 mph).

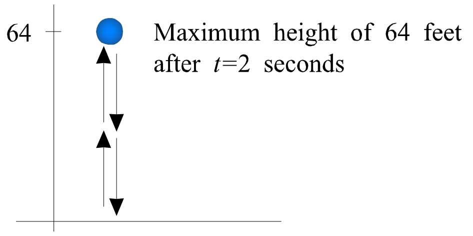
Solution: Its initial height is $r_0 = 0$, so (2.2) implies that its height r at time t is

$$r(t) = 64t - 16t^2$$

since $g = 32$ feet/sec^2 near the surface of the earth. It follows that the velocity of the rock is $v(t) = 64 - 32t$. If we set $v(t) = 0$ and solve, we obtain

$$\begin{aligned} 64 - 32t &= 0 \\ t &= 2 \text{ sec} \end{aligned}$$

That is, the object must have reached a maximum height when $t = 2$. Substituting $t = 2$ into $r(t)$ then yields the rock's maximum height, which is $r(2) = 64(2) - 16(2)^2 = 64$ feet.



1-1: Projectile launched from the earth's surface

EXAMPLE 10 Find the model (2.2), the velocity $v(t)$, and the maximum height of a rock which is projected vertically from the surface of Mars with a velocity of 64 feet per second.

Solution: Near the surface of Mars, the acceleration due to gravity is $g = 12.2$ ft./sec^2. Thus, the rock's height at time t is

$$r(t) = 64t - 6.1t^2$$

The velocity of the rock is $v(t) = 64 - 12.2t$, so that setting $v(t) = 0$ yields

$$\begin{aligned} 64 - 12.2t &= 0 \\ t &= 5.246 \text{ sec} \end{aligned}$$

Substituting $t = 5.246$ into $r(t)$ yields

$$r(5.246) = 64(5.246) - 6.1(5.246)^2 = 168\ feet,\ 10\ inches$$

That is, the rock will rise much higher above the surface of Mars than it would above the surface of the earth.

Exercises:

Find $f'(x)$ and $f''(x)$.

1. $f(x) = x^5$
2. $f(x) = x^7$
3. $f(x) = x^2 + 2x + 3$
4. $f(x) = 2x^2 + 3$
5. $f(x) = 3x^8 + x^2$
6. $f(x) = x^5 - 2x^3$
7. $f(x) = 2x^3 + 5x - 4x^2 - 10$
8. $f(x) = 2x^4 - 2x^3 + x + 1$
9. $f(x) = 3x^5 - 4x^3 + 2x^2 - 23x + 17$
10. $f(x) = 3x^5 - 17x^2 + 2x + 1$
11. $f(x) = (2x + 1)^2$
12. $f(x) = (x^2 + 3)^2$
13. $f(x) = 2 - x$
14. $f(x) = 6 - 5x$
15. $f(x) = 0.3x + 2.5$
16. $f(x) = 3.2 - 0.7x$
17. $f(x) = 0.15x^2 + 0.34x - 0.62$
18. $f(x) = 10.4x - 3.7x^2 + 9.8x^{10}$
19. $f(x) = \left(x^2 + 1\right)\left(x^5 + 2\right)$
20. $f(x) = (x^3 - 5)^2(4x^2 + 0.5x)$
21. $f(x) = \dfrac{x^3 + 2x}{x}$
22. $f(x) = \dfrac{x^4 + 2x^3}{x + 2}$

Find the velocity and acceleration of an object whose position at time t is $r(t)$:

23.	$r(t) = -16t^2$	24.	$r(t) = 64 - 16t^2$
25.	$r(t) = 50t - 4.8t^2$	26.	$r(t) = 75 + 17t - 4.8t^2$
27.	$r(t) = 35t - 10t^4$	28.	$r(t) = 35t^3 - 10t^4$
29.	$r(t) = t(16 - t)^2$	30.	$r(t) = (t+1)(16 - t)^2$

31. Find the specified derivatives. These must be done in the order given.

(a) Find $f^{(4)}(x)$ if $f(x) = x^3 + 3x + 1$
(b) Find $f^{(6)}(x)$ if $f(x) = x^5 + 3x^2 + 2x + 3$
(c) Find $f^{(7)}(x)$ if $f(x) = x^6$
(d) Find $f^{(8)}(x)$ if $f(x) = x^7$
(e) Find $f^{(11)}(x)$ if $f(x) = x^{10}$
(f) Find $f^{(100)}(x)$ if $f(x) = x^{99} + 33x^{95} + 17x^{80} + 3x^{77} - 13x^{27} + 1$

32. What is the general form of a function whose second derivative is identically zero?

33. Using (2.2), find the height $r(t)$ of the object above the ground—i.e., the earth's surface—at time t. Then velocity of the object..

(a) An object is dropped from a window 30 ft. off of the ground.
(b) A rock is thrown vertically upward with an initial velocity of 20 ft./sec. The thrower releases the object 5.5 ft. above the ground.
(c) A ball rolls off of a level table 4 ft. high. (i.e., initial *vertical* velocity is 0)

34. Supposing that the surface gravity of Mars is approximately 12.2 ft./sec.^{2}, use (2.2) to find the function $r(t)$ giving the height of the object above the surface of Mars at time t. Then find the velocity $v(t)$.

(a) An object is dropped from a window 30 ft. off of the ground.
(b) A rock is thrown vertically upward with an initial velocity of 20 ft./sec. The thrower releases the object 5.5 ft. above the ground.
(c) A ball rolls off of a level table 4 ft. high.
(d) An object drops from a window 38 ft. above the ground. It strikes a deck on the first floor, 12 ft. above the ground. It then rolls horizontally for 2 seconds before falling the remaining distance to the ground.

35. Grapher: Suppose that a baseball is thrown upward from a height of 7.5 feet with an initial velocity of 73 feet per second (i.e., about 50 miles per hour). If thrown near the earth's surface, the position of the object at time t is

$$r(t) = 7.5 + 73t - 16t^2$$

while if thrown near the surface of Mars, the position of the object at time t is

$$s(t) = 7.5 + 73t - 6.1t^2$$

Use a grapher to answer the following:

(a) How much higher will the ball's maximum height be on Mars than on the Earth?

(b) How much longer will it be aloft on Mars than on the Earth?

36. Suppose that a baseball is thrown upward from a height of 7.5 feet with an initial velocity of 110 feet per second (i.e., about 75 miles per hour).

(a) What is the position of the ball t seconds after release if the thrower is standing on the earth?
(b) What is the position of the ball t seconds after release if the thrower is standing on Mars?
(c) How much higher will the ball's maximum height be on Mars than on the Earth?
(d) How much longer will it be aloft on Mars than on the earth?

37. Graph the implied height functions above each planet, and then use the graphs to answer the following:

(a) Which remains aloft for the longest period of time—a ball dropped from 100 feet on the earth's surface or a ball dropped from 30 feet on Mars' surface?
(b) Which has the greatest height—a ball thrown upward from the earth's surface with an initial velocity of 100 feet per second or a ball thrown upward from Mars' surface with an initial velocity of 50 feet per second?

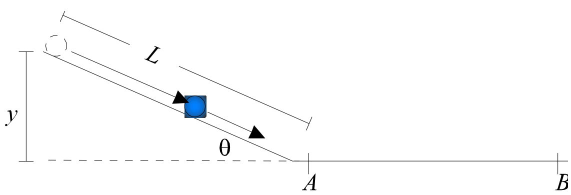
38. * A rock thrown into the air from an initial height of $r_0 = 0$ returns to the earth's surface T seconds later. What is the maximum height of the rock as a function of T?

39. **Grapher:** Compute $f'(x)$ for $f(x) = x^3 - 3x^2 + 1$. Then graph $f(x)$ and $f'(x)$ on $[-1, 3]$. Where does the largest value of $f(x)$ occur? At what value is $f'(x) = 0$? Can you explain the connection here?

40. **Grapher:** Use $f(x) = x^3 - 3x^2 + 1$ to answer the following:

(a) Graph both $f(x)$ and $f'(x)$ on $[-1, 3]$. What is significant about the graph of $f(x)$ on intervals where $f'(x) > 0$?
(b) Graph both $f(x)$and $f''(x)$ on $[-1, 3]$. Are tangent lines to $y = f(x)$ above or below the curve over intervals where $f''(x) > 0$?

41. * Galileo determined that the acceleration due to gravity at the earth's surface is 32 feet per second per second by relating it to the velocity v_F at the bottom of an inclined plane as shown below.



1-2: Galileo's experiment to measure the acceleration due to gravity

If the ball is initially at rest, the acceleration due to gravity g is given by

$$g = \frac{v_F^2}{2y}$$

Derive this formula using the fact that the distance $r(t)$ the ball rolls along the ramp t seconds after being released is

$$r(t) = \frac{1}{2} g t^2 \sin\theta$$

42. Try it out! The final velocity v_F is computed by measuring the time T required from the ball to roll from A to B after it has rolled across the ramp. In this case,

$$g = \frac{(B-A)^2}{2yT^2}$$

By making y very small in comparison to L, the time T is large enough to measure with a stop watch or even a wrist watch. Try it out using values like $L = 4$ feet, $y = 3$ inches and $|B - A| = 4$ feet. (Note: use a heavy ball which will not slow appreciably between points A and B.)

43. Show that if $f(x)$ and $g(x)$ are polynomials, then

$$\frac{d}{dx}(f(x) - g(x)) = \left(\frac{d}{dx} f(x)\right) - \left(\frac{d}{dx} g(x)\right)$$

44. Show that if $f(x)$ is a polynomial and k is constant, then

$$\frac{d}{dx}(kf(x)) = k\frac{d}{dx} f(x)$$

45. Write to Learn: Common sense tells us that the rate of change of a constant is 0. In a short essay, use this common sense fact to justify the derivative rule

$$\text{if } k \text{ is constant, then } \frac{d}{dx} k = 0$$

2.2 The Product and Quotient Rules

The Product Rule

In the last section, we derived rules for differentiating powers and polynomials. In this section, we derive rules for differentiating products and quotients of functions, beginning with the derivative of a product.

If $f(x)$ is a differentiable at x, then

$$f(x+h) = f(x) + f'(x)h + o(h)$$

Likewise, if $g(x)$ is differentiable at x, then $g(x+h) = g(x) + g'(x)h + o(h)$. Thus, the product $f(x+h)g(x+h)$ is of the form

$$f(x+h)g(x+h) = [f(x) + f'(x)h + o(h)][g(x) + g'(x)h + o(h)]$$

Expanding the product on the right leads to the expression

$$f(x)g(x) \quad + \quad [f'(x)g(x) + f(x)g'(x)]\,h \; + \; \text{higher order terms}$$

The derivative of $f(x)g(x)$ is the expression that is the coefficient of h:

The Product Rule: The derivative of a product is given by

$$\frac{d}{dx}\left(f\left(x\right)g\left(x\right)\right)=f'\left(x\right)g\left(x\right)+f\left(x\right)g'\left(x\right)$$

If $u=f\left(x\right)$ and $v=g\left(x\right)$, then in Liebniz notation the product rule is of the form

$$\frac{d\left(uv\right)}{dx}=\frac{du}{dx}v+u\frac{dv}{dx}$$

and in fact, the product rule is often memorized using the mnemonic

$$\left(uv\right)'=u'v+uv' \tag{2.3}$$

Moreover, the product rule is the only rule for differentiating products that will produce the same results as the rules in the last section.

EXAMPLE 1 Evaluate $\frac{d}{dx}x^3$ by applying the product rule to $x^2\cdot x$.

Solution: We apply (2.3) with $u=x^2$ and $v=x$:

$$\begin{array}{ccccccccc} & (uv)' & & u' & \cdot & v & + & u & \cdot & v' \\ \frac{d}{dx} & \left(x^2\cdot x\right) & = & \left(\frac{d}{dx}x^2\right) & \cdot & x & + & x^2 & \cdot & \left(\frac{d}{dx}x\right)\end{array}$$

As a result, the product rule applied to $x^2\cdot x$ yields

$$\frac{d}{dx}\left(x^2\cdot x\right)=2x\cdot x+x^2\cdot 1=2x^2+x^2$$

which reduces to $\frac{d}{dx}\left(x^3\right)=3x^2$, as expected.

EXAMPLE 2 Evaluate $\frac{d}{dx}\,x^3\left(x^3+4x^2\right)$ using the product rule:

Solution: We apply (2.3) with $u=x^3$ and $v=x^3+4x^2$.

$$\begin{array}{cccccccc} & (uv)' & = & u' & v & + & u & v' \\ \frac{d}{dx} & \left[x^3\left(x^3+4x^2\right)\right] & = & \left(\frac{d}{dx}x^3\right) & \left(x^3+4x^2\right) & + & x^3 & \left(\frac{d}{dx}\left(x^3+4x^2\right)\right)\end{array}$$

As a result, the product rule yields

$$\begin{aligned}\frac{d}{dx}\left[x^3\left(x^3+4x^2\right)\right] &= 3x^2\left(x^3+4x^2\right)+x^3\left(3x^2+8x\right)\\ &= 3x^5+12x^4\ +\ 3x^5+8x^4\\ &= 6x^5+20x^4\end{aligned}$$

Check your Reading *Evaluate $\frac{d}{dx}\left(x^6+4x^5\right)$ and explain why it produces the same result as applying the product rule to $x^3\left(x^3+4x^2\right)$.*

More with the Product Rule

The product rule can be applied even when only partial information about a function is given.

EXAMPLE 3 Suppose that $h(x) = x^2 f(x)$ and that it is known that $f(5) = 11$ and $f'(5) = 7$. What is $h'(5)$?

Solution: Since $h(x)$ is the product of $u = x$ and $v = f(x)$, the product rule implies that

$$\begin{array}{rcccccc} (uv)' & = & u' & v & + & u & v' \\ \frac{d}{dx}\left[x^2 f(x)\right] & = & \left(\frac{d}{dx}x^2\right) & f(x) & + & x^2 & \frac{d}{dx}f(x) \\ & = & 2x & f(x) & + & x^2 & f'(x) \end{array}$$

Thus, $h'(x) = 2xf(x) + x^2 f'(x)$, which implies that

$$\begin{aligned} h'(5) &= 2\cdot 5\cdot f(5) + (5)^2\cdot f'(5) \\ &= 10\cdot 11 + 25\cdot 7 \\ &= 285 \end{aligned}$$

In fact, we can use the product rule to develop new rules for differentiation. For example, if $f(x) = x^{-2}$, then $x^2 f(x) = 1$ and

$$\begin{aligned} \frac{d}{dx}x^2 f(x) &= \frac{d}{dx}1 \\ \left(\frac{d}{dx}x^2\right) f(x) + x^2\left(\frac{d}{dx}f(x)\right) &= 0 \\ 2xf(x) + x^2 f'(x) &= 0 \end{aligned}$$

Substituting $f(x) = x^{-2}$ and solving for $f'(x)$ yields

$$\begin{aligned} 2x\cdot x^{-2} + x^2 f'(x) &= 0 \\ x^2 f'(x) &= -2x^{-1} \\ f'(x) &= -2x^{-3} \end{aligned}$$

Since $f'(x) = \frac{d}{dx}x^{-2}$, this means that

$$\frac{d}{dx}x^{-2} = -2x^{-3} \tag{2.4}$$

Indeed, similar methods will be used in the exercises to show that

$$\frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} \tag{2.5}$$

and clearly, both (2.4) and (2.5) are in the form implied by the power rule. That is, the product rule implies that

$$\frac{d}{dx}x^n = nx^{n-1} \tag{2.6}$$

for any rational number n. In chapter 2 it will be shown that (2.6) holds for any real number n.

EXAMPLE 4 Use (2.6) to evaluate $\frac{d}{dx}\frac{1}{x^2}$

Solution: First, we write $\frac{1}{x^2}$ as x^{-2} and then we apply (2.6):

$$\frac{d}{dx}x^{-2} = -2x^{-2-1} = -2x^{-3}$$

Writing the result in terms of fractions then yields

$$\frac{d}{dx}\frac{1}{x^2} = \frac{-2}{x^3}$$

EXAMPLE 5 Use (2.6) to evaluate $\dfrac{d}{dx}\sqrt[3]{x^2}$

Solution: First, we write $\sqrt[3]{x^2}$ as $x^{2/3}$ and then we apply (2.6):

$$\frac{d}{dx}x^{2/3} = \frac{2}{3}x^{2/3-1} = \frac{2}{3}x^{-1/3}$$

Writing the result in terms of fractions then yields

$$\frac{d}{dx}\sqrt[3]{x^2} = \frac{2}{3\sqrt[3]{x}}$$

Check your Reading *Why is* $\sqrt[3]{x^2}$ *the same as* $x^{2/3}$?

The Quotient Rule

Suppose now that $p(x) = \frac{f(x)}{g(x)}$. Then $g(x)\,p(x) = f(x)$ and the product rule implies that

$$g'(x)\,p(x) + g(x)\,p'(x) = f'(x)$$

If we now solve for $p'(x)$, then we obtain

$$\begin{aligned} g(x)\,p'(x) &= f'(x) - g'(x)\,p(x) \\ p'(x) &= \frac{f'(x) - g'(x)\,p(x)}{g(x)} \end{aligned}$$

Since $p(x) = \frac{f(x)}{g(x)}$, this leads to

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x) - g'(x)\,\frac{f(x)}{g(x)}}{g(x)} = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$$

The result is known as the *quotient rule* for differentiating the ratio of two function:

The Quotient Rule: If $f(x)$ and $g(x)$ are polynomials, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2} \qquad (2.7)$$

Another proof of the quotient rule will be presented in the exercises in the next section.

EXAMPLE 5 Use the quotient rule to evaluate

$$\frac{d}{dx}\left(\frac{1}{2x+3}\right)$$

Solution: Comparison with (2.7) reveals that $f(x) = 1$ and $g(x) = 2x+3$. Thus, (2.7) implies that

$$\frac{d}{dx}\left(\frac{1}{2x+3}\right) = \frac{\left[\frac{d}{dx}(1)\right](2x+3) - 1\cdot\left[\frac{d}{dx}(2x+3)\right]}{(2x+3)^2}$$

Evaluating the derivatives and simplifying thus yields

$$\frac{d}{dx}\left(\frac{1}{2x+3}\right) = \frac{0\cdot(2x+3) - 1\cdot 2}{(2x+3)^2} = \frac{-2}{(2x+3)^2}$$

EXAMPLE 6 Use the quotient rule to evaluate

$$\frac{d}{dx}\left(\frac{x^2+2x}{x^4+1}\right)$$

Solution: Comparison with (2.7) reveals that $f(x) = x^2+2x$ and $g(x) = x^4+1$. Thus, (2.7) implies that

$$\frac{d}{dx}\left(\frac{x^2+2x}{x^4+1}\right) = \frac{\left[\frac{d}{dx}(x^2+2x)\right](x^4+1) - (x^2+2x)\left[\frac{d}{dx}(x^4+1)\right]}{(x^4+1)^2}$$

Evaluating the derivatives and simplifying thus yields

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^2+2x}{x^4+1}\right) &= \frac{(2x+2)(x^4+1) - (x^2+2x)(4x^3)}{(x^4+1)^2}\\ &= \frac{(2x^5+2x+2x^4+2) - (4x^5+8x^4)}{(x^4+1)^2}\\ &= \frac{-2x^5-6x^4+2x+2}{(x^4+1)^2}\end{aligned}$$

Check your Reading *How would you use the quotient rule to evaluate* $\frac{d}{dx}x^{-1}$?

Tangent Lines Revisited

Finally, let us not forget that $f'(p)$ is the slope of the tangent line to the graph of $f(x)$ at the input p. To determine the slope of the tangent line to the graph of $f(x)$ at $x = p$, we compute $f'(x)$ and then substitute $x = p$.

EXAMPLE 7 Find the equation of the tangent line to $y = \frac{1}{x}$ at $p = \frac{3}{2}$.

Solution: If we let $f(x) = \frac{1}{x}$, then (2.6) implies that

$$f'(x) = \frac{d}{dx}x^{-1} = -x^{-2}$$

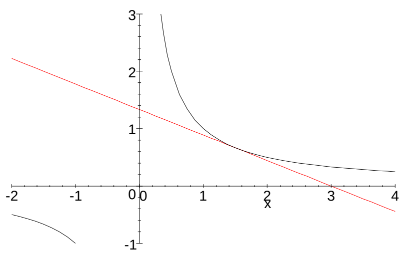
As a result, the slope of the tangent line is

$$m = f'\left(\frac{3}{2}\right) = -\left(\frac{3}{2}\right)^{-2} = \frac{-1}{\left(\frac{3}{2}\right)^2} = -\frac{4}{9}$$

Since $f\left(\frac{3}{2}\right) = \frac{2}{3}$, the tangent line is the line with slope $m = \frac{-4}{9}$ that passes through $\left(\frac{3}{2}, \frac{2}{3}\right)$:

$$y = \frac{2}{3} - \frac{4}{9}\left(x - \frac{3}{2}\right)$$

This simplifies to $y = \frac{4}{3} - \frac{4}{9}x$.



2-1: $y = \frac{4}{3} - \frac{4}{9}x$ is tangent to $y = \frac{1}{x}$

EXAMPLE 8 Find the equation of the tangent line to $y = x^2/(x+6)$ when $p = 3$.

Solution: To find $f'(x)$, we use the quotient rule:

$$f'(x) = \frac{d}{dx}\left(\frac{x^2}{x+6}\right) = \frac{\left(\frac{d}{dx}x^2\right)(x+6) - x^2\frac{d}{dx}(x+6)}{(x+6)^2}$$

Evaluating the derivatives and simplifying yields

$$f'(x) = \frac{2x(x+6) - x^2(1+0)}{(x+6)^2} = \frac{x^2+12x}{(x+6)^2}$$

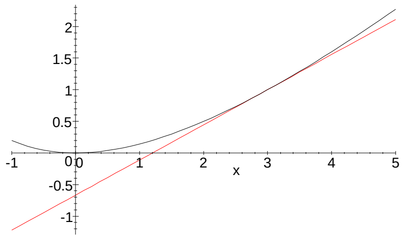
As a result, the slope of the tangent line is

$$m = f'(3) = \frac{3^2 + 12 \cdot 3}{(3+6)^2} = \frac{5}{9}$$

Since $f(3) = \frac{3^2}{3+6} = 1$, the point of tangency is $(3, 1)$. Thus, the equation of the tangent line is

$$y = 1 + \frac{5}{9}(x - 3)$$

which simplifies to $y = -\frac{2}{3} + \frac{5}{9}x$



2-2: Graphs of $f(x) = \frac{x^2}{x+6}$ and $L_3(x) = -\frac{2}{3} + \frac{5}{9}x$

Exercises:

Find $f'(x)$ first by applying the product rule, and then by expanding the product and using the power rule. Both methods should yield the same result.

1. $f(x) = x^2 \cdot x^3$
2. $f(x) = x^4 \cdot x^3$
3. $f(x) = x^4 \cdot 1$
4. $f(x) = x^7 \cdot 2$
5. $f(x) = x(x+1)$
6. $f(x) = x^2(x+3)$
7. $f(x) = (2x+1)(3x+4)$
8. $f(x) = (x+7)(3x+4)$
9. $f(x) = (2x+1)^2$
10. $f(x) = \left(x^2+3\right)\left(3x^3+1\right)$
11. $f(x) = \left(3x^3+1\right)^2$
12. $f(x) = \left(x^2+3x\right)^2$

Find $f'(x)$:

13. $f(x) = \dfrac{1}{x}$
14. $f(x) = x^{-3}$
15. $f(x) = \sqrt[3]{x}$
16. $f(x) = \sqrt[4]{x}$
17. $f(x) = x^{-3}$
18. $f(x) = \dfrac{x^2-1}{x^4}$
19. $f(x) = (\sqrt[4]{x})^3 + \sqrt{x}$
20. $f(x) = x^{-2}(x^2+4x^3)$
21. $f(x) = x^{1/3}(x^2-x)$
22. $f(x) = \dfrac{x^4-1}{x^2}$
23. $f(x) = \dfrac{1}{5x+1}$
24. $f(x) = \dfrac{1}{x^2+1}$
25. $f(x) = \dfrac{x}{5x+1}$
26. $f(x) = \dfrac{x-1}{x+1}$
27. $f(x) = \dfrac{3.5x+2.7}{x^2+1.1}$
28. $f(x) = \dfrac{x^2}{x+1}$
29. $f(x) = \dfrac{x^2+3x}{x+1}$
30. $f(x) = \dfrac{x^3+2x}{x+1}$

Find the equation of the tangent line to the curve for the given value of p. Then graph the curve and the tangent line over $[-4, 4]$.

31. $y = \dfrac{1}{x}, \quad p = 1$
32. $y = \dfrac{1}{x^2}, \quad p = 1$
33. $y = \dfrac{10}{x^2+1}, \quad p = 2$
34. $y = \dfrac{x}{x^2-8}, \quad p = 3$
35. $y = \dfrac{x^2+2}{x^3-5}, \quad p = -1$
36. $y = \dfrac{3x^2+2x}{x^3+2x+1}, \quad p = -2$

Compute the following derivatives where it is assumed the function $g'(x)$ exists.

37.	$\frac{d}{dx}[x^2 g(x)]$	38.	$\frac{d}{dx}[(5x^3 - 8x)g(x)]$
39.	$\frac{d}{dx}[g(x)\sqrt{x}]$	40.	$\frac{d}{dx}[g(x)]^2$
41.	$\frac{d}{dx}\frac{x^2}{g(x)}$	42.	$\frac{d}{dx}\frac{g(x)+2}{x^3 - g(x)}$
43.	$\frac{d}{dx}[x^{3/2}g(x)]$	44.	$\frac{d}{dx}\frac{[xg(x)]}{x^3 - g(x)}$

45. The demand function for selling a particular brand of gloves is given by

$$p(x) = \frac{50}{2 + \sqrt{0.1x}}$$

(a) What price will be charged for a pair of gloves if the demand is $x = 400$?

(b) The function $R(x) = xp(x)$ is the revenue function for selling this particular brand of gloves. What is the revenue function in this case?

(c) In business settings, the derivative $R'(x)$ is often called the *marginal revenue.* What is the marginal revenue for the revenue function in (b)?

(d) What is the marginal revenue of selling 400 pairs of gloves? Give an interpretation of $R'(400)$.

46. Suppose the total cost of making x pairs of gloves is given by $C(x) = 0.02x^2 - 0.2x + 1.5$ in dollars.

(a) What is the cost of making 400 pairs of gloves? What is the average cost per pair of glove when $x = 400$?

(b) In business settings, the derivative $C'(x)$ is often called the *marginal cost.* What is the marginal cost for $C(x)$?

(c) What is the marginal cost of making 400 pairs of gloves? Give an interpretation of $C'(400)$.

47. Profit is defined as $P(x) = R(x) - C(x)$ where $R(x)$ is the revenue function and $C(x)$ is the cost function.

(a) What is the profit function $P(x)$ for producing the gloves discussed in exercises 45 and 46.?

(b) What is the profit in making and selling $x = 400$ pairs of gloves? $x = 500$ pairs of gloves?

(c) What is the marginal profit (i.e., the derivative $P'(x)$) in producing x pairs of gloves? Find $P'(500)$ and give the correct units and an interpretation?

48. In Quantum Mechanics the force between two gas molecules has two components. There is an attractive force proportional to r^{-7} and a repulsive force proportional to r^{-13}, where r is the distance between molecules. In the questions below, assume that the force acting between the two gas molecules is

$$F(r) = 0.01r^{-7} - 0.001r^{-13}$$

(a) Find $F'(r)$.

(b) What is the value of $F(r)$ and $F'(r)$ at $r = 0.5$? Which force is dominating at this value for r?

(c) If we increase r a little past $r = 0.5$, will $F(r)$ increase or decrease in value?

(d) **Grapher:** Graph $F(r)$ and $F'(r)$ for r in $[0.4, 0.8]$. At approximately what value for r do the effects of the two forces cancel one another?

49. In this exercise, we use the product rule to find the derivative of $f(x) = \sqrt{x}$

(a) Explain why $f(x) \cdot f(x) = x$ and then use the product rule to show that
$$2f(x)\, f'(x) = 1 \tag{2.8}$$

(b) Substitute $\sqrt{x}$ for $f(x)$ in (2.8) and then solve for $f'(x)$.

50. Show that if n is a positive integer, then
$$\frac{d}{dx}x^{-n} = -nx^{-n-1}$$
Do so by applying the derivative to both sides of
$$x^n x^{-n} = 1$$
and then solving for $\frac{d}{dx}x^{-n}$. (i.e., assume that $\frac{d}{dx}x^n = nx^{n-1}$ is known for positive integers n).

51. Show that if k is a constant, then
$$\frac{d}{dx}(kg(x)) = k\frac{d}{dx}g(x)$$
by applying the product rule to the product $k \cdot g(x)$.

52. * Assume that we already know that
$$\frac{d}{dx}x^n = nx^{n-1} \tag{2.9}$$
for some positive integer n. Use the product rule and (2.9) to show that
$$\frac{d}{dx}x^{n+1} = (n+1)\, x^n \tag{2.10}$$
(Hint: Write x^{n+1} as $x^n \cdot x$)

Remark 1 *The process in exercise 52 is known as a* proof by induction. *Since $x^0 = 1$ and since $\frac{d}{dx}1 = 0$, we know that the power rule is true when $n = 0$. In our proof by induction, we showed that if (2.9) is true, then so also is (2.10). That is, we showed that true for n implies true for $n + 1$. Thus, if the power rule is true for $n = 0$, then it must also be true for $n = 1$. And if the power rule is true for $n = 1$, then it must also hold for $n = 2$, and so on. We thus conclude that the power rule holds for all non-negative integers, n.*

2.3 The Chain Rule

Compositions of Functions

Although the derivative rules we have learned so far will allow us to compute the derivative of any polynomial, they are not necessarily the most practical means to do so. To illustrate, consider that application of the power rule to

$$r(x) = \left(x^2+1\right)^{20}$$

requires the 21 term expansion implied by Pascal's triangle.

Instead, it is more practical to consider $r(x)$ to be a composition of functions, where the *composition* of f with g, which is denoted by $f \circ g$, is defined to be

Composition means using the output from one function as the input to another function.

$$(f \circ g)(x) = f(g(x))$$

That is, the output from $g(x)$ is used as the input to $f(x)$, which is to say that

$$(f \circ g)(x) = f(\text{input})$$

where the input is $g(x)$. For example, if $f(x) = x^{20}$ and $g(x) = x^2+1$, then

$$(f \circ g)(x) = f(\text{input}) = (\text{input})^{20}$$

where the input is x^2+1. Replacing the input by x^2+1 thus yields

$$(f \circ g)(x) = (\text{input})^{20} = \left(x^2+1\right)^{20}$$

EXAMPLE 1 What is the composition $f \circ g$ of $f(x) = x^3+3x+1$ and $g(x) = x^4+9$?

Solution: First, let us notice that we can write f as

$$f(\text{input}) = (\text{input})^3 + 3(\text{input}) + 1$$

Since $g(x) = x^4+9$, we now replace the "input" by x^4+9, thus yielding

$$f(\text{input}) = \left(x^4+9\right)^3 + 3\left(x^4+9\right) + 1$$

Since the input to f is the output from g, we can write $f(\text{input}) = f(g(x)) = (f \circ g)(x)$. Thus, we have

$$(f \circ g)(x) = \left(x^4+9\right)^3 + 3\left(x^4+9\right) + 1$$

EXAMPLE 2 Write the function $r(x) = \left(x^3+2\right)^{14}$ as a composition of two functions.

Solution: Although there are many ways to do so, notice that if we let $f(x) = x^{14}$, then we can write

$$f(\text{input}) = (\text{input})^{14}$$

Thus, if let $g(x) = x^3 + 2$, then the input is $x^3 + 2$ and we have

$$f(\text{input}) = \left(x^3 + 2\right)^{14}$$

Since the input to f is the output from g, we can write $f(\text{input}) = f(g(x)) = (f \circ g)(x)$. Thus,

$$(f \circ g)(x) = \left(x^3 + 2\right)^{14} = r(x)$$

and we can conclude that $r = f \circ g$ where $f(x) = x^{14}$ and $g(x) = x^3 + 2$.

Check your Reading *If* $f(x) = \left(x^2 + 2\right)^4$ *and* $f(\text{input}) = (\text{input})^4$, *then what is* "input"?

The Chain Rule

The graph of a composition is of the form $y = f(g(x))$, or equivalently, $y = f(u)$ where $u = g(x)$. Differential notation can then be used to motivate a rule for differentiating compositions, in that intuitively we can write

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \tag{2.11}$$

Translating from differential notation to operator notation then yields

$$\frac{dy}{dx} = \frac{d}{dx}f(u), \qquad \frac{dy}{du} = f'(u), \qquad \textit{and} \qquad \frac{du}{dx} = g'(x)$$

As a result, (2.11) can be rewritten in operator form as

$$\frac{d}{dx}f(u) = f'(u)\,g'(x)$$

and since $u = g(x)$, this in turn can be written as

$$\frac{d}{dx}f(g(x)) = f'(g(x))\,g'(x)$$

As will be shown later, we thus have

Chain Rule: The derivative of a composition is given by

$$\frac{d}{dx}f[g(x)] = f'[g(x)]\;\frac{d}{dx}g(x)$$

When using the chain rule, however, we often write it in the form

$$\frac{d}{dx}f(\text{input}) = f'(\text{input})\;\frac{d}{dx}\text{input} \tag{2.12}$$

where the input is $g(x)$.

EXAMPLE 3 Find $r'(x)$ for $r(x) = \left(x^2+1\right)^{20}$

Solution: We can write $r(x)$ in the form

$$r(x) = (\text{input})^{20}$$

where the input is equal to x^2+1. The chain rule (2.12) says that

$$\frac{d}{dx}(\text{input})^{20} = 20\,(\text{input})^{19}\ \frac{d}{dx}\text{input}$$

Replacing "input" with x^2+1 and differentiating leads to

$$\begin{aligned} r'(x) &= 20\left(\, x^2+1 \,\right)^{19}\ \frac{d}{dx}\left(\, x^2+1 \,\right) \\ &= 20\left(x^2+1\right)^{19}\ (2x) \\ &= 40x\left(x^2+1\right)^{19} \end{aligned}$$

Certainly, most of us would prefer the computation in example 1 over the differentiation of the 21 term expansion of $r(x) = \left(x^2+1\right)^{20}$. Indeed, the chain rule is one of the most powerful techniques in calculus.

EXAMPLE 4 Find $f'(x)$ for the function $f(x) = \left(x^3+2x\right)^5+1$.

Solution: To do so, we write $f(x)$ in the form

$$f(x) = (\text{input})^5 + 1$$

where the input is x^3+2x. The chain rule says that

$$\frac{d}{dx}\left[(\text{input})^5+1\right] = \left[5\,(\text{input})^4+0\right]\ \left(\frac{d}{dx}\text{input}\right)$$

which means that

$$\begin{aligned} f'(x) &= 5\,(\text{input})^4\ \frac{d}{dx}\ (\text{input}) \\ &= 5\left(x^3+2x\right)^4\ \frac{d}{dx}\left(x^3+2x\right) \\ &= 5\left(x^3+2x\right)^4\ \left(3x^2+2\right) \end{aligned}$$

Check your Reading *Would you rather find $f'(x)$ in example 4 using the product rule to evaluate*

$$f'(x) = \frac{d}{dx}\left[\left(x^3+2x\right)\left(x^3+2x\right)\left(x^3+2x\right)\left(x^3+2x\right)\left(x^3+2x\right)+1\right]$$

A proof of the chain rule is actually rather straightforward. If g is differentiable at x, then $g(x+h) = g(x) + g'(x)h + o(h)$, so that

$$f(g(x+h)) = f[g(x) + g'(x)h + o(h)]$$

If we let $\mathbf{X} = g(x)$ and $\mathbf{H} = g'(x)h + o(h)$, then

$$f(g(x+h)) = f[g(x) + g'(x)h + o(h)] = f(\mathbf{X}+\mathbf{H})$$

However, supposing that f is differentiable at X implies that

$$f(\mathbf{X}+\mathbf{H}) = f(\mathbf{X}) + f'(\mathbf{X})\mathbf{H} \quad + o(\mathbf{H})$$

Since $\mathbf{H} = g'(x)h + o(h)$, we thus have the expansion

$$f(\mathbf{X}+\mathbf{H}) = f(\mathbf{X}) + f'(\mathbf{X})g'(x)h \quad + f'(\mathbf{X})o(h) + o(\mathbf{H})$$

However, $f(\mathbf{X}+\mathbf{H}) = f(g(x+h))$and $\mathbf{X} = g(x)$, so that

$$f(g(x+h)) = f(g(x)) + f'(g(x))g'(x)\ h \quad + \ldots$$

The derivative of $f(g(x))$ is the coefficient of h, which is $f'(g(x))g'(x)$.

As a result, the chain rule applies to all differentiable functions. For example, the chain rule also applies to algebraic functions, where an *algebraic function* is a function defined using only integer powers, integer roots, and arithmetic.

EXAMPLE 5 Find $f'(x)$ for $f(x) = \sqrt{x^2+1}$.

Solution: First, we write $f(x)$ as

$$f(x) = \left(x^2+1\right)^{1/2}$$

Consequently, we have

$$f'(x) = \frac{d}{dx}(\text{input})^{1/2} = \frac{1}{2}(\text{input})^{-1/2}\ \frac{d}{dx}(\text{input})$$

where the "input" is x^2+1. It follows that

$$\begin{aligned} f'(x) &= \frac{1}{2}\left(x^2+1\right)^{-1/2}\ \frac{d}{dx}\left(x^2+1\right) \\ &= \frac{1}{2}\left(x^2+1\right)^{-1/2}\ 2x \\ &= x\left(x^2+1\right)^{-1/2} \end{aligned}$$

EXAMPLE 6 Find $f'(x)$ for

$$f(x) = \frac{1}{\sqrt[3]{x^2+1}}$$

Solution: We first rewrite the function as

$$f(x) = \left(x^2+1\right)^{-1/3}$$

As a result, we have

$$f'(x) = \frac{d}{dx}(\text{input})^{-1/3}$$

which according to the chain rule simplifies to

$$f'(x) = \frac{-1}{3}(\text{input})^{-4/3}\ \frac{d}{dx}(\text{input})$$

Substituting x^2+1 for the input then yields

$$\begin{aligned} f'(x) &= \frac{-1}{3}\left(x^2+1\right)^{-4/3}\ \frac{d}{dx}\left(x^2+1\right) \\ &= \frac{-1}{3}\left(x^2+1\right)^{-4/3}\ 2x \end{aligned}$$

which simplifies to $f'(x) = \frac{-2}{3}x\left(x^2+1\right)^{-4/3}$.

Check your Reading *Write $f'(x) = \frac{-2}{3}x\left(x^2+1\right)^{-4/3}$ in terms of radicals.*

The Chain and Product Rules Together

The chain and product rules often occur together. Indeed, the derivative of a quotient

$$\frac{d}{dx}\frac{f(x)}{g(x)}$$

can be considered the derivative of a product of $f(x)$ with the composition $[g(x)]^{-1}$. That is,

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{d}{dx}f(x)[g(x)]^{-1}$$

In exercise 44 this will be used to prove the quotient rule. However, for now we will simply illustrate it with an example.

EXAMPLE 7 Evaluate the derivative

$$\frac{d}{dx}\frac{x^2}{3x+1}$$

using a combination of the product and chain rule.

Solution: The ratio $\frac{x^2}{3x+1}$ can be written as $x^2(3x+1)^{-1}$, which can then be written as $x^2(\text{input})^{-1}$ where the input is $3x+1$. This then leads to

$$\frac{d}{dx}x^2(\text{input})^{-1} = \left(\frac{d}{dx}x^2\right)(\text{input})^{-1} + x^2\frac{d}{dx}(\text{input})^{-1}$$

Application of the chain rule thus yields

$$\frac{d}{dx}x^2(\text{input})^{-1} = \left(\frac{d}{dx}x^2\right)(\text{input})^{-1} + x^2(-1)(\text{input})^{-2}\frac{d}{dx}\text{input}$$

Replacing the input by $3x+1$ then implies that

$$\begin{aligned}\frac{d}{dx}x^2(3x+1)^{-1} &= 2x(3x+1)^{-1} - x^2(3x+1)^{-2}\frac{d}{dx}(3x+1)\\ &= 2x(3x+1)^{-1} - x^2(3x+1)^{-2}3\\ &= 2x(3x+1)^{-2} - 3x^2(3x+1)^{-2}\end{aligned}$$

Moreover, the result of the chain rule is a product of two functions. Thus, when the first derivative involves the chain rule, the computation of the second derivative often begins with the product rule.

EXAMPLE 8 Find $f''(x)$ when $f(x) = \sqrt{(x^2+1)^3}$

Solution: Since $f(x) = \left(x^2+1\right)^{3/2}$, we have

$$f'(x) = \frac{d}{dx}(\text{input})^{3/2} = \frac{3}{2}(\text{input})^{1/2}\ \frac{d}{dx}\text{input}$$

where the input is x^2+1. As a result, the first derivative is

$$f'(x) = \frac{3}{2}\left(x^2+1\right)^{1/2}\ (2x) = 3x\left(x^2+1\right)^{1/2}$$

Since the chain rule produced a product, the second derivative begins with the product rule,

$$f''(x) = \left(\frac{d}{dx}(3x)\right)\left(x^2+1\right)^{1/2} + (3x)\frac{d}{dx}\left(x^2+1\right)^{1/2}$$

Application of the chain rule then yields

$$f''(x) = 3\left(x^2+1\right)^{1/2} + 3x\frac{1}{2}\left(x^2+1\right)^{-1/2}\ (2x)$$

which simplifies to

$$f''(x) = 3\sqrt{x^2+1} + \frac{3x^2}{\sqrt{x^2+1}}$$

Since the common denominator is $\sqrt{x^2+1}$, we obtain

$$\begin{aligned}f''(x) &= \frac{3\left(x^2+1\right)^{1/2}\left(x^2+1\right)^{1/2}}{\sqrt{x^2+1}} + \frac{3x^2}{\sqrt{x^2+1}}\\ &= \frac{3\left(x^2+1\right)+3x^2}{\sqrt{x^2+1}}\\ &= \frac{6x^2+3}{\sqrt{x^2+1}}\end{aligned}$$

Exercises:

Find the derivative of each function below. The letters k and R denote constants when they occur.

1. $f(x) = (2x+1)^2$	2. $f(x) = (3x+1)^3$	3. $f(x) = \left(x^2+1\right)^4$
4. $f(x) = (x^3+2)^7$	5. $f(x) = (x^4+1)^{10}$	6. $p(x) = (3x+1)^9$
7. $f(x) = (4x^2+9)^5$	8. $g(x) = \left(x^5+1\right)^3$	9. $f(x) = \sqrt{x^2-1}$
10. $f(x) = \sqrt{x^2+2x}$	11. $f(x) = \sqrt[3]{x^3-1}$	12. $h(x) = \sqrt[3]{x^3+3}$
13. $f(x) = \sqrt[3]{x^4+3x+2}$	14. $g(x) = (\sqrt{x}+1)^6$	15. $f(x) = \sqrt[4]{0.5x^2+1.5}$
16. $f(x) = (kx+1)^2$	17. $f(x) = \left(x^2+k^2\right)^2$	18. $f(x) = \sqrt{R^2-x^2}$
19. $f(x) = \sqrt{kx+1}$	20. $f(x) = \sqrt{x+\sqrt{x}}$	21. $f(x) = \sqrt{1+\sqrt{x}}$

Find $f''(x)$ for each of the following and simplify completely (see 4-12 for $f'(x)$).

22. $f(x) = (x^3+2)^7$	23. $f(x) = (x^4+1)^{10}$	24. $p(x) = (3x+1)^9$
25. $f(x) = (4x^2+9)^5$	26. $f(x) = \left(x^5+1\right)^3$	27. $f(x) = \sqrt{x^2-1}$
28. $f(x) = \sqrt{x^2+2x}$	29. $g(x) = \sqrt[3]{x^3-1}$	30. $h(x) = \sqrt[3]{x^3+3}$

Find $f'(x)$ in terms of $g(x)$ and $g'(x)$.

31. $f(x) = (x^2+g(x))^2$	32. $f(x) = \sqrt{g(x)+20}$
33. $f(x) = x^2 g(5x)$	34. $f(x) = g(3x^2-5)$
35. $f(x) = g(g(x))$	36. $f(x) = g(g(x^2))$

37. In each of the following, use the quotient rule in part *i* and the chain rule in part *ii.* The result should be the same in each case.

(a) *i* $\dfrac{d}{dx}\dfrac{1}{x+1}$ *ii* $\dfrac{d}{dx}(x+1)^{-1}$

(b) *i* $\dfrac{d}{dx}\dfrac{1}{x^2+4}$ *ii* $\dfrac{d}{dx}\left(x^2+4\right)^{-1}$

(c) *i* $\dfrac{d}{dx}\dfrac{1}{x^3+1}$ *ii* $\dfrac{d}{dx}\left(x^3+1\right)^{-1}$

38. Suppose that $f(x) = \left(x^2+3\right)^3$.

(a) Find $f'(x)$ using the chain rule.
(b) Find $f'(x)$ by expanding the sum and applying the power rule term-by-term.
(c) Expand (a) so that it is the same as (b).

39. An object moving in a straight line has a velocity of $v(t) = 100 - 9.8t$. Moreover, the *Kinetic Energy* of a moving object is

$$T = \frac{1}{2}mv^2$$

where m is the mass and v is the velocity of the object. What is $\frac{dT}{dt}$ and what does it represent?

40. The *potential energy* of an object in a certain gravitational field is given by

$$U = \frac{-1}{4.5\,r}$$

where r is the distance from the object to the center of the gravitational source. If the object is rising vertically in the field so that its height at time t is $r = t^{2/3}$, then what is $\frac{dU}{dt}$?

41. Suppose that the CO_2 concentration $c(t)$ in the atmosphere in parts per million (ppm) at time t years after 1939 (i.e., $t = 0$ in the year 1939) is defined explicitly by

$$c(t) = 1.44t + 280$$

Suppose also that the global temperature increase $g(c)$ in degrees Fahrenheit is

$$g(c) = 0.016c - 4.48$$

where c is the CO_2 concentration in the atmosphere in ppm.

(a) Compute $c'(t)$ and $g'(c)$, and then compute their product.

(b) Use composition to write the global temperature increase g as a function of t.

(c) Since the ocean level rises one foot for every 3 degree increase in global temperature, we define

$$R(g) = \frac{1}{3}g$$

where R is the change in ocean level caused by g. Write R as a function of t.

(d) Compute $R'(t)$. Explain why it is a third of the value of the product in (a).

42. An object of constant mass m kg with a position at time t of $r(t) = 50t - 4.9t^2$ has a velocity at time t of $v(t) = 50 - 9.8t$. The total energy of the object is

$$H = \frac{1}{2}mv^2 + 9.8mr$$

What is $H'(t)$? What is significant about this result?

43. In this exercise, we consider the function $f(x) = \sqrt{1 - x^2}$

(a) Show that the graph of $f(x)$ is the upper half of the unit circle. (Hint: let $y = f(x)$ and transform into an equation of a circle).

(b) What is the slope of the tangent line to the graph of $f(x)$ at an input p?

(c) What is the slope of the line through $(0, 0)$ and $\left(p, \sqrt{1 - p^2}\right)$?

(d) How are the slopes in (b) and (c) related? Why would we expect this given what we know about the graph of $f(x)$?

44. Use the chain rule and the product rule to evaluate the following derivative:

$$\frac{d}{dx} f(x)\,[g(x)]^{-1}$$

Then simplify the result into the quotient rule.

45. The parabola $Q_0(x) = 5 - \dfrac{x^2}{10}$ is the *quadratic approximation at* $x = 0$ of the upper half of the circle of radius 5, which is the graph of

$$f(x) = \sqrt{25 - x^2}$$

Answer the following to discover some of the properties of $Q_0(x)$.

(a) Find the tangent line to $f(x)$ when $x = 0$ and the tangent line to $Q_0(x)$ when $x = 0$ and show that they are the same.

(b) **Grapher:** Graph $f(x)$ and $Q_0(x)$ on the domain $[-5, 5]$. Is $Q_0(x)$ above or below $f(x)$ for x near 0?

(c) Since $f(x)$ and $Q_0(x)$ are both practically the same as their tangent line near $x = 0$, then they must also be "practically the same as each other" near $x = 0$. In your opinion, are $Q_0(x)$ and $f(x)$ practically the same over the interval $[-1, 1]$? Why or why not? Use graphs to justify your answer.

(d) Show that $f''(0)$ and $Q_0''(0)$ are the same.

46. The parabola

$$Q_3(x) = \frac{175}{64} + \frac{51}{32}x - \frac{25}{64}x^2$$

is the *quadratic approximation at* $x = 3$ of $f(x) = \sqrt{25 - x^2}$. Answer the following to discover some of the properties of $Q_3(x)$.

(a) Show that $f(x)$ and $Q_3(x)$ share the same tangent line when $x = 3$. That is, find the tangent line to $f(x)$ when $x = 3$ and the tangent line to $Q_3(x)$ when $x = 3$ and show that they are the same.

(b) **Grapher:** Graph $f(x)$ and $Q_3(x)$ on the domain $[-5, 5]$. Is $Q_3(x)$ above or below $f(x)$ for x near 3? (answering may require some zooming)

(c) Show that $f''(3)$ and $Q_3''(3)$ are the same.

47. Write to Learn: The chain rule is closely related to the composition of linear functions. Indeed, show that if

$$L(x) = mx + b \qquad \text{and} \qquad K(x) = nx + c$$

then the slope of the composition of L with K is equal to the product of the slopes. Write a short essay detailing this computation and discussing its relationship to the chain rule.

48. Write to Learn: An object in free fall in with a constant gravitational acceleration of g has a height at time t of

$$r(t) = r_0 + v_0 t - \frac{1}{2}gt^2$$

If its mass m is constant, then its *total mechanical energy* is

$$H = \frac{1}{2}mv^2 + mgr$$

What is $H'(t)$? What does this say about $H(t)$? Write a short essay explaining why we might say that total mechanical energy is *conserved.*

49. * The product rule is a special case of the chain rule. In particular, we can obtain the product rule by applying the chain rule to

$$(\, f(x) + g(x)\,)^2 - (\, f(x) - g(x)\,)^2 \qquad (2.13)$$

(a) Show that (2.13) simplifies to $4f(x)\,g(x)$.
(b) Apply the chain rule to (2.13) and simplify.

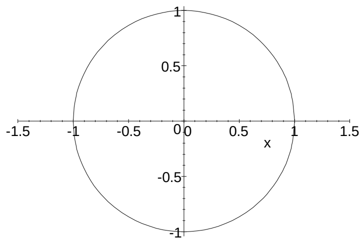
2.4 Implicit Differentiation

Implicitly Defined Functions

In analytic geometry, each point in the xy-plane is assigned an x-coordinate and a y-coordinate. An *algebraic curve* is then defined to be the set of all points whose x and y coordinates satisfy an equation of the form

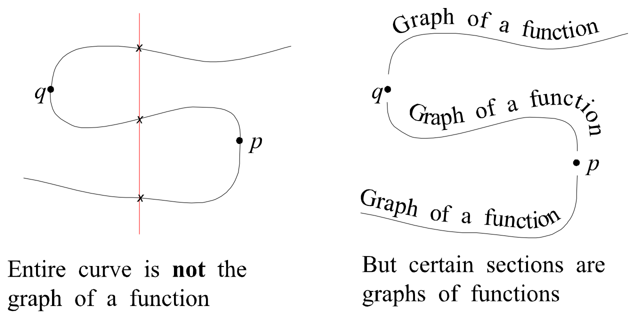
$$g(x, y) = k$$

where k is a number and where $g(x, y)$ is a polynomial in both x and y. For example, if $f(x)$ is a polynomial, then its graph $y = f(x)$ is an algebraic curve. As another example, consider that the unit circle is the set of points in the plane whose coordinates satisfy the equation $x^2 + y^2 = 1$.



4-1: The Unit Circle

However, a function has *only one* output for each input in its domain and thus, the graph of a function is intersected by a vertical line at most once. Consequently, the unit circle is **not** the graph of a function. Instead, it is one of the many algebraic curves that can be divided into sections which are graphs of functions.



4-2: An algebraic curve composed of graphs of functions.

When an algebraic curve can be divided into sections, each of which is the graph of a function, then we say that each section *implicitly defines* y as a function of x.

> **Implicitly Defined Functions**
> If a section of a curve in the xy-plane is the graph of a function, then that section of the curve defines y *implicitly* as a function of x.

In contrast, an algebraic curve expressed in the form $y = f(x)$ is said to define y *explicitly* as a function of x.

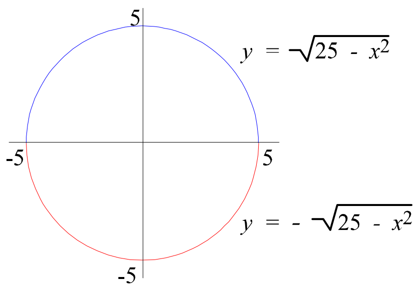
EXAMPLE 1 Determine the functions which are defined implicitly by the equation

$$x^2 + y^2 = 25 \tag{2.14}$$

Solution: If we solve for y in $x^2 + y^2 = 25$, we obtain the *explicitly defined* functions

$$y = \sqrt{25 - x^2} \qquad \text{and} \qquad y = -\sqrt{25 - x^2} \tag{2.15}$$

Thus, the curve (2.15) *implies* the existence of *two* functions, as shown below:

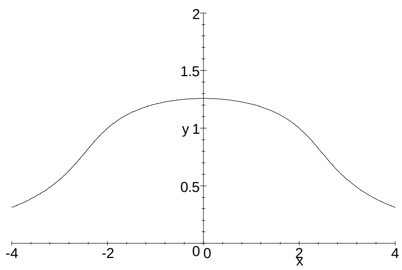


4-3: The two functions defined implicitly by $x^2 + y^2 = 25$

Unfortunately, many algebraic curves have equations that cannot be solved for y. For example, the equation

$$x^2 y = 5 - y^7 \tag{2.16}$$

is an equation of the algebraic curve shown in figure 8-4.



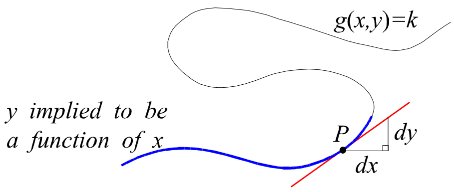
4-4: An implicitly defined function

The curve is clearly the graph of a function and thus (2.16) implicitly defines y as a function of x. However, there are no techniques in algebra which can be used to actually solve for y as a function of x.

Check your Reading *Obtain the explicitly-defined functions in (2.15) by solving for y in $x^2+y^2 = 25$.*

Implicit Differentiation

Suppose a section of a curve $g(x, y) = k$ defines y implicitly to be a function of x and let P be a point on that section of the curve. Then $\frac{dy}{dx}$ is the slope of the tangent line at P.



4-5: Slope of tangent line at P is dy/dx

To compute $\frac{dy}{dx}$ given $g(x, y) = k$, we apply the derivative operator to $g(x, y) = k$, use the chain rule to differentiate expressions in y, and then use the fact that

$$\frac{d}{dx}y = \frac{dy}{dx}$$

Often, we use the notation $y' = \frac{dy}{dx}$, and we use the term *implicit differentiation* to refer to the process for finding y' given $g(x, y) = k$.

EXAMPLE 2 Use implicit differentiation to find y' when $x^2+y^2 = 25$.

Solution: Application of the operator $\frac{d}{dx}$ results in

$$\begin{aligned} \frac{d}{dx}x^2 + \frac{d}{dx}y^2 &= \frac{d}{dx}25 \\ 2x + \frac{d}{dx}y^2 &= 0 \end{aligned} \quad (2.17)$$

Since y is implied to be a function of x, (2.17) is of the form

$$2x + \frac{d}{dx}(\text{input})^2 = 0$$

where the input is y. As a result, we have

$$2x + 2(\text{input})^1 \frac{d}{dx}(\text{input}) = 0$$

Replacing "input" by y then yields

$$\begin{aligned} 2x + 2(y)^1 \frac{d}{dx}(y) &= 0 \\ 2x + 2y\frac{dy}{dx} &= 0 \end{aligned}$$

Letting $y' = \frac{dy}{dx}$ yields $2x + 2yy' = 0$, and solving for y' yields

$$y' = \frac{-x}{y} \quad (2.18)$$

The key to implicit differentiation is the chain rule. In particular, if y is a function of x, then the derivative of an expression in y is the product of a function in y and the derivative y'. For example, in the computation above we saw that

$$\frac{d}{dx}y^2 = 2y\,y'$$

In contrast, the derivative of x^2 is simply $2x$.

Moreover, as shown in (2.18), implicit differentiation produces an equation involving x, y and the derivative y'. Such equations are so important in applications that they are given a name.

Definition 4.2 A *first order differential equation* is an equation in x, y and y' which implicitly defines y as a function of x.

That is, implicit differentiation produces a *differential equation* of a curve.

EXAMPLE 3 Find a differential equation of the curve

$$x^2y = 5 - y^7$$

Solution: We first apply $\frac{d}{dx}$, using the product rule on the left and the power rule on the right:

$$\begin{aligned} \frac{d}{dx}\left(x^2y\right) &= \frac{d}{dx}\left(5-y^7\right) \\ \left(\frac{d}{dx}x^2\right)y + x^2\frac{d}{dx}(y) &= 0 - \frac{d}{dx}(y)^7 \\ 2xy + x^2\frac{dy}{dx} &= -7\,(y)^6\,\frac{d}{dx}y \end{aligned}$$

Since $\frac{d}{dx}(y) = y'$, this simplifies to

$$2xy + x^2y' = -7y^6y' \tag{2.19}$$

We then solve for y' by isolating the terms containing y',

$$x^2y' + 7y^6y' = -2xy \tag{2.20}$$

factoring y' from the terms on the left,

$$\left(x^2+7y^6\right)y' = -2xy$$

and then dividing by $\left(x^2+7y^6\right)$. The result is the differential equation

$$y' = \frac{-2xy}{x^2+7y^6}. \tag{2.21}$$

EXAMPLE 4 Find a differential equation for the curve

$$(xy+1)^3 = x^2$$

Solution: Application of the derivative operator to both sides yields

$$\frac{d}{dx}(\text{input})^3 = \frac{d}{dx}\,x^2$$

where the input is $xy+1$. As a result, we have

$$\begin{aligned} 3\,(\text{input})^2\,\frac{d}{dx}\,(\text{input}) &= 2x \\ 3\,(xy+1)^2\,\frac{d}{dx}\,(xy+1) &= 2x \\ 3\,(xy+1)^2\left(\frac{d}{dx}\,xy\right) &= 2x \end{aligned}$$

We then apply the product rule to the derivative of xy,

$$3\,(xy+1)^2\left[\left(\frac{d}{dx}x\right)y + x\left(\frac{d}{dx}y\right)\right] = 2x$$

which results in the equation

$$3\,(xy+1)^2\,(y+xy') = 2x \qquad (2.22)$$

Check your Reading *Solve for y' in (2.22). (Hint: first divide both sides by $3\,(xy+1)^2$)*

Inverse Functions and Differential Equations

If $f\,(x)$ is differentiable, then $x = f\,(y)$ implicitly defines a collection of functions called *inverses of* f. Inverse functions will be important in chapter 6, where they will be discussed in more detail. For now, however, let us simply explore tangent lines to curves of the form $x = f\,(y)$ using implicit differentiation.

EXAMPLE 5 Find y' if $x = y^2$. Then find the equation of the tangent line to $x = y^2$ at the point $(4,2)$.

Solution: To find y', we use implicit differentiation:

$$\frac{d}{dx}x = \frac{d}{dx}y^2 \qquad \Longrightarrow \qquad 1 = 2yy'$$

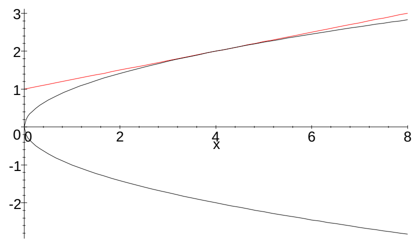
Solving for y' leads to $y' = \frac{1}{2y}$. Thus, the slope of the tangent line to $x = y^2$ at $(4,2)$ is

$$m_{\tan} = \frac{1}{2\,(2)} = \frac{1}{4}$$

and it follows that the equation of the tangent line is

$$y = 2 + \frac{1}{4}\,(x-4)$$

which simplifies to $y = \frac{1}{4}x + 1$.



4-6: $y = x/4 + 1$ is tangent to $x = y^2$

If $x = f(y)$, then implicit differentiation implies that

$$\frac{d}{dx}x = \frac{d}{dx}f(y) \qquad \Longrightarrow \qquad 1 = f'(y)\,y'$$

Thus, the differential equation of a curve of the form $x = f(y)$ is of the form

$$y' = \frac{1}{f'(y)} \tag{2.23}$$

Differential equations of the form (2.23) are very important in applications, as we will see throughout this text.

EXAMPLE 6 What is the differential equation of the curve $x = \sqrt{y}$?

Solution: To find y', we use implicit differentiation:

$$\frac{d}{dx}x = \frac{d}{dx}y^{1/2} \qquad \Longrightarrow \qquad 1 = \frac{1}{2}y^{-1/2}y'$$

Solving for y' results in

$$y' = \frac{1}{\frac{1}{2}y^{-1/2}} = 2y^{1/2}$$

Thus, the differential equation of $x = \sqrt{y}$ is

$$y' = 2\sqrt{y}$$

Check your Reading *Solve for y in $x = \sqrt{y}$. What is the result?*

Tangent Lines to Algebraic Curves

A differential equation of an algebraic curve can be used to find the slope and equation of the line tangent to the curve at a given point. In particular, to find the slope, we simply evaluate y' at a given point.

EXAMPLE 7 Find the equation of the tangent line to the circle

$$x^2 + y^2 = 25$$

at the point $(4, 3)$.

Solution: We showed in example 2 that a differential equation of the curve is

$$y' = \frac{-x}{y}$$

Substituting $x = 4$, $y = 3$ into the equation $y' = -x/y$ yields the slope

$$m = \frac{-4}{3}$$

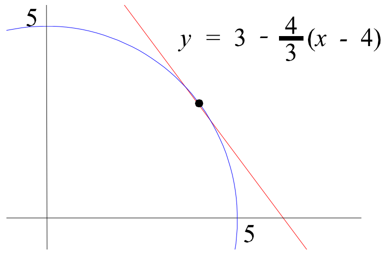
The point-slope equation of the line through (x_1, y_1) with slope m is given by

$$y = y_1 + m(x - x_1) \qquad \text{(point-slope equation)}$$

Thus, the tangent line to $x^2 + y^2 = 25$ at the point $(4, 3)$ is

$$y = 3 - \frac{4}{3}(x - 4)$$

which simplifies to $y = \frac{25}{3} - \frac{4}{3}x$.



4-7: Tangent line to $x^2 + y^2 = 25$ at $(4, 3)$

EXAMPLE 8 Find the slope and equation of the tangent line to

$$x^2 y = 5 - y^7$$

at the point $(2, 1)$.

Solution: A differential equation of the curve is given in (2.21) by

$$y' = \frac{-2xy}{x^2 + 7y^6}$$

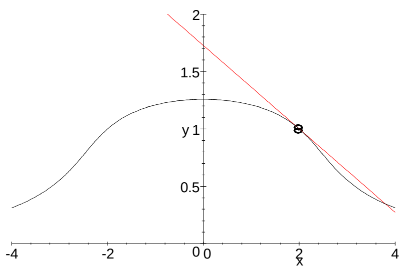
Thus, the slope of the tangent line at the point $(2, 1)$ is

$$m = \frac{-2 \cdot 2 \cdot 1}{2^2 + 7 \cdot 1^6} = -\frac{4}{11}$$

and the equation of the tangent line is

$$y = 1 - \frac{4}{11}(x - 2)$$

which is shown below in figure 8-6:



4-8: Tangent line to the graph of an implicitly defined function

Exercises:

Use implicit differentiation to find a differential equation for each curve.

1. $y = 2x + 3$
2. $y = x^2 + 1$
3. $xy^2 = 2$
4. $y^2 = x^2$
5. $x = y^3$
6. $x = y^4$
7. $x^2 + 4y^2 = 1$
8. $9x^2 + 16y^2 = 25$
9. $x^3 - 3y^2 = 2$
10. $x^2 + y^3 = 3$
11. $x^4 - y^3 - 3 = 0$
12. $y^3 - 2x^2 + 4 = 0$
13. $xy + y^2 = 2$
14. $xy^2 + x^2y = 1$
15. $x^2 + 2xy + y^2 = 1$
16. $3x^2 - 2xy + 3y^2 = 4$
17. $(x + y)^3 = 8xy$
18. $(x + y)^{10} = x + y$
19. $x^2 + y^4 = (xy + 1)^2$
20. $\left(x^2 + y^3\right)^4 = x^4 + (xy + 1)^2$
21. $(xy + 1)^3 = xy$
22. $x^2y^2 = xy$
23. $x = \sqrt{y - 1}$
24. $x = \sqrt{1 + y^2}$

Find the equation of the tangent line to the curve at the point given. You found y' in the exercises above.

25. $xy^2 = 2$ at the point $(2, 1)$.
26. $y^2 = x^2$ at the point $(1, 1)$
27. $x = y^3$ at the point $(1, 1)$
28. $x = y^4$ at the point $(1, 1)$
29. $x^3 - 3y^2 = 2$ at the point $(4.290, -5.065)$
30. $x^2 + y^3 = 3$ at the point $\left(\sqrt{2}, 1\right)$
31. $(x + y)^3 = 8xy$ at the point $(1, 1)$

32 $(x + y)^{10} = x + y$ at the point $(0, 1)$

For each of the curves in 33-40 do the following:

(a) *Find a differential equation of the curve.*

(b) *Solve for y to obtain an explicitly-defined function $y = f(x)$ which is implicitly defined by the given equation.*

(c) *Differentiate $y = f(x)$ and show that the result is the same as that in (a).*

33. $xy^2 = 2$
34. $9x^2 - 18x + 4y^2 + 8y - 23 = 0$
35. $x^2 + 4y^2 = 1$
36. $4x^2 + y^2 - 6y = -5$
37. $x^3 - 3y^2 = 2$
38. $x^2 - 4y^2 + 2x + 8y - 7 = 0$
39. $x^2 + 2xy + y^2 - 1 = 0$
40. $3x^2 - 2xy + 3y^2 = 4$

41. For every value of k, show that the functions $y = kx^2$ satisfy the differential equation

$$xy' = 2y$$

Graph $y = x^2$, $y = 2x^2$, $y = -x^2$, $y = -2x^2$ and $y = 0$ on the domain $[-1, 1]$. What do all of these curves have in common? What point do they all pass through?

42. Show that for every value of k, the curves $y = kx^3 + 1$ satisfy the differential equation

$$xy' = 3y - 3$$

Graph the solutions which correspond to $k = 0$, 1, 2, and 3 on the domain $[-1, 1]$. What do all of these curves have in common? What point do they all pass through?

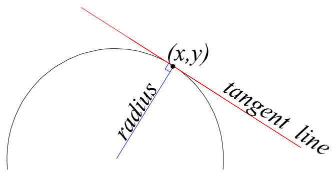
43. In analytic geometry, a curve can be defined by more than one equation (and thus, has more than one differential equation). For example, squaring both sides of $x^2 + y^2 = 25$ results in the equation

$$x^4 + 2x^2y^2 + y^4 = 625 \tag{2.24}$$

Use implicit differentiation to find the differential equation of (2.24), and then show that the slope of the tangent line at $(4, 3)$ is

$$m_{\tan} = \frac{-4}{3}.$$

44. Suppose the circle below has the equation $x^2 + y^2 = R^2$



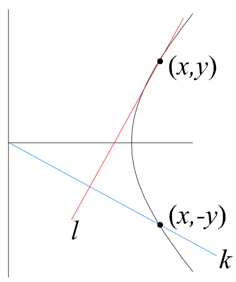
4-9: Tangent line to a circle

(a) Show that $y' = \frac{-x}{y}$
(b) Show that the slope of the radius is $\frac{y}{x}$
(c) Show that the radius and the tangent line are perpendicular

45. * Find the differential equation of the hyperbola

$$x^2 - y^2 = 1$$

and show that the line from the origin through $(x, -y)$ is perpendicular to the tangent line at a point (x, y)on the hyperbola. That is, show that line l is perpendicular to line k in the figure below:



4-10: Tangent line to a hyperbola

2.5 Rates of Change

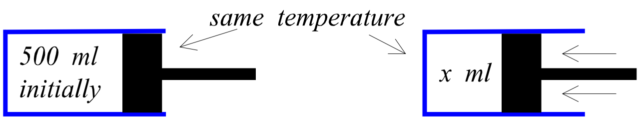
The Derivative as a Rate of Change

In this section, we further explore the derivative of a rate of change, especially as it is used in various scientific and mathematical applications. Moreover, the key to interpreting rates of change in applications is in including the *units of measurement* in the result. Also, reverting to differential notation may help.

EXAMPLE 1 A cylinder initially contains 500 ml of air at a pressure of 1 atmosphere (atm). If the air in the cylinder is compressed by a piston without changing its temperature, then Boyle's law says that the pressure P is related to the volume x by

$$P = \frac{500}{x}$$

What is $P'(250\ ml)$? What does it mean?



5-1: Boyles Law

Solution: Since we can write $P(x) = 500x^{-1}$, the derivative is

$$P'(x) = -500x^{-2}$$

As a result, when $x = 250\ ml$, we have

$$P'(250) = -500(250)^{-2} = \frac{-500}{(250)^2} = -\frac{1}{125}$$

To understand what this result means, let's change to differential notation and include units. Since x is measures in ml, dx is also in ml; and since P is in atmospheres, dP is also in atm. Thus, the pressure is changing with respect to volume at a rate of

$$\frac{dP}{dx} = \frac{-1}{125}\frac{atm}{ml}$$

when the volume is $x = 250$ ml. We interpret this to mean that for volumes close to 250 ml, the pressure increases by about 1 atm for each 125 ml decrease in volume.

EXAMPLE 2 Let y denote the concentration in moles per liter of the chemical CH_3CHO in the chemical reaction

$$CH_3CHO \longrightarrow CH_4 + CO$$

If the initial concentration is 0.3 moles per liter and the temperature is $20°C$, then it can be shown that

$$y(t) = \frac{3}{6t+10}$$

where t is time in seconds since the reaction began. What is the initial rate of the reaction?

Solution: The quotient rule implies that

$$y'(t) = \frac{(6t+10)\frac{d}{dt}3 - 3\frac{d}{dt}(6t+10)}{(6t+10)^2} = \frac{-18}{(6t+10)^2}$$

Thus, the rate of change at $t = 0$ is $y'(0)$, which is

$$y'(0) = \frac{-18}{(0+10)^2} = -0.18$$

Including units and differentials then yields our interpretation.. In particular, y is in moles/liter and t is in seconds, so that

$$\frac{dy}{dt} = -0.18 \, \frac{moles/Liter}{\text{sec}}$$

Check your Reading *What is the initial rate of the reaction in example 3?*

Rates of Change in Economics

In economics, the total cost $C(x)$ incurred in producing x units of a commodity is called a *cost function*, and the instantaneous rate of change of cost with respect to number of units produced is called the *marginal cost.* That is, marginal cost is the derivative of the cost function.

Economists often interpret marginal cost using the difference quotient approximation:

$$C'(p) \approx \frac{C(p+h) - C(p)}{h}$$

Specifically, if $h = 1$, then the difference quotient approximation reduces to

$$C'(p) \approx C(p+1) - C(p)$$

and this in turn implies that

$$C(p+1) \approx C(p) + C'(p)$$

That is, marginal cost can be interpreted to be the cost of making "the next" unit of a commodity.

EXAMPLE 3 The total cost in making x tennis rackets (in thousands of rackets) is given by $C(x) = 20 + 1.2x - 0.01x^2$ in thousands of dollars. What is the marginal cost of making 9,000 tennis rackets?

Solution: Since $C'(x) = 1.2 - 0.02x$, the marginal cost at 9,000 units is

$$C'(9) = 1.2 - 0.02(9) = 1.02$$

To interpret this result, let's first include units (C is in dollars and x is in thousands of rackets):

$$C'(9) = 1.02 \, \frac{thousands\ of\ dollars}{thousand\ rackets} = 1.02 \, \frac{dollars}{racket}$$

The discussion immediately before this example gives us a means for interpreting the result. In particular, after making 9,000 rackets, the cost of the next racket (i.e., the $9{,}001^{st}$) is about 1.02 dollars.

Also in economics, if the price p is a function of the quantity x units of a commodity demanded by a market, then p as a function of x is called a *demand function.* The revenue $R(x)$ for a demand function $p(x)$ is given by $R(x) = xp(x)$, which is the price times the number sold, and the derivative $R'(x)$ of the revenue is called the *marginal revenue* from selling x units of a commodity. The profit $P(x)$ from making and selling x units of a commodity is

$$P(x) = R(x) - C(x)$$

which is to say that profit is the difference between revenue and costs.

EXAMPLE 4 Suppose that the price p in thousands of dollars as a function of the number x of a certain automobile available for sale is

$$p(x) = 2 + 2x^{-0.877}$$

What is the marginal revenue from selling 100 automobiles?

Solution: Since Revenue = price × number sold, we have

$$R(x) = \left(2 + 2x^{-0.877}\right)x = 2x + 2x^{0.123}$$

The marginal revenue function is thus

$$R'(x) = 2 + 2(0.123)\,x^{-0.877}$$

As a result, $R'(100) = 2 + 2(0.123)(100)^{-0.877} = 2.00433.$ Revenue is measured in thousands of dollars, so that the marginal revenue from selling 100 automobiles is

$$R'(100) = \$20,043 \textit{ per car sold}$$

Check your Reading *About how much revenue can we expect from selling the 101st automobile in example 4?*

Rate Problems

Many applications involve functions of the form $y = f(x)$ in which x and y are *implicitly defined* as functions of another variable t. In such applications, a curve $y = f(x)$ is actually a function of the form $y(t) = f[x(t)]$, and the derivative $y'(t)$ is of the form

$$\frac{dy}{dt} = \frac{d}{dt} f(\text{input}) \tag{2.25}$$

where the input is $x(t)$. The chain rule in this case says that

$$\frac{d}{dt} f(\text{input}) = f'(\text{input}) \frac{d}{dt}(\text{input})$$

where again, the input is $x(t)$.

EXAMPLE 5 Suppose that $y = x^3 + 2x$, and suppose that x is a function of t. Find $\frac{dy}{dt}$ when $x = 3$ and $\frac{dx}{dt} = 7$.

Solution: To begin with, we actually have

$$y(t) = [x(t)]^3 + 2[x(t)]$$

As a result, the derivative $\frac{dy}{dt}$ is of the form

$$\frac{dy}{dt} = \frac{d}{dt}\left([\text{input}]^3 + 2[\text{input}]\right)$$

where the input is $x(t)$. Therefore, we have

$$\frac{dy}{dt} = \left(3[\text{input}]^2 + 2\right)\frac{d}{dt}(\text{input})$$

so that replacing the input by $x(t)$ leads to

$$\frac{dy}{dt} = \left(3x^2 + 2\right)\frac{dx}{dt}$$

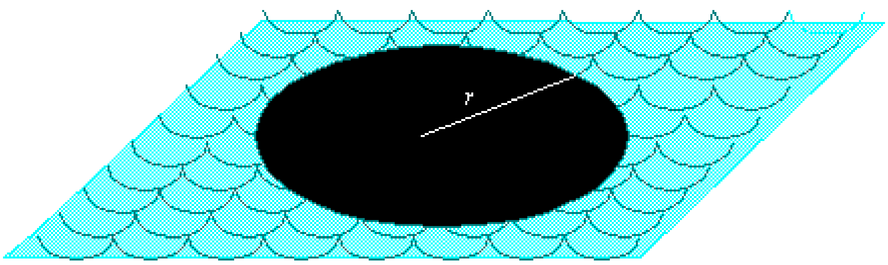
When $x = 3$ and $\frac{dx}{dt} = 7$, we thus have

$$\frac{dy}{dt} = \left(3 \cdot 3^2 + 2\right) \cdot 7 = 203$$

Example 5 illustrates that we must *wait* to substitute numerical values *until the final step.*

Example 5 is also an example of a *rate problem,* which is an application in which x and y are functions of a third variable t, and the goal is to determine $\frac{dy}{dt}$ given information about x, y, and $\frac{dx}{dt}$.

EXAMPLE 6 Suppose a circular oil slick is spreading at a rate of 2 miles per hour when the radius of the oil slick is 10 miles. How fast is the area of the oil slick increasing?



5-2: An oil slick with radius r spreading at a rate of 2 miles per hour

Solution: If $A(t)$ denotes the area of the oil slick at time t and if $r(t)$ denotes its radius at time t, then

$$A(t) = \pi[r(t)]^2$$

As a result, the chain rule says that

$$\frac{dA}{dt} = \pi\frac{d}{dt}[\text{input}]^2$$

where the input is $r(t)$. Thus, we have

$$\frac{dA}{dt} = 2\pi\,[\text{input}]\,\frac{d}{dt}(\text{input}) = 2\pi r(t)\,\frac{d}{dt}r(t)$$

which in differential notation implies that

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

Since $dr/dt = 2$ miles per hour when $r = 10$ miles, the rate of change of the area of the slick when $r = 10$ is

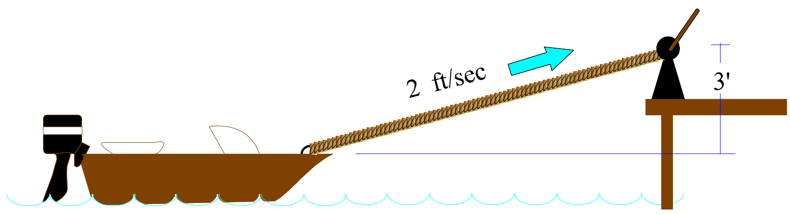
$$\frac{dA}{dt} = 2\pi\cdot 10\cdot 2 = 40\pi = 125.6637\,\frac{(miles)^2}{hour}$$

Check your Reading *What does $A(t)$ reduce to in example 6 when $r(t) = \sqrt{40t}$?*

More Rate Problems

Many rate problems require the use of geometric concepts, such as the Pythagorean theorem, similar triangles, and volume formulas. Also, rate problems may occur as equations of curves rather than graphs of functions.

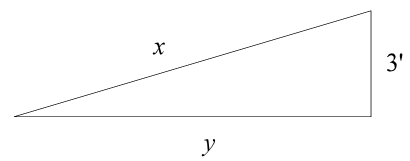
EXAMPLE 7 A boat is pulled toward a dock by a rope attached to its bow.



5-3: A boat being pulled toward a dock

If the top of the winch is 3 feet higher than the bow and if the rope is pulled in at a rate of 2 feet per second, how fast is the boat moving when it is 10 feet from the dock.

Solution: Let us let x denote the length of the rope from the winch to the bow and let y denote the distance of the bow from the deck. Then x, y, and the height 3 feet form a right triangle.



5-4: Right triangle diagram of figure 5-3

As a result, $y^2 + 9 = x^2$. Application of $\frac{d}{dt}$ to both sides of the equation yield

$$\frac{d}{dt}\left(y^2 + 9\right) = \frac{d}{dt}x^2$$

Since x and y are *both* functions of a different variable t, we use the chain rule to differentiate expressions in both x and y:

$$\frac{d}{dt}(output)^2 + \frac{d}{dt}9 = \frac{d}{dt}(output)^2$$

where the output is y and the input is x. As a result, we have

$$\begin{aligned} 2output \cdot \frac{d}{dt}(output) &= 2output \cdot \frac{d}{dt}(output) \\ 2y\frac{d}{dt}(y) &= 2x\frac{d}{dt}(x) \\ y\frac{dy}{dt} &= x\frac{dx}{dt} \end{aligned}$$

When $y = 10$, then $100 + 9 = x^2$, which means that $x = \sqrt{109}$. Since $\frac{dx}{dt} = 2$ feet per second, we have

$$\begin{aligned} 10\frac{dy}{dt} &= \left(\sqrt{109}\right)\frac{dx}{dt} \\ \frac{dy}{dt} &= \frac{\sqrt{109}}{10} \cdot 2\ \frac{ft}{\sec} = 2.088\ \frac{ft}{\sec} \end{aligned}$$

Exercises:

Find $\frac{dy}{dt}$ at given input p. Assume that y is in feet and t is in seconds, and include units in your result.

1. $y = t^2 + 3, \quad p = 2$
2. $y = 0.3t^2 + 2.3t + 1, \quad p = 0.7$
3. $y = 3t + 2\ , \quad p = 2$
4. $y = t^3 + 2t, \quad p = 0.5$
5. $y = \frac{1}{t^2},\ p = 2$
6. $y = \frac{1}{\sqrt{t}},\ p = 1$
7. $y = (3t-2)^{-1},\ p = 1$
8. $y = \left(t^2 - 3\right)^{-2},\ p = 2$
9. $y = \sqrt{6t+7},\ p = 3$
10. $y = \sqrt{t^2+3},\ p = 2$

In exercises 11 through 15, find $\frac{dy}{dt}$ using the given information.

11. $y = 6\pi r^2,\ \frac{dr}{dt} = -0.2,\ r = 1.5$

12. $y = 2x^2 - 5x,\ \frac{dx}{dt} = 1.5,\ x = 2$

13. $y + x^3 = x,\ \frac{dx}{dt} = -1,\ x = 1$

14. $y - (2x+1)^3 = 24,\ \frac{dx}{dt} = 2.1,\ x = 2$

Exercises 15 - 28 ask you to determine a rate of change and interpret the result.

15. A spherical balloon has a volume in cubic inches of

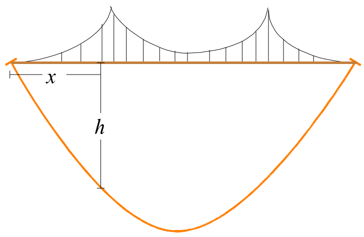
$$V(r) = \frac{4}{3}\pi r^3$$

where r is the radius of the balloon in inches. Find $\frac{dV}{dr}$ when $r = 2$ inches. Interpret the result as a rate of change.

16. A bridge over a canyon has a height h in feet of

$$h(x) = 0.13x(50 - x)$$

where x is the distance in feet from one side of the canyon to a point on the bridge.



5-5: A bridge across a canyon

Find $\frac{dh}{dx}$ when $x = 25$ and interpret the result.

17. The average length L in centimeters of a fish chosen at random from a particular population is given by

$$L(t) = 0.31 + 0.29t - 0.0037t^2$$

where t is time in days since hatching. What is $\frac{dL}{dt}$ when $t = 10$? Interpret the result as a rate of change.

18. On a certain pond, the thickness of the layer of ice on the surface of the pond satisfies

$$y(t) = \sqrt{t}$$

where y is in inches and t is in hours. How fast is the layer of ice increasing after 3 hours? After 5 hours?

19. Let y denote the concentration in moles per liter of the chemical HI in the chemical reaction

$$2HI \longrightarrow H_2 + I_2$$

If the initial concentration is 0.01 moles per liter and the temperature is $20°C$, then it can be shown that

$$y(t) = \frac{1}{-0.045t + 100}$$

where t is time in seconds since the reaction began. What is the initial rate of the reaction?

20. The enzymatic activity $E(T)$ of a certain enzyme at temperature T in degrees Celsius is given by

$$E(T) = 11.760 + 19.1156T - 0.22893T^2$$

where enzymatic activity is measured in activity units **(U).**[1] What is the rate of change $\frac{dE}{dT}$ of enzymatic activity with respect to temperature when the temperature is $20°C$ (which is considered to be "room temperature")

21. Frazier furs are grown for six to ten years before being harvested for sale as Christmas trees. Suppose the total cost of growing x Frazier furs for six years is given by $C(x) = 2100 + 1.5x + 0.001x^2$ in dollars.

[1] An activity unit **U** is the amount of enzyme which will catalyse 1 micromole of a substrate per minute under standard conditions.

(a) What is $C'(x)$? What does it represent?
(b) What is the marginal cost of growing 1,000 Frazier furs for six years? Be sure to include the units.
(c) If 1,000 Frazier fur trees are grown for six years, then about how much would it cost to grow the "next" Frazier fur?

22. In exercise 21, what is the marginal cost of growing 750 Frazier furs for six years? Be sure to include the units.

23. The price p per Frazier fur given a demand for x six year old Frazier furs is given by $p(x) = -0.01x + 35$.

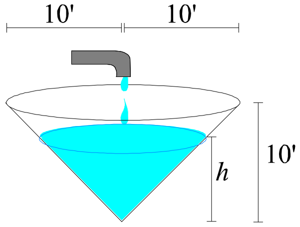
(a) What is the revenue function $R(x)$ for selling six year old Frazier firs?
(b) What are the units of $R'(1000)$?
(c) What is the marginal revenue of selling 1,000 six year old Frazier firs? Give an interpretation of $R'(1000)$.
(d) What is the marginal revenue of selling 750 six year old Frazier firs? Give an interpretation of $R'(750)$.

24. Profit is defined to be $P(x) = R(x) - C(x)$ where $R(x)$ is the revenue function and $C(x)$ is the cost function.

(a) What is the profit function $P(x)$ for growing and selling six year old Frazier firs (see 21 and 23)?
(b) What is $P(1000)$, the profit of growing and selling 1000 six year old Frazier firs?
(c) What are the units of $P'(1000)$?
(d) What is the marginal profit of growing and selling 1,000 six year old Frazier firs? Give an interpretation of $P'(1000)$.

25. Water is being pumped into cylindrical tank which has a radius of 4 feet. The tank is initially empty and the volume of water pumped into the tank at time t in minutes since pumping began is given by $V(t) = 2t - 0.01t^2$ cubic feet. How fast is the height h of the water rising after 100 minutes? How would you interpret this result?

26. Water is being pumped into a tank which is initially empty at a constant rate of 2 cubic feet per minute. The tank is in the shape of a right circular cone that is 10 feet high and 20 feet wide across its top. (Volume of a cone $= \frac{1}{3}\,\pi$ radius^2 $\cdot$ height).



5-6: Water pumped into a cone

How fast is the height h of the water rising after 10 minutes?

27. The acceleration of an object due to the earth's gravitational field is

$$a(r) = \frac{-k}{r^2}$$

where $k = 95,194.14 \; \frac{mi^3}{\sec^2}$ and r is the distance from the center of the earth (assuming the earth is a sphere with uniform density).

(a) What is the acceleration due to gravity at the earth's surface? (mean radius of earth = 3963 miles) What does it become when it is converted into feet per $\sec^2$? (1 mile = 5280 feet)

(b) What is the acceleration due to gravity at 10 miles above the earth's surface (in miles per $\sec^2$)?

(c) What is the rate of change of the acceleration at the earth's surface? How quickly does the acceleration change as an object leaves the earth's surface? How does this relate to (a) and (b)?

28. Repeat exercise 27 for the acceleration due to gravity on the surface of Mars where $k = 36,292.76 \; \frac{mi^3}{\sec^2}$ (Mars has a mean radius of 2,106 miles).

Exercises 29 - 40 are rate problems.

29. A rectangle with a length of l and a width of 2 feet has a perimeter of $P = 2l + 4$. Find the rate at which the perimeter is increasing if the length l is increasing at the rate $\frac{dl}{dt} = 3$ ft./sec.

30. Find the rate at which the area of a rectangle is increasing if its width is constant at 2 feet and if the length l is increasing at the rate of $\frac{dl}{dt} = 3$ ft./sec.

31. If the radius of a spherical balloon is increasing at 1.2 $cm/\sec$ when the radius is 12 cm, at what rate is the volume contained by the balloon increasing? (Recall $V = \frac{4}{3}\pi r^3$)

32. If the volume contained by a spherical balloon is increasing at 10 $cc/\sec$ (a cc is one cubic centimeter) when the radius is 15 cm, at what rate is the radius increasing?

33. The surface area of a melting ice cube is decreasing at a rate of 0.3 $\text{in}^2/\min$ when its dimensions are $2'' \times 2'' \times 2''$. At what rate is the length of a side of the cube decreasing?

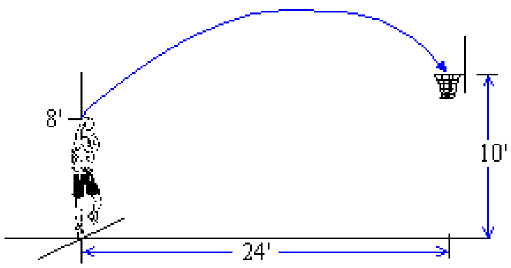
34. The surface area of a melting ice cube is decreasing at a rate of 0.3 $\text{in}^2/\min$ when its dimensions are $2'' \times 2'' \times 2''$. At what rate is its volume decreasing?

35. A boy, $4'8''$ tall, notices his shadow lengthening as he walks away from the base of a $16'$ tall street lamp. If he walks away from the street lamp at a rate of 3 $ft./\sec$, at what rate is the tip of his shadow moving when he is 30 feet from the base of the pole?

36. A girl is approaching an intersection on her bicycle and another is standing still on the other street which forms the intersection. The girl on the bicycle is riding at 15 $ft./\sec$ and the two streets are straight and at right angles to one another. At what rate are the girls approaching one another when the girl riding the bicycle is 50 $feet$ from the intersection and the other girl is 30 $feet$ from the intersection?

37. A Basketball's Velocity: An NBA player shoots a 3-pointer from 24 feet (

3 point line is at 23'9") that passes through the hoop 1.4 seconds later.



5-7: An NBA 3-pointer

It can be shown that the ball had to have traveled along a parabolic arc of the form

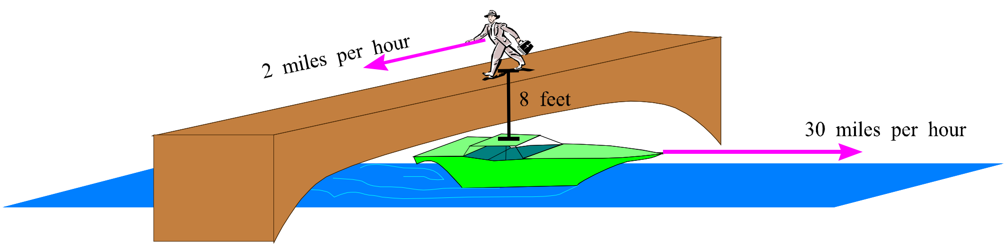
$$y = 8 + 1.39x - 0.0544x^2$$

Use the fact that the ball moved 24 feet horizontally in 1.4 seconds to determine the constant horizontal velocity $\frac{dx}{dt}$. What was the vertical velocity $\frac{dy}{dt}$ of the ball initially (i.e., when $x = 0$)? What was the vertical velocity when the ball fell through the hoop?

38. A wide receiver runs next to the right sideline of a football field at a constant rate of 27 feet per second. When he crosses the 50 yard line, a defender is crossing the 30 yard line running at a constant speed of 24 feet per second *towards him along the same sideline.* On what yard line will they meet and how fast will the distance between them be decreasing at that point?

39. * Suppose the defender in problem 38 above is on the 30 yard line, but is in the middle of the field at about 25 yards from the sideline. If the defender runs at a constant rate of 24 feet per second, what linear path must he pursue in order to do so? How fast is the distance between them decreasing when the defender and the receiver meet?

40. * A man walks across a bridge at a constant rate of 2 miles per hour when a boat traveling at a constant rate of 30 miles per hour passes directly beneath him at a right angle to his path.



5-8: Walking above a speeding boat

If the man is exactly 8 feet above the boat when the boat passes directly underneath, then how far apart are the man and the boat one minute later (assuming each follows a straight path at a constant elevation)? How fast is the distance between the man and the boat increasing at that time?

2.6 The Exponential Function

The Exponential Function

In this section, we introduce the *exponential function*, a function that is in many ways foundational to the study of calculus. In doing so, we also introduce the new constant e, which is called *Euler's constant,* as a universal constant similar in stature to the universal constant π.

To begin with, let's recall that the *compound interest* formula says that if P dollars are invested at an *annual* interest rate r compounded n times each year, then the value of the investment after t years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

For example, let's suppose that we invest \$1 at 100% interest for $t = 1$ years. The compound interest formula becomes

$$A = \left(1 + \frac{1}{n}\right)^{n}$$

where n is the number of compoundings. Moreover, if n increases, then the value of A increases as well.

once per year	$n = 1$	$\Longrightarrow$	$A = \left(1 + \frac{1}{1}\right)^1 = \$2.00$
twice per year	$n = 2$	$\Longrightarrow$	$A = \left(1 + \frac{1}{2}\right)^2 = \$2.25$
once per quarter	$n = 4$	$\Longrightarrow$	$A = \left(1 + \frac{1}{4}\right)^4 = \$2.44$
once per month	$n = 12$	$\Longrightarrow$	$A = \left(1 + \frac{1}{12}\right)^{12} = \$2.61$
once per day	$n = 365$	$\Longrightarrow$	$A = \left(1 + \frac{1}{365}\right)^{365} = \$2.71$
once per day	$n = 8,670$	$\Longrightarrow$	$A = \left(1 + \frac{1}{8760}\right)^{8760} = \$2.718$
once per second	$n = 31,536,000$	$\Longrightarrow$	$A = \left(1 + \frac{1}{31{,}536{,}000}\right)^{31{,}536{,}000} = \$2.718$

That is, as the number of compoundings per year increases, the value of the investment increases accordingly, but only up to *Euler's constant*

$$e = 2.7182818286\ldots$$

Let's suppose now that the annual interest rate is 200%, so that $r = 2$. Then

once per year	$n = 1$	$\Longrightarrow$	$A = \left(1 + \frac{2}{1}\right)^1 = \$3.00$
twice per year	$n = 2$	$\Longrightarrow$	$A = \left(1 + \frac{2}{2}\right)^2 = \$4.00$
once per quarter	$n = 4$	$\Longrightarrow$	$A = \left(1 + \frac{2}{4}\right)^4 = \$5.06$
once per month	$n = 12$	$\Longrightarrow$	$A = \left(1 + \frac{2}{12}\right)^{12} = \$6.36$
once per day	$n = 365$	$\Longrightarrow$	$A = \left(1 + \frac{2}{365}\right)^{365} = \$7.35$
once per day	$n = 8,670$	$\Longrightarrow$	$A = \left(1 + \frac{2}{8760}\right)^{8760} = \$7.387$
once per second	$n = 31,536,000$	$\Longrightarrow$	$A = \left(1 + \frac{2}{31{,}536{,}000}\right)^{31{,}536{,}000} = \$7.389$

The significance of the number 7.389... is that it is very close to

$$e^2 = (2.7182818286\ldots)^2 = 7.389056\ldots$$

That is, as the number n of compoundings approaches ∞, the value of the investment approaches e^r, where r is the annual interest rate. This discussion motivates us to define a new function.

Definition 6.1: The exponential function is defined

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

where each n in the limit is a positive integer.

In particular, e^x is the function in which the number e is raised to the power x.

Let's now calculate the derivative of e^x. To begin with, let's notice that for each n, we have

$$\begin{aligned}\frac{d}{dx}\left(1+\frac{x}{n}\right)^n &= n\left(1+\frac{x}{n}\right)^{n-1}\frac{d}{dx}\left(1+\frac{1}{n}x\right)\\ &= n\left(1+\frac{x}{n}\right)^{n-1}\left(\frac{1}{n}\right)\\ &= \left(1+\frac{x}{n}\right)^{n-1}\end{aligned}$$

That is, we can write

$$\frac{d}{dx}\left(1+\frac{x}{n}\right)^n = \left(1+\frac{x}{n}\right)^n\left(1+\frac{x}{n}\right)^{-1}$$

Now let's assume (rather boldly!) that we can interchange the limit and the derivative. Then

$$\begin{aligned}\frac{d}{dx}\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n &= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{-1}\\ \frac{d}{dx}e^x &= e^x(1+0)^{-1} = e^x\end{aligned}$$

That is, the exponential function is its own derivative:

$$\frac{d}{dx}e^x = e^x$$

EXAMPLE 1 Find the slope and the equation of the tangent line to $y = e^x$ when $x = 1$.

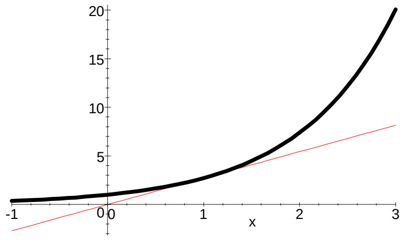
Solution: Since $y = e^x$ implies that $y' = e^x$, the slope of the tangent line is

$$y'(1) = e^1 = e$$

Moreover, $y = e^x$ implies that $y(1) = e^1 = e$, so that the tangent line is the line through $(1, e)$ with slope e:

$$y - e = e(x-1)$$

Thus, $y = ex$, which is approximately $y = 2.7182\,x$.



6-1: A tangent line to $y = e^x$

Check your Reading *Why can we conclude that $y(0) = 1$ when $y = a^x$?*

The Exponential Function and the Chain Rule

In operator notation, the derivative of the exponential function is of the form

$$\frac{d}{dx}e^x = e^x \tag{2.26}$$

The derivative (2.26) can be written in chain rule form as

$$\frac{d}{dx}e^{\text{input}} = e^{\text{input}}\frac{d}{dx}(\text{input})$$

EXAMPLE 2 Find $f'(x)$ for $f(x) = e^{-x^2}$.

Solution: The chain rule tells us that

$$\frac{d}{dx}e^{\text{input}} = e^{\text{input}}\frac{d}{dx}\text{input}$$

where the input is $-x^2$. Replacing "input" by "$-x^2$" results in

$$\frac{d}{dx}e^{-x^2} = e^{-x^2}\frac{d}{dx}\left(-x^2\right) = -2x\,e^{-x^2} \tag{2.27}$$

EXAMPLE 3 Find $f'(x)$ if $f(x) = x^2e^x$.

Solution: Since x^2e^x is the product of x^2 and e^x, we use the product rule:

$$\frac{d}{dx}x^2e^x = \left(\frac{d}{dx}x^2\right)e^x + x^2\left(\frac{d}{dx}e^x\right) = 2xe^x + x^2e^x$$

EXAMPLE 4 Find $f'(x)$ if $f(x) = x\left(e^x+1\right)^{3/2}$

Solution: Here we must use a combination of the product rule *and* the chain rule. Indeed,

$$f'(x) = \frac{d}{dx}\left(x\left(e^x+1\right)^{3/2}\right) = \left(\frac{d}{dx}x\right)\left(e^x+1\right)^{3/2} + x\frac{d}{dx}\left(e^x+1\right)^{3/2}$$

Thus, the chain rule implies that

$$\begin{aligned} f'(x) &= (e^x+1)^{3/2} + x\frac{3}{2}(e^x+1)^{1/2}\frac{d}{dx}(e^x+1) \\ &= (e^x+1)^{3/2} + \frac{3}{2}x(e^x+1)^{1/2}\ e^x \end{aligned}$$

Finally, factoring out $(e^x+1)^{1/2}$ yields

$$f'(x) = (e^x+1)^{1/2}\left(e^x+1+\frac{3}{2}xe^x\right)$$

Second derivatives are similarly evaluated. However, it is worth remembering that if the first derivative involves the chain rule, then the second derivative will often begin with the product rule:

EXAMPLE 5 Find $f''(x)$ for $f(x) = e^{-x^2}$.

Solution: In example 2, we used the chain rule to obtain $f'(x) = -2xe^{-x^2}$. Thus, finding $f''(x)$ begins with the product rule:

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left(-2xe^{-x^2}\right) \\ &= \left[\frac{d}{dx}(-2x)\right]e^{-x^2} + (-2x)\frac{d}{dx}e^{-x^2} \end{aligned}$$

We now use the chain rule to evaluate $\frac{d}{dx}e^{-x^2}$ (see (2.27) for details):

$$\begin{aligned} \frac{d^2}{dx^2}e^{-x^2} &= -2e^{-x^2} + (-2x)\left(-2xe^{-x^2}\right) \\ &= -2e^{-x^2} + 4x^2e^{-x^2} \end{aligned}$$

Factoring out $2e^{-x^2}$ results in

$$\frac{d^2}{dx^2}e^{-x^2} = 2e^{-x^2}\left(2x^2-1\right) \tag{2.28}$$

Check your Reading *Compute the derivative of e^{-x} using the chain rule.*

Properties of the Exponential Function

Since $e = 2.7182818284\ldots$ is a number, the properties of e^x follow from the laws of exponents for a number e raised to a power x.

Properties	**of e^x**
(1) $e^0 = 1$	(4) $e^x > 0$ for all x
(2) $e^{x+a} = e^xe^a$	(5) $e^{-x} = \dfrac{1}{e^x}$
(3) $e^{kx} = (e^x)^k$	

Property (5) is especially useful when simplifying expressions involving exponential functions.

EXAMPLE 6 Simplify the expression

$$\frac{\frac{1}{e^x}}{1-\frac{1}{e^x}} \tag{2.29}$$

to an expression with only one exponential.

Solution: Property (5) implies the more compactly written

$$\frac{e^{-x}}{1-e^{-x}}$$

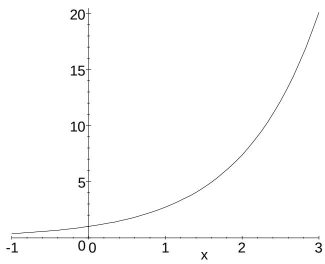
Now let us multiply the numerator and the denominator by e^x:

$$\left(\frac{e^{-x}}{1-e^{-x}}\right)\frac{e^x}{e^x} = \frac{e^{-x}e^x}{(1-e^{-x})\,e^x} = \frac{e^{-x}e^x}{e^x - e^{-x}e^x}$$

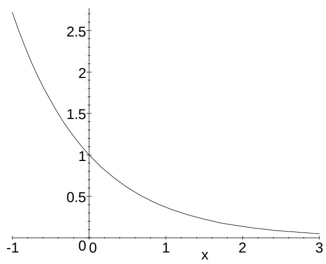
Property (2) implies that $e^{-x}e^x = e^{-x+x}$, so that

$$\frac{e^{-x}}{1-e^{-x}} = \frac{e^{-x+x}}{e^x - e^{-x+x}} = \frac{e^0}{e^x - e^0} = \frac{1}{e^x - 1}$$

Since $e > 1$, the exponential e^x becomes larger and larger without bound as x approaches ∞. Consequently, $e^{-x} = \frac{1}{e^x}$ approaches 0 as x approaches ∞.



6-2a: Graph of $f(x) = e^x$



6-2b: Graph of $f(x) = e^{-x}$

In general, e^{kx} for $k > 0$ becomes arbitrarily large as x approaches ∞. Thus,

$$\lim_{x\to\infty} e^{-kx} = \lim_{x\to\infty} \frac{1}{e^{kx}} = 0 \quad \text{if} \quad k > 0 \tag{2.30}$$

Check your Reading *What does $e^{2x}e^{-x}$ simplify to?*

Applications of the Exponential

Later in this text, we will consider a number of applications of the exponential, as it is foundational to the study of differential equations. For now, let's look at applications that follow immediately from definition 6.1.

If n is very large, then definition 6.1 implies that

$$e^x \approx \left(1+\frac{x}{n}\right)^n \tag{2.31}$$

Raising e^x to a power t and multiplying by P then yields

$$P\left[e^x\right]^t \approx P\left[\left(1+\frac{x}{n}\right)^n\right]^t$$

If we now let $x = r$ and consider n very large, then

$$Pe^{rt} \approx P\left(1+\frac{r}{n}\right)^{nt} \tag{2.32}$$

That is, the compound interest formula for very large values of n is very close to an exponential function. This idea is quite useful in studying investments like real estate and the stock market where the value of the investment is theoretically in a continual state of change.

EXAMPLE 7 What is the approximate value after 3 years of an investment of $\$5,000$ with a projected annual rate of 12% if the number of compoundings is not known specifically but is known to be quite large (such as an investment in the stock market)?

Solution: Since n, the number of compoundings per year, is not known but is very large, we use the fact that

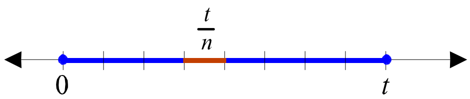
$$A = P\left(1+\frac{r}{n}\right)^{nt} \approx Pe^{rt}$$

If $P = \$5,000$, $r = 0.12$, and $t = 3$, then

$$A \approx 5000e^{0.12\cdot 3} = \$7,166.64$$

The compound interest idea also occurs in non-financial applications. For example, suppose a population is growing at a rate of $k\%$ per unit time, which is to say that if dt is sufficiently close to 0, then between time t and time $t+dt$ the population will increase by kdt %. To illustrate, suppose that $k = 20\%$ per hour. Then from time $t = 2$ hours to time $t + dt = 2.01$ hours, the population increases by approximately $kdt = 0.2$ %. The parameter k is called the *intrinsic growth rate* of the population.

Let's suppose the population has a population of P at time $t = 0$, and let's determine about how large $A(t)$ the population will be after t hours. To do so, we choose a large positive integer n and divide the interval $[0,t]$ into n equal time periods, each of which has a duration of $dt = \frac{t}{n}$.



6.3: The interval $[0,t]$ partitioned into periods of duration $dt = \frac{t}{n}$

The population increases by about $kdt = \frac{kt}{n}$ % over each time period. For example, over the first time period the population increases by about $\frac{kt}{n}$ % of the initial population P, which is $\frac{kt}{n}P$. Thus, at time $\frac{t}{n}$, the population is approximately the sum of the initial population P and the increase during $\left[0,\frac{t}{n}\right]$:

$$A\left(\frac{t}{n}\right) \approx P + \frac{kt}{n}P = P\left(1+\frac{kt}{n}\right)$$

During the period $\left[\frac{t}{n}, \frac{2t}{n}\right]$, the population increases by about $\frac{kt}{n}$ % of $A\left(\frac{t}{n}\right)$, so that

$$A\left(\frac{2t}{n}\right) \approx A\left(\frac{t}{n}\right) + \frac{kt}{n} A\left(\frac{t}{n}\right) = A\left(\frac{t}{n}\right)\left(1 + \frac{kt}{n}\right) = P\left(1 + \frac{kt}{n}\right)^2$$

Similarly, $A\left(\frac{3t}{n}\right) \approx P\left(1 + \frac{kt}{n}\right)^3$ and continuing across each time period leads to

$$A(t) = A\left(\frac{nt}{n}\right) \approx P\left(1 + \frac{kt}{n}\right)^n$$

In the limit as n approaches ∞, we obtain the exact population of

$$A(t) = P \lim_{n\to\infty}\left(1 + \frac{kt}{n}\right)^n = Pe^{kt}$$

Thus, the population at time t is given by $A(t) = Pe^{kt}$.

EXAMPLE 8 A certain bacteria culture is increasing at an intrinsic rate of about 20% per minute. If there are 100 bacteria initially, about how many bacteria will there be after 7 hours?

Solution: Since $P = 100$ and $k = 80\%$ per hour, the population at time t in minutes is given by

$$A(t) = 100e^{0.8t}$$

After $t = 7$ hours the bacteria population will be

$$A(7) = 100e^{0.8(7)} = 27,042 \; bacteria$$

Exercises:

Simplify each expression to an expression involving a single exponential

1. $e^{-x}e^{2x}$
2. $\left(e^{x/2} - 1\right)\left(e^{x/2} + 1\right)$
3. $e^{-x}\left(e^x + e^{2x}\right)$
4. $\left(e^{5x} + e^{9x}\right)e^{-9x}$
5. $\dfrac{e^{2x} - 4}{e^x - 2}$
6. $\dfrac{e^x - 1}{e^{2x} - 1}$
7. $\dfrac{e^{-x} - 1}{e^{-x}}$
8. $\dfrac{e^x}{e^{2x} - e^x}$
9. $\dfrac{e^x - e^{-x}}{e^x - 1}$
10. $\dfrac{e^{2x} - 4e^x + 4}{e^x - 2}$

Find the slope and the equation of the tangent line at the given value of p.

11. $f(x) = e^x, \quad p = 2$
12. $f(x) = e^x + e^{-x}, \quad p = 1$
13. $f(x) = xe^{-x}, p = 1$
14. $f(x) = x^3e^{-x}, \quad p = 0$
15. $f(x) = e^x - e^{-x}, \quad p = 1$
16. $f(x) = e^x - e^{-x}, \quad p = 0$
17. $f(x) = \dfrac{e^{2x} - 1}{e^{2x} + 1}, \quad p = 0$
18. $f(x) = \dfrac{e^{2x} - 1}{e^{2x} + 1}, \quad p = 1$

Find the derivative $f'(x)$ and second derivative $f''(x)$ of each function below.

19. $f(x) = xe^{-x}$ 20. $f(x) = x^2e^{-x}$ 21. $f(x) = e^{-x^2/2}$
22. $f(x) = e^{\sqrt{x}}$ 23. $f(x) = e^{1/x}$ 24. $f(x) = e^{-1/x^2}$
25. $f(x) = \sqrt{x}e^{x/4}$ 26. $f(x) = xe^{-x^2}$ 27. $f(x) = xe^{-x} + e^{2x}$
28. $f(x) = \dfrac{1}{e^x + e^{-x}}$ 29. $f(x) = \dfrac{x+1}{e^x}$ 30. $f(x) = \dfrac{x^2+x}{e^{5x}}$

31. What is the value after 5 years of an investment of \$10,000 at an *annual* rate of 12% if it is compounded monthly? If it is compounded daily? If it is compounded hourly? Compare these values with the value obtained from the exponential approximation of compound interest in (2.32).

32. If n is too large, then roundoff error forces us to use (2.32) instead of the compound interest formula. For example, compute the values in exercise 31 if the investment is compounded once every second, once every millisecond (= 1,000 times per second), once every nanosecond (= 10^9 times each second). Do you trust these calculations? Or do you suspect roundoff error might be making the compound interest formula unreliable?

33. Ace mutuals are an investment that consistently have somewhere between a 9 and a 15 percent annual interest rate. If an investor invests \$20,000 in Ace mutuals, what is approximately the highest and the lowest values they can expect for their investment after 5 years (assuming that the number of compoundings per year is very large)?

34. Suppose that \$1000 is placed in a savings account with an *annual* interest rate of 4% compounded daily. How much difference would there be between the exponential approximation and the compound interest formula in (2.32) after 20 years?

35. The world population in 1950 was 2.52 billion people, and at any given time it has been growing at an intrinsic rate of about 1.875% annually since then. What is the exponential function model for world population? About how many people are in the world at present, based on this model? About how many people will there be in 2050?

36. Repeat exercise 35 for the population of the United States, which was at 150.697 million in 1950 and at any given time has been growing at an intrinsic rate of about 1.236% annually since then.

37. Grapher: In probability, the bell curve, which is also known as the *standard normal density,* is the graph of the function $f(x) = e^{-x^2/2}$.

(a) Find $f'(x)$ and $f''(x)$.
(b) Graph $f(x)$ and $f'(x)$ on the same domain $[-2, 2]$.
(c) Where does the largest value of $f(x)$ occur? At what value is $f'(x) = 0$? What is the connection here?
(d) At what two values is $f''(x) = 0$? (the absolute value of these two numbers is called the *standard deviation* of the standard normal density)

38. Grapher: The function $f(x) = xe^{-x}$ is sometimes called the *alpha* function. It occurs frequently in applications in probability and biology.

(a) Find $f'(x)$.
(b) Graph $y = f(x)$ and $y = f'(x)$ on the same domain $[0, 5]$.
(c) Where does the largest value of $f(x)$ occur? At what value is $f'(x) = 0$? What is the connection here?

39. Numerical: Use the definition of the derivative to explain the identity

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

Complete the following table as a means of verifying the limit above

h	-0.01	-0.001	-0.0001	$\rightarrow$	0	$\leftarrow$	0.0001	0.001	0.01
$\frac{e^h-1}{h}$				$\rightarrow$	?	$\leftarrow$			

40. Grapher: Graph the function e^x along with the function

$$\frac{e^{x+h} - e^x}{h}$$

for $h = 0.01$ over the interval $[-2, 2]$. Why are the two so similar?

41. Show that the tangent line to $y = e^x$ at $x = 0$ is $y = 1 + x$.

42. Let $L_0(x) = 1 + x$ (i.e., the tangent line in 41). Show that the tangent line to $y = e^x$ at $x = p$ is

$$y = e^p L_0(x - p)$$

43. Computer Algebra System: If $n = 1000$, then definition 5.1 implies that

$$e^x \approx \left(1 + \frac{x}{1000}\right)^{1000}$$

Use a computer algebra system to expand the quantity on the right as a polynomial in x, and then let $x = 1$ in the first 5 terms to obtain the approximation

$$e \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}$$

44. Write to Learn: Write an essay in which you use definition 5.1 to explain why

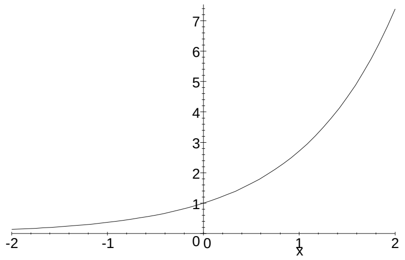
$$\lim_{x\to\infty} \frac{x^m}{e^x} = 0$$

for any positive power m. (Hint: you may want to use (2.31) and assume that $n > m$).

2.7 The Natural Logarithm

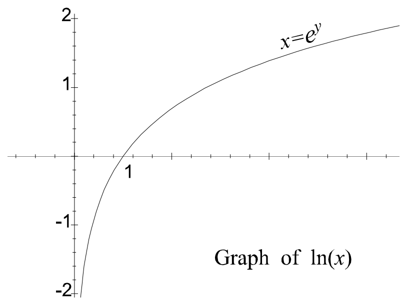
Definition of the Natural Logarithm

The graph of the exponential function $y = e^x$ is shown in figure 7-1.



7-1: Graph of $y = e^x$

Not only is there only one y for each x, but there is also only one x for each $y > 0$. As a result, the curve $x = e^y$ *implicitly* defines y as a function of x. The function defined implicitly by $x = e^y$ is called the *natural logarithm* and is denoted $y = \ln(x)$.



7-2: The curve $x = e^y$ implicitly defines $y = \ln(x)$.

Notice, however, that $\ln(x)$ is defined only for $x > 0$.

> **Definition 7.1:** The *natural logarithm* $y = \ln(x)$ is the function defined implicitly by the curve $x = e^y$.

Since $y = \ln(x)$ is the same as $x = e^y$, replacing y by $\ln(x)$ yields $x = e^{\ln(x)}$. Similarly, $\ln(e^x) = x$. We say that e^x and $\ln(x)$ are *inverses* of each other because of these two properties.

> **Theorem 7.2:** If $x > 0$, then $\ln(x)$ and e^x cancel under composition. That is,
> $$\ln(e^x) = x \quad \text{and} \quad e^{\ln(x)} = x.$$

For example, the fact that $\ln(x)$ is the inverse of e^x allows us to use $\ln(x)$ to isolate k in the function $y = Pe^{kt}$. Solving for k in this fashion is important in many applications.

EXAMPLE 1 Find k given that $y(t) = 3e^{kt}$ and $y(2) = 7$

Solution: Since $y(2) = 7$, we let $t = 2$ and $y = 7$ in $y = 3e^{kt}$:

$$7 = 3\,e^{2k}$$

We then divide both sides of the equation by 3

$$e^{2k} = \frac{7}{3}$$

We take the natural logarithm of both sides to eliminate the exponential:

$$\begin{aligned} \ln\left(e^{2k}\right) &= \ln\left(\frac{7}{3}\right) \\ 2k &= \ln\left(\frac{7}{3}\right) \end{aligned}$$

Finally, we divide both sides by 2 to obtain

$$k = \frac{1}{2}\ln\left(\frac{7}{3}\right) = 0.4236489$$

Check your Reading *What is the value of* $\ln(e)$?

Properties of the Natural Logarithm

The properties of the natural logarithm follow from the laws of exponents. To illustrate, consider that if $a = e^x$ and $b = e^y$, then $\ln(a) = x$, $\ln(b) = y$, and

$$\ln(ab) = \ln(e^x e^y) = \ln\left(e^{x+y}\right) = x + y = \ln(a) + \ln(b)$$

Moreover, if $a = e^x$, then $x = \ln(a)$, so that

$$\ln(a^r) = \ln((e^x)^r) = \ln(e^{rx}) = rx = r\ln(a)$$

An important corollary of these three is that

$$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$$

The derivative of $\ln(x)$ begins with the fact that if $y = \ln(x)$, then $e^y = x$. Implicit differentiation applied to $e^y = x$ yields

$$e^y\,y' = 1$$

Since $x = e^y$, the equation $e^y y' = 1$ reduces to $xy' = 1$, which implies that $y' = 1/x$. That is,

$$\frac{d}{dx}\ln(x) = \frac{1}{x} \qquad (2.33)$$

which in chain rule form is written

$$\frac{d}{dx}\ln(\text{input}) = \frac{1}{\text{input}}\frac{d}{dx}\text{input} \qquad (2.34)$$

In summary, $\ln(x)$ has the following properties.

Properties	**of** $\ln(x)$
(1) $\ln(1) = 0$	(4) $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$
(2) $\ln(ab) = \ln(a) + \ln(b)$	(5) $\frac{d}{dx}\ln(x) = \frac{1}{x}$
(3) $\ln(a^r) = r\ln(a)$	

EXAMPLE 2 Find $f'(x)$ for $f(x) = \ln(e^x + 1)$

Solution: To begin with, we notice that $\frac{d}{dx}\ln(e^x+1)$ is given by

$$\frac{d}{dx}\ln(\text{input}) = \frac{1}{\text{input}}\frac{d}{dx}(\text{input})$$

where the input is $e^x + 1$. As a result, we have

$$\frac{d}{dx}\ln(e^x+1) = \frac{1}{e^x+1}\frac{d}{dx}(e^x+1) = \frac{e^x}{e^x+1}$$

Simplify using the properties of the logarithm before applying the derivative.

Whenever possible, the properties of the logarithm should be used to simplify a function before it is differentiated.

EXAMPLE 3 Find $f'(x)$ for $f(x) = \ln\left(2x^4 - x^3\right)$ and $x > 0$.

Solution: Since $2x^4 - x^3 = x^3(2x-1)$, we can write $f(x)$ as

$$f(x) = \ln\left[x^3(2x-1)\right]$$

after which properties of the logarithm yield

$$f(x) = \ln\left(x^3\right) + \ln(2x-1) = 3\ln(x) + \ln(2x-1)$$

Consequently, the derivative of $f(x)$ is

$$f'(x) = 3\frac{d}{dx}\ln(x) + \frac{1}{2x-1}\frac{d}{dx}(2x-1) = \frac{3}{x} + \frac{2}{2x-1}$$

Check your Reading *What is $f'(x)$ when $f(x) = \ln\left(x^3\right)$?*

Logarithmic Differentiation

The properties of the logarithm can also be used to simplify some derivative computations. In particular, the following algorithm, which is called *logarithmic differentiation,* can be used to compute y' given $y = f(x)$:

i. Apply $\ln(x)$ to both sides of $y = f(x)$
ii. Simplify using the properties of the natural logarithm
iii. Apply $\frac{d}{dx}$ and use the fact that $\frac{d}{dx}\ln[f(x)] = \frac{f'(x)}{f(x)}$
iv. Solve for y'

This process works even when $f(x) \leq 0$, as is shown in exercise 41.

EXAMPLE 4 Find the derivative of

$$y = \frac{x}{x^2+1} \tag{2.35}$$

Solution: We first apply $\ln(x)$ to both sides,

$$\ln(y) = \ln\left(\frac{x}{x^2+1}\right),$$

We then simplify using the properties of the logarithm:

$$\ln(y) = \ln(x) - \ln\left(x^2+1\right)$$

Application of the derivative to both sides yields

$$\begin{aligned} \frac{d}{dx}\ln(y) &= \frac{d}{dx}\ln(x) - \frac{d}{dx}\ln\left(x^2+1\right) \\ \frac{y'}{y} &= \frac{1}{x} - \frac{\frac{d}{dx}\left(x^2+1\right)}{x^2+1} \end{aligned}$$

after which we solve for y':

$$y' = y\left(\frac{1}{x} - \frac{2x}{x^2+1}\right)$$

The result is the derivative of (2.35):

$$\frac{d}{dx}\left(\frac{x}{x^2+1}\right) = \frac{x}{x^2+1}\left(\frac{1}{x} - \frac{2x}{x^2+1}\right)$$

EXAMPLE 5 Find y' when

$$y = \frac{e^x\sqrt{x^2-2}}{(x-1)^4} \tag{2.36}$$

Solution: Application of $\ln(x)$ and simplification with the properties of $\ln(x)$ yields

$$\begin{aligned} \ln(y) &= \ln\left(\frac{e^x\sqrt{x^2-2}}{(x-1)^4}\right) \\ &= \ln(e^x) + \ln\left(x^2-2\right)^{1/2} - \ln(x-1)^4 \\ &= x + \frac{1}{2}\ln\left(x^2-2\right) - 4\ln(x-1) \end{aligned}$$

Application of the derivative yields

$$\begin{aligned} \frac{d}{dx}\ln(y) &= \frac{d}{dx}\left(x + \frac{1}{2}\ln\left(x^2-2\right) - 4\ln(x-1)\right) \\ \frac{y'}{y} &= 1 + \frac{1}{2}\frac{\frac{d}{dx}\left(x^2-2\right)}{x^2-2} - 4\frac{\frac{d}{dx}(x-1)}{x-1} \\ y' &= y\left(1 + \frac{1}{2}\frac{2x}{x^2-2} - 4\frac{1}{x-1}\right) \end{aligned}$$

which reduces to the final result

$$y' = \frac{e^x\sqrt{x^2-2}}{(x-1)^4}\left(1+\frac{x}{x^2-2}-\frac{4}{x-1}\right)$$

Check your Reading *How involved would it be to compute the derivative of (2.36) without logarithmic differentiation?*

Proofs with Logarithms

Even though mathematics frequently involves graphing, computation, and mathematical modeling, a large part of mathematics is devoted to proving that a given mathematical statement is either true or false. This is analogous to a prosecutor proving that a defendant is guilty beyond a reasonable doubt, except that in mathematics the burden of proof is much heavier. In mathematical proofs, a proof must leave no doubt as to the truth or falsity of a statement.

That is, we must assume that certain a priori facts are already known to be true beyond any doubt, and then these facts must be used to test the validity of new statements. Let's conclude by assuming that the properties of the natural logarithm are true beyond any doubt, and let's see if we can use them to show that other mathematical statements are valid.

EXAMPLE 6 Use logarithmic differentiation to $y = \frac{f(x)}{g(x)}$ to prove the quotient rule when f and g are positive and differentiable for all x.

Solution: Since $y = f(x)/g(x)$, the natural logarithm yields

$$\ln(y) = \ln\left(\frac{f(x)}{g(x)}\right) = \ln(f(x)) - \ln(g(x))$$

Application of the derivative then yields

$$\frac{y'}{y} = \frac{d}{dx}\ln(f(x)) - \frac{d}{dx}\ln(g(x)) = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}$$

We then find a common denominator:

$$\frac{y'}{y} = \frac{f'(x)g(x)}{f(x)g(x)} - \frac{f(x)g'(x)}{f(x)g(x)} = \frac{f'(x)g(x)-f(x)g'(x)}{f(x)g(x)}$$

Since $y = \frac{f(x)}{g(x)}$, this implies that

$$y' = y\left(\frac{f'(x)g(x)-f(x)g'(x)}{f(x)g(x)}\right) = \frac{f(x)}{g(x)}\left(\frac{f'(x)g(x)-f(x)g'(x)}{f(x)g(x)}\right)$$

Since $y' = \frac{d}{dx}\frac{f(x)}{g(x)}$, cancellation then yields

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$$

which ends the proof.

Exercises:

Find the value of k for which $y(t) = Pe^{kt}$.

1. $y(t) = e^{kt}, \quad y(1) = 2$
2. $y(t) = 3e^{kt}, \quad y(1) = 9$
3. $y(t) = 1.5e^{kt}, \quad y(5) = 2$
4. $y(t) = 4.5e^{kt}, \quad y(1) = 3$
5. $y(t) = 2e^{kt}, \quad y(1) = 1$
6. $y(t) = 3.2e^{kt}, y(1) = 3$
7. $y(t) = e^{kt}, \quad y(1) = 1$
8. $y(t) = 4e^{kt}, y(4) = 4$

Find $f'(x)$. You may want to simplify before computing the derivative.

9. $f(x) = \ln\left(x^2\right)$
10. $f(x) = \ln\left(x^3 + x\right)$
11. $f(x) = x^2 \ln(x)$
12. $f(x) = x^3 \ln(x)$
13. $f(x) = e^x \ln(x)$
14. $f(x) = \ln\left(x^2\right) \ln(x)$
15. $f(x) = \ln\left(\dfrac{x}{x+1}\right)$
16. $f(x) = \ln\left(\dfrac{4x+5}{2x-1}\right)$
17. $f(x) = \ln\left(\dfrac{x^2}{x+1}\right)$
18. $f(x) = \ln\left(\dfrac{x^2}{3x+6}\right)$
19. $f(x) = \ln\left(\dfrac{e^{5x}}{\sqrt{x^2+1}}\right)$
20. $f(x) = \ln\left(e^{-x}\sqrt{x^4-1}\right)$

Use logarithmic differentiation to find y'.

21. $y = \dfrac{7x+6}{4x^2+1}$
22. $y = \dfrac{2x^2+7}{\sqrt{x-4}}$
23. $y = \dfrac{x^2-1}{e^{5x}}$
24. $y = \dfrac{e^x}{e^{2x}+2}$
25. $y = x(x-1)(x-2)(x-3)$
26. $y = \dfrac{x(x+1)}{(x+2)^2(x+3)}$
27. $y = \dfrac{(3x-5)\sqrt{x^3+1}}{e^x(4x-8)}$
28. $y = \sqrt{\dfrac{e^x+1}{e^{2x}+2}}$
29. $y = \left(1 + \dfrac{1}{x}\right)^x$
30. $y = x\sqrt{x^3+3x^2+3x+1}$

Write to Learn: The applications 31-40 represent a series of short proofs using the properties of logarithms and exponentials. In each, the result should be a short essay explaining and justifying each step in the proof. Assume that $f(x)$ and $g(x)$ are differentiable functions.

31. Use the properties (1), (2), and (3) of the logarithm to prove that

$$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$$

for all $a > 0$ and $b > 0$.

32. Apply logarithmic differentiation to $y = x^r$ to prove that

$$\frac{d}{dx}x^r = rx^{r-1}$$

for all real numbers r.

33. Apply logarithmic differentiation to $y = f(x)g(x)$ and thus obtain the product rule.

34. Use logarithmic differentiation to prove the chain rule formula for the exponential function:

$$\frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$$

35. Use logarithmic differentiation to prove that if $y = f(x)\, g(x)$, then

$$\frac{y'}{y} = \frac{f'}{f} + \frac{g'}{g}$$

36. Use logarithmic differentiation to prove that if $y = \dfrac{f(x)}{g(x)}$, then

$$\frac{y'}{y} = \frac{f'}{f} - \frac{g'}{g}$$

37. Apply the logarithm to $y = a^x$ to show that $a^x = e^{x \ln(a)}$ for all $a > 0$. Then apply logarithmic differentiation to $y = a^x$ to prove that

$$\frac{d}{dx} a^x = a^x \ln(a)$$

38. Let's prove a special case of the chain rule. Apply logarithmic differentiation to $y = [f(x)]^n$ to show that

$$\frac{d}{dx} [f(x)]^n = n [f(x)]^{n-1} f'(x)$$

given that (2.34) is true.

39. In the next chapter, we will prove the theorem that says

"If $h'(x) = 0$ for all x in (a, b), then $h(x)$ is constant on (a, b)"

Use it and the following steps to prove that if $a > 0$ and $x > 0$, then

$$\ln(ax) = \ln(x) + \ln(a) \tag{2.37}$$

(a) Show that $h'(x) = 0$ when

$$h(x) = \ln(ax) - \ln(x)$$

(Note: Since we are proving (2.37), we cannot use it to simplify $\ln(ax)$. Instead, use the chain rule to differentiate $\ln(ax)$).

(b) Show that if C is a constant which satisfies

$$C = \ln(ax) - \ln(x)$$

then $C = \ln(a)$ (Hint: let $x = 1$). Then solve for $\ln(ax)$ in order to obtain (2.37).

40. In the next chapter, we will prove the theorem that says

"If $h'(x) = 0$ for all x in (a, b), then $h(x)$ is constant on (a, b)"

Use it and the following steps to prove that if $x > 0$ and r is constant, then

$$\ln(x^r) = r \ln(x) \tag{2.38}$$

(a) Show that $h'(x) = 0$ when

$$h(x) = r \ln(x) - \ln(x^r)$$

(Note: Since we are proving (2.38), we cannot use it to simplify $\ln(ax)$. Instead, use the chain rule to differentiate $\ln(ax)$.

(b) Show that if C is a constant which satisfies

$$C = r \ln(x) - \ln(x^r)$$

then $C = 0$ (Hint: let $x = 1$). Solve for $\ln(x^r)$ to obtain (2.38).

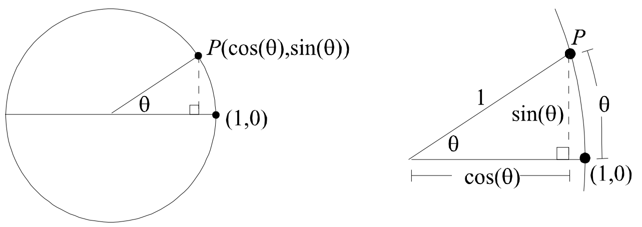
41. **Write to Learn:** In a short essay, apply logarithmic differentiation to $y^2 = [f(x)]^2$ and explain why it justifies the use of logarithmic differentiation even when $f(x) \leq 0$.

2.8 The Sine and Cosine

The Sine and Cosine Functions

In this section, we continue exploring differentiation by developing derivative rules for the trigonometric functions. Before we do so, however, let us first review some of the basic ideas from trigonometry.

To begin with, let θ be the distance along the *unit circle* from the x-axis to a point P on the circle (i.e., θ is in *radians)*. Then the x-coordinate of P is defined to be $\cos(\theta)$ ("cosine **of** theta") and the y-coordinate of P is defined to be $\sin(\theta)$ (sine **of** theta").



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8-1: The Unit Circle

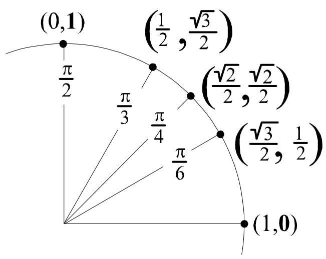
Since small changes in the angle θ cause only small changes in the coordinates, the functions $\sin(\theta)$ and $\cos(\theta)$ are continuous for all θ.

Some common sines and cosines are shown below:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

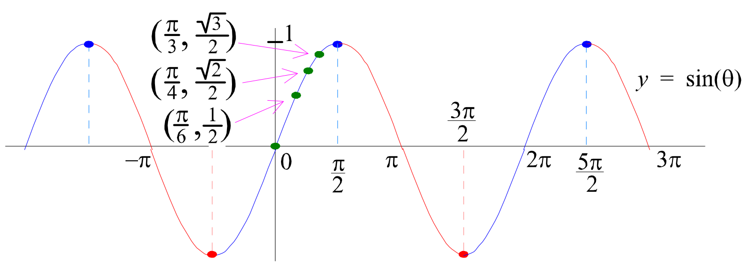
8-2: Some common values of $\sin(\theta)$ and $\cos(\theta)$

These cosine-sine pairs correspond to points on the unit circle:



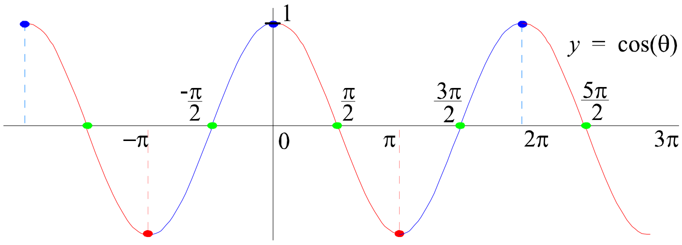
8-3: Some common Cosine-Sine pairs

The unit circle definition implies that $\sin(\theta)$ oscillates between -1 and 1 with a period of 2π.



8-4: The graph of the sine function

Likewise, $-1 \leq \cos(\theta) \leq 1$, and $\cos(\theta + 2\pi) = \cos(\theta)$ for all θ:



8-5: Graph of the cosine function

Moreover, figures 8-4 and 8-5 imply that if n is an integer, then

$$\sin(n\pi) = 0 \qquad \textit{and} \qquad \cos\left(\left(n + \frac{1}{2}\right)\pi\right) = 0$$

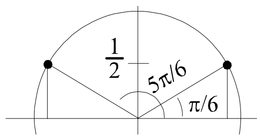
Equivalently, if m is an odd integer, then $\cos\left(\frac{m\pi}{2}\right) = 0$.

EXAMPLE 1 Solve the equation $\cos(x) - 2\sin(x)\cos(x) = 0$.

Solution: Factoring leads to

$$\cos(x)(1 - 2\sin(x)) = 0$$

Thus, $\cos(x) = 0$, which implies that $x = \left(n + \frac{1}{2}\right)\pi$ for any integer n. Also, $1 - 2\sin(x) = 0$, or $\sin(x) = \frac{1}{2}$. However, $\sin(x) = \frac{1}{2}$ occurs twice on the unit circle—once in the first quadrant and once in the second quadrant.



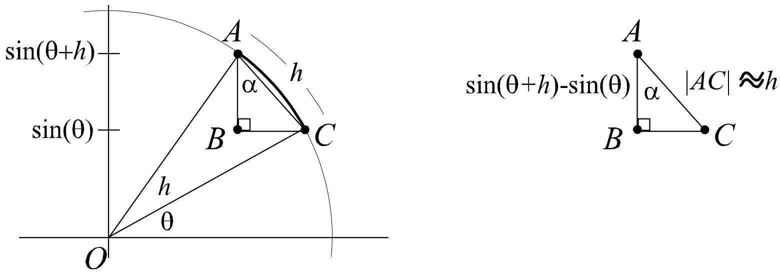
Since $\sin(\theta)$ has a period of 2π, the solutions to $\sin(x) = \frac{1}{2}$ are

$$x = \frac{\pi}{6} + 2n\pi, \qquad x = \frac{5\pi}{6} + 2n\pi$$

Check your Reading *What is the set of all the solutions to the equation in example 1?*

Derivatives and Identities

Let's use the unit circle to obtain the derivative of the sine function. Let θ and h be acute positive angles with h small enough that $\theta + h$ is in the 1st quadrant. Then let A be the point with coordinates $(\cos(\theta + h), \sin(\theta + h))$, and let C be the point with coordinates $(\cos(\theta), \sin(\theta))$.



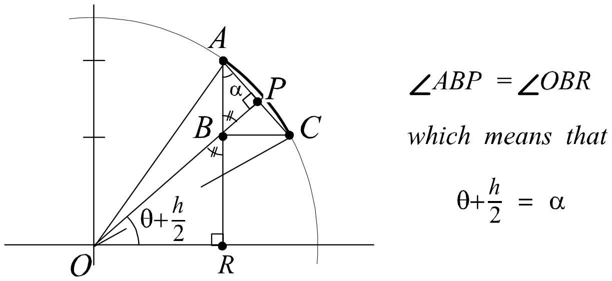
8-6: A is $(\cos(\theta + h), \sin(\theta + h))$ and C is $(\cos(\theta), \sin(\theta))$

If h is sufficiently small, then h is approximately the same as $|AC|$, which is the length of the segment $\overline{AC}$. Since $|AB| = \sin(\theta + h) - \sin(\theta)$, triangle ΔABC implies that

$$\frac{\sin(\theta+h)-\sin(\theta)}{h} \approx \frac{\sin(\theta+h)-\sin(\theta)}{|AC|} = \cos(\alpha)$$

where α is the angle $\angle BAC$.

To determine angle α, we use the fact that the radius $\overline{OP}$ intersects the chord $\overline{AC}$ in a right angle and bisects angle $\angle AOC$.



8-7: Line segment $\overline{OP}$ bisects line segment $\overline{AB}$

Right triangles ΔORB and ΔAPB are similar, which implies that $\angle ABP = \angle OBP$, and thus, that $\alpha = \theta + \frac{h}{2}$. This means that

$$\frac{\sin(\theta+h)-\sin(\theta)}{h} \approx \cos\left(\theta + \frac{h}{2}\right)$$

Moreover, h and $|AC|$ become arbitrarily close as h approaches 0 (a concept we will justify further at the end of this section), so that

$$\lim_{h\to 0} \frac{\sin(\theta+h)-\sin(\theta)}{h} = \lim_{h\to 0} \cos\left(\theta + \frac{h}{2}\right) = \cos(\theta)$$

Thus, the limit definition of the derivative implies that

$$\frac{d}{d\theta}\sin(\theta) = \cos(\theta) \tag{2.39}$$

We often write (2.39) in chain rule form as

$$\frac{d}{d\theta}\sin(\text{input}) = \cos(\text{input})\frac{d}{d\theta}(\text{input}) \tag{2.40}$$

EXAMPLE 2 Find $y'(\theta)$ when $y(\theta) = \sin(4\theta)$.

Solution: The chain rule (2.40) implies that

$$\frac{d}{d\theta}\sin(\text{input}) = \cos(\text{input})\frac{d}{d\theta}(\text{input})$$

where the input of 4θ. As a result,

$$\frac{d}{d\theta}\sin(4\theta) = \cos(4\theta)\frac{d}{d\theta}(4\theta) = 4\cos(4\theta)$$

Likewise, the trigonometric identities

$$\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right) \qquad \text{and} \qquad \sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$$

imply that

$$\frac{d}{d\theta}\cos(\theta) = \frac{d}{d\theta}\sin\left(\frac{\pi}{2} - \theta\right) = -\cos\left(\frac{\pi}{2} - \theta\right) = -\sin(\theta)$$

Don't forget the negative when differentiating the cosine function

Thus, the derivative of $\cos(\theta)$ is the **negative** of the sine function:

$$\frac{d}{d\theta}\cos(\theta) = -\sin(\theta) \tag{2.41}$$

In chain rule form, we write

$$\frac{d}{d\theta}\cos(\text{input}) = -\sin(\text{input})\frac{d}{d\theta}(\text{input}) \tag{2.42}$$

EXAMPLE 3 What is the slope and equation of the tangent line to $y = x\cos\left(x^2\right)$ at $x = 0$?

Solution: The product rule implies that

$$y'(x) = \frac{d}{dx}\left[x\cos\left(x^2\right)\right] = \left(\frac{d}{dx}x\right)\cos\left(x^2\right) + x\frac{d}{dx}\cos\left(x^2\right)$$

The chain rule then implies that

$$\begin{aligned} y'(x) &= \cos\left(x^2\right) - \sin\left(x^2\right)\frac{d}{dx}x^2 \\ &= \cos\left(x^2\right) - 2x\sin\left(x^2\right) \end{aligned}$$

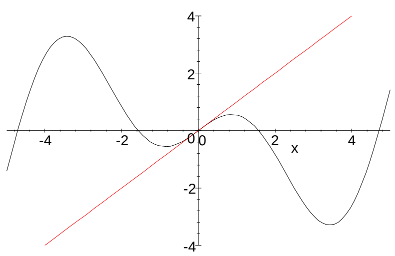
As a result, the slope of the tangent line is

$$y'(0) = \cos(0) - 0\sin(0) = 1$$

Moreover, $y(0) = 0\cos\left(0^2\right) = 0$, so that the equation of the tangent line is

$$y = 0 + 1(x - 0) = x$$

which is shown along with the function $y(x) = x\cos\left(x^2\right)$ in the figure below:



8-8: The line $y = x$ is tangent to $y = x\cos\left(x^2\right)$

The unit circle can also be used to establish the following basic identities:

$$\sin(x+a) = \sin(x)\cos(a) + \cos(x)\sin(a) \tag{2.43}$$
$$\cos(x+a) = \cos(x)\cos(a) - \sin(x)\sin(a) \tag{2.44}$$
$$\sin(-x) = -\sin(x) \tag{2.45}$$
$$\cos(-x) = \cos(x) \tag{2.46}$$

These identities are then used to obtain the *Pythagorean* identities

$$\begin{aligned} \cos^2(x) + \sin^2(x) &= 1 \\ 1 + \tan^2(x) &= \sec^2(x) \\ 1 + \cot^2(x) &= \csc^2(x) \end{aligned}$$

as well as the *double angle* identities

$$\begin{array}{ll} 2\sin(x)\cos(x) = \sin(2x) & 2\cos^2(x) - 1 = \cos(2x) \\ \cos^2(x) - \sin^2(x) = \cos(2x) & 1 - 2\sin^2(x) = \cos(2x) \end{array}$$

The double angle identities then lead to

$$\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x) \qquad \text{and} \qquad \sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x) \tag{2.47}$$

Each of these identities can also be obtained directly from the unit circle.

EXAMPLE 4 Find $f''(x)$ when $f(x) = \sin(x)\cos(x)$.

Solution: We begin with the product rule:

$$\begin{aligned} f'(x) &= \frac{d}{dx}[\sin(x)\cos(x)] \\ &= \left[\frac{d}{dx}\sin(x)\right]\cos(x) + \sin(x)\left[\frac{d}{dx}\cos(x)\right] \\ &= \cos(x)\cos(x) + \sin(x)[-\sin(x)] \\ &= \cos^2(x) - \sin^2(x) \end{aligned}$$

Before computing $f''(x)$, let us apply an identity to $f'(x)$:

$$f'(x) = \cos^2(x) - \sin^2(x) = \cos(2x)$$

As a result, the second derivative is of the form

$$f''(x) = \frac{d}{dx}\cos(\text{input}) = -\sin(\text{input})\,\frac{d}{dx}\,\text{input}$$

where the input is $2x$. Finally, we have

$$f''(x) = -\sin(2x)\frac{d}{dx}(2x) = -2\sin(2x)$$

Check your Reading *What is the derivative of* $g(x) = \cos^2(x) + \sin^2(x)$?

Harmonic Oscillations and Rates of Change

A *harmonic oscillation* is a function of the form

$$y(t) = a\cos(\omega t) + b\sin(\omega t) + M \tag{2.48}$$

where a, b, ω, and M are constants. Often, the variable t denotes time, in which case the Greek letter omega, ω, represents the *angular velocity* of the oscillation.

EXAMPLE 5 **Tides:** The height h in feet above or below sea level of the ocean near Bridgeport, Connecticut, at time t in hours since midnight on Sept. 1, 1991, is given by

$$h(t) = 3.35 - 0.0175\cos(0.506t) + 3.184\sin(0.506t)$$

How fast is the ocean level rising at 10 a.m?

Solution: The derivative of $h(t)$ is given by

$$h'(t) = 0.0175(0.506)\sin(0.506t) + 3.184(0.506)\cos(0.506t)$$

Since 10 a.m. corresponds to $t = 10$ hours, the ocean level at 10 a.m. is changing at a rate of

$$\begin{aligned} h'(10) &= 0.0175(0.506)\sin(5.06) + 3.184(0.506)\cos(5.06) \\ &= 0.54\ \textit{feet per hour} \end{aligned}$$

The *period* of a harmonic oscillation (2.48) is the smallest number $T > 0$ such that $y(t+T) = y(t)$ for all t. It is related to the angular velocity by the formula

$$T = \frac{2\pi}{\omega}$$

and the graph of $y(t)$ over one period is called a *cycle.* The number of cycles that occur per unit time is the *frequency* of the oscillation, which is given by

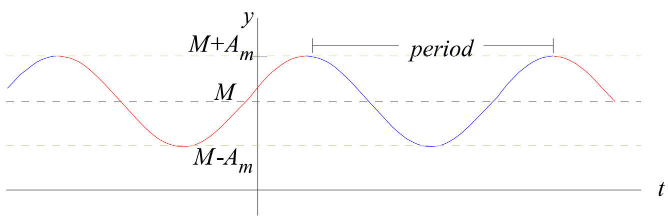
$$f = \frac{1}{T}$$

Frequencies are often measured in *hertz,* where 1 hertz is one cycle per second.

The amplitude of the oscillation is given by

$$A_m = \sqrt{a^2 + b^2}$$

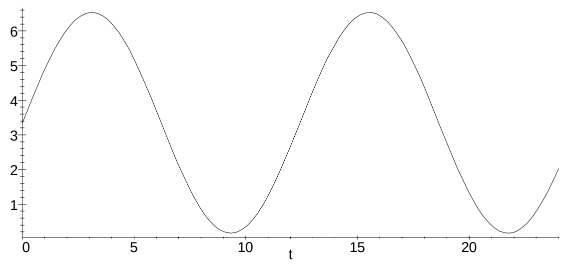
and the graph of the oscillation is a shifted sine wave oscillating about the line $y = M$.



8-9: Simple Oscillation about the mean value M

EXAMPLE 6 What is the period of the tide in example 5? How high is high tide?

Solution: The graph of $h(t)$ over the first 24 hours is shown below:



8-10: Tides are an oscillation in sea level

Since $\omega = 0.506$, the period of the tide is

$$T = \frac{2\pi}{0.506} = 12.42\ hours$$

The amplitude of the oscillation is

$$A_m = \sqrt{(-0.0175)^2 + (3.184)^2} = 3.184\ feet$$

Thus, high tide is the sum of the mean value $M = 3.35$ and the amplitude $A_m = 3.184$ feet, which is

$$M + A_m = 3.35 + 3.184 = 6.534\ feet$$

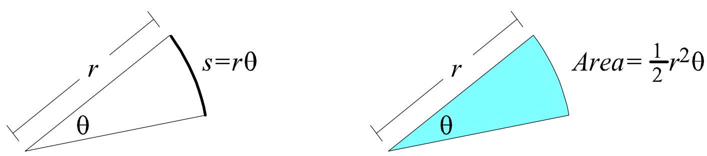
Check your Reading *How low is low tide in example 2?*

Proof that $\frac{d}{d\theta}\sin(\theta) = \cos(\theta)$ using the Unit Circle

The derivative formulas for the trigonometric functions follow from

$$\frac{d}{d\theta}\sin(\theta) = \cos(\theta) \tag{2.49}$$

Thus, we conclude this chapter with a "rigorous" proof of (2.49). To do so, however, we will need the following basic ideas about a sector with length s of a circle with radius r subtending an angle θ.

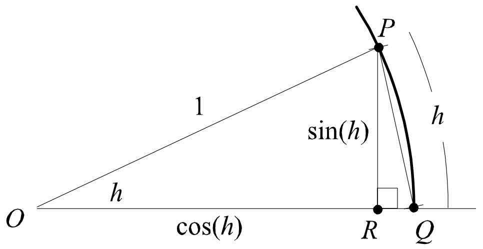


8-11: Arclength and Area for the arc of a circle

EXAMPLE 7 Show that

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1$$

Solution: Suppose $\widehat{OPQ}$ is a sector of the unit circle subtending a small positive angle h:

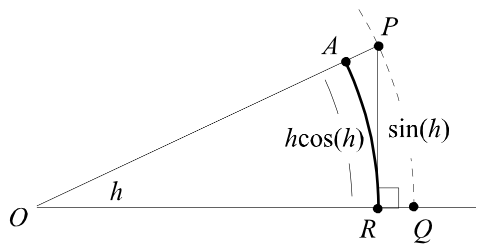


8-12: Relationship of $\sin(h)$ to h

The secant $\overline{PQ}$ is shorter than the arc $\widehat{PQ}$, and the side $\overline{PR}$ is shorter than the hypotenuse $\overline{PQ}$. As a result,

$$\sin(h) < h$$

Suppose now that $\widehat{AR}$ is the arc of a circle centered at O with radius $\cos(h)$ subtending angle $\angle AOR$.



8-13: Relationship of $h\cos(h)$ to $\sin(h)$

The area of sector $\widehat{ORA}$ is less than the area of triangle ΔOPQ, and since segment $\overline{OR}$ is common to both, the respective area formulas imply that $h\cos(h) < \sin(h)$.

Combining the two yields the inequality

$$h\cos(h) < \sin(h) < h$$

Division by h thus implies that

$$\cos(h) < \frac{\sin(h)}{h} < 1$$

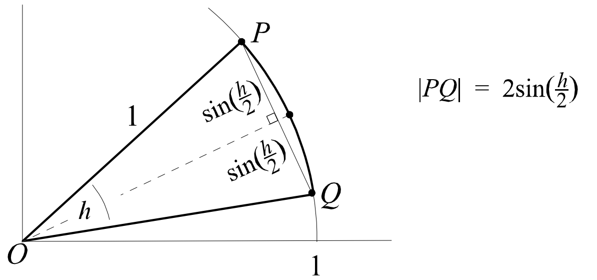
Application of the limit as h approaches 0 then yields

$$\begin{aligned} \lim_{h \to 0} \cos(h) &\leq \lim_{h \to 0} \frac{\sin(h)}{h} \leq 1 \\ 1 &\leq \lim_{h \to 0} \frac{\sin(h)}{h} \leq 1 \end{aligned}$$

By the sandwich theorem, we conclude that

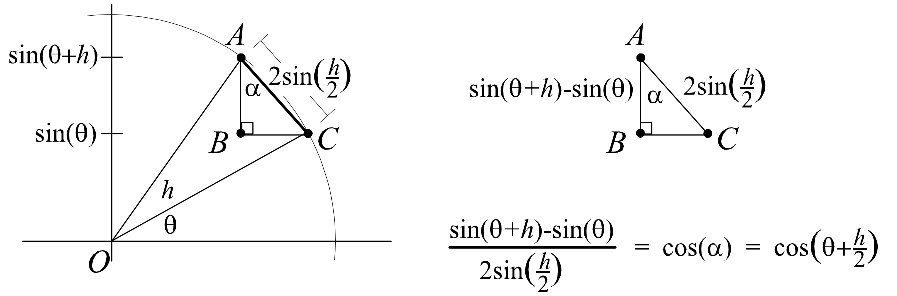
$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1$$

Notice now that the length of a chord that spans an angle h on the unit circle is $2\sin\left(\frac{h}{2}\right)$.



8-14: Length of a Chord of a Circle

As a result, the length of the chord $\overline{AC}$ in figures 7-6 and 7-7 is actually $2\sin(h/2)$.



8-15: Actual values for ΔABC

Now let's prove that the derivative of $\sin(\theta)$ is $\cos(\theta)$. Since $\alpha = \theta + \frac{h}{2}$ as we showed earlier, we have

$$\begin{aligned}
\lim_{h\to 0} \frac{\sin(\theta+h)-\sin(\theta)}{2\sin(h/2)} &= \lim_{h\to 0} \cos\left(\theta + \frac{h}{2}\right) \\
\lim_{h\to 0} \left(\frac{\sin(\theta+h)-\sin(\theta)}{h} \cdot \frac{h}{2\sin(h/2)}\right) &= \lim_{h\to 0} \cos\left(\theta + \frac{h}{2}\right) \\
\left(\lim_{h\to 0} \frac{\sin(\theta+h)-\sin(\theta)}{h}\right)\left(\lim_{h\to 0} \frac{h/2}{\sin(h/2)}\right) &= \lim_{h\to 0} \cos\left(\theta + \frac{h}{2}\right) \\
\left(\frac{d}{d\theta}\sin(\theta)\right)\left(\lim_{h\to 0} \frac{h/2}{\sin(h/2)}\right) &= \cos(\theta)
\end{aligned}$$

Let us now let $u = h/2$. Then u approaches 0 as h approaches 0 and

$$\begin{aligned}
\left(\frac{d}{d\theta}\sin(\theta)\right)\left(\lim_{u\to 0} \frac{u}{\sin(u)}\right) &= \cos(\theta) \\
\left(\frac{d}{d\theta}\sin(\theta)\right)\left(\frac{1}{\lim_{u\to 0} \frac{\sin(u)}{u}}\right) &= \cos(\theta) \\
\left(\frac{d}{d\theta}\sin(\theta)\right)\left(\frac{1}{1}\right) &= \cos(\theta)
\end{aligned}$$

which simplifies to $\frac{d}{d\theta}\sin(\theta) = \cos(\theta)$, which completes the proof.

Exercises

Find $y'(t)$ for each of the following:

1.	$y(t) = \sin(3t)$	2.	$y(t) = \cos(3t)$
3.	$y(t) = \cos(\pi t)$	4.	$y(t) = \sin\left(\pi^2 t\right)$
5.	$y(t) = \sqrt{2}\cos\left(\sqrt{7}\,t\right)$	6.	$y(t) = 12\sin\left(\sqrt{\pi}\,t\right)$
7.	$y(t) = t^2 + \sin(2t)$	8.	$y(t) = 2t - \cos(3t)$
9.	$y(t) = e^{-t}\cos(3t)$	10.	$y(t) = e^{-t/3}\sin(\pi t)$
11.	$y(t) = \sin(t)\sin(3t)$	12.	$y(t) = \sin(t)\cos(2t)$
13.	$y(t) = \sin^3(t)$	14.	$y(t) = \cos^3(t)$

Find $f''(x)$. You may want to simplify with trigonometric identities either initially or after the 1st derivative.

15.	$f(x) = \sin^2(x)$	16.	$f(x) = \cos^2(3x)$
17.	$f(x) = (\cos x + \sin x)^2$	18.	$f(x) = (\cos x - \sin x)^2$
19.	$f(t) = \cos^4(t) - \sin^4(t)$	20.	$f(t) = \cos^4(t) + \sin^4(t)$
21.	$f(x) = 3\sin(x) - \sin^3(x)$	22.	$f(x) = 2\sin^2(x) - 2\sin^4(x)$
23.	$q(t) = \dfrac{\sin(t)}{1 + \cos(t)}$	24.	$r(t) = \dfrac{\cos(t)}{\sin(t) - 1}$

Find the period, frequency, and amplitude of the oscillation. Then find rate of change of the harmonic oscillation at the given input..

25.	$y(t) = \sin(3t), \quad p = 0$	26.	$y(t) = \cos(3t), \quad p = 0$
27.	$y(t) = 5\cos(\pi t), \quad p = \frac{1}{3}$	28.	$y(t) = \sin\left(\pi^2 t\right), \quad p = 0$
29.	$y(t) = \cos(t) - \sin(t), \quad p = \frac{\pi}{2}$	30.	$y(t) = \cos(t) + \sin(t), \quad p = \pi$
31.	$y(t) = \cos(3t) + \sin(3t), \quad p = 0$	32.	$y(t) = \sqrt{8}\cos(t) - \sqrt{8}\sin(t), \quad p = 0$

33. Find the equation of the tangent line to $y = \sin(2x)$ at $p = \pi$, and then graph both the curve and the line over $[0, 2\pi]$.

34. Find the equation of the tangent line to $y = \sin(x) + \cos(x)$ at $p = 0$, and then graph both the curve and the line over $[-\pi, \pi]$.

35. The price p per pound of ground beef at time t in years since 1980 (and up to 1998) is approximately the same as the function

$$p(t) = 1.85 + 0.15\sin\left(\frac{2\pi}{9}t\right)$$

How fast is the price of ground beef growing in 1985? In 1995? [2]

36. Elmo's ice cream shop notes that if y denotes the number of customers per week at t weeks since the beginning of the year, then

$$y(t) = 200 + 20\cos\left(\frac{\pi}{26}t\right) - 20\sqrt{3}\sin\left(\frac{\pi}{26}t\right)$$

How fast is the number of customers per week increasing after 13 weeks?

---

[2] Based on Bureau of Labor Statistics data and on examples from Stefan Wagner and Steven R. Costenoble of Hofstra University.

37. If $y(t)$ denotes the number of hours of daylight in Johnson City, TN, on the day which is t days after the beginning of 1999, then

$$y(t) = 12.2 - 2.2855 \cos\left(\frac{2\pi}{365} t\right) + 0.4036 \sin\left(\frac{2\pi}{365} t\right)$$

What is the amplitude and period of the oscillation? How fast is the length of the day changing on the first day of the year?

38. The average monthly temperatures y in Denver, Colorado at t months after the beginning of the year can be closely approximated by the function

$$y = 51.6 - 10.95\sqrt{3} \cos\left(\frac{\pi}{6} t\right) - 10.95 \sin\left(\frac{\pi}{6} t\right)$$

Which month is the coldest month of the year? Which is the hottest month? How fast is the average monthly temperature changing in June?

39. Household current in the United states typically has a voltage of about 117 volts oscillating at 60 cycles per second. The amplitude is defined to be

$$A_m = \sqrt{2}\ voltage$$

Explain why the voltage at an AC wall outlet can be modeled by the function

$$V(t) = 165 \sin(120\pi t)$$

40. **Write to Learn:** In some parts of Europe, the voltage at a wall outlet is given by $g(t) = 320 \sin(60\pi t)$. Describe the differences in amplitude and frequency between this voltage and the voltage at a wall outlet in the US (as described above). (One can actually discern the flicker in an incandescent lamp powered with the voltage described by $g(t)$)

41. The "A" below middle "C" on a piano produces a sound wave whose fundamental pitch can be modeled by

$$y(t) = 60 \sin(880\pi t)$$

What is the frequency of the fundamental pitch of the "A" below middle "C"? How many complete oscillations occur each second? How fast is the sound wave oscillating initially (i.e., when $t = 0$)?

42. Use trigonometry and example 7 to establish the limit

$$\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0$$

Then use the identity

$$\sin(A + B) = \sin(A)\cos(B) + \sin(B)\cos(A)$$

and the limits

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1, \qquad \lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0$$

to compute the derivative of $f(x) = \sin(x)$ using the limit definition of the derivative.

43. Write to Learn: In figures 8-4, 8-5, and 8-15, the chord $\overline{BC}$ has length

$$|BC| = \cos(\theta) - \cos(\theta + h)$$

for θ in the first quadrant and h a sufficiently small positive angle. Mimic the discussion in the first section to suggest that

$$\frac{\cos(\theta + h) - \cos(\theta)}{h} \approx -\sin\left(\theta + \frac{h}{2}\right)$$

Then mimic the discussion at the end of the section to actually prove that

$$\lim_{h \to 0} \frac{\cos(\theta + h) - \cos(\theta)}{h} = -\sin(\theta)$$

43. Trigonometry Review Exercises: Find all solutions to the following:

(a)	$\cos^2(x) - \sin^2(x) = 0$	(b)	$4\cos^2(x) + 1 = 4\cos(x)$
(c)	$\sin(x) - 2\sin^3(x) = 0$	(d)	$2\sin^2(x) + 1 = 3\sin(x)$

2.9 Additional Functions

Derivatives of Tangent, Secant, Cotangent, and Cosecant

The derivatives of the remaining trigonometric functions follow from the derivatives of the sine and cosine functions. For example,

$$\begin{aligned} \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) &= \frac{\left[\frac{d}{dx}\sin(x)\right]\cos(x) - \sin(x)\left[\frac{d}{dx}\cos(x)\right]}{\cos^2(x)} \\ &= \frac{[\cos(x)]\cos(x) - \sin(x)[-\sin(x)]}{\cos^2(x)} \\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \end{aligned}$$

Simplifying with the identity $\cos^2(x) + \sin^2(x) = 1$ thus yields

$$\frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{1}{\cos^2(x)} \qquad \text{or} \qquad \frac{d}{dx}\tan(x) = \sec^2(x)$$

Derivatives of the remaining trigonometric functions are similarly obtained and are summarized in the table below:

Derivatives of trigonometric functions that start with "c" have a negative sign.

$\frac{d}{dx}\sin(x) = \cos(x)$	$\frac{d}{dx}\cos(x) = -\sin(x)$
$\frac{d}{dx}\tan(x) = \sec^2(x)$	$\frac{d}{dx}\cot(x) = -\csc^2(x)$
$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$	$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$

EXAMPLE 1 Find the equation of the tangent line to $y = x\tan(x)$ at $p = \pi$.

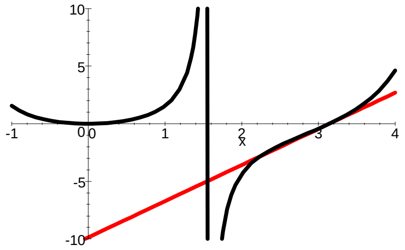
Solution: The product rule implies that the derivative is

$$f'(x) = \left(\frac{d}{dx}x\right)\tan(x) + x\frac{d}{dx}\tan(x)$$
$$= \tan(x) + x\sec^2(x)$$

As a result, $f'(\pi) = \tan(\pi) + \pi\sec^2(\pi) = \pi$. Since $f(\pi) = \pi\tan(\pi) = 0$, the linearization at π is

$$y = 0 + \pi(x - \pi)$$

Thus, $y = \pi x - \pi^2$ is tangent to $y = x\tan(x)$ when $x = \pi$.



9-1: $y = \pi x - \pi^2$ is tangent to $y = x\tan(x)$

Chain rule forms follow by replacing x by a given input and multiplying by the derivative of that input. For example, the chain rule form of the derivative of $\tan(x)$ is

$$\frac{d}{dx}\tan(\text{input}) = \sec^2(\text{input})\frac{d}{dx}(\text{input})$$

EXAMPLE 2 Find $f'(x)$ for $\csc(x^2+1)$.

Solution: The chain rule implies that the derivative is of the form

$$\frac{d}{dx}\csc(\text{input}) = -\csc(\text{input})\cot(\text{input})\frac{d}{dx}(\text{input})$$

where the input is $x^2 + 1$. Thus, we have

$$\frac{d}{dx}\cot(x^2+1) = -\csc(x^2+1)\cot(x^2+1)\frac{d}{dx}(x^2+1)$$
$$= -2x\csc(x^2+1)\cot(x^2+1)$$

Check your Reading *What is the derivative of* $\cos^2(x) + \sin^2(x)$?

Chain Rule and Identities

Often the chain rule must be used more than once in derivatives involving trig functions. In such instances, it will be important to progress through the computation in a neat and organized fashion.

EXAMPLE 3 Find $f'(x)$ for $f(x) = \tan^2(x^3)$.

Solution: Since $\tan^2(x^3) = [\tan(x^3)]^2$, the chain rule leads to

$$\frac{d}{dx}[\text{input}]^2 = 2[\text{input}]^1 \frac{d}{dx}\text{input}$$

where the input is $\tan(x^3)$. The result is

$$\frac{d}{dx}\left[\tan(x^3)\right]^2 = 2\left[\tan(x^3)\right] \frac{d}{dx}\tan(x^3)$$

We apply the chain rule again to obtain

$$\begin{aligned}\frac{d}{dx}\left[\tan(x^3)\right]^2 &= 2\left[\tan(x^3)\right] \frac{d}{dx}\tan(\text{input}) \\ &= 2\left[\tan(x^3)\right] \sec^2(\text{input}) \frac{d}{dx}\text{input}\end{aligned}$$

where the input is x^3. This simplifies to

$$\begin{aligned}\frac{d}{dx}\left[\tan(x^3)\right]^2 &= 2\left[\tan(x^3)\right] \sec^2(x^3) \frac{d}{dx}x^3 \\ &= 2\tan(x^3)\sec^2(x^3) \cdot (3x^2) \\ &= 6x^2\tan(x^3)\sec^2(x^3)\end{aligned}$$

As before, identities are often used to simplify derivative calculations.

EXAMPLE 4 Find $f'(x)$ when $f(x) = \sec^4(x) - \sec^2(x)\tan^2(x)$

Solution: To begin with, we can factor out $\sec^2(x)$ to obtain

$$f(x) = \sec^2(x)\left[\sec^2(x) - \tan^2(x)\right]$$

We then simplify $f(x)$ using the identity $\sec^2(x) = 1 + \tan^2(x)$:

$$f(x) = \sec^2(x)\left[1 + \tan^2(x) - \tan^2(x)\right] = \sec^2(x)$$

Finally, since $\sec^2(x) = (\sec x)^2$, we apply the chain rule with an input of $\sec(x)$:

$$f'(x) = \frac{d}{dx}(\text{input})^2 = 2(\text{input})^1 \frac{d}{dx}(\text{input})$$

Replacing the input by $\sec(x)$ then leads to

$$f'(x) = 2\sec(x)\frac{d}{dx}\sec(x) = 2\sec(x)\left[\sec(x)\tan(x)\right]$$

which simplifies to $f'(x) = 2\sec^2(x)\tan(x)$.

Check your Reading *What is the derivative of* $\cos^2(x) + \sin^2(x)$?

Exponential and Logarithms with Arbitrary Bases

If $y = a^x$ for some fixed base $a > 0$, then $\ln(y) = \ln(a^x) = x\ln(a)$. Thus, $y = e^{x\ln(a)}$ which leads us to define a^x to be

$$a^x = e^{x\ln(a)} \tag{2.50}$$

Moreover, if $y = a^x$, then $\ln(y) = x\ln(a)$ and logarithmic differentiation yields

$$\frac{y'}{y} = \ln(a) \qquad \Longrightarrow \qquad y' = y\ln(a)$$

Since $y = a^x$, this yields the following:

$$\frac{d}{dx}a^x = a^x\ln(a) \tag{2.51}$$

In chain rule form, this rule becomes

$$\frac{d}{dx}a^{\text{input}} = a^{\text{input}}\ln(a)\ \frac{d}{dx}(\text{input})$$

where $a > 0$.

EXAMPLE 5 Find y' when $y = 2^{\sin(x)}$.

Solution: Since $y = 2^{\text{input}}$ if the input is $\sin(x)$, application of (2.51) yields

$$y' = \frac{d}{dx}2^{\text{input}} = 2^{\text{input}}\ln(2)\ \frac{d}{dx}(\text{input})$$

Since the input is $\sin(x)$, this in turn becomes

$$y' = 2^{\sin(x)}\ln(2)\frac{d}{dx}(\sin x) = 2^{\sin(x)}\ln(2)\cos(x)$$

In addition, we define $y = \log_a(x)$ to mean that $x = a^y$. However, $x = a^y$ implies that

$$\ln(x) = y\ln(a), \qquad y = \frac{\ln(x)}{\ln(a)}$$

Consequently, the logarithm base a is given by

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

where $a > 0$.

EXAMPLE 6 What is the slope and equation of the tangent line to $y = \log_{10}(x)$ at $x = 1$?

Solution: To begin with, $\log_{10}(x) = \frac{\ln(x)}{\ln(10)}$, so that

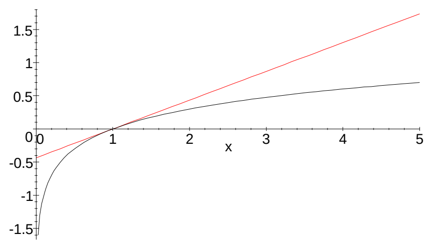
$$y' = \frac{1}{\ln(10)} \frac{d}{dx} \ln(x) = \frac{1}{x \ln(10)}$$

Thus, the slope of the tangent line at $x = 1$ is

$$y'(1) = \frac{1}{1 \ln(10)} = 0.4343$$

Since $y(0) = \log_{10}(1) = 0$, the equation of the tangent line at $x = 1$ is

$$y = 0 + 0.4343(x - 1) = 0.4343(x - 1)$$



9-2: The line $y = 0.4343(x - 1)$ is tangent to $y = \log_{10}(x)$ at $(1, 0)$.

Check your Reading *What is the slope of the tangent line to* $y = \cosh(x)$ *at* $x = 0$?

Derivatives of Functions of the form $[f(x)]^{g(x)}$

The definition (2.50) implies that functions of the form $[f(x)]^{g(x)}$ are defined

$$[f(x)]^{g(x)} = e^{g(x) \ln[f(x)]}$$

Logarithmic differentiation is then used to find the derivative of a function of the form $[f(x)]^{g(x)}$.

EXAMPLE 7 Find y' when $y = (x^2 + 1)^x$.

Solution: To do so, we apply the natural logarithm to obtain

$$\ln(y) = \ln(x^2 + 1)^x = x \ln(x^2 + 1)$$

Thus, via the product rule we obtain

$$\begin{aligned} \frac{y'}{y} &= \frac{d}{dx}\left[x \ln(x^2 + 1)\right] \\ &= \left[\frac{d}{dx} x\right] \ln(x^2 + 1) + x \frac{d}{dx} \ln(x^2 + 1) \\ &= \ln(x^2 + 1) + x \frac{2x}{x^2 + 1} \end{aligned}$$

As a result, we obtain

$$y' = y\left[\ln(x^2 + 1) + \frac{2x^2}{x^2 + 1}\right]$$

which after substituting $y = \left(x^2+1\right)^x$ yields

$$\frac{d}{dx}\left(x^2+1\right)^x = \left(x^2+1\right)^x \left[\ln\left(x^2+1\right) + \frac{2x^2}{x^2+1}\right]$$

Exercises:

Find the first derivative of each of the following: .

1.	$f(x) = \tan(2x)$	2.	$f(x) = \sec(3x)$
3.	$f(x) = x^2 \cot(x)$	4.	$f(x) = x \csc(2x)$
5.	$f(x) = \tan^5\left(x^3\right)$	6.	$f(x) = \sec^2(2x+7)$
7.	$f(x) = e^x \sec(2x)$	8.	$f(x) = \ln(\csc x)$
9.	$f(x) = 2\sin(x)\csc(2x)$	10.	$f(x) = \frac{1}{2}\ln(\cot x)$
11.	$f(x) = \ln(\sec x)$	12.	$f(x) = \frac{1}{2}\ln(\tan x)$
13.	$f(x) = e^{-x}\sin(e^x)$	14.	$f(x) = e^{-x}\sec(e^x)$
15.	$f(x) = \ln(\sec x + \tan x)$	16.	$f(x) = \ln(\csc x + \cot x)$

17.	$y = 3^x$	18.	$y = 5^{-x}$
19.	$y = \pi^{-2x}$	20.	$y = 1^x$
21.	$y = 10^{\ln(x)}$	22.	$y = \pi^{-\sin(x)}$
23.	$y = \log_5(x)$	24.	$y = \log_3(x)$
25.	$y = \log_2\left(x^3+1\right)$	26.	$y = \log_{10}(2x+3)$
27.	$y = x^x$	28.	$y = x^{\ln(x)}$
29.	$y = \left(1+\frac{1}{x}\right)^x$	30.	$y = (x+1)^{\sin(x)}$

31. Find the slope and equation of the tangent line to $y = \tan(x)$ at the following points: $x = -\pi$, $x = 0$, and $x = \pi$

32. Find the slope and equation of the tangent line to $y = 10^x$ at $x = 1$.

33. Find the slope and equation of the tangent line to $y = \sec(x)$ at $p = \pi$. .

34. Find the slope and equation of the tangent line to $y = \tan\left(x^3\right)$ at $p = 0$.

35. Use the definition of $\log_a(x)$ to show that

$$\log_a(b) = \frac{\log_c(b)}{\log_c(a)}$$

for any positive numbers a, b, and c.

36. What is significant about the function $\log_x(x)$? What is its derivative?

37. Consider $f(x) = x^{\sqrt{x}}$.

(a) Determine the domain of $f(x)$ using the fact that $\ln(x)$ is defined only when $x > 0$.

(b) Graph the function on its domain with the grapher.

38. Consider $f(x) = (-x)^{\sqrt{x+10}}$.

(a) Determine the domain of $f(x)$ using the fact that $\ln(x)$ is defined only when $x > 0$.

(b) Graph the function on its domain with the grapher.

39. What is significant about the function $x^{1/\ln(x)}$. What is its derivative?

40. **Mathematical Gibberish**.

(a) Explain why the function $f(x) = \ln(x)^{\sqrt{1-x}}$ is not defined for any x.

(b) Input the function into a grapher and attempt to graph the function. Also, try to evaluate the function at a particular value or generate a table of values. What happens?

41. Use the identity

$$\tan(A+B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$$

and the limit

$$\lim_{h\to 0} \frac{\tan(h)}{h} = 1$$

to compute the derivative of $f(x) = \tan(x)$ using the limit definition of the derivative.

42. **Write to Learn:** A trigonometric function c_f is a *cofunction* of a trig function f if

$$c_f(\phi) = f(\theta)$$

when ϕ is the *complementary angle* of θ (i.e., $\phi+\theta = \frac{\pi}{2}$). Write a short essay in which you use the definition of a cofunction to explain why derivatives of trigonometric functions that start with "c" have a negative sign in front.

Self Test

A variety of questions are asked in a variety of ways in the problems below. Answer as many of the questions below as possible before looking at the answers in the back of the book.

1. Answer each statement as true or false.

   (a) Applying the product rule to $x^2 \cdot x^3$ will yield a different result than applying the power rule to x^5.

   (b) $\frac{d}{dx}x^n = nx^{n-1}$ is called the power rule.

   (c) $\frac{d}{dx}[(x^2-2)f(x)] = 2xf'(x) + (x^2-2)f(x)$.

   (d) If $f(x+h) = \frac{4}{5}x^3 + \frac{12}{5}x^2h + \frac{12}{5}xh^2 + h^3$, then $f'(x) = \frac{12}{5}x^2$.

   (e) The derivative function, $f'(x)$, is the function which gives the rate of change of the function $f(x)$ at the input x.

   (f) If $y = x^2$ where x and y are functions of t, then

   $$\frac{dy}{dt} = 2x\frac{dx}{dt}$$

   (g) The point $(2,1)$ lies on the curve $x^2 + y^3 = 3$.

   (h) The curve given by $y = x^3 - x^2$ satisfies the differential equation $y' + 2x = 3x^2$.

   (i) If $C(x)$ is the cost of producing x objects, then $C'(100)$ is the approximate cost of producing the 101^{st} object.

   (j) The equation $\frac{d}{dx}k = 0$ where k is a constant function says that k has no rate of change at any particular input value. (*Note: Recall that a vertical line has no slope and a horizontal line has* 0 *slope.*)

2. Answer each statement as true or false. If false, determine the reason.

   (a) The natural logarithm, $\ln(x)$, is the inverse function of the exponential function, e^x.

   (b) There is a real number r such that $e^{\ln(r)} \neq \ln(e^r)$.

   (c) The natural logarithm, $\ln(x)$, is defined for all real numbers x.

   (d) One of the law of exponents states that $e^{ab} = e^a e^b$.

   (e) One of the properties of the logarithm is that $\ln(a+b) = \ln(a)\ln(b)$.

   (f) The exponential function is the only function equal to its derivative.

   (g) If $y = \sin(x)$, then $y'' = -\sin(x)$.

   (h) If $y = \cot(x)$, then y' is negative.

3. If $y = \ln\left(x^2\right)$, then y' is **not** equal to which of the following:

   (a) $\frac{1}{x^2}$ (b) $\frac{2}{x}$ (c) $\frac{2x}{x^2}$ (d) $2 \cdot \frac{1}{x}$

4. $\ln(e^{3x^2-7})$ simplifies to

   (a) $\ln(3x^2-7)$ (b) $\ln(e^{3x^2}) + \ln(e^{-7})$ (c) $3x^2 - 7$ (d) e^{3x^2-7}

5. $2\sin\left(\frac{-x}{2}\right)\cos\left(\frac{-x}{2}\right)$ is the derivative of which of the following functions:

(a) $\sin(x)$ (b) $\cos(x)$ (c) $\sin(2x)$ (d) $\cos(2x)$

6. What is $f'(x)$ if $f(x) = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^3$?

(a) $3\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$ (b) $3\left(\frac{1}{2\sqrt{x}} - \frac{1}{2x^{3/2}}\right)^2$

(c) $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)\left(\frac{1}{2\sqrt{x}} - \frac{1}{2x^{3/2}}\right)$ (d) $3\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2\left(\frac{1}{2\sqrt{x}} - \frac{1}{2x^{3/2}}\right)$

7. What is $f'(x)$ if $f(x) = \sqrt{g(x)}$?

(a) $\frac{2}{\sqrt{g(x)}}g'(x)$ (b) $\frac{1}{2\sqrt{g(x)}}g'(x)$ (c)$\frac{1}{2\sqrt{g(x)}}$ (d)$\sqrt{g(x)}g'(x)$

8. Find the slope of the tangent line to the curve given by $x^2 + y^3 = 3$ at the point $(2, -1)$.

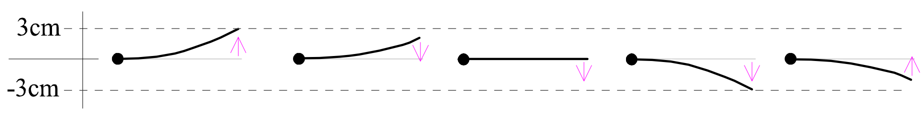
(a) $-4/3$ (b) $-3/4$ (c) $4/3$ (d) $3/4$

9. A cistern is in the shape of an inverted right circular cone with base radius $r = 3$ ft. and height $h = 5$ ft. . If rain is falling such that it is filling the cistern at a rate of 2.35 ft.^{3}/hour (equivalent to 1$''$ of rain per hour) , find the rate at which the water level is rising when the height of the water 2 ft.. (Hint: the volume of a right circular cone is given by $V = \frac{1}{3}\pi r^2 h$.)

(a) about 6 inches per hour (b) about 9 inches per hour (c) about 1 foot per hour (d)

10. What is the tangent line to $f(x) = 3x + 2$?

11. A reed oscillates up and down, causing its free end to have a vertical displacement y cm from horizontal at time t in seconds satisfying $y = 3\sin(440\pi t)$.



What is the amplitude, period, and frequency of the oscillation? How fast is the end of the reed oscillating when $y = 3$? How fast is the reed oscillating when $y = 0$?

12. What is the derivative of $f(x) = \sin\left(x^3 + 1\right)$?

13. What is the derivative of $f(x) = x(x+1)(x+2)(x+3)$?

14. What is the derivative of $y = x^{2x}$?

15. What is the slope and equation of the tangent line to $y = \tan^2(x + \sin(2x))$ at $x = \pi$?

16. Blue Spruce are grown for eight to twelve years before being harvested for sale as Christmas trees. Suppose the total cost of growing x Blue Spruce for eight years is given by $C(x) = 2800 + 1.3x + 0.002x^2$ in dollars. What is the marginal cost of growing 850 Blue Spruce for eight years? Give the units.

17. **Write to Learn:** In a couple of sentences, use the logarithm and the fact that $|x| = \sqrt{x^2}$ to prove that

$$\frac{d}{dx}\ln(|x|) = \frac{1}{x}$$

18. **Write to Learn:** A sphere of radius r has a volume of

$$V = \frac{4}{3}\pi r^3$$

and a surface area of $S = 4\pi r^2$. Derive the equation

$$\frac{dV}{dr} = S$$

and then explain it geometrically by discussing why a small change in the radius should cause a small change in volume that is approximately the same as the surface area of the sphere.

The Next Step... The Q-Derivative

Is the concept of the limit absolutely necessary for calculus? Is there some way to "do" all of calculus similar to the "drop higher powers of h" method in chapter 1? Indeed there is, and our next step is to briefly describe a new type of calculus that does not require limits.

When we say calculus without limits, then we are saying that we can define the derivative, develop analogues of differentiation rules, and recover the interpretation of the derivative as a slope of a tangent line and as a rate of change. To do so, however, requires that we first develop a different type of difference quotient.

To begin with, let's recall that the derivative is defined by the limit

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

For example, if $f(x) = x^2$, then $f(x+h) = (x+h)^2$ and

$$f'(x) = \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0}\frac{2xh + h^2}{h} = \lim_{h\to 0}(2x+h) = 2x$$

Moreover, we can transform the limit by letting $r = x + h$. In particular, $h = r - x$ and r approaches x as h approaches 0, so that the definition of the derivative becomes

$$f'(x) = \lim_{r\to x}\frac{f(r) - f(x)}{r - x} \tag{2.52}$$

For example, if $f(x) = x^2$, then $f(r) = r^2$ and

$$f'(x) = \lim_{r\to x}\frac{r^2 - x^2}{r - x} = \lim_{r\to x}\frac{(r-x)(r+x)}{r - x} = \lim_{r\to x}(r+x) = 2x$$

Let's suppose now that we let $r = qx$, where q is a number. Then $q = r/x$ and as r approaches x, the number q must approach 1. Thus, (2.52) becomes

$$f'(x) = \lim_{q\to 1}\frac{f(qx) - f(x)}{qx - x}$$

For example, if $f(x) = x^2$, then $f(qx) = q^2x^2$ and

$$f'(x) = \lim_{q \to 1} \frac{q^2x^2 - x^2}{qx - x} = \lim_{q \to 1} \frac{(q^2 - 1)x^2}{(q - 1)x} = \left(\lim_{q \to 1} \frac{q^2 - 1}{q - 1}\right) x = 2x$$

Notice that the final limit involves only q, which is unlike the other two calculations of the derivative of $f(x) = x^2$.

It is this separation of the limit from the variable x that allows us to "do" calculus without limits. To begin with, we let q be a fixed number that is close to 1 (e.g., let $q = 1.01$). We then define the q-integers $[n]$ to be numbers of the form

$$[n] = \frac{q^n - 1}{q - 1}$$

and we notice that the q-integers $[0]$ and $[1]$ reduce to 0 and 1, respectively:

$$[0] = \frac{q^0 - 1}{q - 1} = \frac{1 - 1}{q - 1} = 0, \qquad [1] = \frac{q^1 - 1}{q - 1} = \frac{q - 1}{q - 1} = 1$$

However, the other q-integers reduce to the ordinary integers only *in the limit* as q approaches 1. For example,

$$[2] = \frac{q^2 - 1}{q - 1}$$

which is not equal to 2 for any $q \neq 1$. Yet in the limit we have

$$\lim_{q \to 1} [2] = \lim_{q \to 1} \frac{q^2 - 1}{q - 1} = \lim_{q \to 1} \frac{(q + 1)(q - 1)}{q - 1} = \lim_{q \to 1} (q + 1) = 2$$

We now define the q-derivative to be the difference quotient

$$f^q(x) = \frac{f(qx) - f(x)}{qx - x}$$

This is not the ordinary derivative, but it does reduce to the ordinary derivative *in the limit* as q approaches 1. For example, if $f(x) = x^2$, then

$$f^q(x) = \frac{q^2x^2 - x^2}{qx - x} = \frac{(q^2 - 1)x^2}{(q - 1)x} = \left(\frac{q^2 - 1}{q - 1}\right) x = [2]\,x$$

and thus, $f^q(x)$ approaches $f'(x)$ as q approaches 1.

It can be shown that the q-derivative satisfies rules similar to the rules of the ordinary derivative. For example, if we let $\frac{d_q}{d_q x}$ denote the q-derivative operator, then

$$\frac{d_q}{d_q x}(f(x) + g(x)) = f^q(x) + g^q(x)$$

Moreover, the power rule takes on a very familiar form for positive integers n,

$$\frac{d_q}{d_q x} x^n = [n]\,x^{n-1}$$

as does the *q-product rule* for two functions $f(x)$ and $g(x)$:

$$\frac{d_q}{d_q x}[f(x)\,g(x)] = f^q(x)\,g(x) + f(qx)\,g^q(x)$$

There is also a q-chain rule, a q-exponential function, q-logarithms, q-trigonometric functions, and a host of other q-analogues of ordinary calculus.

Moreover, as the examples above show, the limit concept is required only to reduce q-calculus results to results in ordinary calculus—limits are not required for the q-calculus itself! Those interested in learning more might want to read the undergraduate textbook *Quantum Calculus* by Victor Kac and Pokman Cheung (Springer Verlag; ISBN: 0387953418; 1st edition, December 15, 2001).[3]

Write to Learn In a short essay, derive the q-derivative of $f(x) = x^3$ and then explain what it reduces to as q approaches 1.

Write to Learn Write a short essay in which you derive the q-product rule

$$\frac{d_q}{d_q x}[f(x)\,g(x)] = f^q(x)\,g(x) + f(qx)\,g^q(x)$$

(Hint: you might want to start on the right and simplify to the left of the above equation).

Write to Learn Go to the library or search the internet to find out more about the q-calculus. Present your findings in a written report.

Group Learning Even though the q-calculus avoids the limit concept, it will not ever replace the usual calculus. This is because some of the most important results about the q-calculus are stated in terms of the ordinary calculus. For example, if $f(x)$ is a polynomial of degree n, then it can be shown that

$$f^q(x) = f'(x) + \frac{xq-x}{2!}f''(x) + \frac{(xq-x)^2}{3!}f'''(x) + \ldots + \frac{(xq-x)^{n-1}}{n!}f^{(n)}(x) \tag{2.53}$$

where $f^{(n)}(x)$ is the ordinary derivative and $n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$ is the product of the first n positive integers. Have each member of the group perform one of the following steps. Report your results in either a presentation or a formal paper.

(a) Verify (2.53) for $f(x) = 1$ and $f(x) = x$.

(b) Verify (2.53) for $f(x) = x^2$ by showing that

$$\frac{d_q}{d_q x}x^2 = f'(x) + \frac{xq-x}{2}f''(x)$$

(c) Verify (2.53) for $f(x) = x^3$ by showing that

$$\frac{d_q}{d_q x}x^3 = f'(x) + \frac{xq-x}{2}f''(x) + \frac{(xq-x)^2}{6}f'''(x)$$

(d) Verify (2.53) for $f(x) = x^4$ by showing that

$$\frac{d_q}{d_q x}x^4 = f'(x) + \frac{xq-x}{2}f''(x) + \frac{(xq-x)^2}{6}f'''(x) + \frac{(xq-x)^3}{24}f^{(4)}(x)$$

Advanced Contexts:

Logarithms were introduced independent of limits and even calculus itself. In fact, long before computers became commonplace, logarithms were often used in calculations. These computations were based primarily on the properties

$$\ln(ab) = \ln(a) + \ln(b) \qquad \textit{and} \qquad \ln(a^r) = r\ln(a)$$

[3] Much of this material was adapted from the online newsletter "This Week's Finds in Mathematical Physics (Week 183)" by John Baez (http://math.ucr.edu/home/baez/).

In particular, this property allows the product of two numbers a and b to be reduced to the sum of two logarithms, $\ln(a)$ and $\ln(b)$. For example, to compute $2 \cdot 3$, we could first estimate $\ln(2)$ and $\ln(3)$:

$$\begin{aligned} \ln(2) &= 0.69314\,71805 \\ \ln(3) &= 1.09861\,22887 \end{aligned}$$

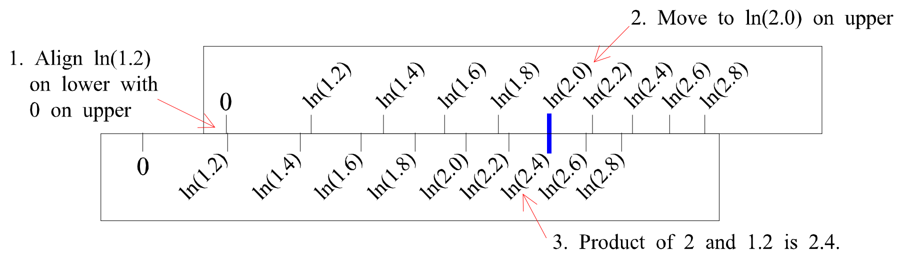
and then add these expressions together:

$$\ln(2) + \ln(3) = 1.79175\,94692$$

The result is $\ln(6)$, which thus implies that $2 \cdot 3 = 6$.

This concept resulted in a mechanical device called a *slide rule,* which was comprised of two "rulers" marked off in units of logarithms of decimal numbers. To compute the product $a \cdot b$, the number "0" on the upper ruler was aligned with the number a on the lower ruler. The number b was then located on the upper ruler, so that the product $a \cdot b$ was the corresponding number on the lower ruler.

For example, to compute the product of 1.2 and 2.0, we would first align the number $\ln(1.2)$ on the lower ruler with "0" on the upper ruler:



NS-1: A Slide Rule is marked in logarithmic units

The number $\ln(2.0)$ would then be located on the upper ruler. Having done so, the product of 1.2 and 2.0 is the corresponding location on the lower ruler. Thus, the slide rule tells us that $1.2 \times 2.0 = 2.4$.

1. Use the two tables below to label two metric rulers

Location	0	4.0 cm	7.3 cm	10.1 cm	12.5 cm	14.6 cm	16.5 cm	18.2 cm
Label	0	$\ln(1.25)$	$\ln(1.5)$	$\ln(1.75)$	$\ln(2)$	$\ln(2.25)$	$\ln(2.5)$	$\ln(2.75)$

Location	19.8 cm	21.2 cm	22.5 cm	23.8 cm	24.9	26.0	27.1
Label	$\ln(3)$	$\ln(3.25)$	$\ln(3.5)$	$\ln(3.75)$	$\ln(4)$	$\ln(4.25)$	$\ln(4.5)$

and then use them to compute 1.5×2.5.

2. Logarithms were originally invented by John Napier. However, Napier defined the logarithm of N to be the number L for which

$$N = 10^7 \left(1 - 10^{-7}\right)^L$$

Show that L is of the form

$$L = \log_a \left(10^{-7} N\right)$$

What is a?

3. How would you do division with a slide rule? For example, how would you compute $2.4 \div 2.0$?

3. APPLICATIONS OF THE DERIVATIVE

Calculus was developed independently in the late seventeenth century by both Sir Isaac Newton of England and Gottfried Leibniz of Germany. However, Newton and Leibniz did not develop calculus in the context of functions. In fact, the function concept was not established until a two volume treatise by Euler published in 1748.[1] Instead, both Newton and Leibniz viewed calculus as a tool to be applied in the study of *analytic geometry.*

Unfortunately, the original formulations of calculus were based on intuition rather than rigor, and as a result, calculus seemed to produce contradictory results. In the early 1800's, the mathematician Augustin-Louis Cauchy sought to rectify this situation by establishing an axiomatic foundation of Calculus similar in form to the axiomatic foundations of geometry established by Euclid. His principal tool in this endeavor was the *Mean Value Theorem,* which even today is considered foundational to much of calculus.

In particular, the Mean Value theorem is the basis for the tools used in graphing functions and in finding their extreme values. In this chapter, we begin with the Mean Value theorem and its implications, which in turn sets the stage for the applications of the derivative in the second part of this chapter. The result will provide valuable insights into how concepts such as tangents and rates of change are applied to real-world problems.

3.1 The Mean Value Theorem

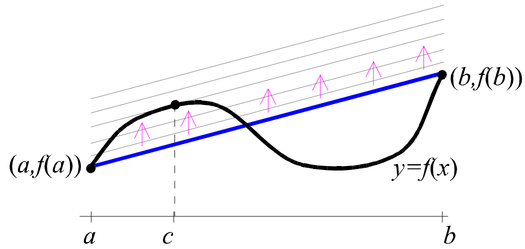
The Mean Value Theorem

While limits and derivatives are fundamental concepts in calculus, they alone are not sufficient to allow calculus to be as applicable and widely-used as it is today. Instead, as we will see in this section, the theoretical foundation of calculus is the *Mean Value Theorem.*

Suppose $f(x)$ is defined on $[a, b]$. If the secant line through $(a, f(a))$ and $(b, f(b))$ is translated vertically, then at least for the function in the figure below, the result is a family of secant lines that becomes closer and closer to a tangent

[1] However, Gottfried Liebniz did use the term "function" in a manner consistent with the function concept we use today.

line at some input c in (a, b).

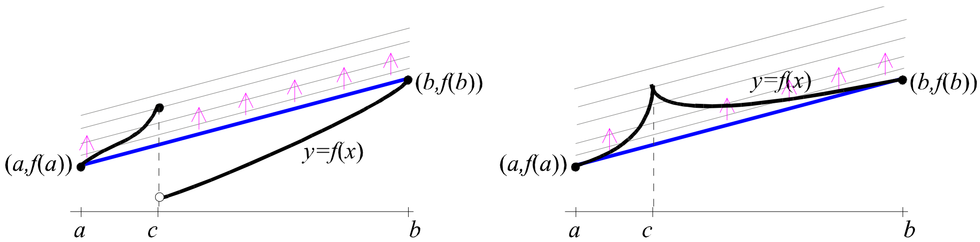


1-1: One of the parallels is a tangent line to the curve

Thus, the slope $f'(c)$ of the tangent line must be the same as the slope of the secant line, which means that

$$\frac{f(b) - f(a)}{b - a} = f'(c) \tag{3.1}$$

However, this concept is not valid for all functions $f(x)$. In the plots below, the secant line through $(a, f(a))$ and $(b, f(b))$ is also translated vertically, but in these cases without implying the existence of a tangent line parallel to the original secant.



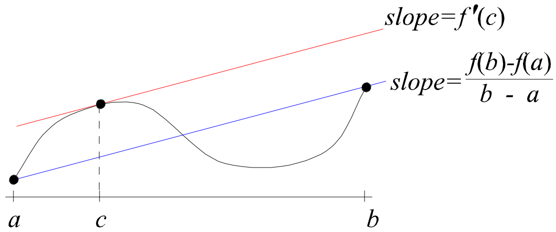
f is not continuous on [*a*,*b*] *f has a cusp at a point in* [*a*,*b*]

1-2: Figure 1-1 requires continuity and differentiability of f

Figure 1-2 illustrates why $f(x)$ must be continuous on $[a, b]$ and differentiable on (a, b) in order for (3.1) to hold. Given these conditions, however, the following theorem can be established (though the proof is beyond this text).

The Mean Value Theorem: If $f(x)$ is continuous on $[a, b]$ and is differentiable on (a, b), then there is a number c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c) \tag{3.2}$$



1-3: The Mean Value Theorem

A more useful form of the Mean Value Theorem follows from 2 observations: First, if $f(x)$ is differentiable on an open interval containing $[a, b]$, then $f(x)$ must be continuous on $[a, b]$. Second, (3.2) is equivalent to $f(b) - f(a) = f'(c)(b - a)$.

The Mean Value Theorem (Restated): If $f(x)$ is differentiable on an open interval containing $[a,b]$, then there is a number c in $[a,b]$ such that

$$f(b) - f(a) = f'(c)(b-a) \tag{3.3}$$

This latter version of the Mean Value theorem (MVT) holds even when $a = b$ (i.e., even when $[a,b]$ is a single point). This makes it quite useful in proving the fundamental theorems in calculus, such as the one below:

Theorem 1.2: If $f'(x) = 0$ for all x in (p,q), then $f(x)$ is constant over (p,q).

Proof. If $[a,b]$ is contained in (p,q), then the MVT implies that

$$f(b) - f(a) = f'(c)(b-a)$$

for some c in $[a,b]$. However, $f'(c) = 0$ since c is in (p,q), which implies that

$$f(b) - f(a) = 0 \qquad or \qquad f(b) = f(a)$$

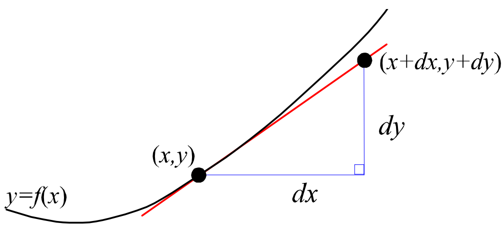
for all a, b in (p,q). Thus, all the outputs of $f(x)$ over (p,q) are the same, which is to say that $f(x)$ is constant. ■

We will not prove the Mean Value theorem here, as the proof requires some remarkable results from the study of the topology of the real line. However, a sketch of the proof is included in the appendix for completeness.

Check your Reading *How do we obtain (3.3) from (3.2)?*

Differentials Revisited

Let's look at another example in which the Mean Value theorem is used to establish an important result in calculus. Recall that the *differentials* of x and y, which are denoted by dx and dy, respectively, are defined to be small changes along the tangent line to the curve at a point (x,y).



1-4: The differentials dx and dy

Thus, if $y = f(x)$ passes through (x,y), then $f'(x)$ is the slope of the tangent line at (x,y) and consequently,

$$dy = f'(x)\,dx \tag{3.4}$$

Equivalently, we can write (3.4) in the form $dy = y'\,dx$.

EXAMPLE 1 Find the differential dy for the curve $y = x^3 + 2x$.

Solution: Since $y' = 3x^2 + 2$, the differential dy is $dy = \left(3x^2 + 2\right) dx$

EXAMPLE 2 Find dy for $y = x \sin(x)$.

Solution: Since $y' = \sin(x) + x\cos(x)$, the differential is

$$dy = [\sin(x) + x\cos(x)]\, dx$$

Suppose now that $f(x)$ has a continuous derivative on a neighborhood of a point p. If $a = p$ and $b = p + dx$, then the MVT implies that

$$f(p + dx) - f(p) = f'(c)\, dx$$

The quantity $\Delta y = f(p + dx) - f(p)$ is the *change* in $y = f(x)$ over $[p, p + dx]$. Moreover, the MVT implies that

$$\Delta y = f'(c)\, dx$$

for some c in $[a, b]$. Since f' is continuous, it follows that $f'(c) \approx f'(p)$ when dx is sufficiently close to 0. As a result, $\Delta y \approx dy$. That is, the MVT implies that the change Δy is closely approximated by the differential dy when dx is close to 0.

EXAMPLE 3 Find Δy and dy for $f(x) = x^3$ when $p = 1$ and $dx = 0.05$.

Solution: Since $\Delta y = f(p + dx) - f(p)$, we have

$$\Delta y = f(1.05) - f(1) = (1.05)^3 - 1^3 = 0.157625$$

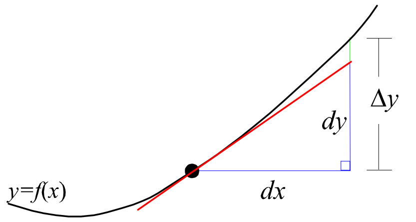
Since $f'(x) = 3x^2$, we have $f'(1) = 3$ and

$$dy = f'(1)\, dx = 3 \cdot 0.05 = 0.15$$

which closely approximates the actual value of $\Delta y = 0.157625$.

Check your Reading *By how much does Δy differ from dy in example 2?*

Graphically, dy is a change in y *along the tangent line,* while Δy is the resulting change in y *along the curve.*



1-5: dy is close to Δy for dx close to 0

The fact that $\Delta y \approx dy$ when dx is sufficiently close to 0 is thus a reflection of the fact that a tangent line is practically the same as a small section of the curve.

EXAMPLE 4 Find Δy and dy for $f(x) = x^3 + 2x$ when $p = 1$ and $dx = 0.1$, and then illustrate them graphically.

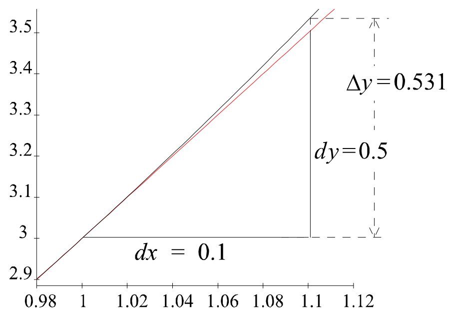
Solution: Since $p = 1$, and $p + dx = 1.1$, we have

$$\begin{aligned} \Delta y &= f(1.1) - f(1) \\ &= (1.1)^3 + 2(1.1) - \left(1^3 + 2 \cdot 1\right) \\ &= 0.531 \end{aligned}$$

To find dy, we first notice that $f'(x) = 3x^2 + 2x$ so that $f'(1) = 5$. As a result, the differential dy is

$$dy = f'(x)\,dx = f'(1)\ (0.1) = 5 \cdot 0.1 = 0.5$$

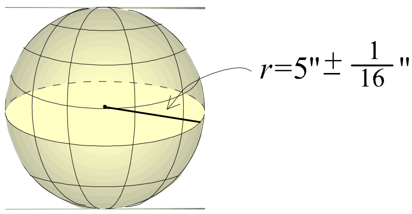
which closely approximates the actual value of $\Delta y = 0.531$.



1-6: dy approximates dy

As a result, dy can be used to approximate Δy in applications where dx is sufficiently close to 0. In particular, a *tolerance* is defined to be the maximum allowable error in the measurement of a quantity, so that if dx is the tolerance for a quantity x, then dy is often used as an approximation of the tolerance in y.

EXAMPLE 5 Find the volume V and approximate the tolerance dV of a sphere whose radius is measured to be 5 inches to within a tolerance of $\frac{1}{16}$ of an inch.



1-7: Sphere with radius of $5 \pm \frac{1}{16}$ inches.

Solution: The volume of a sphere with radius r is given by

$$V = \frac{4}{3}\pi r^3 \tag{3.5}$$

which implies that the volume of a sphere with radius 5 inches is

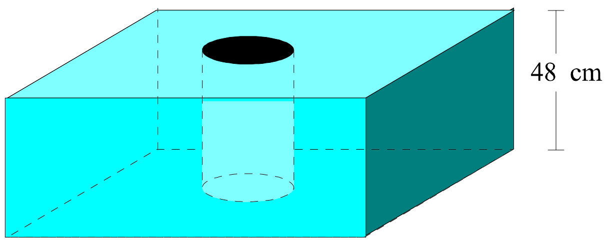
$$V = \frac{4}{3}\pi (5)^3 = \frac{500\pi}{3} \approx 523.6 \text{ in}^3$$

Since $V' = 4\pi r^2$, the differential is $dV = 4\pi r^2 dr$. Thus,

$$dV = 4\pi (5)^2 \frac{1}{16} = 19.6 \; in^3$$

That is, the sphere's volume is 523.6 in^3, give or take about 19.6 in^3.

EXAMPLE 6 A right cylindrical hole is bored through a block of steel with a uniform width of 48 cm. If the radius of the hole is 5 cm to within a tolerance of 1 mm, then what is the tolerance in the volume of the resulting hole?



1-8: Hole drilled through a block

Solution: If r is the radius of the hole, then the volume of the hole is $V = \pi r^2 \cdot 48$ since the height of the hole is 48 cm. Moreover, $V'(r) = 96\pi r$ and thus, $V'(1) = 96 \cdot \pi \cdot 5 = 1507.96$. Since $dr = 1$ mm $= 0.1$ cm, we have

$$dV = V'(1)\, dr = 1507.96 \cdot 0.1 = 150.796 \text{ cm}^3$$

Check your Reading *What is the volume of the hole in example 6?*

Proofs Based on the Mean Value Theorem

The mathematician Augustin-Louis Cauchy tried to base all of calculus on the Mean Value theorem, much like Euclid based geomety on his 5 postulates. Although he was not completely successful, much of his program is still used today. That is, the Mean Value theorem continues to be used in graduate and undergraduate mathematics courses to prove many of the theorems of calculus.

EXAMPLE 7 Use the MVT to prove that if $b > 1$, then

$$\ln(b) \leq b - 1$$

Solution: To do so, we apply the MVT in the form (3.3) to the function $f(x) = \ln(x)$ on $[1, b]$. That is, since $f'(x) = \frac{1}{x}$, there is a number c in $[1, b]$ such that

$$\ln(b) - \ln(1) = \frac{1}{c}(b - 1)$$

However, c in $[1, b]$ implies that $c \geq 1$, which means that $\frac{1}{c} \leq 1$. Since $\ln(1) = 0$, this yields

$$\ln(b) = \frac{1}{c}(b - 1) \leq b - 1$$

Let's conclude with one more example of how the Mean Value Theorem is used as the foundation for Calculus.

EXAMPLE 8 Suppose that $f(x)$ is continuous and differentiable everywhere. Use the MVT to prove that if $f(x)$ is periodic with period T, then $f'(c) = 0$ for infinitely many values of c.

Solution: Actually, the proof is quite easy. Since $f(0) = f(T)$, the MVT implies that there is a number c_1 in $(0, T)$ such that

$$f'(c_1) = \frac{f(T) - f(0)}{T - 0} = 0$$

Likewise, the MVT implies that there is a number c_2 in $(T, 2T)$ such that

$$f'(c_2) = \frac{f(2T) - f(T)}{2T - T} = 0$$

and continuing for $(2T, 3T)$, $(3T, 4T)$, $\ldots$, we select infinitely many numbers c_1, c_2, $\ldots$ at which the derivative vanishes (i.e, is equal to 0).

Exercises:

Compute the differential dy.

1.	$y = x^2$	2.	$y = x^3$
3.	$y = \sin(x)$	4.	$y = \cos(x)$
5.	$y = \tan(x)$	6.	$y = \sec(x)$
7.	$y = e^x$	8.	$y = 2^x$
9.	$y = \sqrt{1-x^2}$	10.	$y = \sin(x^2)$

Find dy and Δy for the given values of p and $\Delta x = dx$. Would you conclude that dy is a good approximation of Δy?

11.	$f(x) = x^2,\ p = 1,\ dx = 0.01$	12.	$f(x) = x^2 + 2x,\ p = 2,\ dx = 0.02$
13.	$f(x) = x^3 + x^2,\ p = 2,\ dx = 0.1$	14.	$f(x) = (x-1)^2,\ p = 0,\ dx = 0.01$
15.	$f(x) = 2x + 3,\ p = 1,\ dx = 0.2$	16.	$f(x) = 3x - 1,\ p = -1.5,\ dx = 0.2$
17.	$y = -0.3x^2 + 0.7x + 0.5$ $p = 0.2,\ dx = 0.001$	18.	$y = 4x^2 + 35x - 15$ $p = 10,\ dx = 0.1$
19.	$xy = 1,\ x = 0.5,\ dx = 0.1$	20.	$y^2 = x,\ x = 1.44,\ dx = 0.1$

Exercises 21-28 involve tolerances and differentials as approximations.

21. A certain rectangle has a length which is twice its width. The length is 1 cm measured to within a tolerance of 0.1 cm. Estimate the maximum error in calculating the area of the rectangle using this length. (Hint: calculate dA.)

22. If a cube with has sides of length $x = 2$ cm to within a tolerance of 0.1 cm, then what is the tolerance in the volume of the cube?

23. A thin circular disk has a radius $r = 10$ cm measured to within a tolerance of 0.3 cm. Estimate the maximum error in calculating the area of the disk using this measurement. (Hint: calculate dA.)

24. The circumference of the top of a soup can is measured to be $9\frac{1}{4}''$ to within a 1/16 of an inch.

(a) Express the area of a circle as a function of the circumference of the circle (Hint: $r = C/2\pi$)

(b) What is the area of the top of the can?

(c) About how much variation in the area computation is possible given that the circumference is accurate only to a sixteenth of an inch?

25. The volume of a pyramid is one-third the height times the area of the base. If a pyramid has a square base with the length of one side being $100m$ measured to within a tolerance of $0.1m$, estimate the maximum error in the volume measurement as a function of the height, h.

26. A cylindrical hole is to be bored in a block of metal. The hole is to have a capacity of $50cc$ (cc is cubic centimeters), a height of $h = 6.351cm$ and a radius of $R = 1.583cm$. The volume must be accurate to within $0.2cc$, that is $|dV| < 0.2$. The volume of a cylinder is given by $V = \pi R^2 h$.

(a) If we assume that the height h is without error, to what tolerance dr must we keep the radius in order for $|dV| < 0.2$.

(b) If we assume that the radius R will be perfect, to what tolerance dh must we keep the height?

27. A copper wire $3\,mm$ in diameter is to be covered with a plastic insulation $0.5\,mm$ thick. Use differentials to estimate the volume of the plastic necessary to coat a roll of copper wire $100m$ $(100,000\ mm)$ in length.

28. A spherical shell has a radius of 100 cm and a thickness of 0.1 cm. Use differentials to approximate the volume of the shell.

Exercises 29-40 use the Mean Value Theorem to prove results in calculus.

29. Use the Mean Value Theorem to prove that if $f'(x) = 1$ for all x in (a, b) and if $f(x)$ is continuous on $[a, b]$, then

$$f(b) - f(a) = b - a$$

30. Use the Mean Value theorem to prove that if $b > a > 1$, then

$$b^{10} - a^{10} > 10(b - a)$$

(Hint: let $f(x) = x^{10}$ and notice that if $c > a$, then also $c > 1$).

31. Use the Mean value theorem to prove that if $b \geq a$, then

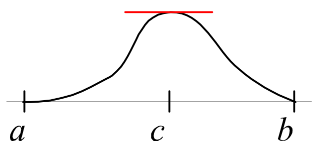
$$\sin(b) - \sin(a) \leq b - a$$

32. Use the Mean value theorem to prove that if $0 < a < b$, then

$$\sqrt{ab} < \frac{a+b}{2}$$

(Hint: let $f(x) = \sqrt{ax}$ and notice that $f(a) = a$, $f(b) = \sqrt{ab}$ and $\sqrt{\frac{a}{c}} < 1$.)

33. **Rolle's Theorem:** Show that if $f(a) = f(b) = 0$, if $f(x)$ is continuous on $[a, b]$, and if $f(x)$ is differentiable on (a, b), then there is a c in (a, b) such that $f'(c) = 0$.



1-9: Rolle's Theorem

34. Suppose that $f(x)$ is periodic with a period $T > 0$, and suppose that $f''(x) = p(x) f(x)$, where $p(x) > 0$ for all x. Show that $f(x)$ must have at least one real zero. (Hint: Show that $f'(x)$ is periodic, and then apply the MVT to $f'(x)$ over $[0, T]$).

35. Use the Mean value theorem to prove the following: If there is a number $\delta < 1$ such that $|f'(x)| \leq \delta$ for all x in an interval (p, q), then

$$|f(b) - f(a)| \leq \delta\,|b - a| \tag{3.6}$$

for all intervals $[a, b]$ in (p, q). (A function that satisfies (3.6) is called a *contraction*).

36. Find the Error: If $f(x) = \tan(x)$, then $f'(x) = \sec^2(x)$ and thus on the interval $[0, b]$, where $b > 2$, we have

$$\tan(b) - \tan(0) = \sec^2(c)(b - 0)$$

for some number c in $(0, b)$. However, $\sec^2(c) \geq 1$, so that

$$\tan(b) = \sec^2(c)(b - 0) \geq b - 0$$

That is, if $b \geq 2$, then $\tan(b) \geq b$ (Note: This conclusion can't be true since if $b = \pi$, then $\tan(\pi) = 0$).

37. Graph $f'(x)$ on the interval given. Estimate $\min[f'(x)]$ (i.e. the minimum value of $f'(x)$ on the given interval) and $\max[f'(x)]$ (the maximum value of $f'(x)$ on the given interval).[2] Compare each to the quantity $\frac{f(b)-f(a)}{b-a}$

(a)	$f(x) = 3x - 2x^3$ on $[-1, 1]$	(b)	$f(x) = 3x^{1/2} - x$ on $[1, 3]$
(c)	$f(x) = x^3 + 3x$ on $[-1, 1]$	(d)	$f(x) = e^x$ on $[0, \ln(2)]$

38. The results in the previous exercise lead to the following conjecture: If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then

$$\min[f'(x)] \leq \frac{f(b) - f(a)}{b - a} \leq \max[f'(x)] \qquad (3.7)$$

Prove this result with the Mean Value Theorem.

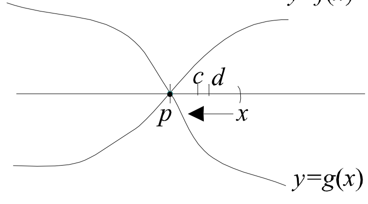
39. Use the theorem in exercise 38 to prove theorem 1.2. (i.e., use (3.7) to show that if $f'(x) = 0$ on (a, b), then $f(x)$ is a constant function on (a, b)).

40. Write to Learn: At 3:00 p.m., the odometer on your automobile reads 15,000 miles. At 4:00 p.m., the odometer on your automobile reads 15,060 miles. Let $r(t)$ denote your odometer reading in miles at time t in hours since noon, and then write a short essay in which you apply and interpret the Mean Value Theorem to $r(t)$ over $[3, 4]$. What instrument measures $r'(t)$? What does the Mean Value theorem imply about $r'(c)$ at some time c in $[3, 4]$?

3.2 L'Hôpital's Rule

L'Hôpital's Rule for $\frac{0}{0}$ Forms

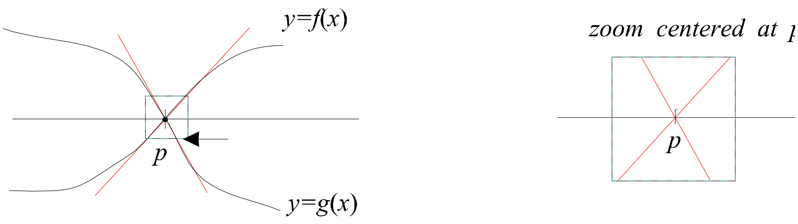
Suppose that $f(x)$ and $g(x)$ are differentiable in a neighborhood of p and that both $f(p) = 0$ and $g(p) = 0$.



2-1: $f(p) = g(p) = 0$

[2]If available, the "trace" option can be used to estimate the maximum and the minimum of $f'(x)$.

If x is close to p, then the curves $y = f(x)$ and $y = g(x)$ are practically the same as their tangent lines at $(p, 0)$.



2-2: Linearization is Used to Evaluate Limits

Since $f(p) = g(p) = 0$, their tangent lines are, respectively,

$$y = f'(p)(x-p) \qquad \textit{and} \qquad y = g'(p)(x-p)$$

If x is approaching p, then $f(x) \approx f'(p)(x-p)$ and $g(x) \approx g'(p)(x-p)$, which if $g'(p) \neq 0$ implies that

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \lim_{x \to p} \frac{f'(p)(x-p)}{g'(p)(x-p)} = \frac{f'(p)}{g'(p)} \tag{3.8}$$

This idea can be generalized into *L'Hôpital's rule* (pronounced "Low-pi-tal").

L'Hôpital's Rule: If $f(x)$ and $g(x)$ are differentiable on an open interval (a, b) containing p, except possibly at p itself, and if $\lim_{x \to p} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$, then

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \lim_{x \to p} \frac{f'(x)}{g'(x)} \tag{3.9}$$

when both limits exist.

Unlike (3.8), L'Hôpital's rule holds even if $g'(p) = 0$. In operator notation, L'Hôpital's rule says that if $\lim_{x \to p} f(x) = \lim_{x \to p} g(x) = 0$, then

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \lim_{x \to p} \frac{\frac{d}{dx} f(x)}{\frac{d}{dx} g(x)} \tag{3.10}$$

EXAMPLE 1 Use L'Hôpital's rule to evaluate

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

Solution: First we notice that the limit is of the form $\frac{0}{0}$. As a result, (3.10) says that

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{\frac{d}{dx}\left(x^2 - 1\right)}{\frac{d}{dx}(x - 1)} = \lim_{x \to 1} \frac{2x}{1} = \frac{2}{1} = 2$$

Notice that at $p = 1$, the tangent line to $y = x^2 - 1$ is $y = 2(x-1)$. Thus, example 1 can be summarized as saying that $x^2 - 1$ can be replaced by $2(x-1)$ for x near 1, so that the limit is the ratio of $2(x-1)$ to $(x-1)$, which is 2.

EXAMPLE 2 Use L'Hôpital's rule to evaluate

$$\lim_{x\to 0}\frac{\sin(x)}{x}$$

Solution: The limit is of the form $\frac{0}{0}$, so that

$$\lim_{x\to 0}\frac{\sin(x)}{x}=\lim_{x\to 0}\frac{\frac{d}{dx}\sin(x)}{\frac{d}{dx}x}=\lim_{x\to 0}\frac{\cos(x)}{1}=\cos(0)=1$$

The last limit is due to the fact that $\cos(x)$ is continuous at $x=0$.

Not surprisingly, the proof of (3.9) follows from the Mean Value theorem. If $x>p$, then the MVT implies that there are numbers c and d in (p,x) such that

$$f(x)-f(p)=f'(c)(x-p)\qquad \textit{and}\qquad g(x)-g(p)=g'(d)(x-p)$$

Since $f(p)=g(p)=0$, then $f(x)=f'(c)(x-p)$ and $g(x)=g'(d)(x-p)$, so that

$$\lim_{x\to p^+}\frac{f(x)}{g(x)}=\lim_{x\to p^+}\frac{f'(c)(x-p)}{g'(d)(x-p)}=\lim_{x\to p^+}\frac{f'(c)}{g'(d)}$$

However, as x approaches p, then c and d also approach p, so that

$$\lim_{x\to p^+}\frac{f(x)}{g(x)}=\lim_{x\to p^+}\frac{f'(x)}{g'(x)}$$

if both limits exist. The proof for $x<p$ is similar.

Check your Reading *How would evaluate the limit in example 1 without using L'Hôpital's rule?*

Infinite Limits and Limits to Infinity

If $f(x)$ and $g(x)$ both approach ∞ as x approaches a number p, then

$$\lim_{x\to p}\frac{f(x)}{g(x)}$$

is *of the form* $\frac{\infty}{\infty}$. Similarly, if $f(x)$ and $g(x)$ both approach ∞ as x approaches ∞, then the limit to ∞ of $\frac{f(x)}{g(x)}$ is also of the form $\frac{\infty}{\infty}$. L'Hôpital's rule also holds for limits of the form $\frac{\infty}{\infty}$ when the limit of $\frac{f'(x)}{g'(x)}$ also exists.

EXAMPLE 3 Evaluate the limit

$$\lim_{x\to\infty}\frac{x^4+1}{x^4+2x+3}\tag{3.11}$$

Solution: As x goes to ∞, the quantity x^4 also goes to ∞. Thus, (3.11) is of the form $\frac{\infty}{\infty}$ and can be evaluated using L'Hôpital's rule :

$$\lim_{x\to\infty}\frac{x^4+1}{x^4+2x+3}=\lim_{x\to\infty}\frac{\frac{d}{dx}\left(x^4+1\right)}{\frac{d}{dx}\left(x^4+2x+3\right)}=\lim_{x\to\infty}\frac{4x^3}{4x^3+2}$$

The limit $\lim_{x\to\infty} \frac{4x^3}{4x^3+2}$ is also of the form $\frac{\infty}{\infty}$, and thus, we apply L'Hôpital's rule again:

$$\lim_{x\to\infty} \frac{4x^3}{4x^3+2} = \lim_{x\to\infty} \frac{\frac{d}{dx}\left(4x^3\right)}{\frac{d}{dx}\left(4x^3+2\right)} = \lim_{x\to\infty} \frac{12x^2}{12x^2} = 1$$

In summary, we showed that

$$\lim_{x\to\infty} \frac{x^4+1}{x^4+2x+3} = \lim_{x\to\infty} \frac{4x^3}{4x^3+2} = \lim_{x\to\infty} \frac{12x^2}{12x^2} = 1$$

EXAMPLE 4 Evaluate the limit

$$\lim_{x\to\infty} \frac{2^x-1}{2^x+1}$$

Solution: Since 2^x approaches ∞ as x approaches ∞, the limit is of the form $\frac{\infty}{\infty}$. Thus,

$$\lim_{x\to\infty} \frac{2^x-1}{2^x+1} = \lim_{x\to\infty} \frac{\frac{d}{dx}\left(2^x-1\right)}{\frac{d}{dx}\left(2^x+1\right)} = \lim_{x\to\infty} \frac{2^x\ln(2)}{2^x\ln(2)} = 1$$

It is important to notice that L'Hôpital's rule is not always helpful. For example,

$$\lim_{x\to\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}} \tag{3.12}$$

is of the form $\frac{\infty}{\infty}$. Application of L'Hôpital's rule yields

$$\lim_{x\to\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}} = \lim_{x\to\infty} \frac{\frac{d}{dx}\left(e^x+e^{-x}\right)}{\frac{d}{dx}\left(e^x-e^{-x}\right)} = \lim_{x\to\infty} \frac{e^x-e^{-x}}{e^x+e^{-x}}$$

The resulting limit is also of the form $\frac{\infty}{\infty}$, and L'Hôpital's rule yields

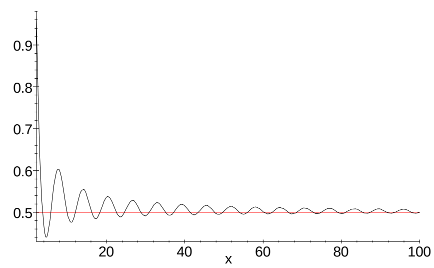
$$\lim_{x\to\infty} \frac{e^x-e^{-x}}{e^x+e^{-x}} = \lim_{x\to\infty} \frac{\frac{d}{dx}\left(e^x-e^{-x}\right)}{\frac{d}{dx}\left(e^x+e^{-x}\right)} = \lim_{x\to\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}}$$

That is, L'Hôpital's rule simply takes us back to where we started!

In addition, application of L'Hôpital's rule to limits which **do** exist may result in limits which **do not** exist. For example, in the exercises, we will show that

$$\lim_{x\to\infty} \frac{x+\sin(x)}{2x-1} = \frac{1}{2} \tag{3.13}$$

as is implied by the graph of $f(x) = \dfrac{x+\sin(x)}{2x-1}$ below:



2-3: L'Hopital's rule does not apply to all $\frac{\infty}{\infty}$ forms

However, even though the limit (3.13) is of the form $\frac{\infty}{\infty}$, L'Hôpital's rule yields

$$\lim_{x\to\infty}\frac{x+\sin(x)}{2x-1}=\lim_{x\to\infty}\frac{\frac{d}{dx}(x+\sin(x))}{\frac{d}{dx}(2x-1)}=\lim_{x\to\infty}\frac{1+\cos(x)}{2}$$

The last limit does not exist because $\cos(x)$ does not have a horizontal asymptote.

Check your Reading *How would we evaluate (3.12)?*

Limits of the form $0\cdot\infty$

L'Hôpital's rule only applies to limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Although L'Hôpital's rule only applies to limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, there are limits in different forms that can be simplified to a valid L'Hôpital's rule form. To illustrate, consider that if

$$\lim_{x\to p}f(x)=0 \qquad \text{and} \qquad \lim_{x\to p}g(x)=\infty$$

then the limit

$$\lim_{x\to p}[f(x)\,g(x)]$$

is said to be in $0\cdot\infty$ form. A limit of the form $0\cdot\infty$ can be converted into a valid form by taking the reciprocal of either $f(x)$ or $g(x)$.

EXAMPLE 5 Evaluate the limit

$$\lim_{x\to\infty}\left(x^2e^{-x}\right) \tag{3.14}$$

Solution: Since x^2 goes to ∞ and e^{-x} goes to 0 as x approaches ∞, the limit (3.14) is of the form $0\cdot\infty$. Thus, we change e^{-x} into $\frac{1}{e^x}$ and apply L'Hôpital's rule:

$$\lim_{x\to\infty}\left(x^2e^{-x}\right)=\lim_{x\to\infty}\frac{x^2}{e^x}=\lim_{x\to\infty}\frac{2x}{e^x}$$

The resulting limit is of the form $\frac{\infty}{\infty}$, so we apply L'Hôpital's rule again:

$$\lim_{x\to\infty}\left(x^2e^{-x}\right)=\lim_{x\to\infty}\frac{2x}{e^x}=\lim_{x\to\infty}\frac{2}{e^x}=0$$

EXAMPLE 6 Evaluate

$$\lim_{x\to 0^+}x\ln(x)$$

Solution: Since $\ln(x)$ has a vertical asymptote at $x=0$, the limit is of the form $0\cdot\infty$. Thus, we convert it to

$$\lim_{x\to 0^+}\frac{\ln(x)}{x^{-1}}$$

which is of the form $\frac{\infty}{\infty}$. Consequently, L'Hôpital's rule implies that

$$\lim_{x\to 0^+}\frac{\ln(x)}{x^{-1}}=\lim_{x\to 0^+}\frac{1/x}{-x^{-2}}=\lim_{x\to 0^+}\frac{1}{x}\div\left(-x^{-2}\right)$$

Since division by x^{-2} is the same as multiplication by x^2, we have

$$\lim_{x\to 0^+}\frac{\ln(x)}{x^{-1}}=-\lim_{x\to 0^+}\frac{1}{x}\cdot x^2=-\lim_{x\to 0^+}x=0$$

Check your Reading *What is the horizontal asymptote of* $f(x) = x^2 e^{-x}$?

Other Indeterminate Forms

The relationship of e^x to $\ln(x)$ can be used to evaluate limits of the form

$$\lim_{x\to p} [f(x)]^{g(x)} \tag{3.15}$$

In particular, if (3.15) is in one of the forms 0^0, ∞^0 or 1^∞, then we apply the following theorem

Theorem 8.2: If $\lim_{x\to p} \ln\left([f(x)]^{g(x)}\right) = L$, then $\lim_{x\to p} [f(x)]^{g(x)} = e^L$

That is, we first evaluate the *logarithm* of (3.15). If (3.15) is in one of the forms 0^0, ∞^0 or 1^∞, then the result will reduce to a valid L'Hôpital's form. We then apply the exponential, if the limit exists.

EXAMPLE 7 Evaluate the limit

$$\lim_{x\to 0} (x+1)^{1/x}$$

Solution: Since the limit is of the form 1^∞, we apply the logarithm to obtain

$$\lim_{x\to 0} \ln(x+1)^{1/x} = \lim_{x\to 0} \frac{1}{x} \ln(x+1) = \lim_{x\to 0} \frac{\ln(x+1)}{x}$$

which is of the form $\frac{0}{0}$. Application of the derivative results in

$$\lim_{x\to 0} \ln(x+1)^{1/x} = \lim_{x\to 0} \frac{\frac{d}{dx}\ln(x+1)}{\frac{d}{dx}x} = \lim_{x\to 0} \frac{\frac{1}{x+1}}{1} = \frac{\frac{1}{0+1}}{1} = 1$$

Theorem 8.2 then implies that

$$\lim_{x\to 0} (x+1)^{1/x} = e^1 = e$$

In reality, there is little need to try to remember the forms 0^0, ∞^0 or 1^∞. Theorem 8.2 applies to any limit of the form $\lim_{x\to p} [f(x)]^{g(x)}$. The forms 0^0, ∞^0 or 1^∞ are simply those which simplify to a valid L'Hôpital's form.

EXAMPLE 8 Apply theorem 2.2 to

$$\lim_{x\to 0^+} x^{1/\ln(x)}$$

Solution: To begin with, we apply the logarithm to obtain

$$\lim_{x\to 0^+} \ln\left[x^{1/\ln(x)}\right] = \lim_{x\to 0^+} \frac{1}{\ln(x)} \ln(x) = \lim_{x\to 0^+} 1 = 1$$

As a result, theorem 2.2 implies that

$$\lim_{x\to 0^+} x^{1/\ln(x)} = e^1$$

Notice that the form of the limit did not come into play.

Exercises:

Determine the form and evaluate the limit if it exists. If the limit does not exist, explain why not.

1. $\displaystyle\lim_{x\to 1} \frac{x^2-1}{x-1}$
2. $\displaystyle\lim_{x\to -3} \frac{x^2-9}{x+3}$
3. $\displaystyle\lim_{x\to 2} \frac{x^2-4x+4}{x^2-4}$
4. $\displaystyle\lim_{x\to -3} \frac{x^2-2x}{x^3+4x^2-3x-18}$
5. $\displaystyle\lim_{x\to \sqrt{2}} \frac{x^2-2}{x-\sqrt{2}}$
6. $\displaystyle\lim_{x\to 1} \frac{x^4+3x-4}{\sqrt{x}-1}$
7. $\displaystyle\lim_{x\to \sqrt{2}} \frac{(x-\sqrt{2})^7}{x^2-2}$
8. $\displaystyle\lim_{x\to 1^+} \frac{(x-1)^{3/2}}{x^4-2x^2+1}$
9. $\displaystyle\lim_{x\to 0} \frac{x}{e^x-e^{-x}}$
10. $\displaystyle\lim_{x\to 0} \frac{e^{2x}-1}{e^{3x}+e^{-x}}$
11. $\displaystyle\lim_{x\to 0} \frac{\ln(2x+1)}{x}$
12. $\displaystyle\lim_{x\to 1} \frac{\ln(x)}{x-1}$
13. $\displaystyle\lim_{x\to 0} \frac{\sin(2x)}{x}$
14. $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{1-\cos(x)}$
15. $\displaystyle\lim_{x\to 0} \frac{1-\cos(2x)}{\sin^2(x)}$
16. $\displaystyle\lim_{h\to 0} \frac{\cos(h)-1}{h^2}$
17. $\displaystyle\lim_{x\to \pi} \frac{2\sin(x)\cos(x)}{x^2-\pi^2}$
18. $\displaystyle\lim_{x\to \pi} \frac{\sin^2(x)}{(x-\pi)^2}$
19. $\displaystyle\lim_{x\to 0} \frac{\sin^2(x)}{1-\cos(4x)}$
20. $\displaystyle\lim_{x\to 0} \frac{\cos^4(x)-\cos(2x)}{\sin^4(x)}$
21. $\displaystyle\lim_{x\to \infty} \frac{2x+5}{x-3}$
22. $\displaystyle\lim_{x\to \infty} \frac{5x-3}{4x+2}$
23. $\displaystyle\lim_{x\to \infty} \frac{x^2+\sqrt{x}}{x^2-\sqrt{x}}$
24. $\displaystyle\lim_{x\to \infty} \frac{\sqrt{x^2-4x}}{x^2-\sqrt{x}-2}$
25. $\displaystyle\lim_{x\to \infty} \frac{\ln(x)}{x}$
26. $\displaystyle\lim_{x\to \infty} \frac{\ln(x)}{\ln(x^2+4)}$
27. $\displaystyle\lim_{x\to \infty} \frac{e^{2x}-1}{e^{2x}+1}$
28. $\displaystyle\lim_{x\to \infty} \frac{e^x}{e^x+e^{-x}}$

29. $\lim_{x\to\infty} x^3e^{-x}$

30. $\lim_{x\to\infty} \sqrt{x}e^{-x}$

31. $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$

32. $\lim_{x\to\infty} x\ln\left(\frac{x}{1+x}\right)$

33. $\lim_{x\to 0^+} \ln(x)\ln(x+1)$

34. $\lim_{x\to\infty} \cos\left(\frac{1}{x}\right)$

35. $\lim_{x\to 0^+} x^{2x}$

36. $\lim_{x\to\infty} x^{1/x}$

37. $\lim_{x\to 0^+} x^{\sin(x)}$

38. $\lim_{x\to 0} (1+x)^{1/x}$

39. $\lim_{x\to\infty} \left(1+x^2\right)^{1/\ln(x)}$

40. $\lim_{x\to\infty} \left(1+\frac{1}{x^2}\right)^x$

41. In this exercise, we explore the limit

$$\lim_{x\to 1} \frac{3x^2-x-2}{x^2-1} \tag{3.16}$$

(a) **Numerical:** Complete the table

x	0.9	0.99	0.999	$\to$	0	$\leftarrow$	1.001	1.01	1.1
$\frac{3x^2-x-2}{x^2-1}$				$\to$	???	$\leftarrow$			

Use it to estimate the value of the limit (3.16).

(b) Evaluate the limit (3.16) using L'hopital's rule.

(c) Evaluate the limit (3.16) by factoring the numerator, canceling and using continuity.

42. In this exercise, we explore the limit

$$\lim_{x\to 0.5} \frac{6x^2+5x-4}{x-0.5} \tag{3.17}$$

(a) **Numerical:** Complete the table

x	0.49	0.499	0.4999	$\to$	0	$\leftarrow$	0.5001	0.501	0.51
$\frac{6x^2+5x-4}{x-0.5}$				$\to$	???	$\leftarrow$			

Use it to estimate the value of the limit (3.17).

(b) Evaluate the limit (3.17) using L'hopital's rule.

(c) Evaluate the limit (3.17) by factoring the numerator, canceling and using continuity.

43. In this exercise, we explore the limit

$$\lim_{h\to 0} \frac{e^{2h}-1}{e^h-1} \tag{3.18}$$

(a) **Numerical:** Complete the table

h	-0.01	-0.001	-0.0001	$\to$	0	$\leftarrow$	0.0001	0.001	0.01
$\frac{e^{2h}-1}{e^h-1}$				$\to$	???	$\leftarrow$			

Use it to estimate the value of the limit (3.18).

(b) Evaluate the limit (3.18) using L'Hôpital's rule.

(c) Evaluate the limit (3.18) by factoring the numerator, canceling and using continuity.

44. In this exercise, we explore the limit

$$\lim_{x\to 3} x^{1/\ln(x)} \tag{3.19}$$

(a) **Numerical:** Complete the table

x	2.9	2.99	2.999	$\rightarrow$	0	$\leftarrow$	3.001	3.01	3.1
$x^{1/\ln(x)}$				$\rightarrow$	???	$\leftarrow$			

Use it to estimate the value of the limit (3.19).

(b) Use theorem 2.2 to evaluate (3.19).

(c) Use the definition of $[f(x)]^{g(x)}$ to simplify $x^{1/\ln(x)}$. What is significant about the function $x^{1/\ln(x)}$?

45. In this exercise, we evaluate the limit

$$\lim_{x\to\infty} \frac{x+\sin(x)}{2x-1} \tag{3.20}$$

(a) Use the fact that $-1 \le \sin(x) \le 1$ and the sandwich theorem to show that

$$\lim_{x\to\infty} \frac{\sin(x)}{x} = 0$$

(b) Divide the numerator and denominator of (3.20) by x, and then use part a to evaluate the limit.

46. In this exercise, we evaluate the limit

$$\lim_{x\to\infty} \frac{x^2+\cos(x)}{(x-1)^2} \tag{3.21}$$

(a) Use the fact that $-1 \le \cos(x) \le 1$ and the sandwich theorem to show that

$$\lim_{x\to\infty} \frac{\cos(x)}{x^2} = 0$$

(b) Divide the numerator and denominator of (3.21) by x, and then use part a to evaluate the limit.

47. Show that if $f'(x)$ is continuous, then

$$\lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h} = f'(x)$$

(hint: take derivatives with respect to h). Then draw a diagram which shows why

$$\frac{f(x+h)-f(x-h)}{2h}$$

can also be considered a secant line approximation of $f'(x)$.

48. Use L'Hôpital's rule to show that if $f''(x)$ is continuous, then

$$\lim_{h\to 0} \frac{f(x+2h)-2f(x+h)+f(x)}{h^2} = f''(x)$$

(hint: take derivatives with respect to h).

49. **Write to learn:** In a short essay, show that

$$\lim_{x\to p}\frac{g(x)}{f(x)}=\lim_{x\to p}\frac{[f(x)]^{-1}}{[g(x)]^{-1}}$$

and explain why if the first limit is of the form $\frac{\infty}{\infty}$, then the second limit is of the form $\frac{0}{0}$.

50. **Write to Learn:** In section 2.6, definition 6.1, the exponential function is defined by the limit

$$e^x=\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

In a short essay, show that $e^x e^h=e^{x+h}$ by explaining why

$$e^x e^h=\lim_{n\to\infty}\left(1+\frac{x+h}{n}+\frac{xh}{n^2}\right)^n$$

and then computing the limit using L'hopital's rule.

51. **Write to Learn:** Let's apply linearization directly to the limit

$$\lim_{x\to 1}\frac{x^3+3x^2-5x+1}{x-1}$$

Show that if $y=x^3+3x^2-5x+1$, then its tangent line at $x=1$ is

$$y=4(x-1)$$

Then in a short essay, explain why

$$\lim_{x\to 1}\frac{x^3+3x^2-5x+1}{x-1}=\lim_{x\to 1}\frac{4(x-1)}{x-1}$$

52. **Write to Learn:** Let's apply linearization directly to the limit

$$\lim_{x\to 2}\frac{x^3+x^2-12}{x^2-4}$$

Show that the tangent lines at $x=2$ of $y=x^3+x^2-12$ and $y=x^2-4$ are, respectively,

$$y=16(x-2)\qquad\text{and}\qquad y=4(x-2)$$

Then in a short essay, explain why

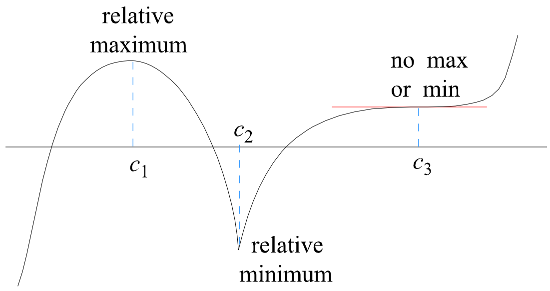
$$\lim_{x\to 2}\frac{x^3+x^2-12}{x^2-4}=\lim_{x\to 2}\frac{16(x-2)}{4(x-2)}$$

3.3 Absolute and Relative Extrema

Relative Extrema

Many applications of calculus require the identification of the *extreme* values—i.e., the highs and lows—of a given function. The derivative can be used to determine where the extreme values, or *extrema*, of a function occur. We begin with some definitions.

Definition 3.1: A function $f(x)$ has a *relative maximum* at $x = p$ if there is a neighborhood (a, b) of p such that $f(p) \geq f(x)$ for all x in (a, b). A function $f(x)$ has a *relative minimum* at $x = p$ if there is a neighborhood (a, b) of p such that $f(p) \leq f(x)$ for all x in (a, b).



3-1: Relative Extrema

Notice that the extrema in the figure above occur at inputs where the tangent line is horizontal (see c_1 above), or where the function has a cusp (see c_2 above).

To see this mathematically, notice that if $f(x)$ is differentiable at $x = p$, then

$$f(p+h) = f(p) + f'(p)\,h + o(h)$$

Subtracting $f(p)$ from both sides thus yields

$$f'(p)\,h + o(h) = f(p+h) - f(p)$$

Suppose now that $f(x)$ has a maximum at $x = p$. Then the quantity $f(p+h) - f(p)$ must be negative for h sufficiently small, which implies that $f'(p)\,h \leq 0$ for all h (positive or negative) sufficiently close to 0. However,

$$h < 0 \Longrightarrow f'(p) \geq 0, \qquad h > 0 \Longrightarrow f'(p) \leq 0$$

and we can have both $f'(p) \leq 0$ and $f'(p) \geq 0$ only if $f'(p) = 0$. Similarly, if $f(x)$ has a minimum at $x = p$ and is differentiable at $x = p$, then $f'(p) = 0$.

Since extrema can also occur at cusps, we are led to the following definition:

Definition 3.2: Let c be in the domain of a function $f(x)$. If $f'(c) = 0$ or if $f'(c)$ does not exist, then c is called a *critical point* of $f(x)$.

In particular, a critical point c for which $f'(c) = 0$ is also known as a *stationary point* of $f(x)$.

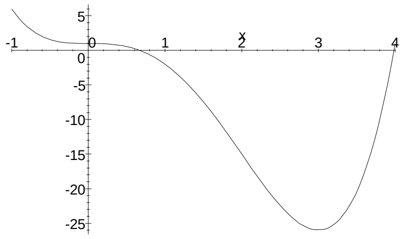
Thus, the extrema of a function must occur at the critical points of a function, although not all critical points lead to extrema as shown by c_3 above. Examination of the graph of $f(x)$ can be used to determine if and what type of extremum occurs at a given critical point..

EXAMPLE 1 Identify any extrema of $f(x) = x^4 - 4x^3 + 1$.

Solution: Since $f'(x) = 4x^3 - 12x^2$ is a polynomial, $f'(x)$ exists for all x. Thus, the critical points occur where $f'(x) = 0$.

$$\begin{aligned} 4x^3 - 12x^2 &= 0 \\ 4x^2(x-3) &= 0 \\ x &= 0, 3 \end{aligned}$$

It follows that the critical points of $f(x) = x^4 - 4x^3 + 1$ are $x = 0$ and $x = 3$. We now graph the function $f(x)$ over an interval containing the critical points 0 and 3.



3-2: A relative minimum at $x = 3$

Clearly, $f(x) = x^4 - 4x^3 + 1$ does not have an extremum at $x = 0$, but it does have a relative minimum at $x = 3$.

Check your Reading *Why can a function have an extremum at an input where a cusp occurs?*

Relative Extrema of Algebraic Functions

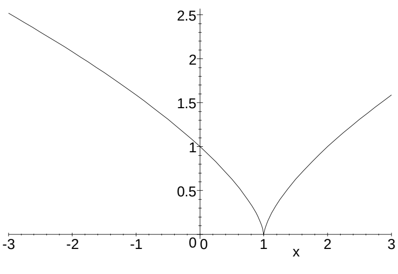
Algebraic functions with exponents between 0 and 1 often have cusps. Moreover, if $f(x)$ is an algebraic function, then a cusp occurs at a point p where $f(p)$ is defined but $f'(p)$ does not exist.

EXAMPLE 2 Identify any extrema of $f(x) = (x-1)^{2/3}$.

Solution: To do so, we first compute the derivative:

$$f'(x) = \frac{2}{3}(x-1)^{-1/3} = \frac{2}{3(x-1)^{1/3}}$$

Notice that $f'(x)$ is not defined at $x = 1$, although $f(1)$ is defined. As a result, $x = 1$ is a nonstationary critical point of $f(x)$, and indeed, a minimum occurs at $x = 1$, as is shown below:



3-3: A relative minimum at $x = 1$

However, if neither $f(p)$ nor $f'(p)$ exist, then p is **not** a critical point of $f(x)$ because p is not in the domain of f.

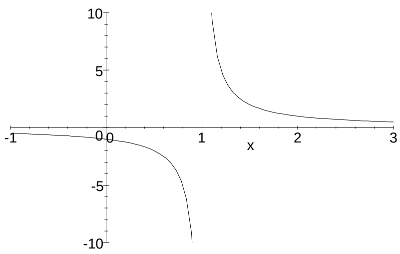
EXAMPLE 3 Identify any extrema of

$$f(x) = \frac{1}{x-1}$$

Solution: $f(x)$ has a derivative of

$$f'(x) = \frac{-1}{(x-1)^2}$$

Clearly, neither the derivative $f'(x)$ nor the function $f(x)$ are defined at $x = 1$. Indeed, $x = 1$ is a *vertical asymptote* of the function $f(x)$, as is shown below:



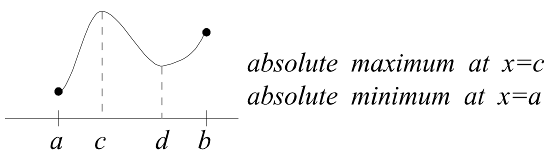
3-4: $f(x) = \frac{1}{x-1}$ has no relative extrema

Since $x = 1$ is **not** a critical point of $f(x)$, the function $f(x)$ has no critical points and thus has no relative extrema.

Check your Reading *Does $f(x) = \frac{1}{x-2}$ have any critical points?*

The Extreme Value Theorem

If $f(p) \geq f(x)$ for all real numbers x in $[a, b]$, then $f(p)$ is called an *absolute maximum* of $f(x)$ over $[a, b]$. Likewise, if $f(p) \leq f(x)$ for all real numbers x in $[a, b]$, then $f(p)$ is called an *absolute minimum* of $f(x)$ over $[a, b]$.



3-5: Absolute Extrema

The *Extreme Value Theorem* says that if a function $f(x)$ is continuous on $[a, b]$, then it attains its absolute maximum and its absolute minimum on $[a, b]$, although an absolute extremum may occur at an endpoint of $[a, b]$.

EXAMPLE 4 Determine the absolute extrema of $f(x) = 2x - x^2$ over $[0, 2]$.

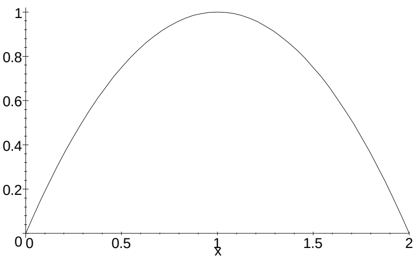
Solution: Since $f(x)$ has a derivative of

$$f'(x) = 2 - 2x$$

the critical points of $f(x)$ are the values of x where $f'(x) = 0$:

$$2 - 2x = 0, \qquad x = 1$$

The graph of $f(x) = 2x - x^2$ shows us that the absolute maximum of $f(x)$ occurs at $x = 1$.



3-6

Moreover, the absolute minimum occurs at both $x = 0$ and $x = 2$.

EXAMPLE 5 Determine the absolute extrema of the function

$$f(x) = (x-7)(x-2)^{2/3}$$

over $[0, 3]$.

Solution: Applying the product rule leads to

$$\begin{aligned} f'(x) &= \left[\frac{d}{dx}(x-7)\right](x-2)^{2/3} + (x-7)\frac{d}{dx}(x-2)^{2/3} \\ &= (x-2)^{2/3} + \frac{2}{3}(x-7)(x-2)^{-1/3} \end{aligned}$$

Transforming from a negative exponent to a fraction yields

$$f'(x) = (x-2)^{2/3} + \frac{2(x-7)}{3(x-2)^{1/3}}$$

Clearly, the derivative does not exist when $x = 2$. Since $f(2) = 0$ implies that $f(2)$ is defined, $x = 2$ is a critical point of $f(x)$.

To find the stationary points, we set $f'(x)$ equal to 0 and solve for x:

$$\begin{aligned} (x-2)^{2/3} + \frac{2(x-7)}{3(x-2)^{1/3}} &= 0 \\ (x-2)^{2/3} &= \frac{-2(x-7)}{3(x-2)^{1/3}} \end{aligned}$$

Multiplying both sides by $3(x-2)^{1/3}$ yields

$$3(x-2)^{1/3}(x-2)^{2/3} = -2x + 14$$

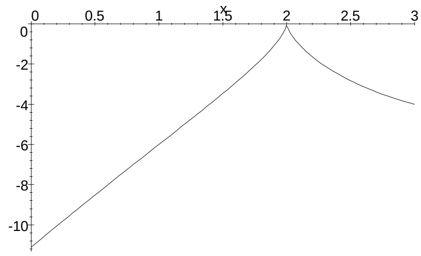
Adding exponents and expanding thus yields

$$\begin{aligned} 3(x-2) &= -2x+14 \\ 3x-6 &= -2x+14 \\ 5x &= 20 \\ x &= 4 \end{aligned}$$

Thus, the critical points of $f(x)$ are $x = 2$ and $x = 4$.

When finding absolute extrema over $[a, b]$, use only the critical points in $[a, b]$.

However, $x = 4$ is **not** in the interval $[0, 3]$, so we ignore it. The graph of $f(x)$ over $[0, 3]$ is shown below:



3-7

Clearly, $f(x) = (x-7)(x-2)^{2/3}$ attains its absolute maximum at $x = 2$, and attains its absolute minimum at $x = 0$.

The extreme value theorem is rather amazing, since it applies even to fractal interpolation functions like the one on page 79! However, because it applies to such functions, the extreme value theorem cannot be proven until a later course.

Check your Reading *Why did we ignore the critical point $x = 4$?*

Applications to Revenue

Finding extrema is a common task in many applications. For example, in business applications we are often given the relationship between the price p of a product and the demand x for that product, and we are asked to find the price and demand which lead to a *maximum revenue.* To do so, we use the fact that

$$\text{Revenue} = \text{ Price } \times \text{ Number Sold}$$

EXAMPLE 6 Suppose Acme Airlines charges a \$300 base price for a seat on an airplane, and suppose they add \$2 for each unsold seat on that airplane. Determine how many seats should be sold on a 200 seat airplane in order to maximize revenue.

Solution: If p denotes the price per seat, then

$$p = 300 + 2 \times \text{“the number of unsold seats”}$$

If we let x denote the number of seats which have been sold, then the number of unsold seats is $200 - x$ and thus p becomes

$$p = 300 + 2\,(200 - x) = 700 - 2x$$

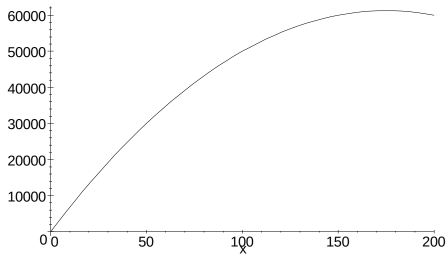
If R denotes the revenue from the sale of seats, then

$$R = px = (700 - 2x)\,x = 700x - 2x^2$$

Since there are only 200 seats on the airplane, we must find the absolute maximum of $R(x) = 700x - 2x^2$ over $[0, 200]$. Since $R'(x) = 700 - 4x$, the critical point(s) are

$$700 - 4x = 0, \qquad x = \frac{700}{4} = 175$$

The graph of $R(x) = 700x - 2x^2$ over $[0, 200]$ is shown below:



3-8

Clearly, the absolute maximum occurs at $x = 175$ seats.

Exercises:

Grapher: *Find the critical points of the following functions, and then graph the function to determine the extremum, if any, that occurs at each critical point.*

1. $f(x) = x^2$
2. $f(x) = x^4$
3. $f(x) = x^3$
4. $f(x) = x^5$
5. $f(x) = x^{2/3}$
6. $f(x) = x^{3/5}$
7. $f(x) = 2x - x^2$
8. $f(x) = 2x^2 - 5x + 3$
9. $f(x) = x^3 - 3x$
10. $f(x) = x^3 - 3x^2 + 2$
11. $f(x) = 2x^3 - 5x^2 + x + 1$
12. $f(x) = -3x^3 - 2x^2 + x - 5$
13. $f(x) = ax^2 + bx + c,\ a > 0$
14. $f(x) = ax^2 + bx + c,\ a < 0$
15. $f(x) = x^2 - \dfrac{5}{x}$
16. $f(x) = \dfrac{1}{x^2 - 2x + 2}$

Find the critical points of $f(x)$ and then determine the absolute extrema of the *continuous function $f(x)$ over the given interval.*

17. $f(x) = x^2 - 4x$ over $[0,4]$
18. $f(x) = x^2 + 3x$ over $[-2,2]$
19. $f(x) = 3x + 2$ over $[1,5]$
20. $f(x) = 2x - 1$ over $[-1,1]$
21. $f(x) = x^3 - 3x^2$ over $[-1,1]$
22. $f(x) = x^4 - 8x^2$ over $[-1,1]$
23. $f(x) = x^4 - 2x^2$ over $[0,3]$
24. $f(x) = x^4 - 8x^3 + 18x^2$ over $[-2,2]$
25. $f(x) = \cos\left(x^2\right)$ over $[0,\pi]$
26. $f(x) = \cos^2(x)$ over $[0,\pi]$
27. $f(x) = x\sin(x)$ over $[-1,1]$
28. $f(x) = \sin^2(x)$ over $[1,3]$
29. $f(x) = (x-3)^{2/3}$ over $[0,4]$
30. $f(x) = \left(x^2 - 2x + 1\right)^{1/3}$ over $[-2,2]$
31. $f(x) = x(x-1)^{2/3}$ over $[0,5]$
32. $f(x) = x^{1/5}(x-1)^{2/3}$ over $[-2,2]$
33. $f(x) = x^{4/3} - 2x^{2/3}$ over $[0,4]$
34. $f(x) = \left(x^{1/3} - 1\right)^2$ over $[-2,0]$

35. Suppose Acme airlines charges \$480 plus \$3 for every unsold seat. On a 200 hundred seat airplane, how many seats should be sold in order to maximize revenue?

36. Suppose Acme airlines charges \$250 plus \$2 for every unsold seat. On a 200 hundred seat airplane, how many seats should be sold in order to maximize revenue?

37. On the interval $[0,1]$,the function $f(x) = x$ is larger than $g(x) = x^2$. At which value of x in the interval $[0,1]$ is $f(x) - g(x)$ the greatest? What is $f(x) - g(x)$ for this x-value?

38. On the interval $[0,1]$, the function $f(x) = x^p$ is larger than $g(x) = x^{p+1}$ when $p > 0$. At which value of x in the interval $[0,1]$ is $f(x) - g(x)$ the greatest? What is $f(x) - g(x)$ for this x-value?

39. Betty starts a summer daycare program for first through third graders in her home. She enrolls 15 students initially, and the enrollment increases by about 5 children per week for the first few weeks. From experience, she knows that the enrollment will peak and then taper to 0 by the end of the 12 week summer session (when the kids return to school). Thus, her enrollment $f(x)$ as a function of the number of weeks x since she began the daycare can be modeled by the function

$$f(x) = 15 + 5x - \frac{25}{48}x^2$$

Use $f(x)$ to answer the following questions.

(a) Show that $f(0) = 15$. How does this relate to Betty's daycare?
(b) Show that $f'(0) = 5$. How does this relate to Betty's daycare?
(c) Find the absolute extrema of $f(x)$ over $[0,12]$. Why does the absolute minimum occur at $x = 12$?
(d) If Betty has space for 30 children in her home, will she have to turn any children away? That is, will the predicted maximum enrollment exceed 30 students?

40. The next summer, Betty enrolls 13 children initially and enrollment increases by about 7 children per week to begin with. This leads to an enrollment function of

$$f(x) = 13 + 7x - \frac{97}{144}x^2$$

Use $f(x)$ to answer the following questions.

(a) Show that $f(0) = 13$. How does this relate to Betty's daycare?

(b) Show that $f'(0) = 7$. How does this relate to Betty's daycare?

(c) Find the absolute extrema of $f(x)$ over $[0, 12]$. Why does the absolute minimum occur at $x = 12$?

(d) If Betty has space for 30 children in her home, will she have to turn any children away? That is, will the predicted maximum enrollment exceed 30 students?

Graphs should be used only after the critical points have been determined.

41. In this exercise, we show that graphs should be used only after the critical points have been determined.

(a) Graph $f(x) = x^3 - 11x^2 - 16x$ over $[-5, 5]$. Does it appear to have any extrema?

(b) Zoom centered on the maximum in (a) until you can estimate the x-coordinate of the maximum to four decimal places. How many zooms were required?

(c) Find the critical point(s) of $f(x) = x^3 - 11x^2 - 16x$ by solving $f'(x) = 0$.

(d) Graph $f(x)$ over an interval containing *all* of the critical points, and then use that graph to explain why graphs alone are not sufficient for identifying the extrema of a function.

42. Graph $f(x) = x^3 - 3x^2 - 9x + 1$ over $[-2, 2]$, and then find its critical points. Are all of the critical points in $[-2, 2]$? Is the graph of $f(x)$ over $[-2, 2]$ an accurate depiction of the function? Explain.

43. Graph $f(x) = x^6 - 81x^4 - x^2 + 81$ over $[-5, 5]$, and then find its critical points. Are all of the critical points in $[-5, 5]$? Is the graph of $f(x)$ over $[-5, 5]$ an accurate depiction of the function? Explain.

44. Show that if $f(x)$ is differentiable at $x = p$ and if $f(x)$ has a minimum at $x = p$, then $f'(p) = 0$.

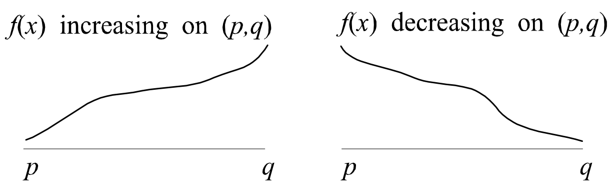
3.4 Monotonicity

Monotonicity

In the last section, we identified extrema by finding critical points and then observing the graph of a function. In this section, we introduce techniques for obtaining information about the graph of a function from its derivatives. We begin this process with the following definition.

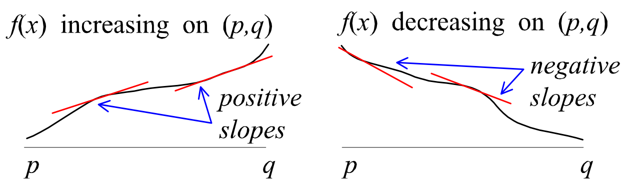
Definition 4.1 If $f(a) < f(b)$ whenever $a < b$ in (p, q), then $f(x)$ is said to be *increasing* on (p, q). If $f(a) > f(b)$ whenever $a < b$ in

(p, q), then $f(x)$ is said to be *decreasing* on (p, q).



4-1: Increasing and Decreasing Functions

When a function is increasing on (p, q), then its tangent lines over (p, q) have positive slopes, and thus, $f'(x) > 0$ for all x in (p, q).



4-2: Increasing functions have tangent lines with positive slopes

Conversely, suppose that $f'(x) > 0$ on (p, q) and that $a < b$ in (p, q). Then $b - a > 0$ and $f'(c) > 0$ for all c in $[a, b]$. As a result, the Mean Value theorem says that

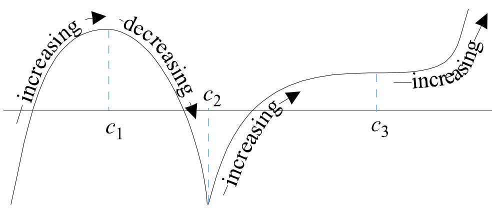
$$f(b) - f(a) = f'(c)(b - a) > 0$$

which implies that $f(b) > f(a)$. Thus, if $a < b$ in (p, q), then $f(a) < f(b)$ in (p, q), so that $f(x)$ is increasing on (p, q).

First Derivative Test: If $f'(x) > 0$ for all x in (p, q), then $f(x)$ is increasing over (p, q). Likewise, if $f'(x) < 0$ for all x in (p, q), then $f(x)$ is decreasing over (p, q).

We say that $f(x)$ is *monotone* on (p, q) if $f'(x)$ does not change signs over (p, q).

To use the first derivative test, we first determine the intervals of monotonicity by finding all the points where $f'(x)$ can change signs and then testing the sign of $f'(x)$ between those points. We will then see that a function increases to a maximum and then decreases immediately thereafter, and likewise, decreases to a minimum and then increases immediately afterward.



4-3: Monotonicity and Extrema

However, there need not be an extremum at each critical point, as is shown at c_3 in figure 6-2.

EXAMPLE 1 Determine the intervals of monotonicity and identify the extrema of

$$f(x) = x^3 - 3x$$

Solution: The points where $f'(x)$ can change signs are the critical points, which are solutions to $f'(x) = 0$. Since $f'(x) = 3x^2 - 3$, we have

$$\begin{aligned} 3x^2 - 3 &= 0 \\ 3\left(x^2 - 1\right) &= 0 \\ 3(x-1)(x+1) &= 0 \end{aligned}$$

As a result, $f'(x) = 0$ if $x = -1, 1$. Since $f'(x)$ is continuous for all x, we need only check a single point in the intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$, respectively, in order to determine the sign of f' on these intervals. Since $x = -2$ is in $(-\infty, -1)$, the calculation

$$f'(-2) = 3(-2)^2 - 3 = 9 > 0$$

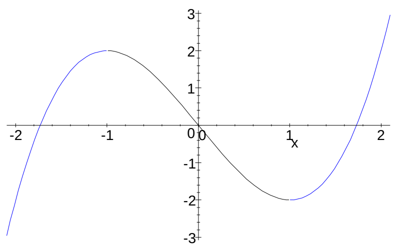
implies that $f'(x) > 0$ for x in $(-\infty, -1)$ and thus, that $f(x)$ is increasing on the interval $(-\infty, -1)$. Moreover,

$$f'(0) = -3 < 0$$

implies that $f'(x) < 0$ for x in $(-1, 1)$ and thus, that $f(x)$ is decreasing on the interval $(-1, 1)$. Since $x = 2$ is in $(1, \infty)$, the calculation

$$f'(2) = \frac{4-1}{4+1} = \frac{3}{5} > 0$$

implies that $f'(x) > 0$ for x in $(1, \infty)$ and thus, that $f(x)$ is increasing on the interval $(1, \infty)$. It follows that $f(x)$ must have a relative maximum at $x = -1$ and a relative minimum at $x = 1$.



4-4: Relative extrema at $x = -1$ and $x = 1$

Check your Reading *Are the extrema in figure 6-3 also absolute extrema? Explain.*

Vertical Asymptotes, Cusps, and Vertical Tangents

The derivative may change signs at points of discontinuity and points of non-differentiability, even though such points may not correspond to extrema.

EXAMPLE 2 Find the intervals of monotonicity and the extrema of

$$f(x) = \frac{4x^2}{(x-2)^2}$$

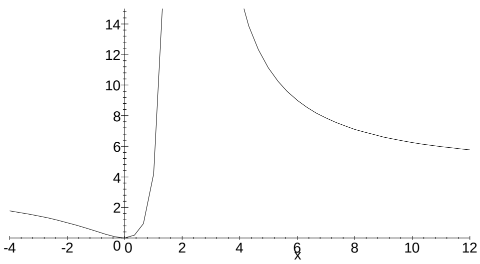
Solution: The quotient rule yields a first derivative of

$$f'(x) = \frac{-16x}{(x-2)^3}$$

The numerator is 0 when $x = 0$, and the denominator is 0 when $x = 2$. Since $x = 0, 2$ are the only points where the derivative can change sign, $f'(x)$ is continuous and single-signed on each of the intervals $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$. Testing in these intervals yields

$$\begin{array}{lll} f'(-1) = -\frac{16}{27} < 0 & \Longrightarrow & f(x) \searrow \text{ on } (-\infty, 0) \\ f'(1) = 16 > 0 & \Longrightarrow & f(x) \nearrow \text{ on } (0, 2) \\ f'(3) = -48 < 0 & \Longrightarrow & f(x) \searrow \text{ on } (2, \infty) \end{array}$$

Since $x = 0$ is a critical point of f, the table above implies that f has a relative minimum at $x = 0$. However, $x = 2$ is **not** a critical point of f since $x = 2$ is a *vertical asymptote* of f. Thus, there is **no** extremum at $x = 2$.



4-5: Vertical asymptote at $x = 2$

However, extrema can occur at points of non-differentiability, as is shown in the next example.

EXAMPLE 3 Find the extrema, the inflection points, and the intervals of monotonicity and concavity of

$$f(x) = x^{1/3}(x-3)^{2/3}$$

Solution: The product rule implies that

$$f'(x) = \frac{1}{3}x^{-2/3}(x-3)^{2/3} + \frac{2}{3}x^{1/3}(x-3)^{-1/3}$$

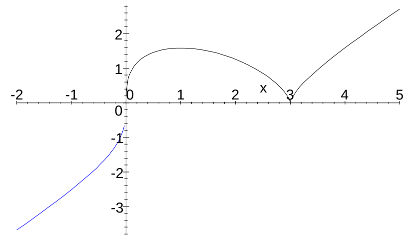
which simplifies to

$$f'(x) = \frac{x-1}{x^{2/3}(x-3)^{1/3}}$$

The numerator is 0 when $x = 1$, and the denominator is 0 when $x = 0, 3$. Thus, we must test for the sign of $f'(x)$ on $(-\infty, 0)$, $(0, 1)$, $(1, 3)$, and $(3, \infty)$.

$$\begin{array}{lll} f'(-1) = 2 > 0 & \Longrightarrow & f(x) \nearrow \text{ on } (-\infty, 0) \\ f'(0.5) = 1.474 < 0 & \Longrightarrow & f(x) \nearrow \text{ on } (0, 1) \\ f'(2) = -0.25 < 0 & \Longrightarrow & f(x) \searrow \text{ on } (1, 3) \\ f'(4) = 1.19 > 0 & \Longrightarrow & f(x) \nearrow \text{ on } (3, \infty) \end{array}$$

Since f is defined and continuous at $x = 1$ and $x = 3$, the function f has a relative maximum at $x = 1$ and a relative minimum at $x = 3$.



4-6: Graph of $\sqrt[3]{x(x-3)^2}$

Check your Reading *What are the points of non-differentiability in figure 4-6?*

Simple Curve Sketching

There are functions whose graphs cannot be produced with a grapher. Fortunately, the extrema and intervals of monotonicity can be used to produce a crude sketch of a function even when graphers fail.

Stationary points correspond to horizontal tangents. Similarly, points of non-differentiability may correspond to cusps or vertical tangents (although they may also correspond to discontinuities). Intervals of monotonicity then lead to a game of "connect the dots" in which tangency and other features of the graph are preserved.

EXAMPLE 4 Sketch the graph of $f(x) = 5x^9 - 9x^5$.

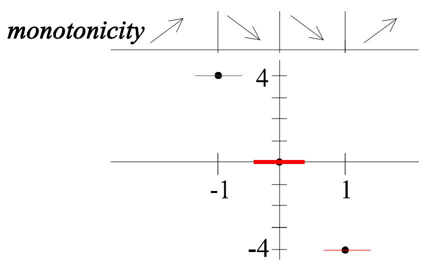
Solution: Since $f'(x) = 45x^8 - 45x^4$, the critical points satisfy

$$\begin{aligned} 45x^8 - 45x^4 &= 0 \\ 45x^4\left(x^4 - 1\right) &= 0 \end{aligned}$$

Since $x^4 = 0$ when $x = 0$ and $x^4 - 1 = 0$ when $x = \pm 1$, the stationary points are at $-1, 0, 1$. Thus, we must test for the sign of $f'(x)$ on $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$.

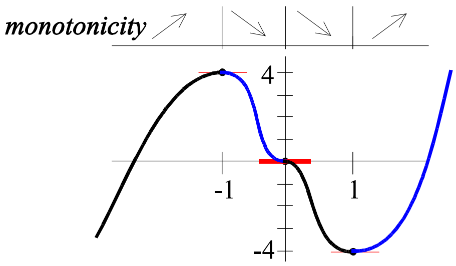
$$\begin{array}{lcl} f'(-2) = 10,800 > 0 & \Longrightarrow & f(x) \nearrow \text{ on } (-\infty, -1) \\ f'(-0.5) = -2.63 < 0 & \Longrightarrow & f(x) \searrow \text{ on } (-1, 0) \\ f'(0.5) = -2.63 < 0 & \Longrightarrow & f(x) \searrow \text{ on } (0, 1) \\ f'(2) = 10,800 > 0 & \Longrightarrow & f(x) \nearrow \text{ on } (1, \infty) \end{array}$$

Notice now that $f(-1) = 4$, $f(0) = 0$, and $f(1) = -4$. Thus, we place short horizontal tangents at the points $(-1, 4)$, $(0, 0)$, and $(1, -4)$.



4-7: Monotonicity and Extrema

The curve increases to tangency at $(-1, 4)$, and then decreases to tangency at $(0, 0)$. To do so, it must have at least an "S" shape between $(-1, 4)$ and $(0, 0)$. Likewise, in decreasing to $(1, -4)$, and then the curve increases afterwards.



4-8: Crude sketch of $y = 5x^9 - 9x^5$

If there are any horizontal asymptotes, they should also be included in the sketch.

EXAMPLE 5 Sketch the graph of $f(x) = x^2 e^{-x}$.

Solution: Since $f'(x) = 2xe^{-x} - x^2 e^{-x}$, the critical points satisfy

$$xe^{-x}(2 - x) = 0$$

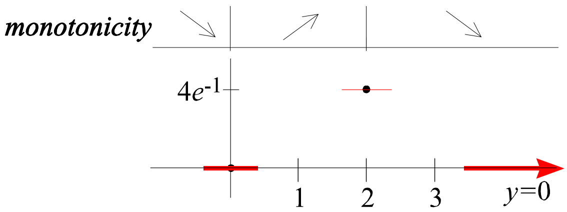
Since $e^{-x} \neq 0$, the critical points are $x = 0$ and $x = 2$. Thus, we must test for the sign of $f'(x)$ on $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$.

$$\begin{array}{lll} f'(-1) = -3e^{-1} < 0 & \Longrightarrow & f(x) \searrow \text{ on } (-\infty, 0) \\ f'(1) = e^{-1} > 0 & \Longrightarrow & f(x) \nearrow \text{ on } (0, 2) \\ f'(3) = -3e^{-1} < 0 & \Longrightarrow & f(x) \searrow \text{ on } (2, \infty) \end{array}$$

Since $f(0) = 0$ and $f(2) = 4e^{-1}$, we place short horizontal tangents at $(0, 0)$ and $\left(2, 4e^{-1}\right)$. In addition,

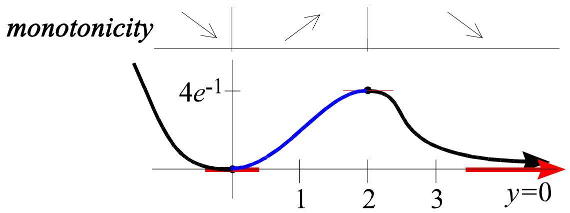
$$\lim_{x \to \infty} x^2 e^{-x} = \lim_{x \to \infty} \frac{x^2}{e^x} \overset{\frac{\infty}{\infty}}{=} \lim_{x \to \infty} \frac{2x}{e^x} \overset{\frac{\infty}{\infty}}{=} \lim_{x \to \infty} \frac{2}{e^x} = 0$$

However, as x approaches $-\infty$, the exponential e^{-x} approaches ∞, so that there is the horizontal asymptote of $y = 0$ applies only as x approaches ∞.



4-9: Monotonicity and Extrema

The graph decreases to $(0, 0)$. Then to preserve tangency, it increases in an "S" shape to $\left(2, 4e^{-1}\right)$. Finally, it decreases to the horizontal asymptote as x approaches ∞.

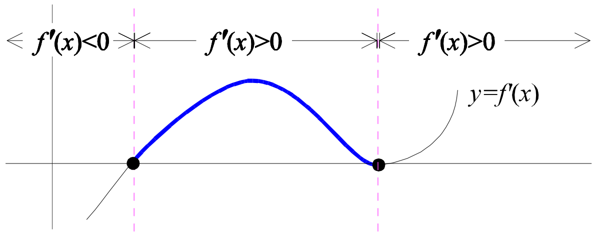


4-10: Crude Sketch of $y = x^2 e^{-x}$

Check your Reading *What does the graph of* $f(x) = 5x^9 - 9x^5$ *over* $[-2, 2]$ *look like on a grapher?*

Monotonicity Given the Graph of $f'(x)$

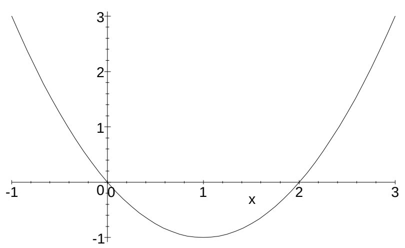
Monotonicity can also be ascertained from the graph of the derivative $f'(x)$ of a function $f(x)$. In particular, when the graph of $f'(x)$ is above the x-axis, then $f'(x) > 0$ and the function itself is increasing. When the graph of $f'(x)$ is below the x-axis, then $f'(x) < 0$ and f is decreasing.



4-11: Graph of $f'(x)$

In addition, stationary points of $f(x)$ are zeroes of $f'(x)$.

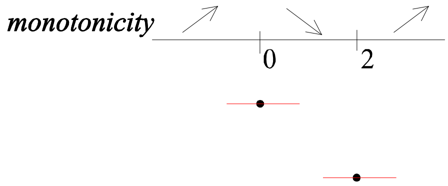
EXAMPLE 6 Sketch the graph of $f(x)$ given the graph of $f'(x)$ shown in figure 4-12.



4-12: Graph of $f'(x)$

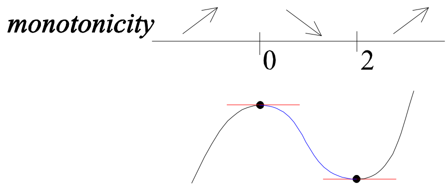
Solution: The graph of $f'(x)$ crosses the x-axis at $x = 0, 2$, which implies that $x = 0, 2$ are stationary points of $f(x)$. Moreover, the graph is *above* the x-axis for x in $(-\infty, 0)$ and $(2, \infty)$, thus implying that $f(x)$ is increasing on $(-\infty, 0)$ and $(2, \infty)$. Likewise, $f'(x) < 0$ on $(0, 2)$ implies that $f(x)$ is decreasing on $(0, 2)$.

Although we don't know the y-values of the stationary points, we can place short horizontal tangents to help determine the graph.



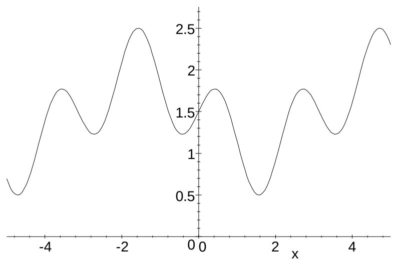
4-13: Monotonicity and Extrema

Connecting these points then provides some indication of what the graph of the function $f(x)$ might look like.



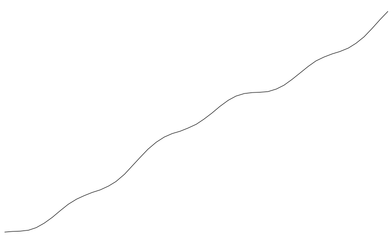
4-14: Crude Sketch of $y = f(x)$

EXAMPLE 7 Sketch the graph of $f(x)$ given the graph of $f'(x)$ shown in figure 4-15.



4-15: Graph of $f'(x)$

Solution: Notice that the graph of $f'(x)$ is completely above the x-axis. Thus, $f'(x) > 0$ for all x and $f(x)$ is always increasing with no horizontal tangents or extrema of any kind. Thus, the graph of $f(x)$ is simply a curve which is always increasing



4-16: Function is increasing everywhere

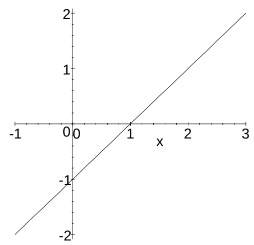
Exercises:

Find the intervals of monotonicity and the extrema of the following functions. Make a crude sketch of the graph using monotonicity and extrema. Verify your results by comparing to a graph of the function produced with a grapher.

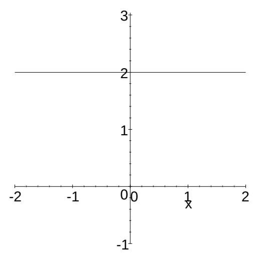
1.	$f(x) = x^2 - 2x$	2.	$f(x) = x^2 - 4x + 2$
3.	$f(x) = x^3 - 3x^2$	4.	$f(x) = x^4 - 2x^2$
5.	$f(x) = 2x - 1$	6.	$f(x) = 4 - 7x$
7.	$f(x) = x^4 - 4x^2 + 4$	8.	$f(x) = 5x^7 - 7x^5$
9.	$f(x) = 3x^5 - 5x^3$	10.	$f(x) = 2x^5 - 5x^2$
11.	$f(x) = x(x-3)^3$	12.	$f(x) = x^5(x-5)^2$
13.	$f(x) = \dfrac{x^3}{x^2+1}$	14.	$f(x) = \dfrac{x^4}{x^2+1}$
15.	$f(x) = (x-2)^{1/3}$	16.	$f(x) = (x-1)^{2/3}$
17.	$f(x) = x^{4/3} - 2x^{2/3}$	18.	$f(x) = x^{4/3} - 4x^{1/3}$
19.	$f(x) = xe^{-x}$	20.	$f(x) = xe^{-3x}$
21.	$f(x) = e^{-x^2/2}$	22.	$f(x) = x - e^x$
23.	$f(x) = \ln\left(x^2+1\right)$	24.	$f(x) = x - \ln(x)$

Determine the intervals of monotonicity and the location of the extrema of the function $f(x)$ given the graph of its derivative $f'(x)$.

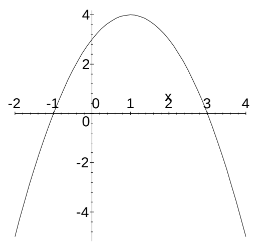
25. Graph of $f'(x)$:



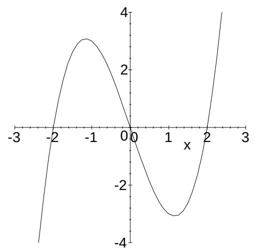
26. Graph of $f'(x)$:



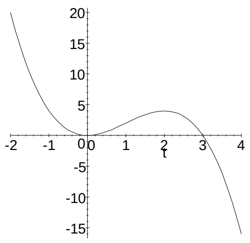
27. Graph of $f'(x)$:



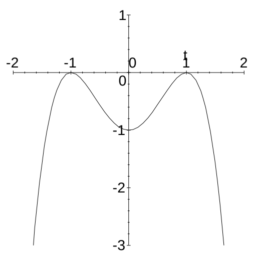
28. Graph of $f'(x)$:



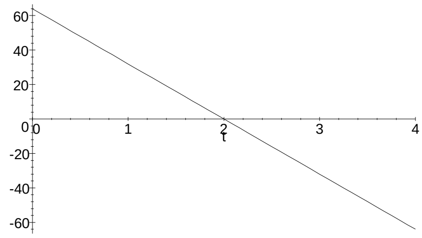
29. Graph of $f'(x)$:



30. Graph of $f'(x)$:



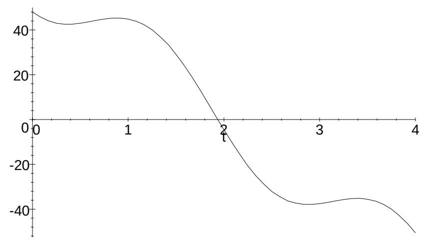
31. Suppose an object in free fall has a height at time t of $r(t)$, and suppose that its velocity function over $[0, 4]$ is as in figure 4-17.



4-17: Graph of $v(t)$

(a) Find the intervals of monotonicity and extrema of $r(t)$.
(b) Sketch the graph of $r(t)$. When does it attain its maximum height?
(c) If $v(t)$ is linear, then what does the acceleration $a(t)$ look like?

32. Suppose an object in free fall has a height at time t of $r(t)$, and suppose that its velocity function over $[0, 4]$ is as in figure 4-18.



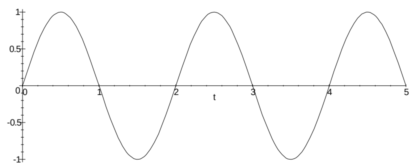
4-18: Graph of $v(t)$

(a) Find the intervals of monotonicity and extrema for t in $[0, 4]$.
(b) Sketch the graph of $r(t)$. When does it attain its maximum height?
(c) What does the acceleration $a(t)$ look like?

33. Find the intervals of monotonicity and extrema of the function $f(x) = x^x$ for $x \geq 0$. What is significant about the extrema of f?

34. If $n > 2$ is a positive integer, then how many maxima does the function $f(x) = x^n e^{-x}$ have? How many minima?

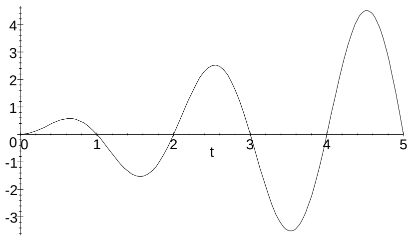
35. Write to Learn: Suppose an object traveling along a line has a position of $r(t)$ on the line at time t and suppose its velocity function is as shown in figure 4-19,



4-19: Graph of velocity $v(t)$

Write a short essay describing its motion between time $t = 0$ and $t = 5$, and explain how you arrived at that description.

36. Write to Learn: Suppose an object traveling along a line has a position of $r(t)$ on the line at time t and suppose its velocity function is as shown in figure 4-20,



4-20: Graph of velocity $v(t)$

Write a short essay describing its motion between time $t = 0$ and $t = 5$, and explain how you arrived at that description.

37. Use monotonicity and extrema to sketch the graph of $y(x) = ax^2 + bx + c$ with a, b, c positive.

38. Use monotonicity and extrema to sketch the graph of $y(x) = -ax^2 + bx + c$ with a, b, c positive.

39. Use monotonicity and extrema to sketch the graph of $y(x) = x^3 + ax$ with a positive.

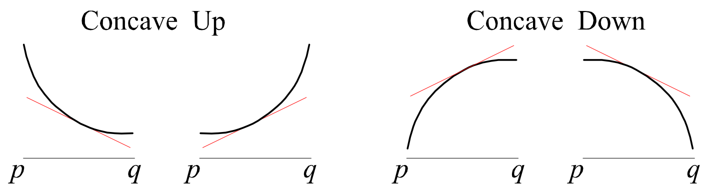
40. Use monotonicity and extrema to sketch the graph of $y(x) = x^3 - ax$ with a positive.

3.5 Concavity

Concavity and Inflection Points

The use of monotonicity to identify extrema requires that all intervals of monotonicity be identified. However, the points where a derivative changes sign may be difficult to calculate or even to estimate. Thus, in this section, we introduce a method for identifying extrema known as the *second derivative test* that uses the second derivative *at* the critical point instead.

If the graph of a function $f(x)$ is *above* its tangent lines over (a, b), then we say that $f(x)$ is *concave up* over (a, b). Similarly, if the graph of $f(x)$ is *below* its tangent lines over (a, b), then we say that $f(x)$ is *concave down* over (a, b).



5-1: Concavity

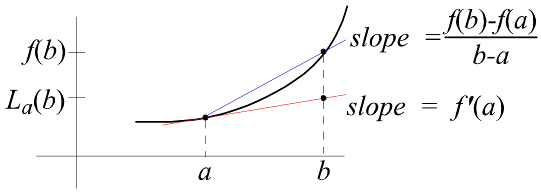
Suppose now that $f''(x) > 0$ on (p,q) and let a and b be in (p,q) such that $a < b$. The Mean Value theorem says that there is a number c in (a,b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

However, $f''(x) > 0$ on (p,q) implies that $f'(x)$ is increasing on (p,q). Thus, $f'(c) > f'(a)$ and

$$\frac{f(b) - f(a)}{b - a} > f'(a)$$

Thus, the line through $(a, f(a))$ with slope $\frac{f(b)-f(a)}{b-a}$ is *above* the tangent line $y = L_a(x)$, which is the line through $(a, f(a))$ with slope $f'(a)$.



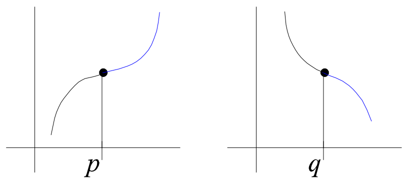
(3.22)

5-2: Tangent line is below the secant line

It follows that $f(b) > L_a(b)$ for all $a < b$ in (p,q), which is to say that the graph of $f(x)$ is above its tangent lines for all x in (p,q).

Theorem 5.1: If $f''(x) > 0$ on (p,q), then $f(x)$ is concave up over (p,q), and if $f''(x) < 0$ on (p,q), then $f(x)$ is concave down over (p,q).

In addition, an *inflection point* is a point on the graph of a function where there is a change in concavity:



5-3: Inflections points

An inflection point must be in *dom* (f). Thus, a vertical asymptote cannot be an inflection point.

EXAMPLE 1 Find the intervals of concavity and the inflection points of

$$f(x) = x^3 - 3x$$

Solution: Since $f'(x) = 3x^2 - 3$, the second derivative is $f''(x) = 6x$. Since $f''(x)$ can only change signs at $x = 0$, we test $f''(x)$ at points on either side of 0:

$$\begin{array}{lll} f''(-1) = -6 < 0 & \Longrightarrow & f(x) \text{ is concave down on } (-\infty, 0) \\ f''(1) = 6 > 0 & \Longrightarrow & f(x) \text{ is concave up on } (0, \infty) \end{array}$$

Moreover, since $f''(x)$ changes signs at $x = 0$, an inflection point must occur at $x = 0$.

We often use CU and CD to denote concave up and concave down, respectively.

EXAMPLE 2 Find the intervals of concavity and the inflection points of

$$f(x) = xe^{-x}$$

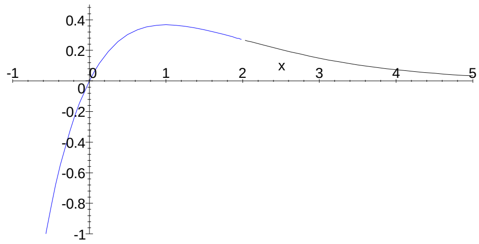
Solution: Since $f'(x) = e^{-x} - xe^{-x}$, the second derivative is

$$f''(x) = -e^{-x} - e^{-x} + xe^{-x} = (x-2)e^{-x}$$

Thus, we test for the sign of $f''(x)$ on $(-\infty, 2)$ and $(2, \infty)$:

$$\begin{array}{lll} f''(0) = (0-2)e^0 = -2 < 0 & \Longrightarrow & f(x)\ CD \text{ on } (-\infty, 2) \\ f''(3) = (3-2)e^3 = e^3 > 0 & \Longrightarrow & f(x)\ CU \text{ on } (2, \infty) \end{array}$$

As a result, the inflection point is at $x = 2$.

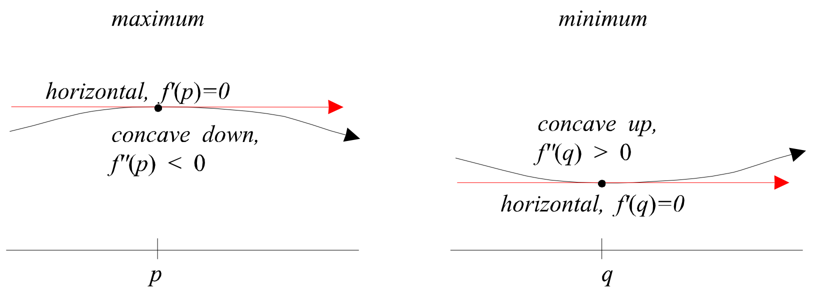


5-4: Graph of $f(x) = xe^{-x}$

Check your Reading *Is it possible for $f''(p) = 0$ and p not be an inflection point of f?*

Extrema and Concavity

Let us observe that if $f(x)$ is second differentiable at $x = p$ and if $f(x)$ has a relative maximum at $x = p$, then $f'(p) = 0$ and the graph of $f(x)$ must be concave down in the vicinity of p.



5-5: Concavity can be used to identify extrema

Likewise, if $f(x)$ has a relative minimum at $x = p$, then $f'(p) = 0$ and the graph of $f(x)$ is concave up in the vicinity of $x = p$. The converses of these statements are also true, as is given by the following theorem.

The Second Derivative Test: If $f'(p) = 0$ and $f''(p) < 0$, then $f(x)$ has a maximum at $x = p$. If $f'(q) = 0$ and $f''(q) > 0$, then $f(x)$ has a minimum at $x = q$.

Unfortunately, if $f'(q) = f''(q) = 0$, then the second derivative test provides no information and another method must be used.

EXAMPLE 3 Use the second derivative test to identify the extrema of $f(x) = x^3 - 3x^2$.

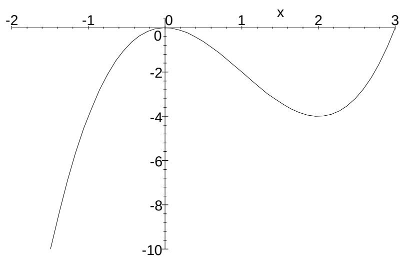
Solution: Since $f'(x) = 3x^2 - 6x$, setting $f'(x) = 0$ yields

$$3x^2 - 6x = 0, \qquad 3x(x-2) = 0$$

Thus, the stationary points are $x = 0$ and 2. Moreover, the second derivative is $f''(x) = 6x - 6$, so that the second derivative test yields

$$\begin{array}{lcl} f''(0) = 6 \cdot 0 - 6 = -6 < 0 & \Longrightarrow & f(x) \text{ has a maximum at } x = 0 \\ f''(2) = 6 \cdot 2 - 6 = 6 > 0 & \Longrightarrow & f(x) \text{ has a minimum at } x = 0 \end{array}$$

These results are confirmed by the graph of $f(x) = x^3 - 3x^2$ below:



5-6

EXAMPLE 4 Identify the extrema of $f(x) = xe^{2x}$.

Solution: Since $f'(x) = e^{2x} + 2xe^{2x}$, the stationary points satisfy

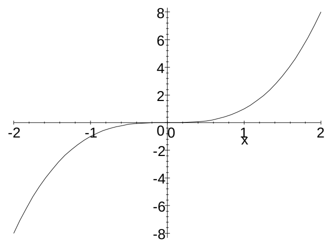
$$e^{2x}(1 + 2x) = 0$$

Since $e^{2x} > 0$ for all x, the stationary point is the solution to

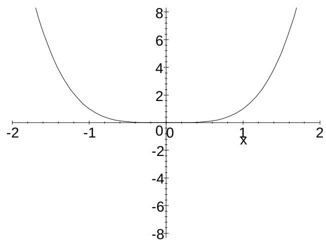
$$1 + 2x = 0$$

which is $x = -1/2$. In addition, $f''(x) = 4e^{2x} + 4xe^{2x}$ and $f''\left(\frac{-1}{2}\right) = 4e^{-1} - 2e^{-1} = 0.74 > 0$. Thus, $f(x) = xe^{2x}$ has a minimum at $x = -1/2$.

If p is a stationary point of $f(x)$ and if $f''(p) = 0$, then the second derivative test provides no information. This is because there may be a maximum, a minimum, or an inflection point at p. For exaample, if $f(x) = x^3$, then $f''(0) = 0$ and there is a point of inflection at $x = 0$, but if $g(x) = x^4$, then $g''(0) = 0$ and there is a minimum at $x = 0$.



5-7a: Graph of $f(x) = x^3$



5-7b: Graph of $g(x) = x^4$

EXAMPLE 5 Find the extrema of

$$f(t) = 3\cos\left(t^3\right)$$

Solution: Using the chain rule, it can be shown that

$$f'(t) = -3\sin\left(t^3\right)\,\frac{d}{dt}t^3 = -9t^2\sin\left(t^3\right)$$

Critical points occur when $\sin\left(t^3\right) = 0$, so the critical points are

$$t^3 = 0, \pi, 2\pi, \ldots \qquad or \qquad t = 0, \sqrt[3]{\pi}, \sqrt[3]{2\pi}, \ldots$$

The second derivative begins with the product rule,

$$\begin{aligned} f''(t) &= \frac{d}{dt}\left(-9t^2\sin\left(t^3\right)\right) \\ &= -\left(\frac{d}{dt}9t^2\right)\sin\left(t^3\right) - 9t^2\frac{d}{dt}\sin\left(t^3\right) \end{aligned}$$

after which we apply the chain rule:

$$\begin{aligned} f''(t) &= -18t\sin\left(t^3\right) - 9t^2\cos\left(t^3\right)\frac{d}{dt}t^3 \\ &= -18t\sin\left(t^3\right) - 27t^4\cos\left(t^3\right) \end{aligned}$$

At the stationary point $t = \sqrt[3]{\pi}$ we have

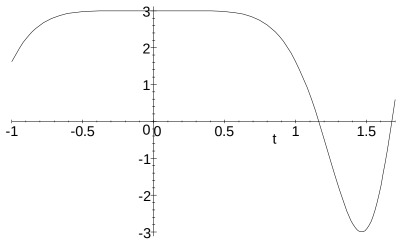
$$f''\left(\sqrt[3]{\pi}\right) = -18\sqrt[3]{\pi}\sin(\pi) - 27\left(\sqrt[3]{\pi}\right)^4\cos(\pi) = 27\left(\sqrt[3]{\pi}\right)^4 > 0$$

Thus, there is a relative minimum at $t = \sqrt[3]{\pi}$. Likewise, it is easy to show that

$$f''\left(\sqrt[3]{2\pi}\right) < 0, \quad f''\left(\sqrt[3]{3\pi}\right) > 0, \quad \ldots$$

so there is a maximum at $t = \sqrt[3]{2\pi}$, a minimum at $\sqrt[3]{3\pi} < 0$, and so on.

At the critical point $t = 0$, we have $f''(0) = 0$ and thus, the second derivative test yields no information about $f(t)$ at $t = 0$. However, the graph of $f(t) = \cos\left(t^3\right)$ has a maximum, and not an inflection point, at $t = 0$.



5-8: $f''(0) = 0$ at a maximum

Check your Reading *Where is* $\frac{9\pi}{4}$ on the x-axis in figure 5-7?

Graphs of Harmonic Oscillations

Recall that a *harmonic oscillation* is a function of the form

$$y(t) = a\cos(\omega t) + b\sin(\omega t) + M$$

where a, b, ω, and M are constants. The extrema of a harmonic oscillation are determined using the second derivative test, which in turn often requires the use of the following table:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan(\theta)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	VA

5-3: Reference angles for $\tan(\theta)$

In the table, VA stands for vertical asymptote.

EXAMPLE 6 **Tides:** The height h in feet above or below sea level of the ocean near Bridgeport, Connecticut, at time t in hours since midnight on Sept. 1, 1991, is approximated by

$$h(t) = 3.35 - 3\cos\left(\frac{t}{2}\right) + 3\sqrt{3}\sin\left(\frac{t}{2}\right)$$

How long after $t = 0$ does the first high tide occur?

Solution: Since $h'(t) = 0$ when $\frac{-3}{2}\sin\left(\frac{t}{2}\right) + \frac{3\sqrt{3}}{2}\cos\left(\frac{t}{2}\right) = 0$, the stationary points are the solution to

$$\begin{aligned} \frac{3}{2}\sin\left(\frac{t}{2}\right) &= \frac{3\sqrt{3}}{2}\cos\left(\frac{t}{2}\right) \\ \frac{\sin(t/2)}{\cos(t/2)} &= \frac{3\sqrt{3}/2}{3/2} \\ \tan\left(\frac{t}{2}\right) &= \sqrt{3} \end{aligned}$$

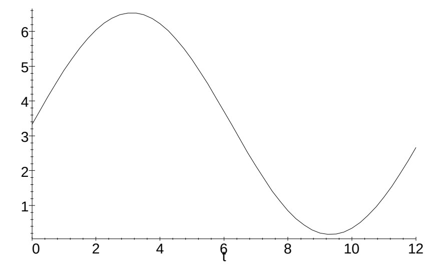
Thus, table 6.3 implies that $\frac{t}{2} = \frac{\pi}{3} + n\pi$, where n is any integer, and as a result, the critical points are of the form

$$t = \frac{2\pi}{3} + 2n\pi$$

The second derivative is $h''(t) = \frac{-3}{4}\cos\left(\frac{t}{2}\right) - \frac{3\sqrt{3}}{4}\sin\left(\frac{t}{2}\right)$. The first positive critical point occurs when $n = 0$, which yields $t = \frac{2\pi}{3}$ and

$$h''\left(\frac{2\pi}{3}\right) = \frac{-3}{4}\cos\left(\frac{1}{2}\frac{2\pi}{3}\right) - \frac{3\sqrt{3}}{4}\sin\left(\frac{1}{2}\frac{2\pi}{3}\right) = -1.5 < 0$$

Thus, $h(t)$ has a maximum at $t = \frac{2\pi}{3} = 2.0944$, so that the first high tide occurs at 2:06 a.m.



5-9: High Tide first occurs at 2:06 a.m.

The extrema, the amplitude, and the period of an oscillation can be used to construct its graph. In fact, periodicity can be used to extend the graph of the oscillation over a single period to the graph of the function on the whole line.

EXAMPLE 7 Determine the amplitude, period, extrema and graph of the function

$$y(t) = 2\sqrt{3}\cos(4\pi t) + 2\sin(4\pi t)$$

Solution: Since $\omega = 4\pi$, the period is

$$T = \frac{2\pi}{4\pi} = \frac{1}{2} \text{ sec}$$

Moreover, since $a = 2\sqrt{3}$ and $b = 2$, the amplitude is

$$A_m = \sqrt{2^2 \cdot 3 + 2^2} = \sqrt{16} = 4$$

Since $y'(t) = -8\pi\sqrt{3}\sin(4\pi t) + 8\pi\cos(4\pi t)$, the critical points of $y(t)$ are solutions to the following equation:

$$\begin{aligned} 8\pi\sqrt{3}\sin(4\pi t) &= 8\pi\cos(4\pi t) \\ \sqrt{3}\frac{\sin(4\pi t)}{\cos(4\pi t)} &= 1 \\ \tan(4\pi t) &= \frac{1}{\sqrt{3}} \end{aligned}$$

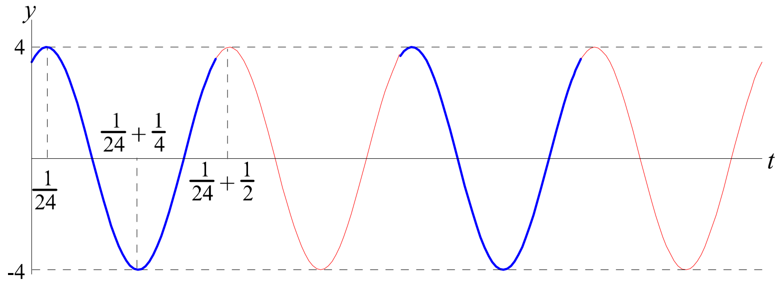
Table 6.1 implies that

$$4\pi t = \frac{\pi}{6} + n\pi$$

where n is an integer. Thus, the extrema are located at

$$t = \frac{1}{24} + \frac{n}{4}$$

Since $y\left(\frac{1}{24}\right) = 4$, there is a maximum at $t = \frac{1}{24}$, a minimum at $t = \frac{1}{24} + \frac{1}{4}$, a maximum again at $t = \frac{1}{24} + \frac{1}{2}$, and so on.



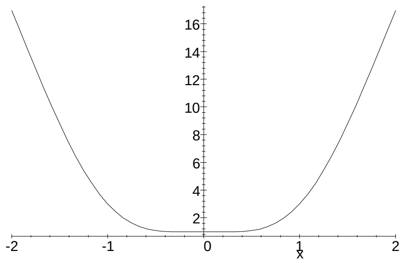
5-10: Extrema and Period determines a harmonic oscillation

Check your Reading *Show that if $f(x) = x^4$, then $f''(0) = 0$. What type of extremum does $f(x) = x^4$ have at $x = 0$?*

Extrema that Cannot be Easily Visualized

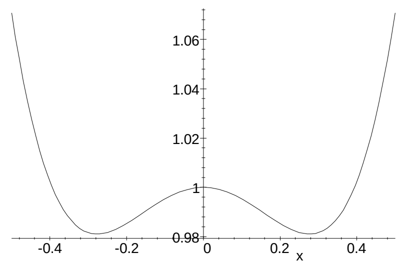
The graph of a function may have features which cannot be observed visually.

There are functions whose extrema cannot be easily visualized. For example, the graph of the function $f(x) = 4x^2 + \cos(3x)$ appears to reach a minimum when $x = 0$.



5-11

However, this is not the case. Zooming centered at the origin reveals that there is actually a maximum at $x = 0$.



5-12

Fortunately, the second derivative test can be employed when visualization and graphing proves to be unreliable.

EXAMPLE 8 Given $f(x) = 4x^2 + \cos(3x)$, show that $f'(0) = 0$ and then determine if $f(x)$ has a maximum or a minimum when $x = 0$.

Solution: The derivative of $f(x) = 4x^2 + \cos(3x)$ is

$$f'(x) = \frac{d}{dx}\left[4x^2 + \cos(3x)\right] = 8x - \sin(3x)\left(\frac{d}{dx}3x\right) = 8x - 3\sin(3x)$$

and if we let $x = 0$, then

$$f'(0) = 8(0) - 3\sin(0) = 0$$

thus verifying that $f(x)$ has a stationary point at $x = 0$. Moreover, the second derivative is

$$f''(x) = 8 - 9\cos(3x)$$

and at $x = 0$ we have

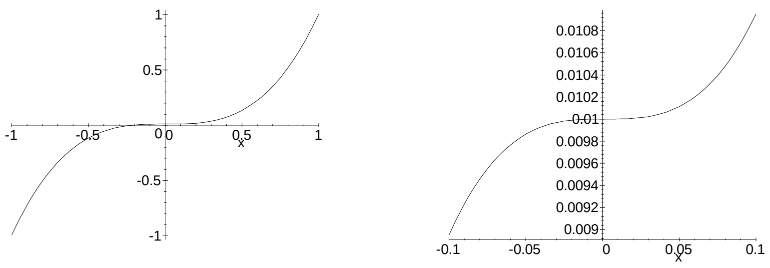
$$f''(0) = 8 - 9\cos(0) = 8 - 9 = -1 < 0$$

Thus, the second derivative test confirms that $f(x) = 4x^2 + \cos(3x)$ has a *maximum at* $x = 0$.

Even zooming does not guarantee that all the features of a graph will be revealed. For example, the function

$$f(x) = x^3 + 0.01\cos(x)$$

does not appear to have any extrema, and zooming does not reveal any new information about the function.



5-13: Zooming does not reveal any extrema

But there is more here than meets the eye!

EXAMPLE 9 Show that $x = 0$ is a stationary point of $f(x) = x^3 + 0.01\cos(x)$ and then determine if $f(x)$ has a maximum or a minimum at $x = 0$.

Solution: The first derivative is

$$f'(x) = 3x^2 - 0.01\sin(x)$$

and clearly, $f'(0) = 0$. Since $f''(x) = 6x - 0.01\cos(x)$, we have

$$f''(0) = -0.01$$

Thus, by the second derivative test, there is a maximum at $x = 0$, even though zooming may never reveal its presence.

Exercises:

Find the intervals of concavity and the inflection points of the following functions. (Note: You found intervals of monotonicity of these functions in the previous section.)

1. $f(x) = x^2 - 2x$
2. $f(x) = x^2 - 4x + 2$
3. $f(x) = x^3 - 3x^2$
4. $f(x) = x^4 - 2x^2$
5. $f(x) = x^4 - 4x^2 + 4$
6. $f(x) = 5x^7 - 7x^5$
7. $f(x) = 3x^5 - 5x^3$
8. $f(x) = 2x^5 - 5x^2$
9. $f(x) = x(x-3)^3$
10. $f(x) = x^5(x-5)^2$

Use the second derivative test to find the extrema of each function.

11. $f(x) = x^3 - 3x + 1$
12. $f(x) = x^4 - 4x^2$
13. $f(x) = x(x-4)^3$
14. $f(x) = x^2(x-4)^3$
15. $f(x) = xe^{-x}$
16. $f(x) = x^2e^{-x}$
17. $f(x) = x + \ln(x)$
18. $f(x) = x\ln(x)$
19. $f(t) = \sin(t^2)$
20. $f(t) = \cos(\pi\sqrt{t})$
21. $f(t) = \sin^2(t)$
22. $f(t) = \cos^2(t)$
23. $f(t) = t + 2\cos(t)$
24. $f(t) = t + \cos(t)$
25. $f(x) = e^x\sin(x)$
26. $f(x) = e^{-x}\cos(x)$

For each of the following, show that $f'(0) = 0$. Then compute $f''(0)$ to determine if the function has a maximum or a minimum when $x = 0$. Graph $f(x)$ on $[-1, 1]$. Does the graph reveal the extremum at $x = 0$?

27. $f(x) = 12x^2 + \cos(5x)$
28. $f(x) = 24x^2 + \cos(7x)$
29. $f(x) = e^x - x + \cos(1.01x)$
30. $f(x) = e^x - x + \cos(0.99x)$
31. $f(x) = x\sin(x) + 2.001\cos(x)$
32. $f(x) = x\sin(x) + 1.999\cos(x)$
33. $f(x) = \sin(x^3) + 0.01\cos(x)$
34. $f(x) = 4\sin(x^2) + \cos(3x)$

Find the period, frequency, and amplitude of the oscillation. Locate the extrema and use them to sketch the graph of the oscillation. Check your work by comparing to a plot produced by a grapher.

35. $y(t) = \sin(3t)$
36. $y(t) = \cos(3t)$
37. $y(t) = 5\cos(\pi t)$
38. $y(t) = \sin(\pi^2 t)$
39. $y(t) = \sqrt{2}\cos(\sqrt{7}\,t)$
40. $y(t) = 12\sin(\sqrt{\pi}\,t)$
41. $y(t) = \cos(t) - \sin(t)$
42. $y(t) = \cos(t) + \sin(t)$
43. $y(t) = \cos(3t) + \sin(3t)$
44. $y(t) = \sqrt{8}\cos(t) - \sqrt{8}\sin(t)$
45. $y(t) = \sqrt{3}\cos(2\pi t) - \sin(2\pi t)$
46. $y(t) = 2\cos(5t) - \sqrt{12}\sin(5t)$

47. The price p per pound of ground beef at time t in years since 1980 (and up to 1998) is approximately the same as the function

$$p(t) = 1.85 + 0.15\sin\left(\frac{2\pi}{9}t\right)$$

What is the period of the oscillation? When does the price of hamburger first reach its peak? What is the maximum price of ground beef during this time?[3]

48. Elmo's ice cream shop notes that if y denotes the number of customers per week at t weeks since the beginning of the year, then

$$y(t) = 200 + 20\cos\left(\frac{\pi}{26}t\right) - 20\sqrt{3}\sin\left(\frac{\pi}{26}t\right)$$

How fast is the number of customers per week increasing after 13 weeks? When does Elmo have the most customers? What is the maximum number of customers he can expect?

49. The average monthly temperatures y in Denver, Colorado at t months after the beginning of the year can be closely approximated by the function

$$y = 51.6 - 10.95\sqrt{3}\cos\left(\frac{\pi}{6}t\right) - 10.95\sin\left(\frac{\pi}{6}t\right)$$

Which month is the coldest month of the year? Which is the hottest month?

50. The current through a certain LC circuit is given by

$$I(t) = 2\cos(1890t) + 2\sin(1890t) \quad amps$$

Determine the period, amplitude, frequency, and extrema of the oscillation. Sketch the graph of $I(t)$.

51. **Grapher:** If y denotes the time the sun rises in Johnson City, TN, on the day which is t days after the beginning of 1999, then

$$y(t) = 6.4403 + 1.3903\cos\left(\frac{2\pi}{365}t\right) - 0.1803\sin\left(\frac{2\pi}{365}t\right)$$

(a) Graph $y(t)$ on the interval $[0, 365]$. Use the graph to estimate the relative maximum and relative minimum of $y(t)$.

(b) What does the model predict will be the earliest sunrise of the year in eastern standard time, and on which day will it occur? What does the model predict will be the latest sunrise of the year, and which day will it occur?

(c) Graph $y'(t)$ on $[0, 365]$. Where does it cross the t-axis? How is that related to (a) and (b)?

52. **Grapher:** If $y(t)$ denotes the number of hours of daylight in Johnson City, TN, on the day which is t days after the beginning of 1999, then $y(t)$ is closely approximated by

$$y(t) = 12.2 - 2.2855\cos\left(\frac{2\pi}{365}t\right) + 0.4036\sin\left(\frac{2\pi}{365}t\right)$$

[3] Based on Bureau of Labor Statistics data and on examples from Stefan Wagner and Steven R. Costenoble of Hofstra University.

(a) Graph $y(t)$ on the interval $[0, 365]$. Use the graph to estimate the relative maximum and relative minimum of $y(t)$.

(b) What does the model predict will be the longest day of the year and how long will it be? What does the model predict will be the shortest day of the year and how short will it be?

(c) Graph $y'(t)$ on $[0, 365]$. Where does it cross the t-axis? How is that related to (a) and (b)?

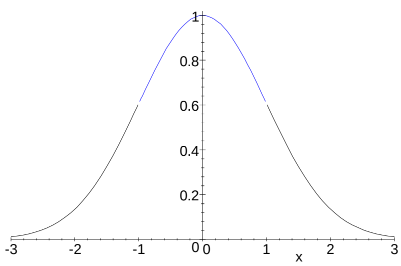
53. Under what condition does

$$f(x) = x^2 + \varepsilon \cos(\omega x)$$

have a maximum at the origin?

54. **Write to Learn:** In a short essay, explain why the second derivative yields no information about the extrema of $f(t) = \sin^2(t^2)$ when $t = 0$. Then explain why $\sin^2(t^2)$ must have a minimum at $t = 0$.

55. The graph of $f(x) = e^{-x^2/2}$ is the familiar "bell curve" of statistics.



5-14

The *standard deviation* of $f(x)$ is the distance from the origin to an inflection point of $f(x)$. What is the standard deviation of $f(x) = e^{-x^2/2}$?

56. If $\sigma > 0$ is a constant, then what is the standard deviation (see exercise 55) of

$$f(x) = e^{-x^2/(2\sigma^2)}$$

3.6 Optimization

Optimization with Constraints

Optimization problems are applications in which the desired answer is a maximum or minimum of a function. For example, a businessman might ask "What price maximizes revenue?" or an ecologist might ask "How do we minimize the impact on the environment?" In this section, we explore optimization problems to illustrate how calculus is used to answer questions like those above.

The typical optimization begins with a single output variable subject to one or more *constraints,* where a constraint is an equation involving two or more input variables. The constraints are used to reduce the output variable to a function of only one input variable, and then the *absolute extrema* of that function are determined, although in some applications relative extrema are sufficient.

EXAMPLE 1 Find the two nonnegative numbers x and y whose sum is 10 and whose product is a maximum.

Solution: The constraint is $x+y=10$. If we let P denote the product of x and y, then our optimization problem is

$$\textit{Maximize}\ \ P=xy \qquad \textit{subject to} \quad x+y=10$$

We must first write P as a function of a *single* input variable. Solving for y in the constraint yields

$$y=10-x$$

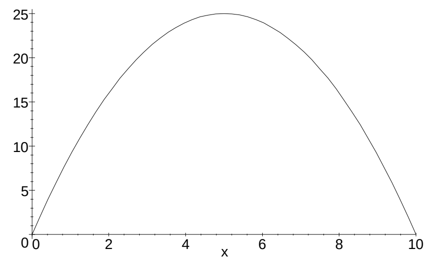
Substituting for y results in

$$P(x)=x(10-x)$$

Since x and y are nonnegative, x is in $[0,10]$. Thus, we must find the absolute maximum of $P(x)$ over $[0,10]$. To do so, we first identify any critical points in $[0,10]$. Setting $P'(x)=10-2x$ equal to 0 yields

$$\begin{aligned} 10-2x &= 0 \\ x &= 5 \end{aligned}$$

The graph of $P(x)$ over $[0,10]$ implies an absolute maximum at $x=5$.



6-1: Maximum occurs at $x=5$

Finally, since $y=10-x$, the value of y corresponding to $x=5$ is

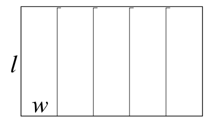
$$y=10-5=5$$

Check your Reading *What is the maximum value of P in example 1?*

Optimization Word Problems

Output variables are often called *dependent variables,* and the input variables are often called *independent variables.* When the optimization problem is presented as a word problem, the dependent variable is the quantity to be either maximized or minimized. The remaining variables are typically independent variables that are used to form constraints.

EXAMPLE 2 John wants to start a kennel by building 5 identical adjacent rectangular runs out of 400 feet of fencing (see figure 6-2),



6-2

Find the dimensions of a rectangular run which yield a maximum area for each run.

Solution: We first notice that the area of each run is to be maximized, so that our dependent variable is

$$A = \text{ area of a run in square feet}$$

The independent variables are the dimensions of the individual runs:

$$\begin{aligned} l &= \text{length of each run in feet (see figure above)} \\ w &= \text{width of each run in feet (see figure above)} \end{aligned}$$

Each run is a rectangle, so that A is the product of the length and the width:

$$A = lw \tag{3.23}$$

Since 400 feet of fence will be used for 10 sections of length w and for 6 sections of length l (see the figure), the constraint is

$$10w + 6l = 400 \tag{3.24}$$

Solving for w in (3.24) yields

$$w = 40 - 0.6l$$

after which substitution into (3.23) yields

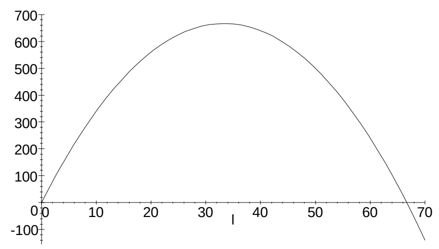
$$A(l) = l(40 - 0.6l) = 40l - 0.6l^2$$

Since l, w, and A must be nonnegative, l must be restricted to the interval $\left[0, \frac{200}{3}\right]$.

To find the critical point(s) of $A(l)$, we set $A'(l) = 40 - 1.2l$ equal to zero:

$$\begin{aligned} 40 - 1.2l &= 0 \\ l &= 33\frac{1}{3} \text{ feet} \end{aligned}$$

The graph of $A(l) = 40l - 0.6l^2$ reveals a maximum at $l = 33$ feet, 4 inches.



6-3

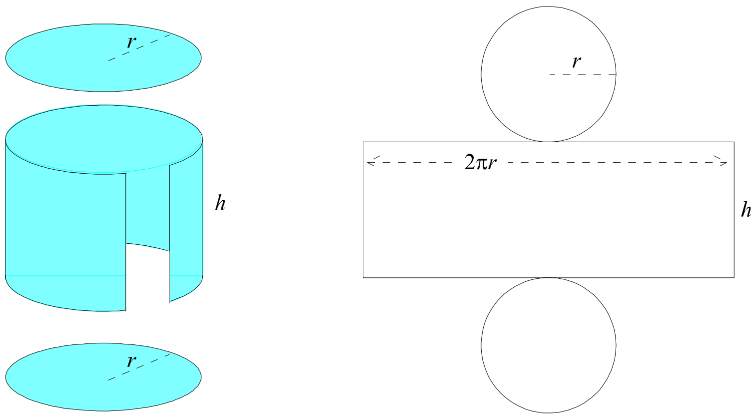
The corresponding width which maximizes the area must be

$$w = 40 - 0.6\left(33\frac{1}{3}\right) = 20 \text{ feet}$$

Thus, a length of 33 feet, 4 inches and a width of 20 feet maximizes the area of each run.

Some of the most important optimization problems are those in which the dependent variable is *minimized*—minimizing costs, minimizing error, minimizing energy, minimizing distance traveled, minimal surfaces, etcetera.

EXAMPLE 3 A certain cylindrical can is to have a volume of 0.25 cubic feet (approximately 2 gallons). Find the height h and radius r of the can that will minimize surface area of the can. What is the relationship between the resulting r and h?



6-4

Solution: Figure 6-4 shows us that the surface area S is the sum of the areas of 2 circles of radius r and a rectangle with height h and width $2\pi r$. Thus,

$$S = 2\pi r^2 + 2\pi rh$$

Moreover, the volume of 0.25 ft^3 is a constraint that is equal to the product of the area πr^2 of the base and the height h, which we then solve for h:

$$\pi r^2 h = 0.25, \qquad h = \frac{0.25}{\pi r^2}$$

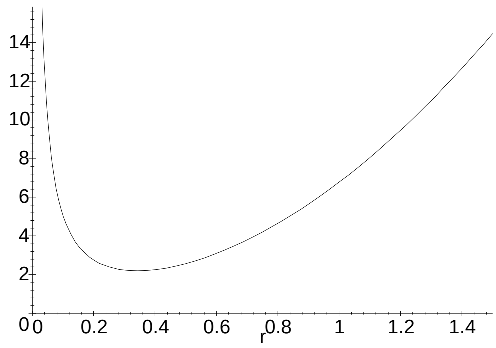
Substitution for h reduces the surface area to a function of 1 variable:

$$S(r) = 2\pi r^2 + 2\pi r\left(\frac{0.25}{\pi r^2}\right) = 2\pi r^2 + 0.5r^{-1}$$

Since $S'(r) = 4\pi r - 0.5r^{-2}$, the critical points are solutions to

$$\begin{aligned} 4\pi r - 0.5r^{-2} &= 0 \\ 4\pi r &= 0.5r^{-2} \\ r^3 &= \frac{1/2}{4\pi} \\ r &= \sqrt[3]{\frac{1}{8\pi}} = \frac{1}{2\sqrt[3]{\pi}} \end{aligned}$$

The graph of $S(r)$ reveals that a minimum occurs.



6-5: Graph of $S(r)$

It can also be shown that $S'(r) < 0$ for all $0 < r < \frac{1}{2\sqrt[3]{\pi}}$ and that $S'(r) > 0$ for all $r > \frac{1}{2\sqrt[3]{\pi}}$, so that the minimum must be an absolute minimum for r in $(0, \infty)$. Thus, the can has the least surface area when $r = \frac{1}{2\sqrt[3]{\pi}}$ and

$$h = \frac{0.25}{\pi \left(\frac{1}{2\sqrt[3]{\pi}} \right)^2} = \frac{1}{\sqrt[3]{\pi}}$$

Moreover, notice that $h = 2r$, which implies that the can with the least surface area is the one in which the height is the same as the diameter of the base—that is, the one whose cross-section is a square.

Check your Reading *Why do we only consider r in $(0, \infty)$ in example 3?*

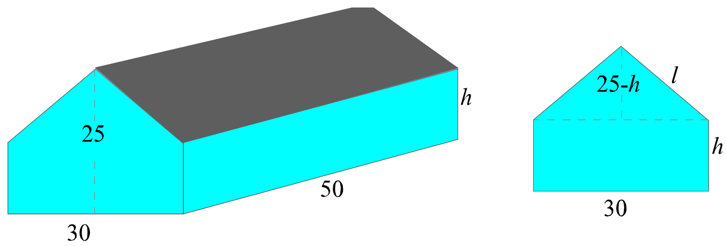
Computer Algebra Systems

A computer is useful only if we understand the results it produces.

In the fifties and sixties, slide rules made multiplication and division easier to perform, but they did not eliminate the need to learn arithmetic. In the seventies and eighties, scientific calculators became commonplace in evaluating scientific functions, but they did not eliminate the need to learn trigonometry. In the nineties, graphing calculators became the tool of choice for graphing functions, but they did not eliminate the need to learn calculus. In each of these cases, technology enhanced mathematics, but it did not replace the mathematics itself.

Likewise, computer algebra systems can be used to differentiate functions and estimate critical points, but they are of little value in identifying constraints or determining which quantity is to be optimized. However, they are a useful tool when calculation by hand is either tedious or prohibitively time-consuming.

EXAMPLE 4 Fred wants to build a house that is 50 feet long, 30 feet wide, and 25 feet tall at its tallest, as shown in figure 6-6 below:



6-6: Height at center of side is 25 feet

If siding costs \$3 per square foot and roofing costs \$10 per square foot, what dimensions h and l minimize the cost of the exterior of the house?

Solution: The area of the roof is $2 \cdot 50 \cdot l$, so the cost of the roof is $10 \cdot 100l = 1000l$. The area of the front and back is $2 \cdot 50 \cdot h$, so that price of the front and back is $3 \cdot 100h = 300h$. Each end is a rectangle with area $30h$ subtended by an isosceles triangle with area $\frac{1}{2}(25-h)\,30$, so the total area of the 2 ends is $2(30h + 15(25-h))$ and the cost of the two ends is $3 \cdot 2 \cdot (15h + 375) = 90h + 2250$. Thus, the total cost C is

$$C = 1000l + 300h + 90h + 2250$$

The constraint relating l to h follows from the Pythagorean theorem, which says that

$$(25-h)^2 + (15)^2 = l^2 \quad or \quad l = \sqrt{(25-h)^2 + 225}$$

Substituting l into the cost then implies that

$$C(h) = 1000\sqrt{(25-h)^2 + 225} \quad + \quad 390h + 2250$$

The derivative follows from the chain rule and is given by

$$C'(h) = \frac{1000(h-25)}{\sqrt{(25-h)^2 + 225}} + 390$$

We then use a computer algebra system to estimate the zeroes of $C'(h)$ in $[0, 18]$.

$$\frac{1000(h-25)}{\sqrt{(25-h)^2 + 225}} + 390 = 0 \quad \Longrightarrow \quad h = 18.644\ ft$$

We also use the computer algebra system to compute $C(h)$ at the endpoints and the critical point:

$$\begin{aligned} C(0) &= 1000\sqrt{25^2 + 225} + 2250 = \$31,404.76 \\ C(18.644) &= 1000\sqrt{(25-18.644)^2 + 225} + 390 \cdot 18.644 + 2250 = \$25,812.22 \\ C(18) &= 1000\sqrt{0^2 + 225} + 390 \cdot 25 + 2250 = \$27,000 \end{aligned}$$

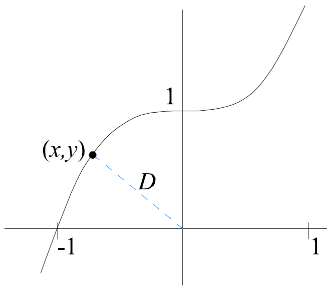
Thus, the cost of the exterior of the house is minimized when $h = 18.644$ feet.

Check your Reading *A standard roof has a pitch of* 30°. *For the house in example 4, this would result in a height of* $h = 16.34$ *feet. What is the total cost in example 4 when* $h = 16.34$ *feet?*

Distance from a Point to a Curve

A common optimization problem is that of finding the shortest distance from a curve to a given point. Often, the resulting minimum distance is called the distance from the point to the curve.

EXAMPLE 5 Find the point on the curve $y = x^3 + 1$ which is closest to the origin.



6-7: D is the distance from (x, y) to $(0, 0)$

Solution: The dependent variable is the distance D from the origin to a point (x, y) on the line, which by the distance formula is

$$D = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$$

Moreover, inspection of the graph reveals that the minimum distance must occur when x is in $[-1, 0]$. Since the constraint is the curve itself $y = x^3 + 1$, substituting for y yields

$$D(x) = \sqrt{x^2 + (x^3 + 1)^2}$$

Now let us use a computer algebra system to calculate and simplify $D'(x)$:

$$\frac{d}{dx}\sqrt{x^2 + (x^3+1)^2} \quad \overset{solver}{\longrightarrow} \quad \frac{3x^5 + 3x^2 + x}{\sqrt{x^2 + (x^3+1)^2}}$$

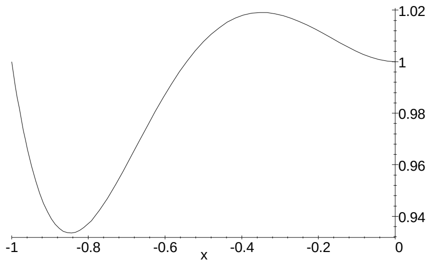
The critical points occur when $D'(x) = 0$, which is when

$$3x^5 + 3x^2 + x = 0$$

Although one solution clearly is $x = 0$, we must use a computer algebra system to numerically estimate other solutions:

$$3x^5 + 3x^2 + x = 0 \quad \overset{solver}{\longrightarrow} \quad x = -0.846, -0.348, 0$$

The graph of $D(x)$ over $[-1, 0]$ is of the form



6-8

and $D(x)$ is minimized when $x = -0.846$. If $x = -0.846$, then

$$y = (-0.846)^3 + 1 = 0.395$$

Thus, the point $(-0.846, 0.395)$ is the point on the curve $y = x^3 + 1$ that is closest to the origin..

Exercises:

1. Maximize $A = xy$ subject to the constraint $3x + y = 30$.

2. Maximize $A = x^2y$ subject to the constraint $x + y = 30$.

3. Maximize $A = x^2y$ subject to the constraint $x + y = 30$.

4. Minimize $A = xy$ subject to the constraint $x^2 + y = 30$.

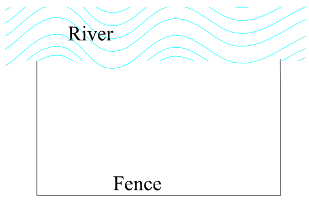
5. Minimize $E = x^2 + y^2$ subject to the constraint $y = x - 1$.

6. Minimize $E = x^2 + y^2$ subject to the constraint $y^2 = 1 - x$.

7. Maximize the product of two positive numbers whose sum is 36.

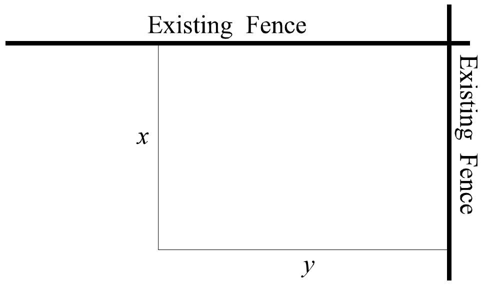
8. Minimize the sum of two positive numbers whose product is 36.

9. A farmer has 400 feet of fence with which to enclose a rectangular field bordering a river. What dimensions of the field maximize the area if the field is to be fenced on only three sides (see picture below)?



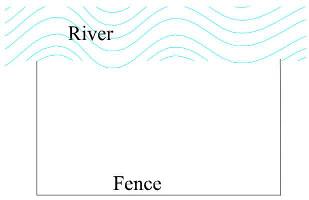
6-9

10. A farmer has 400 feet of fence with which to fence in a rectangular field adjoining two existing fences which meet at a right angle. What dimensions maximize the area of the field?



6-10

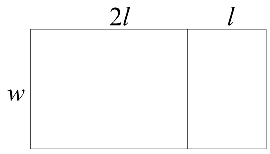
11. A farmer wants to enclose a 4,000 ft^2 field by using as little fence as possible. If the field adjoins a river so that only three sides will be fenced, then what dimensions of the field minimize the amount of fencing required?



6-11

12. Redo example 4 when the cost of siding is \$5 per square foot and the price of roofing is \$8 per square foot.

13. Suppose the two enclosures shown below are to be fenced using 3000 feet of fence



6-12

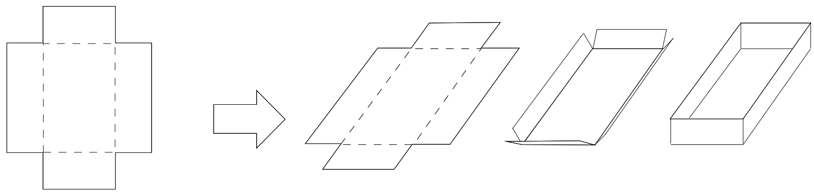
What values of w and l maximize the total area of the two enclosures. What is the total area enclosed with respect to these values?

14. Suppose the two enclosures in exercise 13 are to enclose a total area of 41,000 square feet. What dimensions w and l lead to the shortest possible fence? What is the total length of fence needed?

15. Acme fast food sells 500 megawhoppers a day when the price of a megawhopper is \$2, and they sell 750 megawhoppers a day when the price of a megawhopper is \$1. If the number sold each day is a linear function of the price, what should the price of a megawhopper be in order to maximize the daily revenue from megawhoppers?

16. Acme sporting goods sells 10 tennis rackets each week when the price of each racket is \$60, and it sells 15 tennis rackets each week when the price is \$55. Assuming the relationship between price and the number of rackets sold is linear, what price will maximize weekly revenue?

17. In each of (a) and (b) below, a box with an open top is made by cutting squares from the corners of a sheet of paper and then folding the result into a box.

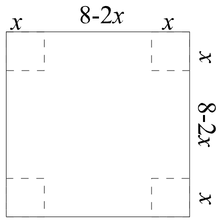


6-13

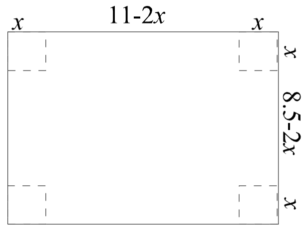
In each case, x is the length of the side of the square cut from the corner and the goal is to maximize the volume.

(a) Suppose the squares of length x are cut from the corners of an 8 inch by 8 inch sheet of paper (see figure below left). What value of x maximizes the volume of the box?

(b) Suppose the squares of length x are cut from the corners of an $8\frac{1}{2}$ inch by 11 inch sheet of paper (see figure below right). What value of x maximizes the volume of the box?

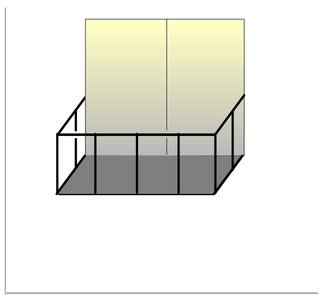


Exercise (a)

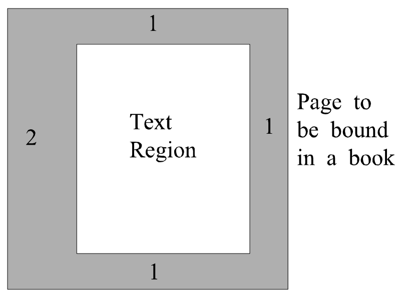


Exercise (b)

18. What dimensions of a 80 ft^2 balcony minimize the length of the rail around it?



Exercise 18

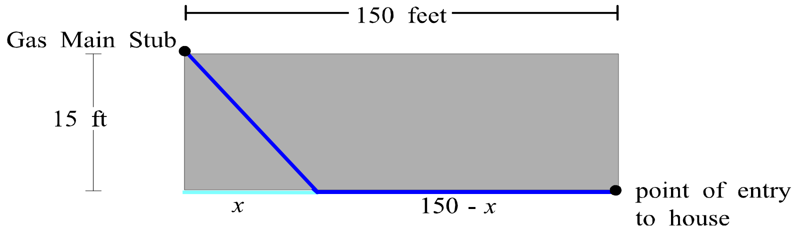


Exercise 19

19. Suppose a bookbinder must leave margins of 2 inches on the left and 1 inch on the top, right and bottom of a page. If the text region is to have an area of 12 square inches, what dimensions of a page make the area of the page a minimum?

20. A box with a square base and an open top is to have a fixed volume of 60 in^3. What dimensions minimize the surface area of the box?

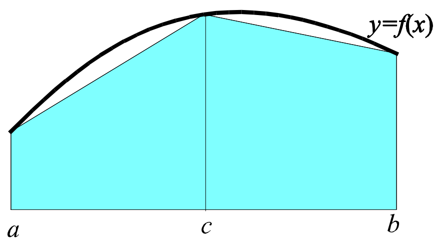
21. You wish to have a gas line run to your house. The gas line, shown in blue below, will begin at a gas main stub that is on the opposite side of your driveway, as shown below:



6-14

The contractor charges \$3.50 per foot to install the line under earth only and \$5.50 per foot to install the line under your driveway since it will have to be patched. What value of x minimizes the cost of the gas line? What is the minimum cost?

22. Suppose that $f(x)$ is continuous, differentiable and concave down on $[a, b]$.

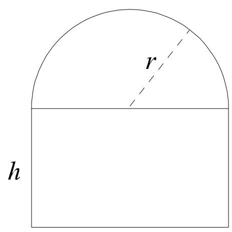


6-15

Show that the area of the shaded region in figure 6-15 is a maximum when c is the number in $[a, b]$ which satisfies the Mean Value Theorem,

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

23. The Norman Window Problem. The window below is called a Norman window.



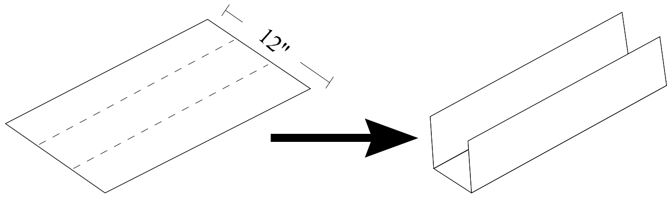
6-16

(a) Find a formula for the area of the Norman window in terms of r and h.

(b) If the perimeter of the Norman window is to remain a constant, P, find the area of the window as a function of r.

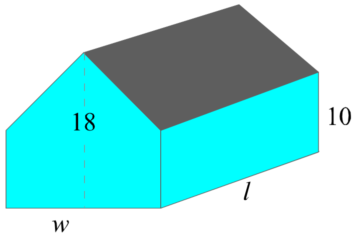
(c) Find the radius which maximizes the area given the fixed perimeter P.

24. A long sheet of metal which is 12 inches wide is to be made into a rain gutter by turning up two sides at right angles to the sheet. How many inches should be turned up in order to give the gutter its greatest capacity?



6-17

25. A house with width w and length l is be 18 feet tall at its tallest and 10 feet tall at each corner.



6-18

What dimensions for a 2000 square foot house (i.e., $wl = 2000$) minimize the area of the roof and sides of the house?

26. What dimensions would minimize the cost of the exterior of the house in exercise 25 if siding is \$3 per square foot and roofing is \$10 per square foot?

In exercises 27-33, find the point on the curve closest to the given point. You may want to use a Computer Algebra system to assist in the calculation.

27. The point on $y = 2x + 3$ closest to $(0, 0)$

28. The point on $y = 1 + x$ closest to $(3, 5)$

29. The point on $y = x^3 - 2x - 3$ closest to $(0, 0)$

30. The point on $y = x^4 - 2x^2 + 1$ closest to the origin.

31. The point on $x^2 + y^2 = (1 - 2y)^2$ closest to the origin.

32. The point on $2x^2 + y^2 = 5$ closest to $(1, 1)$

33. The point on $x^2 + 4y^2 = 4$ closest to the point $(2, 1)$.

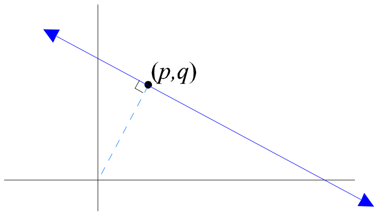
34. The point on $x^2y = 1$ closest to the origin

35. **Computer Algebra System.** The moon's orbit about the earth is well-approximated by the curve

$$x^2 + y^2 = (238,957 - 0.0549y)^2$$

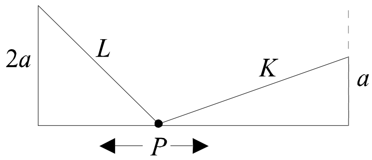
where distance is in miles. How close is the moon to the earth at its closest point? What is the greatest distance between the moon and the earth?

36. Show that if (p, q) is the point on the line $y = mx + b$ closest to the origin, then the line from the origin through (p, q) is *perpendicular* to $y = mx + b$.



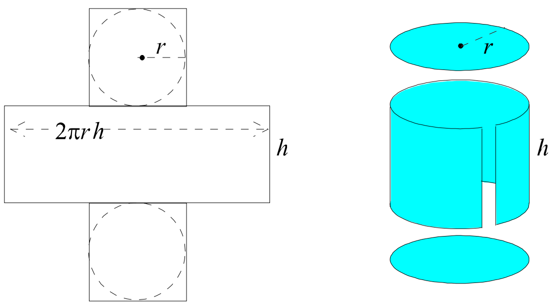
6-19

37. **Computer Algebra System.** For $a > 0$, consider L and K as shown in figure 6-20. Where on the horizontal should the point P be in order for $L+K$ to be a minimum?



6-20

38. Suppose that a certain cylindrical can is to have a volume of 0.25 cubic feet (approximately 2 gallons) but that the top and bottom of the can are both punched out of squares with sides of length $2r$ (see figure 6-21 below). Find the height h and radius r of the can that will minimize surface area of the can. What is the relationship between the resulting r and h?



6-21

39. **Computer Algebra System.** Suppose in exercise 38 that the cost of producing the side of the container is \$0.20 per square foot and the cost of producing the lids is \$0.30 per square foot. The cylindrical container produced will have a volume of 0.25 cubic feet (approximately 2 gallons).

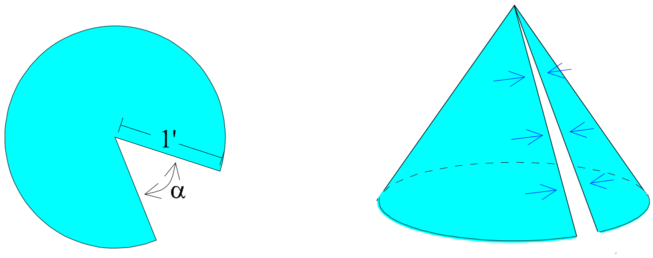
(a) Find the cost function, $C(r)$, for producing this container in terms of the radius of the cylinder, r.

(b) Find the value of r which minimizes the cost function $C(r)$.

(c) What is the cost of producing such a container?

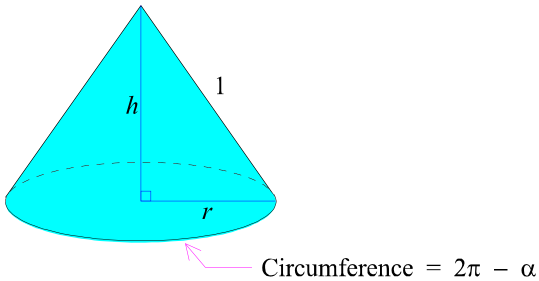
40. **Write to Learn:** Write a short essay showing that of all the cylindrical cans with a fixed volume V, the one with the least surface area is the one in which the cross-section of the can is a square.

41. To make a right circular cone from a flat circular disk with a 1 foot radius, a pie shaped wedge is cut from the disk and the two radial edges are connected to form an open cone.



6-22

Let α denote the angle of the pie shaped wedge cut from the circular disk:



6-23

(a) Explain why the circumference of the base of the cone is $2\pi - \alpha$. What is the height h and the radius of the base r of the cone?

(b) Show that the volume of the cone as a function of α is

$$V(\alpha) = \frac{\pi}{3}\left(1 - \frac{1}{2\pi}\alpha\right)^2 \sqrt{1 - \left(1 - \frac{1}{2\pi}\alpha\right)^2}$$

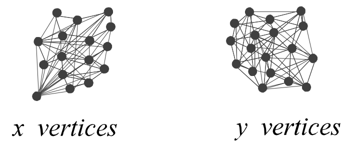
(Recall: Volume formula for cone is $V = \frac{1}{3}\pi r^2 h$).

(c) What angle α maximizes the volume of the cone?

42. Graph Theory: A collection of points called *vertices* with a set of connections called *edges* is called a *graph* if each pair of vertices has at most one edge between them and no vertex is connected to itself.

6-24: Graphs have at most one edge between any two vertices

Suppose that an even number n vertices are to be distributed between a graph with x vertices and another graph with y vertices. Suppose also that every vertex in the graph with x vertices is connected to every other vertex in the graph with x vertices, and every vertex in the graph with y vertices is connected to every other vertex in the graph with y vertices, but there are no edges between the two graphs. (e.g., two separate networks of fully connected computers).



6-25: Minimize the connectivity

What values for x and y minimize the total number of edges in the two graphs?

3.7 Least Squares

Averages and Total Squared Error

Statistical measures are often based on the principle that the *best* measure of a data set is the one with the *least* error. As a result, optimization is used to develop some of the most important techniques in statistics.

For example, if x denotes an arbitrary approximation of the set of numbers

$$a_1, \ldots, a_n$$

then for each $j = 1, \ldots, n$, the *individual error* or *residual* in approximating a_j by x is defined to be the number

$$\varepsilon_j = x - a_j$$

However, ε_j can be either positive or negative, which means the smallest possible error is when all the residuals approach $-\infty$. In addition, adding up the individual errors might lead to undesirable cancelation.

As a result, we work with the *squares* of the residuals, in that we define the *total squared error* $E(x)$ to be the sum of the squares of the individual errors:

$$E(x) = \varepsilon_1^2 + \varepsilon_2^2 + \ldots + \varepsilon_n^2 = (x - a_1)^2 + (x - a_2)^2 + \ldots + (x - a_n)^2$$

In particular, $E(x)$ small means that each of the ε_j must also be small in magnitude.

EXAMPLE 1 Suppose a certain student has test scores of 72, 82 and 79. What number x best represents those test scores in that it minimizes the total squared error?

Solution: To begin with, the total squared error function for the set $\{73, 82, 79\}$ is

$$E(x) = (x - 73)^2 + (x - 82)^2 + (x - 79)^2$$

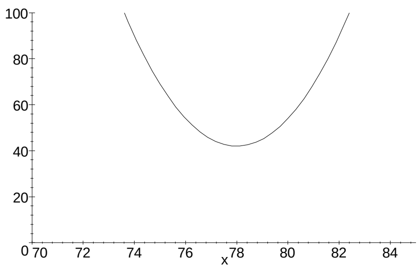
Our goal is to find the number x which minimizes the total squared error. To do so, we compute the derivative of $E(x)$,

$$E'(x) = 2(x - 73) + 2(x - 82) + 2(x - 79)$$

and set it equal to 0:

$$\begin{aligned}
2(x-73+x-82+x-79) &= 0 \\
3x-(73+82+79) &= 0 \\
3x &= 73+82+79 \\
x &= \frac{73+82+79}{3} = 78 \qquad (3.25)
\end{aligned}$$

The graph of $E(x)$ reveals a minimum when $x = 78$,



7-1: Minimum error when $x = 78$

In example 1, the best approximation is the average or *mean* of the numbers in the data set, which we denote by $\bar{x}$. In general, the mean $\bar{x}$ of a data set minimizes total squared error and is thus considered the best approximation of the data set in many applications.

Moreover, the *standard deviation* σ of the data set is the *root-mean-square* error, which is the square root of the average of the squared error in approximating the data set with the mean $\bar{x}$:

$$\sigma = \sqrt{\frac{(\bar{x}-a_1)^2 + (\bar{x}-a_2)^2 + \ldots + (\bar{x}-a_n)^2}{n}}$$

In terms of the total squared error, the standard deviation is given by

$$\sigma = \sqrt{\frac{E(\bar{x})}{n}}$$

In example 1, the standard deviation is

$$\sigma_x = \sqrt{\frac{E(78)}{3}} = \sqrt{\frac{42}{3}} = 3.741657$$

It gives us a sense of the average distance of each of the individual values from the average value of 78. That is, a point in the set $\{73, 82, 79\}$ is on average about 3.74 units away from the average value of 78.

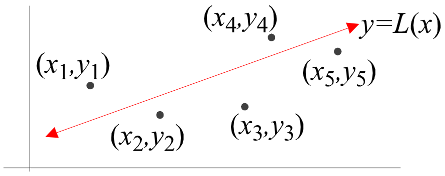
Check your Reading *Explain why $E(m)$ must always be a parabola which opens upward.*

The Least Squares Line.

A 2-dimensional data set is a collection of points of the form

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

We often desire to approximate a 2-dimensional data set with a straight line $y = L(x)$ that passes through the means $(\bar{x}, \bar{y})$ of the data.



7-2: The Least Squares Line

Sinc the line passes through $(\bar{x}, \bar{y})$, then $y = L(x)$ must be of the form

$$L(x) = \bar{y} + m(x - \bar{x}) \tag{3.26}$$

where the slope is a variable. Our goal is to find the slope m for which $y = L(x)$ *best approximates* the data set.

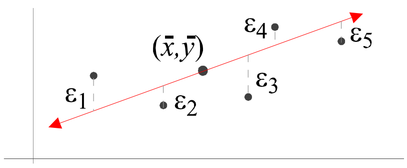
The j^{th} error, or *residual,* in approximating y_j by $L(x_j)$ is

$$\varepsilon_j = L(x_j) - y_j = \bar{y} + m(x_j - \bar{x}) \quad - \quad y_j \tag{3.27}$$

for $j = 1, 2, \ldots, n$. Moreover, (3.27) simplifies to

$$\varepsilon_j = m(x_j - \bar{x}) + \bar{y} - y_j \tag{3.28}$$

Geometrically, $|\varepsilon_j|$ is the vertical distance from the line to the j^{th} data point.



7-3: Errors in approximating the data set with the line

If we define the total squared error to be $E = \varepsilon_1^2 + \ldots + \varepsilon_n^2$, then it follows that

$$E(m) = (m(x_1 - \bar{x}) + \bar{y} - y_1)^2 + \ldots + (m(x_n - \bar{x}) + \bar{y} - y_n)^2$$

The line $y = L(x)$ whose slope is the minimum of $E(m)$ is called the *least squares line*, because it is the line that minimizes the total squared error between itself and the data set.

EXAMPLE 2 John scored a 60 on his first calculus test, a 78 on his second test, an 84 on his third test and an 86 on his fourth test. Find the least squares line for the test score data set.

Solution: If we pair the test number with the test score, we obtain the data set

$$(1, 60), (2, 78), (3, 84), (4, 86) \tag{3.29}$$

The means of the x-coordinate and y-coordinate are

$$\bar{x} = \frac{1+2+3+4}{4} = 2.5, \qquad \bar{y} = \frac{60+78+84+86}{4} = 77$$

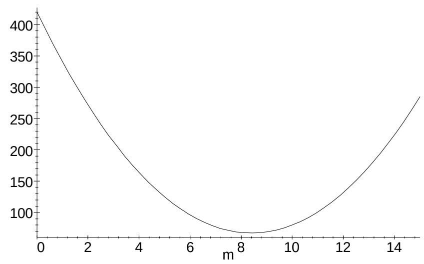
To find the residuals as functions of m, we complete the table below. It follows that the total squared error is the sum of the last column:

x_j	y_j	$x_j - \bar{x}$	$\bar{y} - y_j$	ε_j		ε_j^2
1	60	-1.5	17	$-1.5m + 17$		$2.25m^2 - 51.0m + 289$
2	78	-0.5	-1	$-0.5m - 1$		$0.25m^2 + 1.0m + 1$
3	84	0.5	-7	$0.5m - 7$		$0.25m^2 - 7.0m + 49$
4	86	1.5	-9	$1.5m - 9$		$2.25m^2 - 27.0m + 81$
					$E(m) =$	$5m^2 - 84m + 420$

Since $E(m) = 5m^2 - 84 + 420$, its derivative is $E'(m) = 10m - 84$. Setting $E'(m) = 0$ yields

$$10m - 84 = 0, \qquad m = 8.4$$

The graph of $E(m)$ reveals that it must have a minimum at $m = 8.4$:

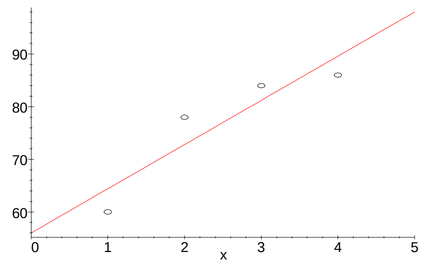


7-4

As a result, the least squares line for (3.29) is the line which passes through $(\bar{x}, \bar{y}) = (2.5, 77)$ with a slope of $m = 8.4$, which is

$$L(x) = 77 + 8.4(x - 2.5)$$

and which simplifies to $L(x) = 56 + 8.4x$.



7-5: The least squares line for the test score data set

That is, John's score's increased by about 8.4 points per test.

The process of obtaining a least squares linear approximation to a data set is called *linear regression*, and it is often calculated mechanically. For example, if the data set (3.29) is entered into a graphing calculator, then the calculator will return the least squares line

$$L(x) = 56 + 8.4x$$

Moreover, such devices also commonly provide the *correlation coefficient* r of the fit, which in terms of the least squares line slope M is given by

$$r = \frac{\sigma_x}{\sigma_y} M$$

where σ_x and σ_y are the standard deviation of the x-coordinates and y-coordinates, respectively. It can be shown that $-1 \leq r \leq 1$, and typically (though not always) the closer $|r|$ is to 1, the better the data can be approximated by a straight line.

Check your Reading *In example 2, what might the y-intercept of 56 suggest about John's level of preparation coming into the class?*

Least Squares Fits of Transformed Data Sets

Often data sets are transformed to fit linear models, thus allowing us to use the least squares line to fit data sets to a wider collection of curves. In particular, defining new independent and dependent variables often leads to a linear model.

EXAMPLE 3 If an object is released from rest, then its height r in feet at time t in seconds is

$$r = a + bt^2 \tag{3.30}$$

Moreover, observations of a falling object lead to the following (t, r) data set:

$$(1, 484), \quad (2, 437), \quad (3, 357), \quad (4, 244)$$

Use this data set to estimate a and b.

Solution: The formula $T = t^2$ transforms (3.30) into the straight line model

$$r = a + bT$$

Thus, letting $T = t^2$ results in a new data set that can be fit to a straight line:

t	r
1	484
2	437
3	357
4	244

is transformed into

$T = t^2$	r
1	484
4	437
9	357
16	244

If we now apply the least squares method to the (T, r) data, then we obtain the least squares line

$$r = 500.6163 - 16.0155\,T$$

Thus, the original data is best approximated (in some sense) by the curve

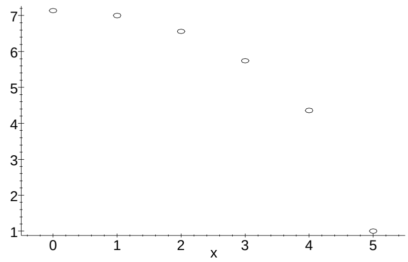
$$r = 500.6163 - 16.0155\,t^2 \tag{3.31}$$

EXAMPLE 4 Determine k and C for which the ellipse

$$y^2 = C + kx^2 \tag{3.32}$$

best fits the data set listed in the table and graphed below.

x	y
0	7.14
1	7
2	6.56
3	5.74
4	4.36
5	1



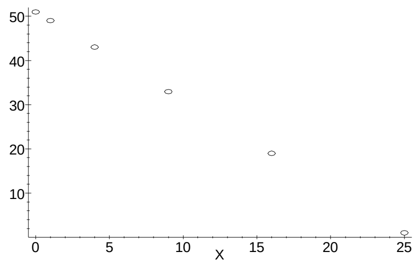
7-6

Solution: We let $X = x^2$ and $Y = y^2$, so that (3.32) is transformed into

$$Y = C + kX$$

The old data set is transformed into the new data set shown below.

X	Y
0	50.98
1	49
4	43.03
9	32.95
16	19.01
25	1



7-7

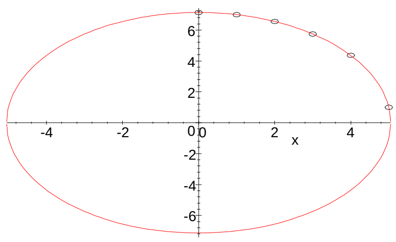
The least squares line for the new (X, Y) data set is

$$Y = 50.99 - 2X$$

As a result, the ellipse which *best fits* the *original* data set is

$$y^2 = 50.99 - 2x^2$$

In figure 7-8, the ellipse is shown along with the data.



7-8

This last example also illustrates the origins of the least squares method itself. The asteroid Ceres was first discovered on January 1, 1800, only to be lost again a few weeks later when astronomers failed to relocate it in the night sky. It remained lost until 1807, when the mathematician Carl Gauss developed the least squares method and applied it to the observational data from seven years earlier. As a result, he was able to determine Ceres' elliptical orbit and predict where in the night sky the "planet" would re-appear. Ceres was rediscovered in 1807 in agreement with Gauss' prediction.

Check your Reading *What is the acceleration of the object whose height at time t is given by (3.31)?*

Curve Fitting and Optimization

It may not always be possible or advantageous to transform a curve-fitting problem into a linear regression. An alternative approach is to approximate a data set

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

with a class of functions $f_m(x)$ parametrized by m. The *total squared error* for this alternative is given by

$$E(m) = (f_m(x_1) - y_1)^2 + \ldots + (f_m(x_n) - y_n)^2$$

If m^* denotes the value of m which minimizes $E(m)$, then the curve

$$y = f_{m^*}(x)$$

is called the *least squares fit* of the data.

EXAMPLE 5 An object launched into the air with an initial velocity of m has a height r in feet at time t in seconds of

$$r = mt - 16t^2 \tag{3.33}$$

neglecting air resistance. Heights are measured at various times to produce the following data set:

time: $t =$	1	2	3	4
height: $r =$	85	135	155	145

Use a least squares fit of the the data to estimate the initial velocity.

Solution: Since the parameter in (3.33) is m, we have

$$r(1) = m-16,\ r(2) = 2m-64,\ r(3) = 3m-144, \text{ and } r(3) = 4m-256$$

As a result, the total squared error is

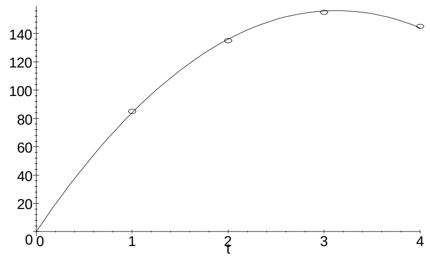
$$\begin{aligned} E(m) &= (m-16-85)^2 + (2m-64-135)^2 + (3m-144-155)^2 + (4m-256-145)^2 \\ &= (m-101)^2 + (2m-199)^2 + (3m-299)^2 + (4m-401)^2 \end{aligned}$$

To find $E'(m)$, we use the chain rule:

$$E'(m) = 2(m-101) + 4(2m-199) + 6(3m-299) + 8(4m-401)$$

This simplifies to $E'(m) = 60m - 6000$, and clearly, $E'(m) = 0$ when $m = 100$. Moreover, $E'' = 60 > 0$, so $E'(m)$ has a minimum at $m = 100$. Thus, the least squares estimate of the initial velocity is $m = 100$ feet/sec.

In example 5, the curve $r = 100t - 16t^2$ is the *least squares fit* of the data, as is shown below.



7-9: A least squares quadratic fit

Moreover, $E(100) = (100 - 101)^2 + (200 - 199)^2 + (300 - 299)^2 + (400 - 401)^2 = 4$, so that the *rms* error is

$$\sigma = \sqrt{\frac{E(100)}{4}} = 1$$

Thus, the initial velocity is 100 feet per second, give or take about 1 foot per second.

Exercises:

Find the number x that best approximates the given set of numbers by minimizing the total squared error. Then find the standard deviation for that approximation..

1. $1, 3, 5, 2$
2. $1, 9, -2, 5$
3. $2, 2, 4, 6, 6$
4. $1, 2, 3, 4, 5$
5. $4.1, 2.8, 5.7, 9.2$
6. $5, 5, 5, 5$

Numerical: *For each given data set, use a table of the form below to compute* $E(m)$.

x_j	y_j	$x_j - \bar{x}$	$\bar{y} - y_j$	$\varepsilon_j = (x_j - \bar{x})m + \bar{y} - y_j$	ε_j^2
				$E(m) =$	total

Then minimize $E(m)$ *and determine the least squares line.*

7. $(1,1), (2,2), (3,3)$
8. $(1,72), (2,97), (3,83)$
9. $(0,3.1), (1,5), (2,6.9)$
10. $(0,4.1), (1,3.9), (2,4.1)$
11. $(1,75), (2,79), (3,85), (4,81)$
12. $(1,75), (2,79), (3,81), (4,85)$

Numerical: *Fit the following curves to the given data sets using the supplied transformations:*

13. $y = C + kt^2$, $T = t^2$,

t	y
0	160
1	144
2	96
3	16

14. $y = C + kt^2$, $T = t^2$,

t	y
0	0
1	1
2	4
3	9

15. $y = C + kt^2$, $T = t^2$,

t	y
0	134
1	107
2	65
3	10

16. $y = \sqrt{C + kt}$, $Y = y^2$,

t	y
135	0
105	1
72	2
30	3

17. $y^2 = kx^2 + C$, $Y = y^2,\ T = t^2$,

t	y
2	9.798
4	9.165
6	7.941
8	6.012

18. $y^2 = kx^2 + C$, $Y = y^2,\ T = t^2$,

t	y
0	10
6	8
8	6
10	0

19. $y = \dfrac{1}{C + kt^2}$, $T = t^2, \quad Y = \dfrac{1}{y}$,

t	y
0.0	1.000
0.5	0.800
1.0	0.500
1.5	0.308
2.0	0.200
2.5	0.138

20. $y = 1 + \sqrt{kt^2 + C}$, $Y = (y-1)^2, \quad T = t^2$,

t	y
0	3.236
1	3.646
2	4.606
3	5.796

21. Numerical: Dropping a rock from a height of 150 feet generates the data set

t = Time (sec)	r = Height (feet)
0	150
1	135
2	85
3	7

Dropping the rock implies an initial velocity of 0, which means that the height r is related to time t via a parabola of the form

$$r = C + kt^2$$

Transform the data so that a least squares line can be used to estimate C and k. Why is k close to 16 and C close to 150?

22. Numerical: Dropping a rock from a height of 150 feet *above the surface of Mars* generates the data set

t = Time (sec)	r = Height (feet)
0	150
1	144
2	125
3	95

Using the fact that an initial velocity of 0 leads to

$$r = C + kt^2$$

transform the data so that linear regression can be used to estimate C and k. What is the acceleration of the rock?

23. Numerical: If a projectile with an initial height of 0 is launched from the surface of the earth, the projectile's position $r(t)$ is of the form

$$r(t) = Ct + kt^2$$

If we let $y = \dfrac{r}{t}$, then $y = C + kt$. Find the values of C and k which best describe the data

t = time in seconds	1	2	3	4
r = height in feet	983	1937	2856	3744

by transforming the data using $y = \frac{r}{t}$ and applying linear regression. What is the initial velocity of the projectile?

24. Numerical: If a projectile with an initial height of 0 is launched from the surface of *Mars*, the projectile's position $r(t)$ is of the form

$$r(t) = Ct + kt^2$$

If we let $y = \dfrac{r}{t}$, then $y = C + kt$. Find the values of C and k which best describe

$t =$ time in seconds	1	2	3	4
$r =$ height in feet	994	1976	2945	3902

by transforming the data using $y = \frac{r}{t}$ and applying linear regression. What is the initial velocity of the projectile?

25. Numerical: The points in the data set

$$(0,5), (3,4), (4,3), (5,0)$$

are points on an ellipse of the form

$$y^2 = C + kx^2$$

Transform the data set and use a least squares line to estimate C and k.

26. Numerical: The points in the data set

$$(1,7), (2,6.8), (3,6.4), (5,5)$$

are points on an ellipse of the form

$$y^2 = C + kx^2$$

Transform the data set and use a least squares line to estimate C and k.

27. Numerical: Find the value of m for which $y = m^x$ best fits the three points $(0,1)$, $(1,2)$, and $(2,4)$ by directly minimizing the least squared error function.

28. Computer Algebra System: Find the value of m for which $y = m^x$ best fits the three points $(0,1)$, $(1,2)$, and $(2,4)$ by directly minimizing the least squared error function.

x	0.1	0.2	0.3
y	1.11	1.25	1.39

(Hint: Estimate the zero of $E'(m)$ graphically).

29. Numerical: A object dropped from a height of 50 feet above the surface of the moon has a height of

$$r = 50 - \frac{1}{2}gt^2 \tag{3.34}$$

where g is the acceleration due to gravity on the moon's surface. If the following heights are measured at the given times,

$t =$ time in seconds	1	2	3	4
$r =$ height in ft	47.4	39.4	26.2	7.6

(3.35)

then what value of g yields the *least total squared error* in fitting the curve (3.34) to the data (3.35)?

30. Numerical: Repeat exercise 29 for the function $r(t) = 200 - \frac{1}{2}gt^2$ (i.e., dropped from 150 feet) and the data set

$t =$ time in seconds	2	4	6	8
$r =$ height in ft	189.4	157.6	104.6	30.4

31. Numerical: Suppose that Acme sporting goods collected the following set of data relating price charged for a racket, x, to the number of rackets per week sold at that price.

$x =$ price	\$50	\$55	\$60	\$65
$y =$ weekly sales	18	15	10	6

(a) Use linear regression to determine the least squares line of y as a function of x.

(b) Explain why the weekly revenue R from the sale of tennis rackets is a function of the form

$$R = xy$$

Using the line of best fit in (a) as a constraint, write R as a function of the variable x.

(c) Graph $R(x)$ with a graphing calculator and determine the price x which maximizes weekly revenue.

32. Numerical: Acme fast food changes the price of its megawhopper sandwich once each week for 4 weeks, and correspondingly, it collects weekly megawhopper sales data for those weeks. The result is the data set below:

$p =$ price per megawhopper	\$2	\$1	\$1.50	\$1.25	\$1.75
$x =$ weekly sales	3500	6000	4500	5000	4000

If the number sold each day is a linear function of the price, what should the price of a megawhopper be in order to maximize the daily revenue from megawhoppers? (Hint: See instructions for exercise 31.)

33. Write to Learn: If course grades were to reflect both the average test score and the average change in the test scores, then the final grade might be based on the formula

$$\text{Final Score} = \text{ Average Score} + \text{ Average Change in Score} \qquad (3.36)$$

In example 2, John's final numerical grade would be $77 + 8.4 = 85.4$, thus giving him a "B" for the course.[4] What would the final numerical grade be for a student with test scores of 93, 63, 63 and 58, respectively. Write a short essay describing how you as an instructor might explain this grading scale to John and how you came up with his scores under this scale.

34. Numerical: The author has found that most of his students do not have significant trends upward or downward. Use (3.36) to determine the final numerical grade of a student with test scores of 87, 82, 78 and 84, respectively, under the author's grading system described above.

[4] *Author's Note:* For most students, a simple average is a sufficient measure of performance. Moreover, when the average does have a large error, it is usually due to an overall decline in performance. However, there have been students like John who have received grades higher than their average because of their marked improvement throughout a course.

35. Since $E(m)$ is a quadratic function, it must be in the form

$$E(m) = am^2 + bm + c$$

In this exercise, we determine the values of a, b, and c. (see the "Check your reading" question on page 253).

(a) Use the fact that

$$E(m) = (m(x_1 - \bar{x}) + \bar{y} - y_1)^2 + \ldots + (m(x_n - \bar{x}) + \bar{y} - y_n)^2$$

to explain why $E(0) = n\sigma_y^2$. What is the value of c?

(b) Let M denote the slope of the least squares line. Use the fact that $E'(M) = 0$ to show that

$$\frac{-b}{2a} = M$$

(c) Differentiate $E(m)$ twice with respect to m to show that

$$E''(m) = 2(x_1 - \bar{x})^2 + \ldots + 2(x_n - \bar{x})^2$$

Use the result to show that $a = n\sigma_x^2$ and then use this and the result in (b) to find the value of b.

36. In exercise 35 we show that the total squared error for the least squares method is of the form

$$E(m) = \sigma_x^2 m^2 - 2\sigma_x \sigma_y r\, m + \sigma_y^2$$

where $r = \frac{\sigma_x}{\sigma_y} M$ is the correlation coefficient of the data set and M is the least squares slope.

(a) Use the fact that $M = \frac{\sigma_y}{\sigma_x} r$ to show that

$$E(M) = \sigma_y^2 \left(1 - r^2\right)$$

(b) Use the result in (a) to explain why $r = \pm 1$ implies that $E(M) = 0$. What does this say about the least squares line approximation?

(c) Use the result in (a) and the fact that $E(m) \geq 0$ to show that $-1 \leq r \leq 1$.

37. Find r for the least squares linear approximation of

$$(1, 60),\ (2, 78),\ (3, 84),\ (4, 86)$$

using the formula $r = \frac{\sigma_x}{\sigma_y} M$. Compare the result to the result produced by a calculator or computer.

38. Find r for the least squares linear approximation of

$$(1, 96), (2, 63), (3, 63), (4, 58)$$

using the formula $r = \frac{\sigma_x}{\sigma_y} M$. Compare the result to the result produced by a calculator or computer.

39. **Write to Learn:** Even when r is very close to 1, it does not guarantee that the least squares line is a good approximation of the data set. For example, the set of data

$$(1, 1),\ (2, 4),\ (3, 9),\ (4, 16)$$

was generated from the function $f(x) = x^2$. Find the least squares line and the correlation coefficient for this data set. Does the value of r justify the use of a straight line to approximate the data? Why or why not?

40. How would we interpret the case that $r = 0$? What does it say about the least squares line?

41. Show that if n is an integer, then $\frac{a+b}{2}$ minimizes the function

$$E(x) = (x-a)^{2n} + (x-b)^{2n}$$

42. The *harmonic mean* $\tilde{R}$ of a set of positive numbers $\{R_1, R_2, \ldots, R_n\}$ is defined by

$$\frac{1}{\tilde{R}} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_n}$$

Show that the harmonic mean minimizes the *total relative error function*

$$E(x) = \left(1 - \frac{x}{R_1}\right)^2 + \left(1 - \frac{x}{R_2}\right)^2 + \ldots + \left(1 - \frac{x}{R_n}\right)^2$$

43. Suppose that $\{x_1, x_2, \ldots, x_n\}$ is a set of possible random outcomes for a given event and suppose that p_j is the probability that x_j is chosen at random from the set. If the set is approximated by a number x, then there is a probability of p_j of the squared error term $\varepsilon_j^2 = (x - x_j)^2$ occurring. Thus, the total squared error in this case is defined

$$E(x) = (x - x_1)^2 p_1 + (x - x_2)^2 p_2 + \ldots + (x - x_n)^2 p_n$$

The value of x which minimizes the total squared error is called the *expected value* of the event. Find a formula for expected value, and then show that it reduces to the mean when each of the x_j are equally likely.

44. The possible outcomes for the sum of two rolled dice are $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, where there is

a probability of $\frac{1}{36}$ of rolling a 2 or a 12
a probability of $\frac{1}{18}$ of rolling a 3 or an 11
a probability of $\frac{1}{12}$ of rolling a 4 or a 10
a probability of $\frac{1}{9}$ of rolling a 5 or a 9
a probability of $\frac{5}{36}$ of rolling a 6 or an 8
a probability of $\frac{1}{6}$ of rolling a 7

What is the *expected value* of the sum of two rolled dice? (see exercise 43)

Self Test

A variety of questions are asked in a variety of ways in the problems below. Answer as many of the questions below as possible before looking at the answers in the back of the book.

1. Answer each statement as true or false. If false, determine the reason.

   (a) If $f'(x) = g'(x)$ for all x in (a, b), then there is a constant C such that $f(x) = g(x) + C$ for all x in (a, b).

   (b) Limits of the form $\frac{0}{0}$ cannot exist.

   (c) If $x = p$ is a critical point of $f(x)$, then $f(x)$ has a relative extremum at $x = p$.

   (d) If $f(x)$ has a relative extremum at $x = p$, then $x = p$ is a critical point of $f(x)$.

   (e) A function cannot be both concave down *and* increasing on an interval.

   (f) The exponential function, e^x, is increasing and concave down on the real line.

   (g) If $f'(p) = 0$ and $f''(p) > 0$, then $f(x)$ has a minimum at $x = p$.

2. Evaluate $\lim_{x \to 0} \dfrac{\sin(x) - x}{x^3}$

   (a) $\dfrac{-1}{6}$ (b) 0 (c) $\dfrac{1}{6}$ (d) ∞

3. Evaluate $\lim_{x \to 1} \dfrac{x - 2}{x^2 - 2}$

   (a) 0 (b) $\dfrac{1}{4}$ (c) $\dfrac{1}{2}$ (d) 1

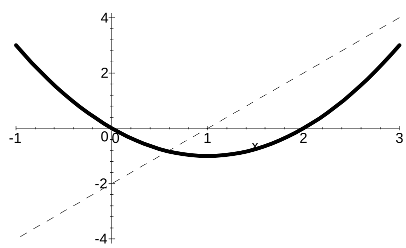
4. Evaluate $\lim_{x \to \infty} \dfrac{(3e^x - 1)^2}{e^{2x} - x^2}$

   (a) 0 (b) 1 (c) 3 (d) 9

5. Evaluate $\lim_{x \to 0^+} [\sin(x) \ln(x)]$

   (a) -1 (b) 0 (c) 1 (d) does not exist

6. The first and second derivative of a function $f(x)$ are graphed below: (The thicker curve is the graph of $f'(x)$).



   Which of the following is **not** true?

(a) $f(x)$ has a maximum at $x = 0$
(b) $f(x)$ has a minimum at $x = 1$
(c) $f(x)$ has a minimum at $x = 2$
(d) $f(x)$ has an inflection point at $x = 1$

7. Consider the function $f(t) = t^3 - 3t^2$, $-3 \leq t < \infty$. On what interval is this function concave up?

   (a) $(0, \infty)$ (b) $(1, \infty)$ (c) $(0, 2)$ (d) $f(t)$ is always concave down

8. What does the graph of $f(x) = 3x^2 - \pi \sin(x^2)$ have at $x = 0$?

   (a) *local* maximum (b) *local* minimum
   (c) inflection point (d) cusp

9. If $y = a\cos(\omega t) + b\sin(\omega t) + M$ has maxima of $(0, 3), (2, 3), (4, 3)$ and so on, and minima of $(1, 1), (3, 1), (5, 1)$, and so on, then what is $y'(0.5)$?

   (a) -1 (b) 0 (c) 1 (d) $1/\pi$ (e) $-\pi$
   (f) -2 (g) 2 (h) does not exist

10. In an optimization problem the constraints are used to

    (a) define variables.
    (b) find the optimal values.
    (c) make the problem difficult.
    (d) reduce the number of variables in the equation to be optimized.

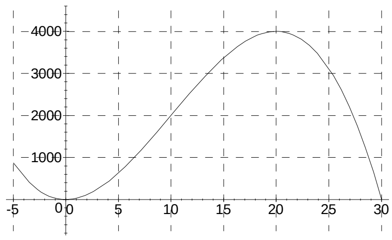
11. The slope of the least-squares line minimizes the

    (a) sum of the errors.
    (b) sum of the squares of the errors.
    (c) square of the sum of the errors.
    (d) total error.

12. What are the absolute extrema of $f(x) = x^4 - 8x^2$ over $[-1, 1]$?

13. Produce a sketch of $f(x) = 2x^3 - 9x^2 + 12x$ which details the critical points, monotonicity, extrema, and inflection points.

14. Use the graph of $f(x)$ below to answer (a)-(e)



    (a) What is the largest interval over which $f'(x)$ is positive?

(b) At what two points on the graph is $f'(x) = 0$?

(c) If $f''(10) = 0$, then what is the largest interval on the domain shown where $f''(x) < 0$?

(d) If $f(x)$ represents the velocity of an object, then at which time(s) x is the object at rest?

(e) If $f(x)$ represents the velocity of an object starting at time $x = 0$, does the object ever get back to its starting position during the trip from time $x = 0$ to $x = 30$?

15. Can a function have both a minimum and an inflection point at the same input? Explain why or why not.

16. Find the period, frequency, amplitude, and extrema of the oscillation given by $y(t) = 3\cos(3t) - 4\sin(3t)$.

17. Maximize $z = y^3 - x^3$ subject to $y = x - 1$.

18. A box has a square base with the constraint that the girth plus the height cannot exceed 36". What dimensions yield the maximum volume?

19. The production P at an Acme widget factory is modeled by a Cobb-Douglass function

$$P = 500x^{0.4}y^{0.8}$$

where x is the number of units of labor and y is the number of units of capital expenditure. If labor costs \$60 per unit and capital expenditure is \$35 per unit, then what is the least costly combination of capital and labor that will produce 10,000 widgets?

20. Find the hyperbola of the form $y^2 = C + kx^2$ which best fits the data by transforming the data below and then computing the least squares line of the transformed data.

x	2	3	4	5
y	1.2	2	2.7	3.5

21. **Write to Learn:** A certain stock is offered initially at \$20 a share at 10:00 a.m. one morning, and then the price increases exponentially throughout the day, thus producing the following data set:

time	10 a.m.	11 a.m.	12 p.m.	1 p.m.	2 p.m.	3 p.m.	4 p.m.
price	20	21	24	26	29	32	35

In a few short sentences, explain how we might use a least squares approximation to find the value of the parameter k for which $y = 20e^{kt}$ best fits the data.

In section 3-6, we studied oscillations of the form

$$y(t) = M + a\cos(\omega t) + b\sin(\omega t)$$

But can a set of data be used to determine a and b in the oscillation? Certainly! Let's take the next step and examine how least squares can be used to study oscillations.

In particular, let's suppose we have a set of data which fits a *seasonal process*—i.e., a process which repeats itself once each year. If time t is measured in months, then the seasonal process can be modeled by

$$y = M + a\cos\left(\frac{2\pi}{12}t\right) + b\sin\left(\frac{2\pi}{12}t\right) \tag{3.37}$$

where M is the *average value* of the oscillation. If time is measured in days, then the model is

$$y = M + a\cos\left(\frac{2\pi}{365}t\right) + b\sin\left(\frac{2\pi}{365}t\right) \tag{3.38}$$

In both cases, the average value M and the angular velocity ω are known a priori.

For time in days, we subtract M from y and divide by $\cos\left(\frac{2\pi}{365}t\right)$, resulting in

$$\frac{y-M}{\cos\left(\frac{2\pi}{365}t\right)} = a + b\frac{\sin\left(\frac{2\pi}{365}t\right)}{\cos\left(\frac{2\pi}{365}t\right)}$$

This in turn reduces to

$$(y-M)\sec\left(\frac{2\pi}{365}t\right) = a + b\tan\left(\frac{2\pi}{365}t\right) \tag{3.39}$$

If we now let

$$Y = (y-M)\sec\left(\frac{2\pi}{365}t\right) \quad \text{and} \quad T = \tan\left(\frac{2\pi}{365}t\right) \tag{3.40}$$

then (3.39) is transformed into the linear model

$$Y = a + bT \tag{3.41}$$

Thus, a and b can be estimated by applying least squares to the transformed data.

For example, the following table lists several sunrise times for the city of Johnson City, TN in the year 1999. The variable t represents the number of days since January 1, and the variable y denotes the decimal time of the sunrise (for example, 7:30 a.m. becomes 7.5) in Eastern Standard time.

Date	$t = \#$ of days	$y =$ Sunrise time
Jan. 15	15	7.6667
Feb. 15	46	7.3000
March 1	60	7.0000
March 15	74	6.6833
April 15	105	5.9333

The average sunrise for Johnson City in 1999 is $M = 6.4403$ (i.e., about 6:26 a.m.).[5]

[5] $M = 6.4403$ was obtained from the average of the sunrise times of the first and fifteenth of each month.

To fit the data with the curve (3.38), we thus use the transformation

$$T = \tan\left(\frac{2\pi}{365}\,t\right) \qquad \text{and} \qquad Y = (y - 6.4403)\sec\left(\frac{2\pi}{365}\,t\right)$$

For example, the data for January 15 is transformed into

$$\begin{aligned} T &= \tan\left(\frac{2\pi}{365}\cdot 15\right) = 0.26411 \\ Y &= (7.6667 - 6.4403)\sec\left(\frac{2\pi}{365}\cdot 15\right) = 1.2685 \end{aligned}$$

The remaining data is likewise transformed, resulting in

T	Y
0.26411	1.2685
1.0130	1.2237
1.6761	1.0924
3.2681	0.8305
−4.1456	2.1621

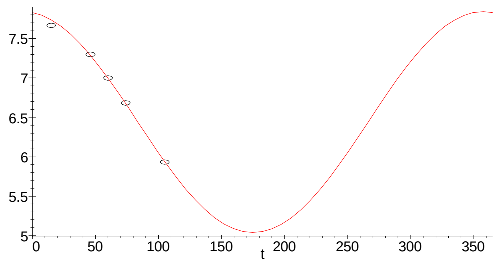
Application of the least squares method to (T, Y) data yields

$$Y = 1.3903 - 0.1803\,T$$

Comparing to (3.41) reveals that $a = 1.3903$ and $b = -0.1803$. Thus, the seasonal model for sunrise times in Johnson City is

$$y = 6.4403 + 1.3903\cos\left(\frac{2\pi}{365}\,t\right) - 0.1803\sin\left(\frac{2\pi}{365}\,t\right) \tag{3.42}$$

The original data is graphed along with the curve (3.42) in the figure below:



NS-1: Predicted sine wave and actual data

It appears that the model is a good fit of the data. Moreover, on May 1, 1999, which is $t = 121$ days since the beginning of the year, the sun rose in Johnson City at 5:37 a.m. Eastern standard time. Substituting $t = 121$ into (3.42) results in

$$y = 6.4403 + 1.3903\cos\left(\frac{2\pi}{365}\cdot 121\right) - 0.1803\sin\left(\frac{2\pi}{365}\cdot 121\right) = 5.6019$$

That is, the model predicts that the sun rose about 5:36 a.m.

Write to Learn The following data set lists the 30 year average normal temperatures by month for Juneau, Alaska, where t is in months and y is in $^\circ F$. (we assume the average corresponds to the middle of each month).

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
t	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5	10.5	11.5
y	22	28	31	39	46	53	56	55	49	42	33	27

The average yearly temperature is $M = 40.1°F$. Transform the data using (3.40), and then estimate a and b by applying least squares to the transformed data. Graph the data and the seasonal curve (3.37) which best fits the data. Include the graph in a short paper which describes your methods and conclusions.

Write to Learn Here are several sunset times in Eastern Standard time for Johnson City, TN in the year 1999.

Date	$t = \#$ of days	$y =$ Sunset time
Jan. 15	15	5.6333
Feb. 15	46	6.1667
March 1	60	6.4000
March 15	74	6.6167
April 15	105	7.0500

The average sunset time is $M = 6.5417$ (i.e., about 6:32 p.m.). Transform the data using (3.40), and then estimate a and b by applying least squares to the transformed data. On June 15, which is $t = 166$ days after the beginning of the year, the sun set at 7:49 eastern standard time. Use the model to predict the sunset on June 15. Write a short paper describing your results and how they were obtained.

Write to Learn Go to the library or search the internet to find more on fitting data to oscillatory models. Then write a report presenting your research.

Group Learning *Try it out!* Record both sunrise or sunset data from your local paper for an extended period of time, and then determine the number of hours of daylight in each of those days. Assume the average number of hours of daylight is $M = 12.2$ hours, and then transform the data using (3.40). Estimate a and b by applying least squares to the transformed data, and use the resulting model to estimate the lengths of the longest and shortest days of the year where you live. Then have the first member of the group present the data, the second present the model, the third present the fit to the data, and the fourth present the conclusions.

Advanced Contexts:

Much of our work in this chapter depends on the fact that

$$y'' = -y, \quad y(0) = p, \quad y'(0) = q \tag{3.43}$$

has only one solution for each choice of p and q. In the following exercises, we develop a proof of this fact along with some other interesting ideas about the differential equation (3.43). However, our proof is only for the case that $p \neq 0$.

1. Before we begin, we need to prove the following: If u and v are non-zero differentiable functions satisfying

$$\frac{u'(x)}{u(x)} = \frac{v'(x)}{v(x)} \tag{3.44}$$

then there is a constant R such that

$$u(x) = Rv(x)$$

(a) Let $h(x) = \frac{u(x)}{v(x)}$ and show that

$$h'(x) = \frac{u'(x)}{v(x)} - \frac{u(x)}{v(x)}\frac{v'(x)}{v(x)}$$

(b) Use (3.44) to show that $h'(x) = 0$, and thus that $h(x) = R$ for some constant R. Why does this imply that $u(x) = Rv(x)$?

2. Now let's suppose that $u(x)$ and $v(x)$ are both solutions to (3.43)—that is, that

$$u'' = -u, \qquad u(0) = p, \qquad u'(0) = q \tag{3.45}$$

and that

$$v'' = -v, \qquad v(0) = p, \qquad v'(0) = q \tag{3.46}$$

Our goal is to show that $u(x) = v(x)$ for all x, which we do using the *Wronskian* of u and v given by

$$W(x) = u(x)\,v'(x) - u'(x)\,v(x)$$

(a) Show that $W'(x) = 0$, which implies that $W(x)$ is constant.

(b) Let $x = 0$ and compute $W(0)$. Why does this imply that

$$u'(x)\,v(x) = u(x)\,v'(x)$$

(c) Show that step (b) implies that

$$\frac{u'(x)}{u(x)} = \frac{v'(x)}{v(x)}$$

and explain why this implies that $u(x) = Rv(x)$ for some constant R.

(d) Let $x = 0$ and show that $R = 1$.

3. Repeat the steps in the previous exercise to show that

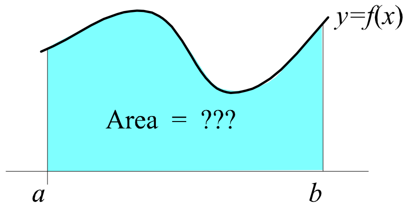
$$y'' + \omega^2 y = 0, \quad y(0) = p, \quad y'(0) = q$$

has only one solution. Here ω^2 is a positive constant.

4. INTEGRATION

Gottfried Leibniz of Germany independently developed the calculus about the same time that Newton did. Indeed, it is Leibniz who introduced much of the notation used in calculus, including the symbols used for differentials and for antiderivatives (we say that $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$). However, Leibniz's development of calculus was not motivated by a desire to study velocities and accelerations.

Instead, Leibniz was motivated in his development of calculus by the problem of finding the area of the region under a curve.



Indeed, the desire to approximate and calculate areas leads us to a new tool and a new concept in calculus—that of the *definite integral.*

Like Leibniz, our goal in this chapter is to develop the definite integral into a tool which can be used to explore the geometry of a region or a solid. We will begin by exploring a method for approximating a function over an interval $[a, b]$. We will then develop this method of approximation into the tool known as the *definite integral.* Most importantly, we will see that there is a connection between antiderivatives and definite integrals, a connection we now call the *Fundamental Theorem of Calculus.*

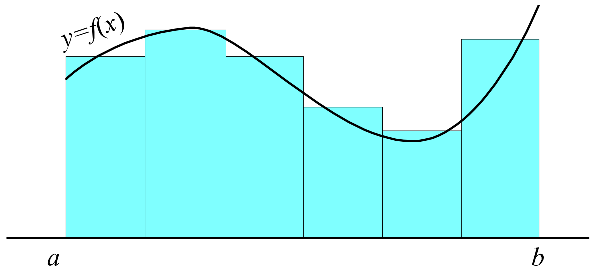
Finally, we will see that the concept of the definite integral is fundamental to many scientific and mathematical pursuits, including differential equations, probability, geometry, and mechanics. Indeed, much of twentieth century science is based on the concept of the definite integral, and much of twentieth century mathematics has been devoted to expanding the definition of the definite integral. And nearly all of twentieth century mathematics and science has utilized and benefited from the fundamental theorem of calculus.

4.1 Simple Function Approximation

Bar Graphs and Simple Functions

Derivatives and linearization approximate a function only near a given point. In many applications, it is necessary to approximate a function $f(x)$ over an entire interval $[a, b]$.

The main tool for constructing such approximations is the *bar graph.* Given a function $y = f(x)$ over $[a, b]$, a bar graph is a collection of non-overlapping rectangular bars covering $[a, b]$ whose tops intersect the graph.

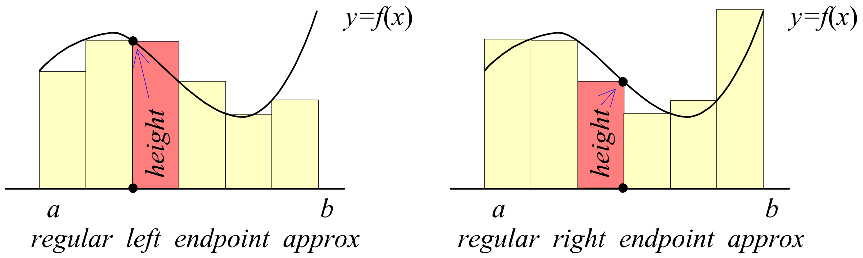


1-1: Bar graph approximation of $y = f(x)$ over $[a, b]$

Mathematically, a bar graph is known as a *simple function approximation* of a given function $f(x)$ over an interval $[a, b]$.

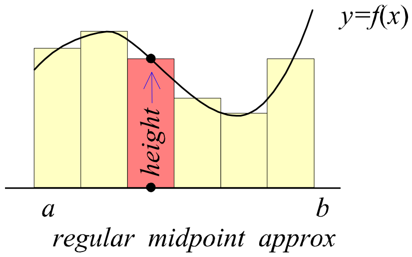
To construct a simple function approximation (i.e., a bar graph) of a function $f(x)$ over an interval $[a, b]$, we first divide the interval into n sections called *subintervals,* where n is the number of rectangles to be used in the bar graph. If all the subintervals have the same width, then the approximation is said to be *regular.*

Three types of simple function approximations are particularly important in calculus. If the height of each bar corresponds to the value of f at the left of the subinterval, then the approximation is a *left endpoint approximation.* If the height corresponds to the right, it is a *right endpoint* approximation.



1-2: Left and right endpoint approximations

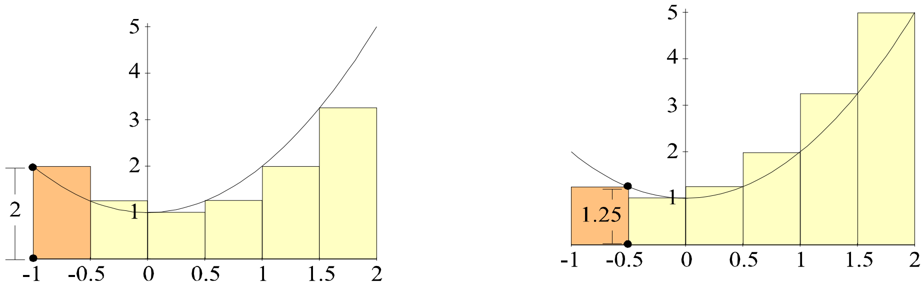
Similarly, if the height of each bar corresponds to the value of f over the midpoint of the subinterval, then the approximation is called a *midpoint approximation.*



1-3: A midpoint approximation

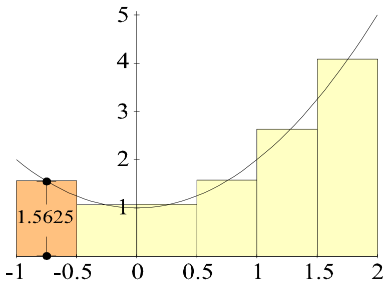
EXAMPLE 1 Draw left endpoint, right endpoint, and midpoint approximations to $y = x^2 + 1$ over $[-1, 2]$ using 6 bars.

Solution: In order to have 6 bars over $[-1, 2]$ which is 3 units long, each bar will have a width of $\frac{1}{2}$ unit. The left endpoint approximation has heights that correspond to the left endpoints. For example, on the first interval $[-1,-1/2]$, the bar will have a height of $y=(-1)^2+1=2$.



1-4: Left and right endpoint approximations

Conversely, the right endpoint approximation has heights corresponding to right endpoints. For example, on the first interval $[-1,-1/2]$, the bar will have a height of $y=(-1/2)^2+1=1.25$.
In the midpoint approximation, the height corresponds to the midpoint of the interval. For example, on the first interval $[-1,-1/2]$, the bar will have a height of $y=(-3/4)^2+1=1.5625$.



1-5: A midpoint approximation

Check your Reading *What is the height of the last rectangle in the right endpoint approximation in example 1?*

Tagged Partitions and Simple Functions

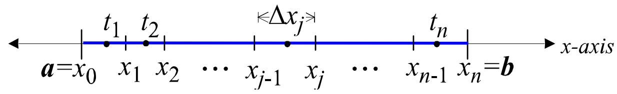
Let's develop some terminology for constructing a simple function approximation of a function $f(x)$ over an interval $[a,b]$. First, we divide $[a,b]$ into a collection of n smaller intervals $[x_{j-1},x_j]$, $\quad j=1,\ldots,n$. This is known as *partitioning* the interval. We then choose a single number t_j from each subinterval $[x_{j-1},x_j]$. This is known as *tagging* the partition.

Definition 1.1: A set of numbers $\{x_j,t_j\}_{j=1}^n$ is called a *tagged partition* of $[a,b]$ if

$$a=x_0<x_1<\ldots<x_n=b \tag{4.1}$$

and if t_j is in $[x_{j-1},x_j]$ for all $j=1,\ldots,n$..

The intervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ are called the *subintervals* of the partition, and the numbers $t_1, t_2, \ldots, t_n$ are called the *tags* of the subintervals. The quantities $\Delta x_j = x_j - x_{j-1}$ are the *widths* of the subintervals, respectively.



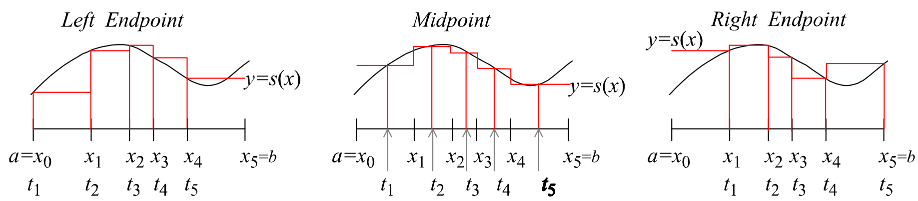
1-6: A tagged partition of $[a, b]$.

The values of f at the tags are then used to generate the heights of the "bars" of the simple function approximation.

Definition 1.2: The *simple function approximation* of $f(x)$ *generated by a tagged partition* $\{x_j, t_j\}_{j=1}^n$ is defined

$$s(x) = \begin{cases} f(t_1) & if \quad x_0 \le x < x_1 \\ f(t_2) & if \quad x_1 \le x < x_2 \\ \vdots & \qquad \vdots \qquad \vdots \\ f(t_n) & if \quad x_{n-1} \le x \le x_n \end{cases} \tag{4.2}$$

If the tags are the *midpoints* of the subintervals of the partition, then $s(x)$ is called a *midpoint approximation* of $f(x)$. Likewise, if the tags are the left endpoints, then $s(x)$ is called a *left endpoint approximation*, and if the tags are the right endpoints, then $s(x)$ is a right endpoint approximation.



1-7: Three types of Simple Function Approximations

Notice that the vertical lines in figure 1–7 are not part of the graph of $s(x)$. They are simply included for emphasis and to illustrate the connection of a simple function approximation to a bar graph.

EXAMPLE 2 Construct the midpoint approximation of $f(x) = x^2 + 1$ over the interval $[-1, 2]$ given the partition

$$x_0 = -1, \; x_1 = -0.5, \; x_2 = 0, \; x_3 = 0.5, \; x_4 = 1, \; x_5 = 1.5, \; x_6 = 2 \tag{4.3}$$

Solution: Since (4.3) forms a partition of $[0, 1]$, a midpoint approximation dictates that the tags are the midpoints

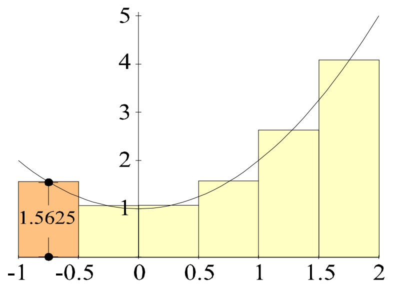
$$t_1 = -0.75, \; t_2 = -0.25, \; t_3 = 0.25, \; t_4 = 0.75, \; t_5 = 1.25, \; t_6 = 1.75 \tag{4.4}$$

Moreover, $f(t_1) = (-0.75)^2 + 1 = 1.5625$. Thus, $s(x) = 1.5625$ when x is in $[-1, -0.5)$. Likewise, $f(t_2) = (-0.25)^2 + 1 = 1.0625$, so $s(x) =$

1.0625 when x is in $[-0.5, 0)$. The midpoint approximation of $f(x) = x^2$ for the partition (4.3) with tags (4.4) is

$$s(x) = \begin{cases} 1.5625 & if \quad -1 \le x < -0.5 \\ 1.0625 & if \quad -0.5 \le x < 0 \\ 1.0625 & if \quad 0 \le x < 0.5 \\ 1.5625 & if \quad 0.5 \le x < 1 \\ 2.5625 & if \quad 1 \le x < 1.5 \\ 4.0625 & if \quad 1.5 \le x \le 2 \end{cases} \tag{4.5}$$

The graph of $s(x)$ versus $f(x) = x^2+1$ over $[-1, 2]$ is shown in the same as figure 1-5 above. However, we repeat the graph here for emphasis..



1-5: A midpoint approximation

Check your Reading *Sketch the graph of the simple function*

$$s(x) = \begin{cases} 1 & if \quad 0 \le x < 2 \\ 3 & if \quad 2 \le x < 4 \\ 2 & if \quad 4 \le x < 6 \\ 5 & if \quad 6 \le x < 8 \end{cases}$$

Tabular Construction of Simple Functions

We often use tables to organize our construction of simple function approximations. For example, let us construct the midpoint approximation of $f(x) = xe^{-x}$ over the interval $[0, 1.5]$ with the partition

$$x_0 = 0,\ x_1 = 0.1,\ x_2 = 0.3,\ x_3 = 0.6,\ x_4 = 1.0,\ x_5 = 1.5 \tag{4.6}$$

To do so, we place the data in (4.6) into a table, and then define the column of tags to be the *average* of the columns labeled x_{j-1} and x_j,

j	x_{j-1}	x_j	t_j	$f(t_j)$
1	0.0	0.1	0.05	
2	0.1	0.3	0.2	
3	0.3	0.6	0.345	
4	0.6	1.0	0.8	
5	1.0	1.5	1.25	

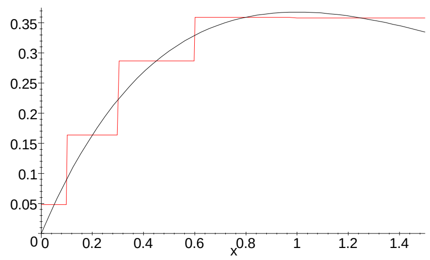
We complete the last column by evaluating $f(x)$ on the tags (to 3 decimal places of accuracy):

j	x_{j-1}	x_j	t_j	$f(t_j)$
1	0.0	0.1	0.05	$0.05e^{-0.05} = 0.048$
2	0.1	0.3	0.2	$0.2e^{-0.2} = 0.164$
3	0.3	0.6	0.45	$0.45e^{-0.45} = 0.287$
4	0.6	1.0	0.8	$0.8e^{-0.8} = 0.359$
5	1.0	1.5	1.25	$1.25e^{-1.25} = 0.358$

As a result, the midpoint approximation is

$$s(x) = \begin{cases} 0.048 & if \quad 0.0 \le x < 0.1 \\ 0.164 & if \quad 0.1 \le x < 0.3 \\ 0.287 & if \quad 0.3 \le x < 0.6 \\ 0.359 & if \quad 0.6 \le x < 1.0 \\ 0.358 & if \quad 1.0 \le x \le 1.5 \end{cases}$$

which is shown below along with the function $f(x) = xe^{-x}$:



1-9: Midpoint approximation of $f(x) = xe^{-x}$ over $[0, 1.5]$

EXAMPLE 2 Approximate $f(x) = \sin(x)$ over the interval $[0, \pi]$ using the partition

$$x_0 = 0, \quad x_1 = \frac{\pi}{6}, \quad x_2 = \frac{\pi}{4}, \quad x_3 = \frac{\pi}{3}, \quad x_4 = \frac{\pi}{2}, \quad x_5 = \pi \tag{4.7}$$

Solution: For a left endpoint approximation, the column of tags is the same as the column labeled x_{j-1}:

j	x_{j-1}	x_j	t_j	$c_j = f(t_j)$
1	0	$\frac{\pi}{6}$	0	$\sin(0) = 0$
2	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$	$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$
3	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$
4	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$
5	$\frac{\pi}{2}$	π	$\frac{\pi}{2}$	$\sin\left(\frac{\pi}{2}\right) = 1$

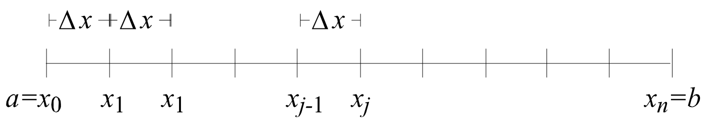
As a result, the left endpoint approximation is

$$s(x) = \begin{cases} 0 & if \quad 0 \le x < \frac{\pi}{6} \\ \frac{1}{2} & if \quad \frac{\pi}{6} \le x < \frac{\pi}{4} \\ \frac{\sqrt{2}}{2} & if \quad \frac{\pi}{4} \le x < \frac{\pi}{3} \\ \frac{\sqrt{3}}{2} & if \quad \frac{\pi}{3} \le x < \frac{\pi}{2} \\ 1 & if \quad \frac{\pi}{2} \le x \le \pi \end{cases} \tag{4.8}$$

Check your Reading *Which two columns are the same in a right endpoint approximation?*

Regular Partitions

A *regular partition* of an interval $[a, b]$ is a partition of $[a, b]$ in which each subinterval has the same width, Δx.



1-10: A regular partition of $[a, b]$

A regular partition with subintervals of width Δx is also known as a Δx*-partition of* $[a, b]$.

If $x_0, x_1, \ldots, x_n$ is a regular partition of $[a, b]$ with n subintervals, then

$$\Delta x = \frac{b-a}{n}$$

and $x_j = a + j\Delta x$ for all $j = 0, 1, \ldots, n$.

EXAMPLE 3 Construct the right endpoint approximation of $\ln(x)$ over a regular partition of $[1, 2]$ with $\Delta x = 0.2$ (i.e., over a 0.2-partition of $[1, 2]$.

Solution: It follows that the partition is

$$x_0 = 1,\ x_1 = 1.2,\ x_2 = 1.4,\ x_3 = 1.6,\ x_4 = 1.8,\ x_5 = 2.0$$

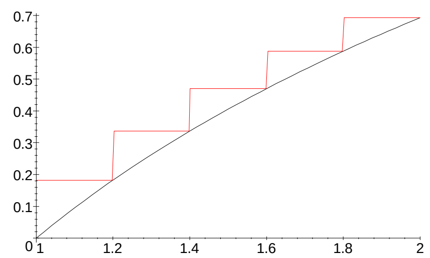
We thus complete the following table (to 5 decimal places of accuracy),

j	x_{j-1}	x_j	t_j	$f(t_j)$
1	1.0	1.2	1.2	$\ln(1.2) = 0.18232$
2	1.2	1.4	1.4	$\ln(1.4) = 0.33647$
3	1.4	1.6	1.6	$\ln(1.6) = 0.47000$
4	1.6	1.8	1.8	$\ln(1.8) = 0.58779$
5	1.8	2.0	2.0	$\ln(2.0) = 0.69315$

which leads to the following right endpoint approximation:

$$s(x) = \begin{cases} 0.18232 & if \quad 1 \le x < 1.2 \\ 0.33647 & if \quad 1.2 \le x < 1.4 \\ 0.47000 & if \quad 1.4 \le x < 1.6 \\ 0.58779 & if \quad 1.6 \le x < 1.8 \\ 0.69315 & if \quad 1.8 \le x \le 2.0 \end{cases} \tag{4.9}$$

The graph of $s(x)$ is shown versus $\ln(x)$ below:



1-11: Right endpoint approximation of $f(x) = \ln(x)$ over $[1, 2]$

EXAMPLE 4 Construct the midpoint approximation of x^2 over a regular partition of $[0, 2]$ with $\Delta x = 0.5$.

Solution: It follows that the partition is

$$x_0 = 0,\ x_1 = 0.5,\ x_2 = 1,\ x_3 = 1.5,\ x_4 = 2$$

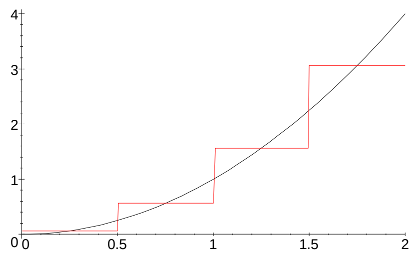
We thus complete the following table

j	x_{j-1}	x_j	t_j	$f(t_j)$
1	0	0.5	0.25	$(0.25)^2 = 0.0625$
2	0.5	1.0	0.75	$(0.75)^2 = 0.5625$
3	1.0	1.5	1.25	$(1.25)^2 = 1.5625$
4	1.5	2.0	1.75	$(1.75)^2 = 3.0625$

which leads to the following right endpoint approximation:

$$s(x) = \begin{cases} 0.0625 & if \quad 0.0 \le x < 0.5 \\ 0.5625 & if \quad 0.5 \le x < 1.0 \\ 1.5625 & if \quad 1.0 \le x < 1.5 \\ 3.0625 & if \quad 1.5 \le x \le 2.0 \end{cases} \tag{4.10}$$

The graph of $s(x)$ is shown versus $f(x) = x^2$ below:

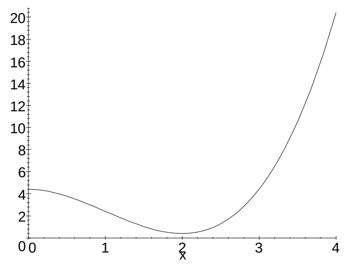


1-12: Midpoint approximation of $f(x) = x^2$ over $[0, 2]$

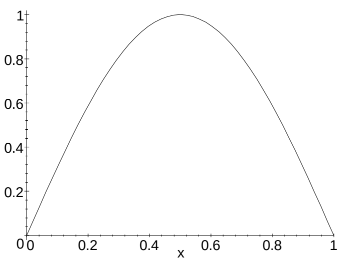
Exercises:

Sketch left endpoint, right endpoint, and midpoint approximations to the given curves over the intervals implied by the graph. Use n rectangles, where n is given.

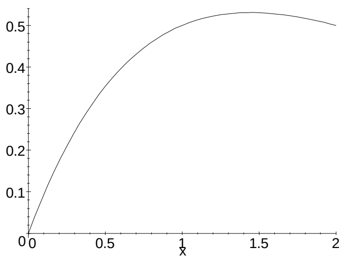
1. $y = x^3 - 3x^2 + 1$, $n = 4$



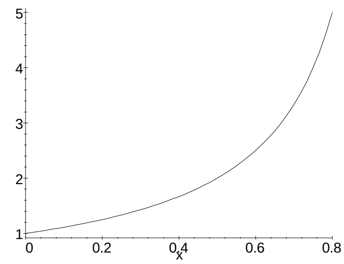
2. $y = \sin(\pi x)$, $n = 5$



3. $y = x \cdot 2^{-x}$, $n = 4$



4. $y = \dfrac{1}{1-x}$, $n = 4$



Graph the following simple functions.

5. $s(x) = \begin{cases} 3 & if \quad 0 \le x < 1 \\ 1 & if \quad 1 \le x < 3 \\ 2 & if \quad 3 \le x \le 4 \end{cases}$

6. $s(x) = \begin{cases} -2 & if \quad 0 \le x < 2 \\ 1 & if \quad 2 \le x < 3 \\ -1 & if \quad 3 \le x \le 4 \end{cases}$

7. $s(x) = \begin{cases} 1.5 & if \quad 0 \le x < 2 \\ 1.2 & if \quad 2 \le x < 3 \\ 1.8 & if \quad 3 \le x \le 4 \end{cases}$

8. $s(x) = \begin{cases} 1.5 & if \quad 0 \le x < 2 \\ -0.4 & if \quad 2 \le x < 4 \\ 1.8 & if \quad 4 \le x \le 6 \end{cases}$

Construct the midpoint approximation to each function below over the given partition, and then graph the function and its midpoint approximation.

9. $f(x) = -\frac{1}{2}x + 4$ $\quad x_0 = -1,\ x_1 = 0,\ x_2 = 1,\ x_3 = 2$
10. $f(x) = 2x + 1$ $\quad x_0 = 0,\ x_1 = 0.5,\ x_2 = 1.0,\ x_3 = 1.5$
11. $f(x) = x^2 - 1$ $\quad x_0 = 1,\ x_1 = 1.2,\ x_2 = 1.4,\ x_3 = 1.6,\ x_4 = 1.8$
12. $r(x) = 2^{-x}$ $\quad x_0 = 0,\ x_1 = 0.3,\ x_2 = 0.6,\ x_3 = 0.9,\ x_4 = 1.2$
13. $f(x) = e^{-x/2}$ $\quad$ regular partition of $[0, 2]$ with $\Delta x = 0.4$
14. $f(x) = e^x - e^{-x}$ $\quad$ regular partition of $[1, 3]$ with $\Delta x = 0.4$
15. $f(x) = \cos(x)$ $\quad$ regular partition of $[0, \pi]$ with $\Delta x = \frac{\pi}{6}$
16. $f(x) = \ln(x + 1)$ $\quad$ regular partition of $[0, e]$ with $\Delta x = \frac{e}{4}$

Construct the left endpoint approximations on the given partition, and then graph the function and the resulting approximations.

17. $f(x) = -\frac{1}{2}x + 4$ $\quad x_0 = -1,\ x_1 = 0,\ x_2 = 1,\ x_3 = 2$
18. $f(x) = 2x + 1$ $\quad x_0 = 1,\ x_1 = 1.2,\ x_2 = 1.4,\ x_3 = 1.6,\ x_4 = 1.8$
19. $f(x) = x^2 - 1$ $\quad$ regular partition of $[0, 1.5]$ with $\Delta x = 0.5$
20. $f(x) = x^2$ $\quad$ regular partition of $\left[0, \sqrt{32}\right]$ with $\Delta x = \sqrt{2}$
21. $f(x) = e^{-x/2}$ $\quad$ regular partition of $[0, 2]$ with $\Delta x = 0.4$
22. $f(x) = \cos(x)$ $\quad$ regular partition of $[0, \pi]$ with $\Delta x = \frac{\pi}{6}$

Construct the simple function approximation of the given function by completing the table listing the partition data. Then graph the function and its simple function approximation.

23. $f(x) = x^3$

j	x_{j-1}	x_j	t_j	$f(t_j)$
1	1	3	2	
2	3	4	3	
3	4	5	5	
4	5	7	6	

24. $f(x) = \ln(x)$

j	x_{j-1}	x_j	t_j	$f(t_j)$
1	1	e^2	e	
2	e^2	e^4	e^3	
3	e^4	e^6	e^5	
4	e^6	e^8	e^7	

25. Define a simple function which models the following process. For the first fifteen minutes of a certain hour, John drove his car at 60 m.p.h., but for the next twenty minutes, he drove at only 40 m.p.h. For ten minutes after that, he drove only 30 m.p.h., and then for the last fifteen minutes, he drove at 65 m.p.h.

26. Define a simple function which models the following process. A certain stock begins the day at \$35, but after an hour the price climbs to \$40. Thirty minutes later, the price has risen to \$45, where it stays for an hour. It then drops back to \$42, remaining there for an hour and a half. It then rallies and sells for \$50 for the next hour, after which it drops to \$48 for an hour. The price then drops to \$40 for another hour, and then finishes the last hour of the day at \$45.

27. Define a simple function which models the following process. The unemployment rate for the first month of the year is 5%, but it rises to 6.5% for the next two months. The fourth month sees a slight decline to 5.8%. For the next month and a half, the unemployment rate is 5.5%, after which it drops to 4.9% for the final half of the sixth month.

28. A flight of steps represents a simple function. Draw the simple function that corresponds to 13 steps each with a width of 12 inches and a rise of 6 inches.

29. Prove the following: If a simple function is multiplied by a constant, then the result is also a simple function.

30. Prove the following: The sum of two simple functions is a simple function.

31. Prove the following: The difference of two simple functions is a simple function.

32. Is the product of simple functions also a simple function? Try it out. Define

$$s_1(x) = \begin{cases} 0.048 & if \quad 0.0 \le x < 0.1 \\ 0.164 & if \quad 0.1 \le x < 0.3 \\ 0.287 & if \quad 0.3 \le x < 0.6 \\ 0.359 & if \quad 0.6 \le x < 0.9 \\ 0.358 & if \quad 0.9 \le x \le 1.0 \end{cases} \qquad s_2(x) = \begin{cases} 2 & if \quad 0.0 \le x < 0.1 \\ 3.2 & if \quad 0.1 \le x < 0.2 \\ 4.352 & if \quad 0.2 \le x < 0.3 \\ 4.916 & if \quad 0.3 \le x < 0.4 \\ 4.999 & if \quad 0.4 \le x \le 1.0 \end{cases}$$

and then graph their product over $[0,1]$. Is it a simple function?

33. Here we explore the simple function

$$s(x) = \begin{cases} 1 & if \quad 0 \le x < 2 \\ 3 & if \quad 2 \le x < 4 \\ 2 & if \quad 4 \le x < 6 \\ 5 & if \quad 6 \le x \le 8 \end{cases}$$

(a) Explain why $s'(x) = 0$ as long as x is not an even integer in $[0,10]$.

(b) Clearly, $s(x)$ is not constant, and yet its derivative is 0 except at four points. Why does this not contradict theorem 1.2 in section 1 of chapter 3?

(c) Graph the two piecewise-defined functions shown below:

$$S_1(x) = \begin{cases} x+1 & if \quad 0 \le x < 2 \\ 3x+2 & if \quad 2 \le x < 4 \\ 2x+3 & if \quad 4 \le x < 6 \\ 5x+4 & if \quad 6 \le x \le 8 \end{cases} \qquad S_2(x) = \begin{cases} x+4 & if \quad 0 \le x < 2 \\ 3x+2 & if \quad 2 \le x < 4 \\ 2x+3 & if \quad 4 \le x < 6 \\ 5x+5 & if \quad 6 \le x \le 8 \end{cases}$$

Explain why $S_1'(x) = S_2'(x) = s(x)$ except when x is an even integer in $[0,10]$. Do $S_1(x)$ and $S_2(x)$ differ by a constant? Why does this not contradict what we proved earlier in the text?

34. Let $S(x)$ denote the piecewise defined function

$$S(x)=\begin{cases} c_1x+b_1 & if \quad x_0\leq x<x_1 \\ c_2x+b_2 & if \quad x_1\leq x<x_2 \\ \vdots & \qquad \vdots \qquad\qquad \vdots \\ c_nx+b_n & if \quad x_{n-1}\leq x\leq x_n \end{cases}$$

What is $S'(x)$? Why does $S'(x)$ not depend on the values of $b_1, b_2, \ldots, b_n$?

The floor and ceiling functions are defined by

$$\begin{aligned} floor(x) &= \lfloor x\rfloor = \text{largest integer } \leq x \\ ceil(x) &= \lceil x\rceil = \text{smallest integer} \geq x \end{aligned}$$

For example, $\lfloor 2.3\rfloor = 2$ *and* $\lceil 2.3\rceil = 3$. *If* $f(x)$ *is continuous on* $[a,b]$ *and if* $h>0$, *then the functions*

$$L_h(x)=h\left\lfloor\frac{f(x)}{h}\right\rfloor \qquad and \qquad U_h(x)=h\left\lceil\frac{f(x)}{h}\right\rceil$$

are simple functions called lower and upper approximations, respectively. In exercises 35-42, we explore some of the properties of upper and lower approximations

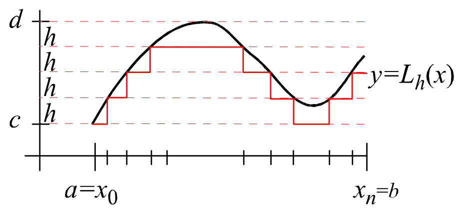
35. Grapher: Graph the upper and lower approximations of $f(x)=3x+1$ on $[0,2]$ using $h=0.25$. That is, graph f over $[0,2]$ along with the functions

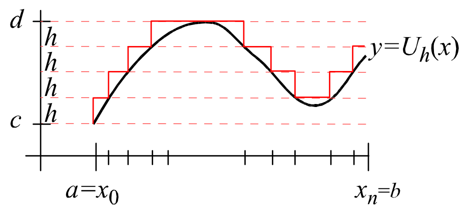
$$L_{0.25}(x)=0.25\left\lfloor\frac{3x+1}{0.25}\right\rfloor \qquad and \qquad U_{0.25}(x)=0.25\left\lceil\frac{3x+1}{0.25}\right\rceil$$

36. Grapher: Graph the upper and lower approximations of $f(x)=x^2$ on $[0,1]$ using $h=0.1$. That is, graph f over $[0,1]$ along with the functions

$$L_{0.1}(x)=0.1\left\lfloor\frac{x^2}{0.1}\right\rfloor \qquad and \qquad U_{0.1}(x)=0.1\left\lceil\frac{x^2}{0.1}\right\rceil$$

37. Write to Learn: Let $[c,d]$ be an interval on the y-axis that contains the range of a bounded function $f(x)$ over $[a,b]$, and suppose we partition $[c,d]$ into a regular partition in which each subinterval has a width of h. Show that $L_h(x)$ is the largest simple function with outputs defined by the partition of $[c,d]$ that lies *below* $f(x)$ over $[a,b]$.





Similarly, show that $U_h(x)$ is the smallest simple function *above* $f(x)$ over $[a,b]$ whose range corresponds to the h-regular partition of $[c,d]$.

38. Construct $L_{0.1}(x)$ and $U_{0.1}(x)$ for $f(x)=\sqrt{x}$ on $[0,4]$, $h=0.25$ using a 0.25-partition of the range of $\sqrt{x}$ over $[0,4]$ and the discussion in exercise 37.

39. * **Write to Learn:** Show that if f is continuous on $[a,b]$ and x is in $[a,b]$, then

$$\lim_{h\to 0} L_h(x) = f(x)$$

(Hint: Explain why if $\frac{f(x)}{h}$ has a decimal expansion of $d_0.d_1d_2\ldots$ where d_0 is an integer and $d_1, d_2, \ldots$ are digits from 0 up to 9, then

$$\frac{f(x)}{h} - \left\lfloor \frac{f(x)}{h} \right\rfloor = 0.d_1d_2\ldots < 1$$

Then multiply by $|h|$ and use the squeeze theorem.)

40. * **Write to Learn:** Show that if f is continuous on $[a,b]$ and x is in $[a,b]$, then

$$\lim_{h\to 0} U_h(x) = f(x)$$

(Hint: Similar to exercise 39).

41. Show that $L_{-h}(x) = U_h(x)$. What is $U_{-h}(x)$?

42. **Write to Learn:** Consider the function

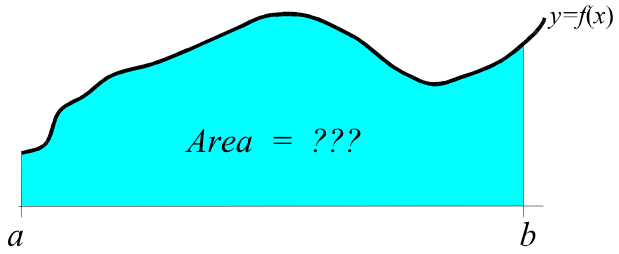
$$f(x) = \begin{cases} 0 & if \quad x \textit{ is a rational number} \\ 1 & if \quad x \textit{ is an irrational number} \end{cases}$$

For each h, what are $L_h(x)$ and $U_h(x)$ when x is rational? When x is irrational? Are $L_h(x)$ and $U_h(x)$ simple functions? Explain.

4.2 Riemann Sum Approximations

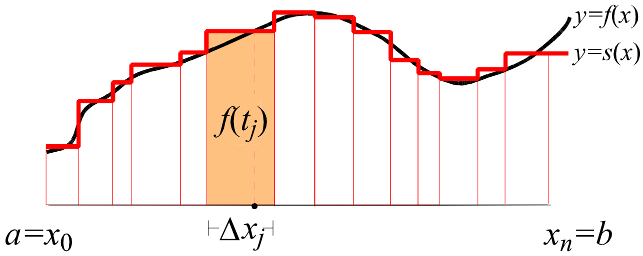
Area Under a Curve

Let's consider some applications of simple function approximation. We begin with the use of simple functions to approximate the area of the region under $y = f(x)$ and over $[a,b]$.



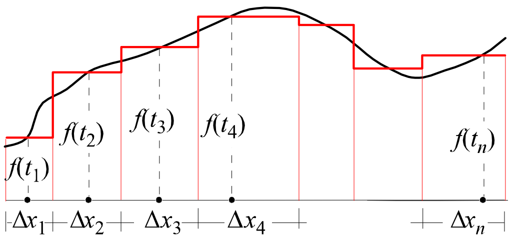
2-1: The region under $y = f(x)$ and over $[a,b]$.

To do so, we approximate $f(x)$ with a simple function and notice that the graph of a simple function is a collection of n rectangles with heights $f(t_j)$ and widths Δx_j, where t_j is the tag and Δx_j is the width of the subinterval $[x_{j-1}, x_j)$.



2-2: Area of j^{th} rectangle is $f(t_j)\,\Delta x_j$

That is, the first rectangle has area $f(t_1)\,\Delta x_1$, the second rectangle has area $f(t_2)\,\Delta x_2$, and so on



2-3: Approximation with rectangles

The area of the region is thus approximated by the sum of the areas of the rectangles:

$$Area \approx f(t_1)\,\Delta x_1 + f(t_2)\,\Delta x_2 + \ldots + f(t_n)\,\Delta x_n \tag{4.11}$$

Often we use the notation Σ, which is pronounced "Sigma," to denote the operation of summation, so that we can write the sum in (4.11) in compact form as

$$\sum_{j=1}^{n} f(t_j)\,\Delta x_j = f(t_1)\,\Delta x_1 + f(t_2)\,\Delta x_2 + \ldots + f(t_n)\,\Delta x_n$$

Definition 2.1: If $\{x_j, t_j\}_{j=1}^{n}$ is a tagged partition of $[a,b]$ and $f(x)$ is defined on $[a,b]$, then

$$\sum_{j=1}^{n} f(t_j)\,\Delta x_j$$

is called a *Riemann sum* approximation of $f(x)$ *over* $[a,b]$.

Thus, if $f(x) \geq 0$ on $[a,b]$, then the area of the region under $y = f(x)$ over $[a,b]$ is approximated by a Riemann sum.

EXAMPLE 1 Find the Riemann sum approximation of the area of the region under $y = x^2$ over $[0,1]$ using the right endpoint approximation over the partition

$$x_0 = 0,\ x_1 = 0.3,\ x_2 = 0.5,\ x_3 = 0.7,\ x_4 = 0.9,\ x_5 = 1.0 \tag{4.12}$$

Solution: To compute a Riemann sum approximation, we add a column of areas $f(t_j)\,\Delta x_j$ to the tabular calculations of simple function approximations in the last section.

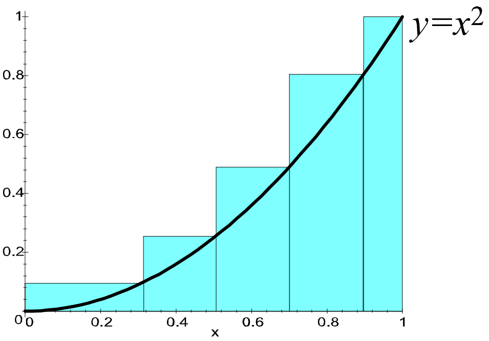
j	x_{j-1}	x_j	t_j	$f(t_j)$	Δx_j	$f(t_j)\,\Delta x_j$
1	0.0	0.3	0.3			
2	0.3	0.5	0.5			
3	0.5	0.7	0.7			
4	0.7	0.9	0.9			
5	0.9	1.0	1.0			
						$\sum_{k=1}^{5} f(t_j)\,\Delta x_j$

Since $f(t_j) = t_j^2$, the result is as follows:

j	x_{j-1}	x_j	t_j	t_j^2	Δx_j	$t_j^2 \Delta x_j$
1	0.0	0.3	0.3	0.09	0.3	$(0.09)(0.3) = 0.027$
2	0.3	0.5	0.5	0.25	0.2	$(0.25)(0.2) = 0.050$
3	0.5	0.7	0.7	0.49	0.2	$(0.49)(0.2) = 0.098$
4	0.7	0.9	0.9	0.81	0.2	$(0.81)(0.2) = 0.162$
5	0.9	1.0	1.0	1.0	0.1	$(1.0)(0.1) = 0.1$
						$\sum_{k=1}^{5} t_j^2 \Delta x_j = \text{sum}$

Summing over the last column then yields our approximation

$$Area \approx \sum_{k=1}^{5} t_j^2 \Delta x_j = 0.027 + 0.050 + 0.098 + 0.162 + 0.1 = 0.437$$



2-4: A Riemann Sum Approximation of $y = x^2$

A calculator or computer can be used to approximate areas by using similar methods to approximate a *definite integral* $\int_a^b f(x)\,dx$. For example, using Maple© to estimate $\int_0^1 x^2 dx$ to seven decimal places results in

$$\int_0^1 x^2 dx = 0.3333333$$

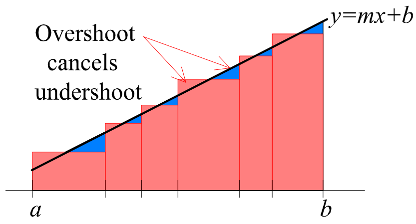
This is the actual area of the region under $y = x^2$ over $[0, 1]$ to 7 decimal places.

Check your Reading *How close can your calculator come to the exact value for example 1 of*

$$Area = \frac{1}{3}?$$

Midpoint Approximations of Area

If $f(x)$ is linear, then the overshoot on one side of the midpoint is exactly the same as the undershoot on the other side of the midpoint.



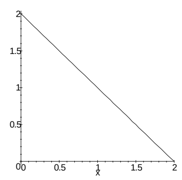
2-5: Midpoint Approximation of a Linear Function

Thus, the midpoint approximation is exact when $y = f(x)$ is a straight line.

EXAMPLE 2 Calculate the area of the region under $y = 2 - x$ and over $[0, 2]$ first by using simple geometry and then by using a midpoint approximation.

Solution: Since the region under $y = 2 - x$ on the interval $[0, 2]$ is a right triangle with base 2 and height 2, we have

$$Area = \frac{1}{2} \cdot 2 \cdot 2 = 2$$



Let's apply the midpoint approximation using the irregular partition

$$x_0 = 0,\ x_1 = 0.2,\ x_2 = 1.0,\ x_3 = 1.4,\ x_4 = 1.6,\ x_5 = 2.0$$

To do so, we construct the table below:

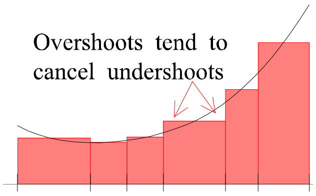
j	x_{j-1}	x_j	t_j	$2 - t_j$	Δx_j	$(2 - t_j)\,\Delta x_j$
1	0.0	0.2	0.1	1.9	0.2	0.38
2	0.2	1.0	0.6	1.4	0.8	1.12
3	1.0	1.4	1.2	0.8	0.4	0.32
4	1.4	1.6	1.5	0.5	0.2	0.10
5	1.6	2.0	1.8	0.2	0.4	0.08
						$\sum_{j=1}^{5} (2 - t_j)\,\Delta x_j$

The midpoint approximation thus yields

$$Area \approx \sum_{j=1}^{5} (2 - t_j)\,\Delta x_j = 0.38 + 1.12 + 0.32 + 0.10 + 0.08 = 2$$

As expected, the midpoint approximation yields the exact value of the integral.

Moreover, if $f(x)$ is differentiable and the subintervals of the partition are sufficiently short, the graph of $f(x)$ is practically a straight line over each subinterval, in which case the midpoint approximation is the best of the three methods we've seen thus far.



2-6: Undershoots tend to cancel overshoots

EXAMPLE 3 Find the Riemann sum approximation of the area of the region under $y = x^2$ over $[0, 1]$ using the left endpoint approximation over the partition

$$x_0 = 0,\ x_1 = 0.2,\ x_2 = 0.4,\ x_3 = 0.6,\ x_4 = 0.8,\ x_5 = 1.0$$

Solution: The calculation is similar to those in examples 1 and 2:

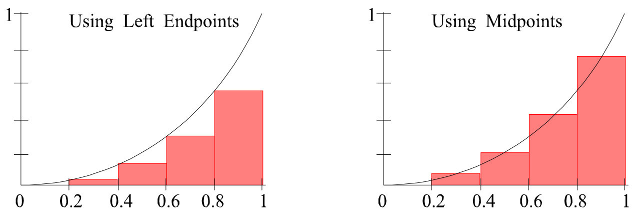
j	x_{j-1}	x_j	t_j	t_j^2	Δx_j	$t_j^2 \Delta x_j$
1	0.0	0.2	0.1	$(0.1)^2 = 0.01$	0.2	$0.01 \cdot 0.2 = 0.002$
2	0.2	0.4	0.3	$(0.3)^2 = 0.09$	0.2	$0.09 \cdot 0.2 = 0.018$
3	0.4	0.6	0.5	$(0.5)^2 = 0.25$	0.2	$0.25 \cdot 0.2 = 0.050$
4	0.6	0.8	0.7	$(0.7)^2 = 0.49$	0.2	$0.49 \cdot 0.2 = 0.098$
5	0.8	1.0	0.9	$(0.9)^2 = 0.81$	0.2	$0.81 \cdot 0.2 = 0.162$
						$\sum_{j=1}^{n} t_j^2 \Delta x_j$

Summing over the last column then yields our approximation

$$Area \approx \sum_{j=1}^{n} t_j^2 \Delta x_j = 0.002 + 0.018 + 0.05 + 0.098 + 0.162 = 0.33$$

Notice that this is much closer to the actual value of $Area = \frac{1}{3}$ than is the right endpoint approximation in example 1.

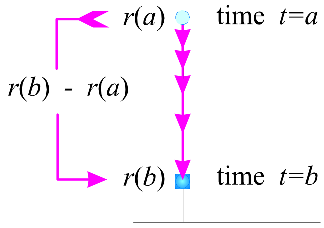
Check your Reading *Use figure 2-7 to explain why the midpoint approximation in example 3 is better than the left endpoint approximation.*



2-7: Which one is better?

Displacement and Velocity

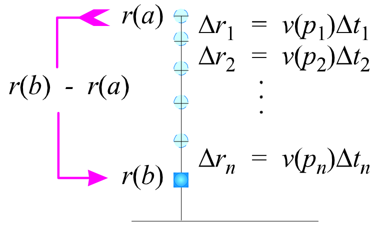
If an object moving in a straight line has a velocity of $v(t)$ at time t, then it moves from a position $r(a)$ to a position $r(b)$ over the time interval $[a,b]$. The difference $r(b)-r(a)$ is called the *displacement* of the object and can be either positive or negative.



2-8: Displacement of a falling object

We can use Riemann sums to estimate the displacement.

To do so, let us partition the time interval $[a,b]$ into a tagged partition $\{t_j, p_j\}$ where the tags are denoted p_j. If the subintervals are small enough, then the velocity will be nearly constant across each subinterval.



2-9: Partition so that $v(t)$ is nearly constant over each subinterval

For a constant velocity over $[t_0,t_1]$, the displacement Δr_1 is the product of the velocity $v(p_1)$ and the duration $\Delta t_1 = t_1 - t_0$. Thus, the new position $r(t_1)$ is approximately the sum of the old position $r(t_0)$ and the change in position Δr_1:

$$r(t_1) \approx r(t_0) + \Delta r_1 = r(t_0) + v(p_1)\,\Delta t_1$$

During the next time interval $[t_1,t_2]$, the object will move approximately $\Delta r_2 = v(p_2)\,\Delta t_2$, so that $r(t_2)$ is approximately

$$r(t_2) \approx r(t_1) + \Delta r_2 \approx r(t_0) + v(p_1)\,\Delta t_1 + v(p_2)\,\Delta t_2$$

Similarly for $[t_2,t_3]$ and so on, until once we reach $[t_{n-1},t_n]$, we obtain

$$r(t_n) \approx r(t_{n-1}) + \Delta r_n \approx r(t_0) + v(p_1)\,\Delta t_1 + \ldots + v(p_n)\,\Delta t_n$$

However, since $t_0 = a$ and $t_n = b$, this yields

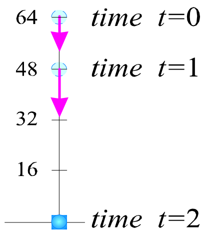
$$r(b) \approx r(a) + v(p_1)\,\Delta t_1 + \ldots + v(p_n)\,\Delta t_n$$

so that the displacement $r(b) - r(a)$ is approximated by

$$r(b) - r(a) \approx \sum_{j=1}^{n} v(p_n)\,\Delta t_n \tag{4.13}$$

Moreover, the displacement may be negative (such as if the object is falling).

EXAMPLE 4 An projectile falls from a height of 64 feet to the ground over the time interval $[0,2]$ with velocity $v(t) = -32t$.



2-10: Ball drops from 64 feet to the ground

Use a midpoint approximation to estimate the displacement of the projectile over $[1.5, 2]$ over the regular partition

$$1.5, 1.6, 1.7, 1.8, 1.9, 2.0$$

Solution: Because $v(t)$ is linear, the midpoint approximation will be exact.

j	t_{j-1}	t_j	p_j	$v(p_j)$	Δt_j	$v(p_j)\Delta t_j$
1	1.5	1.6	1.55	-49.6	0.1	-4.96
2	1.6	1.7	1.65	-52.8	0.1	-5.28
3	1.7	1.8	1.75	-56.0	0.1	-5.60
4	1.8	1.9	1.85	-59.2	0.1	-5.92
5	1.9	2.0	1.95	-62.4	0.1	-6.24
				$\sum_{j=1}^{5}$	$v(p_j)\Delta t_j \approx$	-28

Thus, we estimate that the projectile will fall approximately

$$r(2) - r(1.5) \approx \sum_{j=1}^{5} v(p_j)\Delta t_j = -28\ feet$$

during its last half second of flight.

Check your Reading *The position function for example 4 is* $r(t) = 64 - 16t^2$. *How does the displacement* $r(2) - r(1.5)$ *compare with the estimate in example 4?*

The Trapezoidal Rule

We can also approximate areas by *averaging* the left and right endpoint methods. Indeed, over a regular partition whose subintervals have a width Δx, a left endpoint Riemann sum is of the form

$$\begin{aligned} Area &\approx f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x \\ &\approx \Delta x\left[f(x_0) + f(x_1) + \ldots + f(x_{n-1})\right] \end{aligned} \tag{4.14}$$

Likewise, the right endpoint Riemann sum approximation is of the form

$$\begin{aligned} Area &\approx f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x \\ &\approx \Delta x\left[f(x_1) + \ldots + f(x_{n-1}) + f(x_n)\right] \end{aligned} \tag{4.15}$$

Since (4.14) and (4.15) have $f(x_1) + \ldots + f(x_{n-1})$ in common, the average is

$$Area \approx \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)\right] \tag{4.16}$$

The result (4.16) is called the *trapezoidal rule* and tends to be as accurate as the midpoint approximation since averaging also causes undershoots to cancel overshoots. In fact, the trapezoidal rule is exact when $f(x)$ is linear.

EXAMPLE 5 Approximate the area of the region under $y = 2x + 3$ over $[1, 5]$ using the trapezoidal rule and the partition

$$x_0 = 1,\ x_2 = 2,\ x_3 = 3,\ x_4 = 4,\ x_5 = 5$$

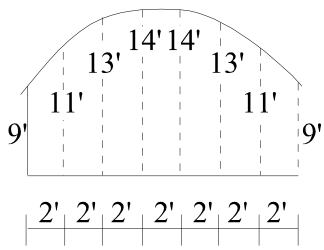
Solution: Clearly, $\Delta x = 1$ and since $f(x) = 2x + 3$, we have

$$f(1) = 5, \quad f(2) = 7, \quad f(3) = 9,\ f(4) = 11,\ f(5) = 13$$

Moreover, $f(x) = 2x + 3$ is linear, so that (4.16) is exact.

$$\begin{aligned} Area &= \frac{\Delta x}{2}\left[f(1) + 2f(2) + 2f(3) + 2f(4) + f(5)\right] \\ &= \frac{1}{2}\left[5 + 2\cdot 7 + 2\cdot 9 + 2\cdot 11 + 13\right] \\ &= 36 \end{aligned}$$

In many applications, however, we are given only measurements and do not have a coordinate system. In such instances, the trapezoidal rule can be used to produce estimates of area. For example, suppose that "John" wants to paint the side of a shed which has a hip roof. He measures heights of the side every 2 feet, so that he obtains a partition of the shed's side.



2-11: Side of Shed with Hip Roof

Doubling the interior measurements, adding the left and right end measurements and multiplying by $\frac{\Delta x}{2}$ yields the trapezoidal rule estimate of the area:

$$Area \approx \frac{2}{2}\left[9 + 2\cdot 11 + 2\cdot 13 + 2\cdot 14 + 2\cdot 14 + 2\cdot 13 + 2\cdot 11 + 9\right] = 170\ ft^2$$

Exercises:

Numerical: *Compute the Riemann sum approximation for the area of the region under the given curve and over the given interval using midpoints as tags. Construct a table similar to those used in the examples. Compare your answer with* *the result of a numerical integration.*[1]

1. $y = x$ over $[0, 1]$
   $0, 0.25, 0.5, 0.75, 1$
2. $y = 4 - \frac{1}{2}x$ over $[-1, 2]$
   $-1, 0, 1, 2$
3. $y = 2x^2 - 1$ over $[0, 1]$
   $1, 2, 3, 4$
4. $y = -x^2 + 3x$ over $[0, 4]$
   $0, 1, 2, 3, 4$
5. $y = x^2$ over $[0, 1]$
   $0, 0.4, 0.6, 0.8, 0.9, 1$
6. $y = x^2$ over $[0, 1]$
   $0, 0.5, 0.8, 0.9, 1$
7. $y = \sqrt{1 - x^2}$ over $[-1, 1]$
   $1, -0.5, 0, 0.5, 1$
8. $y = \sqrt{1 + x^2}$ over $[-1, 1]$
   $-1, -0.5, 0, 0.5, 1$
9. $y = 2\sin(x)\cos(x)$ over $\left[0, \frac{\pi}{2}\right]$
   $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$
10. $y = \sin(2x)$ over $\left[0, \frac{\pi}{2}\right]$
    $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$
11. $y = \dfrac{1}{x^2 + 1}$ over $[0, 1]$
    $0, 0.2, 0.4, 0.6, 0.8, 1.0$
12. $y = \dfrac{1}{1 + x^2/16}$ over $[0, 1]$
    $0, 1, 2, 3, 4$
13. $y = \cos(\pi x)$ over $\left[0, \frac{1}{2}\right]$
    $0, 0.1, 0.2, 0.3, 0.4, 0.5$
14. $y = \sin(\pi x)$ over $[0, 1]$
    $0, 0.2, 0.4, 0.6, 0.8, 1.0$

Numerical: *The following functions are velocities of objects moving in a straight line. Use the given partition and midpoints as tags to construct a Riemann sum estimate of the displacement of the object over the given interval.*

15. $v(t) = 5\ \frac{ft}{\sec}$ over $[0, 1.2]$
    0, 0.4, 0.8, 1.2
16. $v(t) = 0\ \frac{ft}{\sec}$ over $[0, 1.2]$
    0, 0.4, 0.8, 1.2
17. $v(t) = -2\ \frac{ft}{\sec}$ over $[0, 1]$
    0, 0.25, 0.5, 0.75, 1.0
18. $v(t) = -3\ \frac{ft}{\sec}$ over $[0, 1]$
    0, 0.25, 0.5, 0.75, 1.0
19. $v(t) = 2t\ \frac{ft}{\sec}$ over $[0, 1]$
    0, 0.2, 0.4, 0.6, 0.8, 1.0
20. $v(t) = 5t + 1\ \frac{ft}{\sec}$ over $[0, 1]$
    0, 0.2, 0.4, 0.6, 0.8, 1.0
21. $v(t) = \sin(t)\ \frac{ft}{\sec}$ over $[\pi, 2\pi]$
    $\pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$
22. $v(t) = e^{-t}\ \frac{ft}{\sec}$ over $[0, 1]$
    0, 0.1, 0.2, 0.4, 0.7, 1.0

Numerical: *Approximate the area of the region below $y = f(x)$ and above $[a, b]$ for each of the following using the trapezoidal rule. Check your work using a numerical integration algorithm on a calculator or a computer.*

23. $y = x$ over $[0, 1]$
    $0, 0.25, 0.5, 0.75, 1$
24. $y = 4 - \frac{1}{2}x$ over $[-1, 2]$
    $-1, 0, 1, 2$
25. $y = 2x^2 - 1$ *over* $[1, 4]$
    $1, 2, 3, 4$
26. $y = -x^2 + 3x$ over $[0, 4]$
    $0, 1, 2, 3, 4$

Numerical: *In each graph below, a partition of the graph's domain is implied by the labeled points on the x-axis. Sketch the midpoint approximation and use it to*

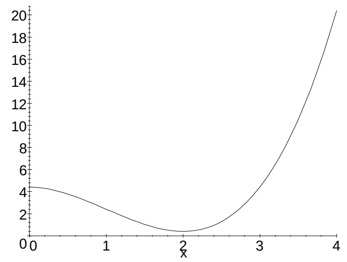
[1] In exercise 7, the partition is denoted by the list of numbers

$$0,\ 0.25,\ 0.5,\ 0.75,\ 1$$

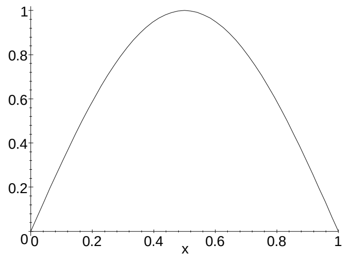
That is, $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$. Likewise, for simplicity, the partitions in exercises 8-24 are written in the same form.

estimate the area under $y = f(x)$ over the graph's domain.

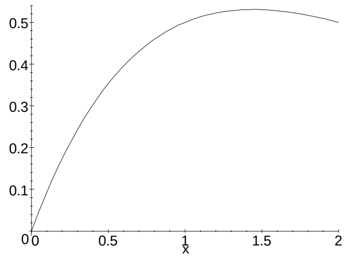
27. $y = x^3 - 3x^2 + 1$



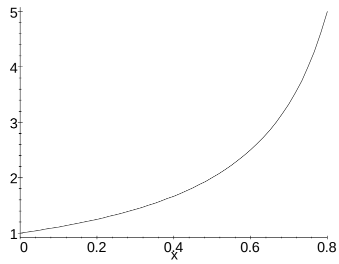
28. $y = \sin(\pi x)$



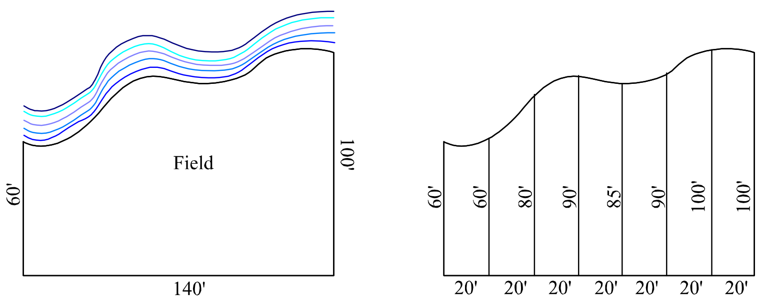
29. $y = x \cdot 2^{-x}$



30. $y = \dfrac{1}{1-x}$



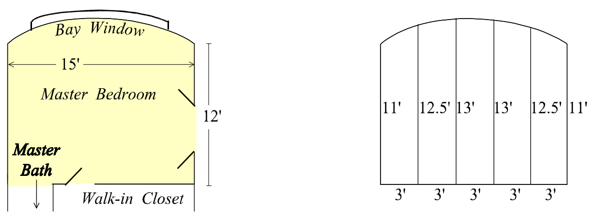
31. Numerical: A river bounds a field, as shown below left. A surveyor measures distances at right angles to the opposite boundary of the field, as shown below right.



2-12: Measurements of a field

Use the trapezoidal rule to estimate the area of the field. What is the size of the field in acres? (Note: 1 acre = 43,560 feet^2).

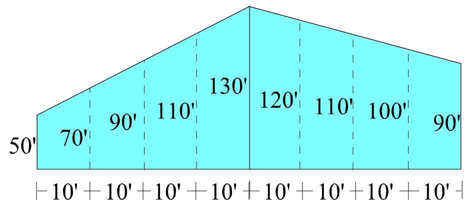
32. Numerical: Fred's flooring is installing parquet tiles in a master bedroom in which one side is irregularly shaped. Fred measures across the room at three foot intervals to produce the diagram below right.



2-13: Measurements of a Room

Use the trapezoidal rule to estimate the square footage of the room.

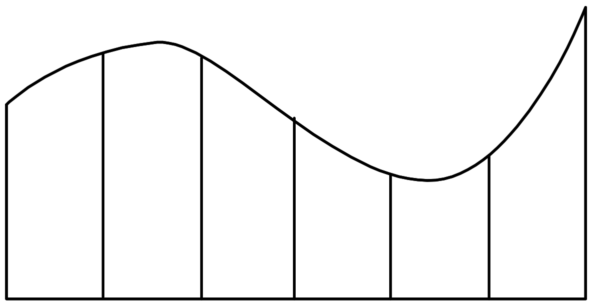
33. **Numerical:** John's commercial painting is hired to paint the side of a building. John measures altitudes at ten foot intervals to produce the diagram below:



2-14: Measurements of the side of a building

Use the trapezoidal rule to find the area in square feet of the side of the building. Why is the result exact?

34. **Write to Learn:** Use a ruler to measure the lengths of the vertical lines in the figure shown below. Measure and label the distance between each pair of vertical lines.



2-15: What is the approximate area of this region in square inches?

Write a short essay in which you report these measurements and use the trapezoidal rule to estimate the area of the region.

35. Suppose that $s(x)$ is the simple function

$$s(x) = \begin{cases} 1 & if \quad 0 \leq x < 2 \\ 3 & if \quad 2 \leq x < 4 \\ 2 & if \quad 4 \leq x < 6 \\ 5 & if \quad 6 \leq x \leq 8 \end{cases}$$

(a) What is the definition of the simple function $3s(x)$?

(b) What is the area under $y = s(x)$ over $[1, 8]$?

(c) What is the area under $y = 3s(x)$ over $[1, 8]$? How is related to the result in (b)?

36. Suppose that $s(x)$ and $p(x)$ are the simple functions.

$$s(x) = \begin{cases} 1 & if \quad 0 \leq x < 2 \\ 3 & if \quad 2 \leq x < 4 \\ 2 & if \quad 4 \leq x < 6 \\ 5 & if \quad 6 \leq x \leq 8 \end{cases} \qquad p(x) = \begin{cases} 2 & if \quad 0 \leq x < 1 \\ 1 & if \quad 1 \leq x < 3 \\ 5 & if \quad 3 \leq x < 7 \\ 3 & if \quad 7 \leq x \leq 8 \end{cases}$$

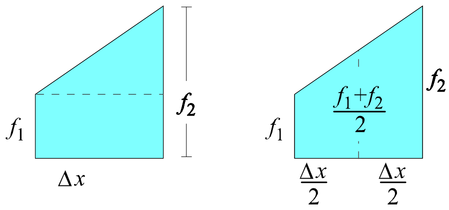
(a) What is the definition of the simple function $s(x) + p(x)$?

(b) What is the area under $y = s(x)$ over $[1, 8]$?

(c) What is the area under $y = p(x)$ over $[1, 8]$?

(d) What is the area under $y = s(x) + p(x)$ over $[1, 8]$? How is it related to (b) and (c)?

37. Write to Learn: Compute the area of the trapezoid below left by summing the area of the right triangle with the area of the rectangle.



2-16: Area of a Trapezoid

Then apply the midpoint method to the trapezoid on the right. In a short essay, explain why all three computations produce the same result.

38. Write to Learn: Use a circle to determine the exact area under $y = \sqrt{4 - x^2}$ over the interval $[0, 2]$, and then use a midpoint approximation to approximate the area over the partition

$$0, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, 2.0$$

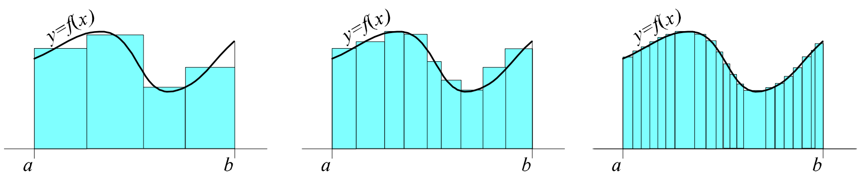
In a short essay, explain why the approximation of the integral is an approximation of π.

4.3 The Definite Integral

Definition of the Integral

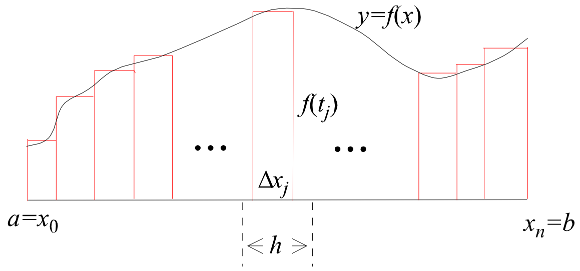
In the previous section, we saw that Riemann sums $\sum f(t_j)\,\Delta x_j$ can be used to approximate areas and displacements. In this section, we use a limit to obtain exact values from Riemann Sum approximations, thus resulting in the important new concept of a *definite integral*.

To begin with, if $f(x)$ is continuous on $[a, b]$, then simple function approximations become better and better as all of the widths Δx_j, $j = 1, \ldots, n$ become shorter and shorter:



3-1: Simple Function Approximations

In particular, if $\Delta x_j < h$ for all $j = 1, \ldots, n$ and if h approaches 0, then all of the widths Δx_j to approach 0 as well.



3-2: In an h-fine partition, each subinterval has a width less than h.

Consequently, for a given $h > 0$, let us say that a tagged partition $\{x_j, t_j\}_{j=1}^{n}$ of $[a, b]$ is *h-fine* if $\Delta x_j < h$ for all $j = 1, \ldots, n$. Then for each $h > 0$, let us construct a Riemann sum $\sum f(t_j)\,\Delta x_j$, so that if h approaches 0 and if $f(x)$ is continuous over $[a, b]$, then the Riemann sums approach an exact value. This exact value is called the *definite integral* of $f(x)$ over $[a, b]$ and is denoted $\int_a^b f(x)\,dx$ where $\int$ is an elongated "S".

Definition 3.1: The *definite integral* of $f(x)$ over $[a, b]$ is defined

$$\int_a^b f(x)\,dx = \lim_{h\to 0}\left[f(t_1)\,\Delta x_1 + f(t_2)\,\Delta x_2 + \ldots + f(t_n)\,\Delta x_n\right]$$

where the limit is over h-fine partitions of $[a, b]$, when the limit exists.

The numbers a, b are called the *limits of integration* and the function $f(x)$ is called the *integrand* of the definite integral. If $\int_a^b f(x)\,dx$ exists, then we say that $f(x)$ is *Riemann integrable* over $[a, b]$. For example, a continuous function over $[a, b]$ is Riemann integrable on $[a, b]$.

Some key observations follow immediately from the definition. First, definition 3.1 can be written using sigma notation as

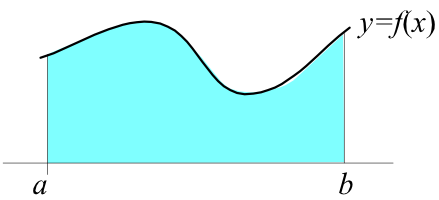
$$\int_a^b f(x)\,dx = \lim_{h\to 0}\sum_{j=1}^{n} f(t_j)\,\Delta x_j$$

A definite integral is a number.

Second, the definite integral $\int_a^b f(x)\,dx$ is a **number**, which implies that the variable used in a definite integral is a *dummy variable*—it is there only for convenience. For example, the definite integral $\int_0^1 x^9 dx$ is the same as $\int_0^1 u^9 du$, and in fact,

$$\int_0^1 x^9 dx = \int_0^1 u^9 du = \int_0^1 t^9 dt = \int_0^1 s^9 ds = \int_0^1 \tau^9 d\tau = \int_0^1 z^9 dz$$

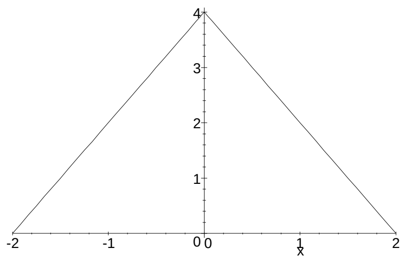
Third, if $f(x) \geq 0$ over $[a, b]$, then $\int_a^b f(x)\,dx$ is the area of the region under $y = f(x)$ over $[a, b]$.



3-3: *Area* $= \int_a^b f(x)\,dx$

EXAMPLE 1 Evaluate $\int_{-2}^{2} (4 - 2|x|)\, dx$.

Solution: To begin with, let's notice that $y = 4 - 2|x|$ over $[-1, 1]$ defines an isosceles triangle with base 4 and height 4.



3-4: $y = 4 - 2|x|$ over $[-2, 2]$

Thus, the definite integral is equal to the area of the triangle, which is

$$\int_{-2}^{2} (4 - 2|x|)\, dx = \frac{1}{2} base \cdot height = 8$$

Check your Reading *If* $\int_0^1 x^9 dx = 0.1$*, then what is the value of* $\int_0^1 \Lambda^9 d\Lambda$*?*

Linearity of the Definite Integral

Definition 3.1 is used to develop the properties of the definite integral. For example, suppose f and g are Riemann integrable on $[a, b]$. For $h > 0$, let $\{x_j, t_j\}_{j=1}^{n}$ be an h-fine tagged partition of $[a, b]$. Then

$$\begin{aligned}\int_a^b [f(x) + g(x)]\, dx &= \lim_{h \to 0} \sum_{j=1}^{n} [f(t_j) + g(t_j)]\, \Delta x_j \\ &= \lim_{h \to 0} \sum_{j=1}^{n} f(t_j)\, \Delta x_j \quad + \quad \lim_{h \to 0} \sum_{j=1}^{n} g(t_j)\, \Delta x_j \\ &= \int_a^b f(x)\, dx + \int_a^b g(x)\, dx\end{aligned}$$

Thus, the integral of a sum is the sum of the integrals:

$$\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx \tag{4.17}$$

Likewise, if k is a constant, then

$$\int_a^b k f(x)\, dx = k \int_a^b f(x)\, dx \tag{4.18}$$

Finally, definition 3.1 allows a *definite integral to be negative*, and more importantly, allows a definite integral to represent quantities other than area, such as displacements or volumes.

EXAMPLE 2 If $\int_0^1 f(x)\,dx = 1.4$ and $\int_0^1 g(x)\,dx = 2.1$, then what is

$$\int_0^1 (f(x) - 2g(x))\,dx$$

Solution: Use (4.17) and (4.18), we obtain

$$\int_0^1 (f(x) - 2g(x))\,dx = \int_0^1 f(x)\,dx - 2\int_0^1 g(x)\,dx = 1.4 - 2(2.1) = -2.8$$

However, it is possible to use areas of familiar regions to calculate definite integrals.

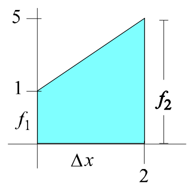
EXAMPLE 3 Evaluate $\int_0^2 \left(2x + 1 - 3\sqrt{4 - x^2}\right) dx$

Solution: To begin with, (4.17) implies that

$$\int_0^2 \left(2x + 1 - 3\sqrt{4 - x^2}\right) dx = \int_0^2 (2x + 1)\,dx - 3\int_0^2 \sqrt{4 - x^2}dx$$

The first integral is the area of the region below the line $y = 2x + 1$. The region is a trapezoid, and the formula for the area of a trapezoid is

$$Area = \left(\frac{f_1 + f_2}{2}\right)\Delta x$$



Thus, we have

$$\int_0^2 (2x + 1) = \left(\frac{1 + 5}{2}\right) 2 = 6$$

In contrast, the region under $y = \sqrt{4 - x^2}$ is a quarter of a circle with radius 2. Thus, its area is simply π, so that we have

$$\int_0^2 \left(2x + 1 - 3\sqrt{4 - x^2}\right) dx = 6 - 3\pi = -3.424$$

Check your Reading *Given that* $\int_0^1 x^9 dx = 0.1$ *and* $\int_0^1 x^4 dx = 0.2$, *what is*

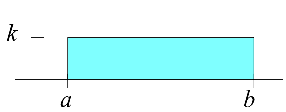
$$\int_0^1 \left(x^9 + x^4\right) dx$$

Properties of the Definite Integral

When $f(x) \geq 0$ on $[a, b]$, then definition 3.1 implies that $\int_a^b f(x)\,dx$ is the exact area of the region under $y = f(x)$ and over $[a, b]$. Moreover, many properties are motivated by the interpretation of a definite integral as an area under a curve.

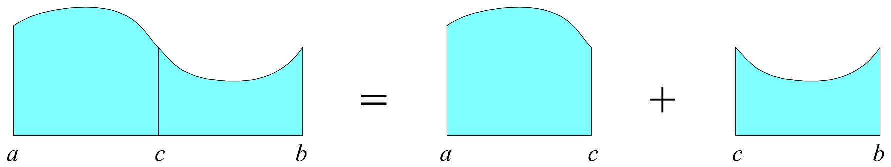
For example, when $k > 0$, then $\int_a^b kdx$ is the area of the rectangle with height k and width $b - a$. It follows that for any real number k that

$$\int_a^b kdx = k(b-a)$$



3-5 : Definite integral of a constant function

Another property of the integral follows from the fact that the region under the graph of $f(x)$ over $[a, b]$ can be divided into two subregions—the region over the subinterval $[a, c]$ and the region over the subinterval $[c, b]$.



3-6: Subdivision of a region

Thus, the area of the entire region is equal to the sum of the areas of the subregions, which leads to

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \tag{4.19}$$

In addition, since the "region" under $f(x)$ over $[a, a]$ has a width of zero, we have

$$\int_a^a f(x)\,dx = 0 \tag{4.20}$$

Geometry and the original definition of the integral also motivate the definition of $\int_b^a f(x)\,dx$ when $b < a$. In particular, we define

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \tag{4.21}$$

in correspondence with the definition, with (4.19), and with (4.20).

EXAMPLE 4 If $\int_1^{-2} f(x)\,dx = 7$ and $\int_3^1 f(x)\,dx = 5$, then what is $\int_{-2}^3 f(x)\,dx$?

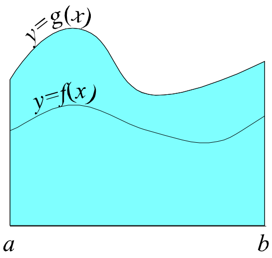
Solution: Property (4.19) implies that

$$\int_{-2}^3 f(x)\,dx = \int_{-2}^1 f(x)\,dx + \int_1^3 f(x)\,dx$$

We now use property (4.20) to rewrite $\int_{-2}^1 f(x)\,dx$ as $-\int_1^{-2} f(x)\,dx$:

$$\begin{aligned}\int_{-2}^3 f(x)\,dx &= -\int_1^{-2} f(x)\,dx + \int_1^3 f(x)\,dx \\ &= -7+5 \\ &= -2\end{aligned}$$

We can also use areas to motivate inequalities involving integrals. To illustrate consider that if $0 \leq f(x) \leq g(x)$ over $[a,b]$, then the region under $y = f(x)$ and over $[a,b]$ is contained in the region under $y = g(x)$ and over $[a,b]$.



This concept leads to the following property:

$$if\ f(x) \leq g(x)\ on\ [a,b], \quad then \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx \tag{4.22}$$

Property (4.22) in turn implies a very useful inequality. Notice that both $f(x) \leq |f(x)|$ and $-f(x) \leq |f(x)|$, so that both $\int_a^b f(x)\,dx$ and $-\int_a^b f(x)\,dx$ are less than or equal to $\int_a^b |f(x)|\,dx$. As a result, we have that

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx \tag{4.23}$$

Property (4.23) is sometimes called the *triangle inequality* for integrals. The triangle inequality is useful because it can be used to obtain key estimates of integrals of certain functions.

EXAMPLE 5 Show that for all $b > a$, we have

$$\int_a^b \sin\left(x^2\right) dx < b - a$$

Solution: Since $\left|\sin\left(x^2\right)\right| \leq 1$ for all x, the triangle inequality implies that

$$\left|\int_a^b \sin\left(x^2\right) dx\right| \leq \int_a^b \left|\sin\left(x^2\right)\right| dx \leq \int_a^b 1dx$$

However, $\int_a^b 1dx$ is the area of a rectangle with height 1 and width $b - a$. Thus, $\int_a^b 1dx = b - a$ and

$$\int_a^b \sin\left(x^2\right) dx \leq \left|\int_a^b \sin\left(x^2\right) dx\right| \leq b - a$$

Check your Reading *If $\int_1^2 f(x)\,dx = 4$ and $\int_2^3 f(x)\,dx = 5$, then what is $\int_1^3 f(x)\,dx$?*

Calculations using the Definition

Definition 3.1 can also be used to compute $\int_a^b f(x)\,dx$ directly. However, to do so often requires a clever choice of the tags. Indeed, if $f(x)$ is continuous, then a proper choice of tags makes the limit in definition 3.1 trivial.

EXAMPLE 6 Evaluate $\int_2^3 xdx$ directly from definition 3.1 using the midpoints as the tags.

Solution: If $\{x_j, t_j\}$ is a tagged partition of $[2,3]$ and if each tag t_j is the midpoint of the interval $[x_{j-1}, x_j]$, then it follows that

$$t_j = \frac{1}{2}(x_j + x_{j-1})$$

Since $f(x) = x$ implies that $f(t_j) = t_j$, we have

$$t_j \Delta x_j = \frac{1}{2}(x_j + x_{j-1})(x_j - x_{j-1}) = \frac{1}{2}\left(x_j^2 - x_{j-1}^2\right)$$

However, definition 3.1 says that

$$\begin{aligned}
\int_2^3 xdx &= \lim_{h\to 0} \sum_{j=1}^{n} t_j \Delta x_j \\
&= \lim_{h\to 0} (t_1 \Delta x_1 + t_2 \Delta x_2 + \ldots + t_n \Delta x_n) \\
&= \lim_{h\to 0} \left(\frac{1}{2}\left(x_1^2 - x_0^2\right) + \frac{1}{2}\left(x_2^2 - x_1^2\right) + \ldots + \frac{1}{2}\left(x_n^2 - x_{n-1}^2\right)\right) \\
&= \lim_{h\to 0} \frac{1}{2}\left(x_n^2 - x_{n-1}^2 + \ldots + x_3^2 - x_2^2 + x_2^2 - x_1^2 + x_1^2 - x_0^2\right)
\end{aligned}$$

All but two of the terms inside the parentheses cancel, thus yielding

$$\int_2^3 xdx = \lim_{h\to 0} \frac{1}{2}\left(x_n^2 - x_0^2\right)$$

However, $x_n = 3$ and $x_0 = 2$, so that we obtain

$$\int_2^3 xdx = \lim_{h\to 0} \frac{1}{2}\left(3^2 - 2^2\right) = \frac{5}{2}$$

EXAMPLE 7 Use geometric means as tags to evaluate

$$\int_1^2 \frac{1}{x^2} dx$$

Solution: Suppose $\{x_j, t_j\}$ is a tagged partition of $[1,2]$ where each tag t_j is the geometric mean of the endpoints of $[x_{j-1}, x_j]$. That is, $t_j = \sqrt{x_{j-1} x_j}$, which implies that

$$\frac{1}{t_j^2} = \frac{1}{x_{j-1} x_j}$$

Since $f(t_j) = 1/t_j^2$, areas of rectangles are given by

$$\begin{aligned}
\frac{1}{t_j^2} \Delta x_j &= \frac{1}{x_{j-1} x_j}(x_j - x_{j-1}) \\
&= \frac{x_j}{x_{j-1} x_j} - \frac{x_{j-1}}{x_{j-1} x_j} \\
&= \frac{1}{x_{j-1}} - \frac{1}{x_j}
\end{aligned}$$

Definition 3.1 then says that

$$\begin{aligned}
\int_1^2 \frac{1}{x^2} dx &= \lim_{h\to 0} \sum_{j=1}^{n} \frac{1}{t_j^2} \Delta x_j \\
&= \lim_{h\to 0} \left(\frac{1}{t_1^2}\Delta x_1 + \frac{1}{t_2^2}\Delta x_2 + \frac{1}{t_3^2}\Delta x_3 + \ldots + \frac{1}{t_n^2}\Delta x_n \right) \\
&= \lim_{h\to 0} \left(\frac{1}{x_0} - \frac{1}{x_1} + \frac{1}{x_1} - \frac{1}{x_2} + \frac{1}{x_2} - \frac{1}{x_3} + \ldots + \frac{1}{x_{n-1}} - \frac{1}{x_n} \right) \\
&= \lim_{h\to 0} \left(\frac{1}{x_0} - \frac{1}{x_n} \right)
\end{aligned}$$

However, $x_0 = 1$ and $x_n = 2$, so that

$$\int_1^2 \frac{1}{x^2} dx = \lim_{h\to 0} \left(\frac{1}{1} - \frac{1}{2} \right) = \frac{1}{2}$$

Exercises:

Given the following definite integrals

$$\begin{aligned}
\int_0^2 f(x)\, dx &= 2, \quad \int_1^3 f(x)\, dx = 4, \quad \int_2^3 f(x)\, dx = 2.5 \\
\int_0^2 g(x)\, dx &= 4, \quad \int_1^4 g(x)\, dx = 5, \quad \int_2^4 g(x)\, dx = 7
\end{aligned}$$

evaluate the integrals below using the properties of the integral. State the property or properties used in each evaluation.

1. $\int_0^0 f(x)\, dx$
2. $\int_1^1 g(s)\, ds$
3. $\int_0^3 f(x)\, dx$
4. $\int_0^4 g(x)\, dx$
5. $\int_2^0 f(r)\, dr$
6. $\int_4^2 g(x)\, dx$
7. $\int_1^2 f(x)\, dx$
8. $\int_1^2 g(x) dx$
9. $2\int_0^2 f(u)\, du$
10. $\int_0^1 f(x)\, dx$
11. $\int_0^1 g(x)\, dx$
12. $3\int_0^2 g(t)\, dt$

13. $\int_0^2 [f(x) - 2g(x)]\, dx$
14. $\int_0^2 [2f(x) - 3g(x)]\, dx$
15. $\int_0^1 [f(x) + g(x)]\, dx$
16. $\int_0^1 [f(x) - g(x)]\, dx$

Use the properties of the integral to simplify to areas under circles or lines, and *then use geometry to determine the integrals.*

17. $\int_0^2 3x dx$

18. $\int_1^3 (-5x)\, dx$

19. $\int_0^3 (4x+3)\, dx$

20. $\int_2^5 (2t-12)\, dt$

21. $\int_0^5 \sqrt{25-t^2} dt$

22. $\int_0^{\sqrt{3}} 5\sqrt{3-x^2} dx$

23. $\int_0^1 \left(x+\sqrt{1-x^2}\right) dx$

24. $\int_0^{\sqrt{3}} \left(x+\sqrt{3-x^2}\right) dx$

25. $\int_{-1}^1 |x|\, dx$

26. $\int_{-2}^3 |x|\, dx$

27. $\int_1^3 \sqrt{1-(x-2)^2} dx$

28. $\int_1^2 \sqrt{4-(u-1)^2} du$

29. $\int_{-1}^1 (x+|x|)\, dx$

30. $\int_{-1}^1 (x-|x|)\, dx$

31. Use definition 3.1 to compute the integral $\int_0^3 2dx$.

32. Use definition 3.1 with tags as the midpoints to compute $\int_0^2 xdx$.

33. Use definition 3.1 with tags as the geometric means of the endpoints of the subinterval (see example 8) to compute

$$\int_{1/2}^1 \frac{1}{x^2} dx$$

34. In this exercise we evaluate the definite integral

$$\int_1^2 \frac{1}{\sqrt{x}} dx$$

(a) Let the tag t_j on a given subinterval $[x_{j-1}, x_j]$ be of the form

$$t_j = \left(\frac{\sqrt{x_{j-1}}+\sqrt{x_j}}{2}\right)^2$$

Explain why t_j is a number in $[x_{j-1}, x_j]$

(b) Use the tags in (a) and definition 3.1 to compute the integral.

35. Use definition 3.1 with midpoints as the tags to derive the formula

$$\int_a^b xdx = \frac{b^2}{2} - \frac{a^2}{2} \tag{4.24}$$

36. Use (4.24) to do the following:

(a) Verify that $\int_a^a xdx = 0$

(b) Show that

$$\int_a^c xdx + \int_c^b xdx = \int_a^b xdx$$

37. Use the triangle inequality for integrals to show that if $|f(x)| \leq K$ for all x in $[a, b]$, then

$$\int_a^b f(x)\, dx \leq K(b-a)$$

38. Use the properties of integrals to show that if $f(x)$ is continuous on $[a, b]$, then

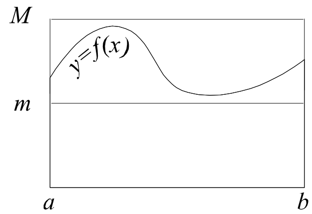
$$\int_a^b L_h(x)\, dx \leq \int_a^b f(x)\, dx \leq \int_a^b U_h(x)\, dx$$

where $h > 0$ and $L_h(x)$ and $U_h(x)$ are respectively the lower and upper simple function approximations of $f(x)$ on $[a, b]$. (Hint: see exercises 35-42 in section 5-1 for the definition of $L_h(x)$ and $U_h(x)$)

39. Graph $f(x) = 2 - x$ and $g(x) = \sqrt{4 - x^2}$ over $[0, 2]$. Which one is larger? Use (4.22) to relate $\int_0^2 (2-x)\, dx$ to $\int_0^2 \sqrt{4-x^2} dx$. Check your work by evaluating both numerically. Use the area formula for trapezoids to derive the formula

40. Graph $f(x) = 2 + x$ and $g(x) = \sqrt{4 + x^2}$ over $[0, 2]$. Which one is larger? Use (4.22) to relate $\int_0^2 (2+x)\, dx$ to $\int_0^2 \sqrt{4+x^2} dx$. Check your work by evaluating both numerically.

41. **Write to Learn:** Write a short essay in which you use the diagram



3-7: Bounds for $\int_a^b f(x)\, dx$

to explain why if $m \leq f(x) \leq M$, then

$$m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a) \tag{4.25}$$

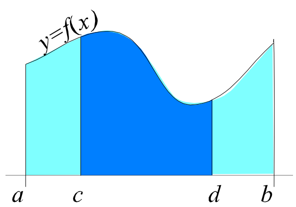
42. In this exercise, we use (4.25) to show that

$$3 \leq \int_{-0.5}^{2.5} \left(x^3 - 3x^2 + 5\right) dx \leq 15$$

(a) Graph $f(x) = x^3 - 3x^2 + 5$ on $[-0.5, 2.5]$, and then identify the greatest integer m which is below the graph of $f(x)$ and the smallest integer M which is above the graph of $f(x)$.

(b) Use the values in (a) in (4.25) to produce the estimate above.

43. **Write to Learn:** Write a short essay in which you use the following diagram to motivate another property of the integral. (Hint: notice that $f(x) \geq 0$ in the diagram).



3-8: What property does this figure suggest?

44. Derive (4.21) from (4.19) and (4.20).

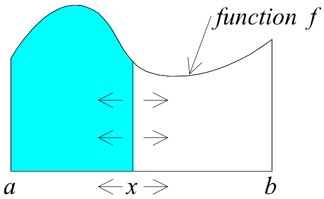
4.4 Derivatives of Indefinite Integrals

Functions Defined by Integrals

Definite integrals can also be used to define new functions. In particular, if $f(x)$ is continuous on $[a,b]$, then we can define a new function $F(x)$ to be

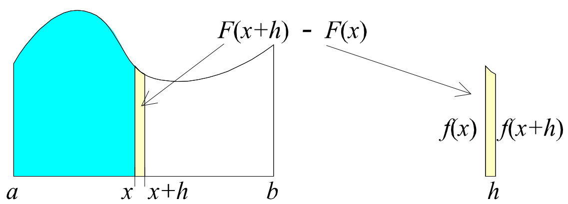
$$F(x) = \int_a^x f(t)\,dt \tag{4.26}$$

where x is in $[a,b]$. The function $F(x)$ is known as an *indefinite integral* of $f(x)$. Moreover, when $f(x)$ is positive, then $F(x)$ represents the area under $f(x)$ over $[a,x]$



4-1: $F(x)$ is a function because the endpoint x is variable

To compute the derivative of $F(x)$ we notice that $F(x+h) - F(x)$ is the area of the small trapezoidal "sliver" shown below.



4-2: Sliver has area of approximately $f(x)\,h$

If $f(x)$ is continuous on $[a,b]$ and h is close to 0, then $f(x+h) \approx f(x)$, so that the area of the "sliver" is practically $f(x) \cdot h$ and thus

$$F(x+h) - F(x) \approx f(x)\,h$$

Dividing by h yields the difference quotient

$$\frac{F(x+h) - F(x)}{h} \approx f(x)$$

Applying the limit as h approaches 0 then yields

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)$$

That is, $F'(x) = f(x)$. Moreover, this result holds for all continuous f on $[a,b]$ and leads to the following theorem:

The Fundamental Theorem: If $f(x)$ is continuous on a neighborhood (p,q) of a, and if

$$F(x) = \int_a^x f(t)\,dt$$

then $F'(x) = f(x)$ for all x in (p,q).

The fundamental theorem in operator notation takes the form

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Rigorous proofs of the fundamental theorem are given in the exercises.

EXAMPLE 1 Find $F'(x)$ for $F(x) = \int_a^x \cos(t)\,dt$

Solution: The fundamental theorem tells us that

$$F'(x) = \frac{d}{dx}\int_a^x \cos(t)\,dt = \cos(x)$$

When the upper limit of integration is a function, we often apply the fundamental theorem in the chain rule form

$$\frac{d}{dx}\int_a^{\text{input}} f(t)\,dt = f(\text{input})\,\frac{d}{dx}(\text{input}) \tag{4.27}$$

EXAMPLE 2 Evaluate and simplify completely the following derivative:

$$\frac{d}{dx}\int_0^{\sin(x)} \frac{dt}{1-t^2}$$

Since the input is $\sin(x)$, we recognize that (4.27) is of the form

$$\frac{d}{dx}\int_0^{\text{input}} \frac{dt}{1-t^2} = \frac{1}{1-(\text{input})^2}\,\frac{d}{dx}(\text{input})$$

Replacing the input by $\sin(x)$ then results in

$$\frac{d}{dx}\int_0^{\sin(x)} \frac{dt}{1-t^2} = \frac{1}{1-\sin^2(x)}\,\frac{d}{dx}(\sin(x))$$

Since $1-\sin^2(x) = \cos^2(x)$ and since $\frac{d}{dx}\sin(x) = \cos(x)$, this simplifies to

$$\frac{d}{dx}\int_0^{\sin(x)} \frac{dt}{1-t^2} = \frac{1}{\cos^2(x)}\cos(x) = \frac{1}{\cos(x)} = \sec(x)$$

Check your Reading *If* $F(x) = \int_1^x \ln(t)\,dt$, *then what is* $F'(x)$?

Curve Sketching of Indefinite Integrals

The fundamental theorem says that if $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$. Since $f(x)$ is the derivative of $F(x)$, we say that $F(x)$ is an *antiderivative* of $f(x)$. That is, *an indefinite integral* of $f(x)$ is also known as *an antiderivative* of $f(x)$. In fact, if $F(x) = C + \int_a^x f(t)\,dt$, then $F'(x) = f(x)$ for any constant C, which shows that a function $f(x)$ has infinitely many antiderivatives. The fundamental theorem allows us to use monotonicity, concavity, and the fact that $F(a) = \int_a^a f(t)\,dt = 0$ to sketch the graph of an antiderivative of a given function $f(x)$.

EXAMPLE 3 Sketch the graph of the function

$$F(x) = \int_0^x \frac{t^2-1}{t^2+1}dt$$

Solution: We begin by noticing that the derivative is

$$F'(x) = \frac{x^2-1}{x^2+1}$$

The critical points occur when the numerator is equal to 0:

$$x^2 - 1 = 0, \qquad x = \pm 1$$

Since the denominator x^2+1 is always positive, $F'(x)$ can change signs only when $x = 1$ or -1. Thus, we need to test points the intervals $(-\infty,-1)$, $(-1,1)$ and $(1,\infty)$, respectively, to determine the monotonicity of $F(x)$:

$$F'(-2) = \frac{4-1}{4+1} = \frac{3}{5} > 0 \quad \Longrightarrow \quad F(x) \nearrow \text{ on } (-\infty,-1)$$
$$F'(0) = \frac{0-1}{0+1} = -1 < 0 \quad \Longrightarrow \quad F(x) \searrow \text{ on } (-1,1)$$
$$F'(2) = \frac{4-1}{4+1} = \frac{3}{5} > 0 \quad \Longrightarrow \quad F(x) \nearrow \text{ on } (1,\infty)$$

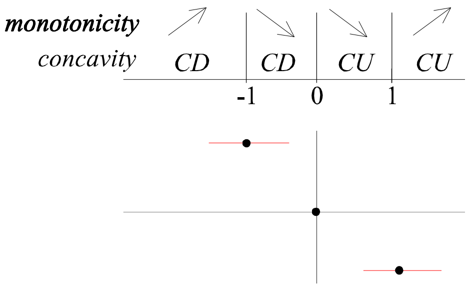
In order to incorporate concavity, we first must compute $F''(x)$:

$$\begin{aligned} F''(x) &= \frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right) \\ &= \frac{\left(x^2+1\right)\frac{d}{dx}\left(x^2-1\right) - \left(x^2-1\right)\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2} \\ &= \frac{\left(x^2+1\right)(2x) - \left(x^2-1\right)(2x)}{\left(x^2+1\right)^2} \\ &= \frac{4x}{\left(x^2+1\right)^2} \end{aligned}$$

Since $F''(x) = 0$ only when $x = 0$, we need only determine the sign of $F''(x)$ over $(-\infty,0)$ and $(0,\infty)$:

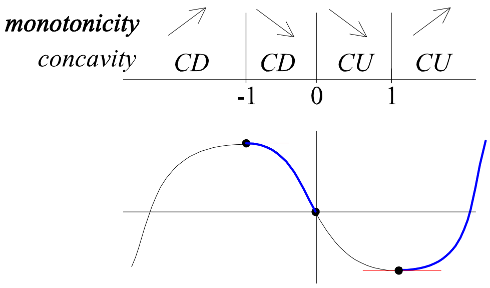
$$F''(-1) = \frac{-4}{(1+1)^2} = -1 > 0 \quad \Longrightarrow \quad F(x) \text{ CD on } (-\infty,0)$$
$$F''(1) = \frac{4}{(1+1)^2} = 1 < 0 \quad \Longrightarrow \quad F(x) \text{ CU on } (0,\infty)$$

In addition, $F(0) = \int_0^0 \frac{t^2-1}{t^2+1} dx = 0$, which leads to the diagram



4-3: Monotonicity and concavity of $F(x)$

Over $(-\infty, -1)$, the graph is increasing and concave down. Over $(-1, 0)$ the graph is decreasing and concave down. Over $(0, 1)$ the graph is decreasing and concave up. Over $(1, \infty)$ the graph is increasing and concave up.



4-4: Graph of $F(x)$

In addition, the graph reveals that $F(x)$ has an inflection point at $x = 0$.

Check your Reading *What led us to place a point at the origin?*

Position and Velocity

In the applications following the skill-building exercises, we consider a variety of uses of the fundamental theorem. For example, suppose that $r(t)$ is the height in feet of an object at time t in seconds. Since the velocity $v(t) = r'(t)$ is the derivative of the height, the height $r(t)$ is an *antiderivative* of the velocity. Thus, if we are given the velocity $v(t)$ and the height r_0 at time $t = 0$, then the position is given by

$$r(t) = r_0 + \int_0^t v(\tau)\, d\tau$$

where the dummy variable τ is the Greek letter "tau."

EXAMPLE 4 The atmosphere applies a drag force to an object with an initial altitude of 100 feet, so that its vertical velocity at time t is given by

$$v(t) = -4\left(\frac{e^{8t-8} - e^{-8t+8}}{e^{8t-8} + e^{-8t+8}}\right)$$

At what time does the object reach a maximum altitude?

Solution: The altitude is given by

$$r(t) = 100 - \int_0^t 4\left(\frac{e^{8\tau-8} - e^{-8\tau+8}}{e^{8\tau-8} + e^{-8\tau+8}}\right) d\tau$$

so that, as expected, the derivative of the altitude is

$$r'(t) = -\frac{d}{dt}\int_0^t 4\left(\frac{e^{8\tau-8} - e^{-8\tau+8}}{e^{8\tau-8} + e^{-8\tau+8}}\right) d\tau = -4\left(\frac{e^{8t-8} - e^{-8t+8}}{e^{8t-8} + e^{-8t+8}}\right)$$

The critical points occur when $r'(t) = 0$, which is when the numerator vanishes:

$$\begin{aligned} e^{8t-8} - e^{-8t+8} &= 0 \\ e^{8t-8} &= e^{-8t+8} \\ \left(e^{8t-8}\right)^2 &= 1 \\ e^{16t-16} &= e^0 \\ 16t - 16 &= 0 \\ t &= 1 \end{aligned}$$

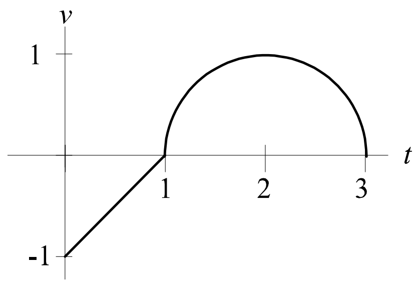
Thus, the stationary point is $t = 1$. Moreover, the second derivative is

$$r''(t) = -4\frac{d}{dt}\left(\frac{e^{8t-8} - e^{-8t+8}}{e^{8t-8} + e^{-8t+8}}\right) = \frac{-128}{\left(e^{8t-8} + e^{-8t+8}\right)^2}$$

which is always negative. Thus, $r(t)$ is concave down everywhere, so that the object reaches its maximum altitude after 1 second.

It is also common for the *graph* of the velocity to be known and used to determine the properties of the position of an object.

EXAMPLE 5 Suppose an object has an initial height of $r(0) = 0$ and a velocity at time t given by the graph in figure 4-5:



4-5: Graph of the velocity of the object

Assuming the curves in the figure are either straight lines or arcs of circles, do the following:

(a) Determine the height of the object at time $t = 1$ seconds.

(b) Determine the height of the object at time $t = 3$ seconds.

(c) Sketch the graph of the object's height as a function of time t over $[0,3]$.

Solution: The height of the object is given by

$$r(t) = \int_0^t v(\tau)\, d\tau$$

where $v(t)$ is as shown in figure 4-5. Thus, $r(1) = \int_0^1 v(\tau)\, d\tau$, which is the negative of the area of the triangle under the interval $[0,1]$ (The definite integral is negative because the integrand is negative). Thus,

$$r(1) = \int_0^1 v(\tau)\, d\tau = -\frac{1}{2} base \cdot height = -\frac{1}{2}$$

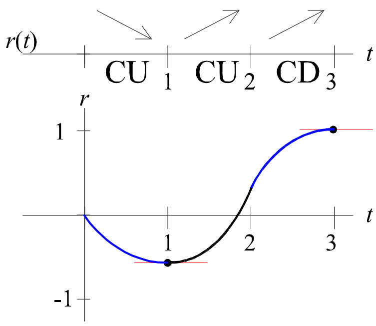
Similarly, $r(3) = \int_0^3 v(\tau)\, d\tau$, which we calculate by writing

$$r(3) = \int_0^1 v(\tau)\, d\tau + \int_1^3 v(\tau)\, d\tau = -\frac{1}{2} + \int_1^3 v(\tau)\, d\tau$$

Since the graph of $v(\tau)$ over $[1,3]$ is a semi-circle of radius 1, the area is $\frac{1}{2}\pi$ and thus

$$r(3) = -\frac{1}{2} + \int_1^3 v(\tau)\, d\tau = \frac{-1}{2} + \frac{\pi}{2} = 1.0708$$

Finally, notice that $r'(t)$ is negative over $[0,1]$ and positive over $[1,3]$, with critical points at 1 and 3. Also, $r'(t)$ is increasing on $[0,2]$ and decreasing over $[2,3]$, so that $r''(t)$ is positive over $[0,2]$ and negative over $[2,3]$. This leads us to the following graph:



4-6: Graph of height $r(t)$ at time t

Check your Reading *What function do we use the most when driving an automobile—velocity or position?*

Indefinite Integrals with Horizontal Asymptotes

A function defined by an indefinite integral may possibly have either one or two horizontal asymptotes. If it does, then monotonicity and concavity dictate how the asymptotes relate to the graph of the indefinite integral.

EXAMPLE 6 Sketch the graph of the function

$$F(x) = \int_0^x \frac{1}{t^2+1} dt$$

given that it has horizontal asymptotes of $y = \frac{-\pi}{2}$ and $y = \frac{\pi}{2}$.

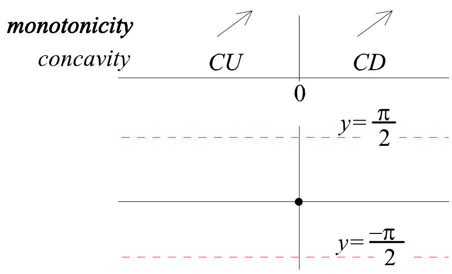
Solution: We begin by noticing that the derivative is

$$F'(x) = \frac{1}{x^2+1}$$

which is positive for all x. Thus $F(x)$ is increasing for all x. The second derivative is

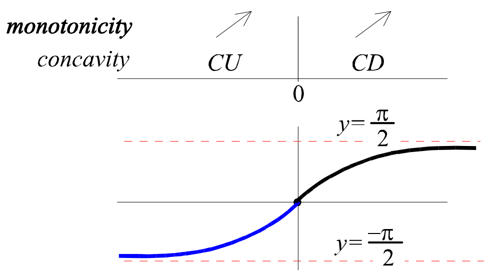
$$F''(x) = \frac{-2x}{(x^2+1)^2}$$

so that $F''(x) > 0$ on $(-\infty, 0)$ and $F''(x) < 0$ on $(0, \infty)$. This leads to the following diagram



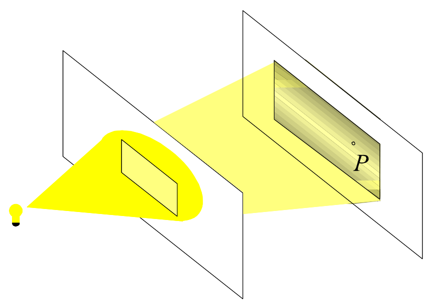
4-7: Monotonicity, Concavity, and Asymptotes of $F(x)$

Since $F(x)$ is concave up on $(-\infty, 0)$, it must be above the asymptote $y = \frac{-\pi}{2}$, and since it is concave down on $(0, \infty)$, it must be below the asymptote $y = \frac{\pi}{2}$:



4-8: Graph of $F(x)$

A special case of Fresnel diffraction in optics is to determine the intensity of light at a given point P in an *observation plane* after it has passed through a rectangular slit in a plane parallel to the observation plane.



4-9: Fresnel integrals measure flux densities near a point P

It can be shown that the flux density at P (flux density is similar to the concept of intensity) is proportional to sums and products of *Fresnel integrals*, which are integrals of the form

$$\mathcal{C}(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt \qquad \textit{and} \qquad \mathcal{S}(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right) dt$$

EXAMPLE 7 Sketch the graph of $\mathcal{C}(x)$ given that it has horizontal asymptotes of $y = \pm\frac{1}{2}$.

Solution: Since $\mathcal{C}'(x) = \cos\left(\frac{\pi}{2}x^2\right)$, the critical points are where $\cos\left(\frac{\pi}{2}x^2\right) = 0$, which is when

$$\frac{\pi}{2}x^2 = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots, \frac{(2n-1)\pi}{2}, \ldots$$

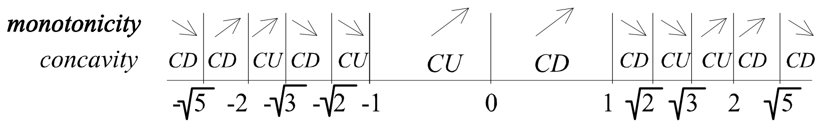
where n is an integer. Solving for x then leads to

$$\begin{aligned} x^2 &= 1, 3, 5, \ldots, 2n-1, \ldots \\ x &= \pm 1, \pm\sqrt{3}, \pm\sqrt{5}, \ldots, \pm\sqrt{2n-1}, \ldots \end{aligned}$$

If $-1 < x < 1$, then $\mathcal{C}'(x) > 0$, and the sign of $\mathcal{C}'(x)$ alternates about the critical points. Concavity then follows from the fact that $\mathcal{C}''(x) = -\pi x \sin\left(\frac{\pi}{2}x^2\right)$ is equal to 0 when

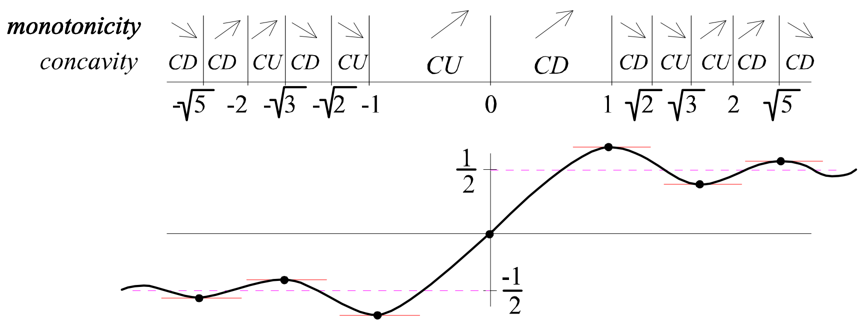
$$x = 0, \pm\sqrt{2}, \pm 2, \pm\sqrt{6}, \ldots, \pm\sqrt{2n}, \ldots$$

This leads to the following diagram:



4-10: Monotonicity and Concavity of $\mathcal{C}(x)$

Incorporating the horizontal asymptotes then leads to the graph below:



4-11: Graph of $\mathcal{C}(x)$

Exercises:

Find $F'(x)$ using the fundamental theorem and simplify completely.

1. $F(x) = \int_0^x 3t\,dt$	2. $F(x) = \int_1^x 3t\,dt$	3. $F(x) = \int_0^x \sqrt{t}\,dt$
4. $F(x) = \int_{-1}^x t^2\,dt$	5. $F(x) = \displaystyle\int_1^x \sin\left(\theta^2\right) d\theta$	6. $F(x) = \displaystyle\int_2^x e^{-t/5}\,dt$

7. $F(x) = \int_1^{e^{-x}} \frac{dt}{t}$ 8. $G(z) = \int_1^{x^2} \frac{dt}{t}$

9. $F(x) = \int_0^{\tan(x)} \frac{dt}{t^2+1}$ 10. $F(x) = \int_1^{\ln(x)} e^t dt$

11. $F(x) = \int_0^{\sin(x)} \frac{dt}{\sqrt{1-t^2}}$ 12. $F(x) = \int_0^{\cos(x)} \frac{1}{\sqrt{1-t^2}} dt$

Use monotonicity and concavity to sketch the graph of each function. Identify all extrema and points of inflection.

13. $F(x) = \int_0^x (2t-1)\, dt$ 14. $F(x) = \int_0^x (3-4t)\, dt$

15. $F(x) = \int_2^x \left(t^2-2t\right) dt$ 16 $F(x) = \int_{-2}^x \left(t^4-3t^2\right) dt$

17. $F(x) = \int_1^x \ln\left(t^2+2\right) dt$ 18. $F(x) = \int_1^x \ln\left(t^2+1\right) dt$

19. $F(x) = \int_1^x \frac{t}{t^2+1} dt$ 20. $F(x) = \int_4^x \frac{2-t}{t^2+1} dt$

21. $F(x) = \int_0^x \cos\left(t^2\right) dt$ 22. $F(x) = \int_0^x \sin\left(t^2\right) dt$

23. $F(x) = \int_1^x \frac{\sin(t)}{2-\cos(t)} dt$ 24. $\int_1^x \frac{\cos(t)}{2+\cos(t)} dt$

25. $F(x) = \int_1^x \left(t^2-1\right) e^{-t^2/3} dt$ 26. $F(x) = \int_1^x \left(t^2-4\right) e^{-t^2/5} dt$

27. The *error function* is defined by

$$\operatorname{erf}(x) = \int_0^x \frac{2}{\sqrt{\pi}} e^{-t^2} dt$$

and has two horizontal asymptotes of $y = \pm 1$. Sketch the graph of $\operatorname{erf}(x)$.

28. Sketch the graph of the function

$$\int_0^x \frac{t}{t^4+3} dt$$

given that it has a horizontal asymptote of $y = \frac{\pi}{4}$.

29. The function $F(x) = \int_0^x \sqrt[3]{t} e^{-t}\, dt$ has a horizontal asymptote of $y = 0.8930$ to 4 decimal places. Sketch the graph of $F(x)$.

30. The function $F(x) = \int_0^x e^{-t} \sin(t)\, dt$ has a horizontal asymptote of $y = \frac{1}{2}$. Sketch the graph of $F(x)$.

31. **An Exponential Distribution:** If the average wait at a certain restaurant is 5 minutes and waiting times are exponentially distributed, then the function

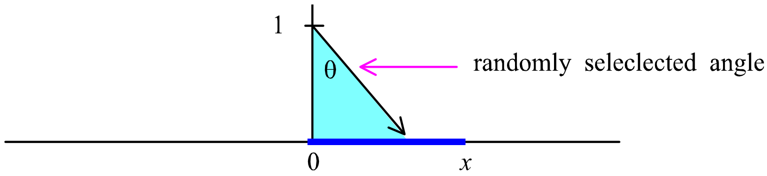
$$P(x) = \int_0^x 0.2e^{-0.2t} dt$$

represents the probability that a customer selected at random will have to wait between 0 and x minutes for service. Moreover, the probability of service approaches 1 as x approaches ∞, which is to say that

$$\lim_{x\to\infty} P(x) = 1 \tag{4.28}$$

Sketch the graph of $P(x)$ for $x > 0$ using the fundamental theorem.

32. **The Cauchy Distribution:** Suppose an angle θ is chosen at random from between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$, and then a line is drawn from the point $(0, 1)$ to the x-axis.



4-12: The Cauchy Distribution

For $x > 0$, the probability that the line will intersect the x-axis in the interval $[0, x]$ is given by

$$P(x) = \int_0^x \frac{1}{\pi(1+t^2)} dt$$

and for $x < 0$, the probability the line will intersect the x-axis in the interval $[x, 0]$ is $-P(x)$. Since there is a probability of $\frac{1}{2}$ of the line falling to the right of the y-axis, we have both

$$\lim_{x\to\infty} P(x) = \frac{1}{2} \qquad \textit{and} \qquad \lim_{x\to-\infty} P(x) = \frac{-1}{2}$$

Sketch the graph of $P(x)$.

33. Sketch the graph of

$$\mathcal{S}(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right) dt$$

given that it has horizontal asymptotes of $y = \pm\frac{1}{2}$.

34. The *cosine integral function* is defined to be

$$\mathrm{Ci}(x) = \int_1^x \frac{\cos(t)}{t} dt$$

It has a horizontal asymptote of $y = 0$ and a vertical asymptote of $x = 0$. Sketch its graph for $x > 0$ using monotonicity and asymptotes.(You might want to use the second derivative test, but intervals of concavity will be difficult to determine directly).

35. **Write to Learn:** It can be shown that the *sine integral function*

$$\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t} dt$$

has horizontal asymptotes of $y = \pm\frac{\pi}{2}$. Write a short essay explaining why the limit

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

implies that $\mathrm{Si}(x)$ does not have a critical point at the origin. Then sketch the graph of $\mathrm{Si}(x)$ using monotonicity and horizontal asymptotes. (You might want to use the second derivative test, but intervals of concavity will be difficult to determine directly).

36. Write to Learn: Write a short essay which uses the limit

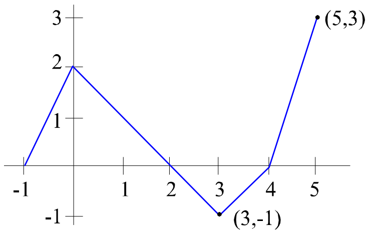
$$\lim_{x \to 1} \frac{\ln(x)}{1-x} = -1$$

to explain why the function

$$\operatorname{dilog}(x) = \int_1^x \frac{\ln(t)}{1-t} dt$$

has no asymptotes, no extrema and no inflection points.

37. Let $F(x) = \int_{-1}^{x} f(t)\, dt$, where f is the piecewise linear function whose graph is shown below:



4-13: Exercise 37

(a) Compute $F(0)$ and $F(2)$

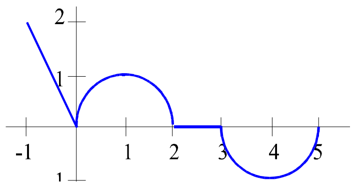
(b) Compute $F(5) - F(4)$. What does this difference represent?

(c) What is the instantaneous rate of change of F with respect to x when $x = 3$?

(d) What are the critical points of $F(x)$ over $[-1, 5]$?

(e) Sketch the graph of $F(x)$.

38. Let $r(t) = \int_0^t v(\tau)\, d\tau$, where v is graphed below.



4-14: Exercise 38

Assume that the arcs in the graph are arcs of circles.

(a) Compute $r(-1)$ and $r(2)$

(b) Compute $r(5)$.

(c) What is the instantaneous rate of change of r with respect to t when $x = 2.5$?

(d) What does the graph of $r(t)$ look like over $[2, 3]$?

(e) Sketch the graph of $r(t)$.

39. * The Extreme Value Theorem says that if $f(t)$ is continuous on $[x, x+h]$, then there are numbers u, v in $[x, x+h]$ such that

$$f(u) \leq f(t) \leq f(v)$$

for all t in $[x, x+h]$. Use this to prove the fundamental theorem by showing that

$$f(u) \leq \frac{1}{h} \int_x^{x+h} f(t)\, dt \leq f(v)$$

and then explaining why $f(u)$ and $f(v)$ approach $f(x)$ as h approaches 0. Explain why if $F(x) = \int_a^x f(t)\, dt$, then $F'(x) = f(x)$.

40. * **Write to Learn:** A function $f(x)$ is *uniformly* continuous on $[a, b]$ if for each $\varepsilon > 0$, there exists an $h > 0$ such that if x is in $(a, b-h)$, then

$$|f(t) - f(x)| < \varepsilon$$

for all t in $(x, x+h)$. Write a short essay in which you prove the fundamental theorem for uniformly continuous functions in the following four steps:

1. Let $F(x) = \int_a^x f(t)\, dt$ for some fixed x, and let $\varepsilon > 0$ be given. Suppose that h is chosen so that $x + h < b$ and such that t in $(x, x+h)$ implies that $|f(t) - f(x)| < \varepsilon$. Use the properties of the integral to show that

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\, dt \tag{4.29}$$

2. By letting $f(t) = f(x) + f(t) - f(x)$, show that

$$\int_x^{x+h} f(t)\, dt = hf(x) + \int_x^{x+h} [f(t) - f(x)]\, dt$$

(Hint: $f(x)$ is treated as a constant since t is the variable of integration).

3. Combine steps 1 and 2 to show that

$$\frac{F(x+h) - F(x)}{h} - f(x) = \frac{1}{h} \int_x^{x+h} [f(t) - f(x)]\, dt$$

4. Finally, use the property $\left| \int_c^d g(t)\, dt \right| \leq \int_c^d |g(t)|\, dt$ to show that

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| \leq \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|\, dt < \frac{1}{h} \int_x^{x+h} \varepsilon dt$$

which leads you to conclude that

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)$$

thus proving the fundamental theorem.

4.5 Antiderivatives

The Antiderivative

The fundamental theorem in the last section introduced us to one of the most important concepts in calculus, that of the *antiderivative.* In this section, we define, explore, and introduce a new notation for this very important concept.

In the last section, we saw that if $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$. As a result, we said that $F(x)$ is an *antiderivative* of $f(x)$. Let's now state this as a definition.

Definition 5.1 A function $F(x)$ is an *antiderivative* of $f(x)$ over the interval (p,q) if $F'(x) = f(x)$ for all x in (p,q).

For example, the function $F(x) = x^2$ is an antiderivative of $f(x) = 2x$ because

$$\frac{d}{dx}x^2 = 2x$$

Moreover, the function $G(x) = x^2 + 1$ is also an antiderivative of $f(x) = 2x$ because

$$\frac{d}{dx}\left(x^2+1\right) = 2x$$

Indeed, if $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ on (p,q) and if we define $h(x) = F(x) - G(x)$, then

$$h'(x) = F'(x) - G'(x) = f(x) - f(x) = 0$$

By theorem 1.2 in section 3.1, there is a number C such that $h(x) = C$ for all x in (p,q), thus implying that

$$F(x) - G(x) = C$$

That is, any two antiderivatives of $f(x)$ differ by a constant.

Theorem 5.2 Every antiderivative of $f(x)$ on (p,q) is of the form $F(x) + C$, where C is a constant and $F'(x) = f(x)$.

The symbol $\int f(x)\,dx$ is used to represent the set of all possible antiderivatives of a function $f(x)$. Thus, if $F(x)$ is an antiderivative of $f(x)$, then

$$\int f(x)\,dx = F(x) + C$$

since $\int f(x)\,dx$ denotes the set of all antiderivatives of $f(x)$.

For example, the antiderivatives of $2x$ are of the form $x^2 + C$. Thus, we write

$$\int 2x\,dx = x^2 + C \tag{4.30}$$

Likewise, the derivative $\frac{d}{dx}x^3 = 3x^2$ implies that antiderivatives of $3x^2$ are

$$\int 3x^2 dx = x^3 + C \tag{4.31}$$

It follows almost immediately that antiderivatives have the following properties:

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx \tag{4.32}$$

$$\int [f(x) - g(x)]\, dx = \int f(x)\, dx - \int g(x)\, dx$$

$$\int kf(x)\, dx = k \int f(x)\, dx \tag{4.33}$$

where k is a constant. These properties are very important in both applications and theory alike..

EXAMPLE 1 Evaluate the antiderivative

$$\int \left(2x + 3x^2\right) dx$$

Solution: The results (4.30) and (4.31) imply that

$$\int \left(2x + 3x^2\right) dx = \int 2x dx + \int 3x^2 dx = x^2 + x^3 + C \tag{4.34}$$

Notice also that only one constant is necessary, even though two antiderivatives were used to obtain the result.

Check your Reading *What is the derivative of $x^2 + x^3 + C$? How does this relate to (4.34)?*

Antiderivatives in Closed Form

Often we desire to evaluate $\int f(x)\, dx$ in closed form, where

Definition 5.3: An antiderivative $\int f(x)\, dx$ is said to be in *closed form* if it is expressed as a finite combination of elementary functions.

Rules for evaluating antiderivatives in closed form are obtained from derivative computations. Indeed, when $n \neq -1$, the power rule implies that

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + C\right) = \frac{1}{n+1}(n+1)\, x^{n+1-1} = x^n$$

This, in turn, implies our first rule for evaluating antiderivatives:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1 \tag{4.35}$$

EXAMPLE 2 Evaluate $\int x^2 dx$ in closed form.

Solution: Letting $n = 2$ in (4.35) yields

$$\int x^2 dx = \frac{x^{2+1}}{2+1} + C = \frac{x^3}{3} + C$$

which can be verified by applying the derivative to the result

$$\frac{d}{dx}\left(\frac{x^3}{3} + C\right) = \frac{1}{3}\frac{d}{dx}x^3 = \frac{1}{3}3x^2 = x^2$$

The rule (4.35) also works for nonintegral values of n.

EXAMPLE 3 Evaluate $\int \sqrt{x} dx$

Convert to numerical exponents when computing antiderivatives using (4.35).

Solution: Writing $\sqrt{x}$ as $x^{1/2}$ allows us to use (4.35):

$$\int \sqrt{x} dx = \int x^{1/2} dx = \frac{x^{1/2+1}}{1/2+1} + C = \frac{x^{3/2}}{3/2} + C = \frac{2}{3} x^{3/2} + C$$

EXAMPLE 4 Evaluate $\int \frac{1}{x^3} dx$.

Solution: Consider that $1/x^3$ is the same as x^{-3}, so that

$$\int \frac{1}{x^3} dx = \int x^{-3} dx = \frac{x^{-3+1}}{-3+1} + C = \frac{x^{-2}}{-2} + C \tag{4.36}$$

The antiderivative of a polynomial is evaluated using a combination of (4.32), (4.33) and (4.35) above.

EXAMPLE 5 Evaluate $\int \left(x^3 + 2x + 1\right) dx$

Solution: We use (4.32) and (4.33) to break the antiderivative of $x^3 + 2x + 1$ into three antiderivatives,

$$\int \left(x^3 + 2x + 1\right) dx = \int x^3 dx \ + 2 \int x dx + \int 1 dx$$

Since x is the same as x^1 and since 1 is the same as x^0, the result is

$$\begin{aligned} \int \left(x^3 + 2x + 1\right) dx &= \int x^3 dx \ + 2 \int x^1 dx + \int x^0 \, dx \\ &= \frac{x^{3+1}}{3+1} + 2 \frac{x^{1+1}}{1+1} + \frac{x^{0+1}}{0+1} + C \\ &= \frac{x^4}{4} + x^2 + x + C \end{aligned}$$

Check your work by applying $\frac{d}{dx}$ to the result.

To check our work, we apply the derivative operator to the result:

$$\frac{d}{dx}\left(\frac{x^4}{4} + x^2 + x + C\right) = \frac{4x^3}{4} + 2x + 1 = x^3 + 2x + 1$$

Check your Reading *Evaluate the antiderivative* $\int (x+1) \, dx$.

Logarithms, Exponentials, Sines and Cosines

The second rule for evaluating antiderivatives is that

$$\int \frac{1}{x} dx = \ln |x| + C \tag{4.37}$$

which complements the first rule by allowing us to evaluate the antiderivative of x^n when $n=-1$. To verify (4.37), we use the fact that $|x| = \sqrt{x^2}$:

$$\frac{d}{dx}\left(\ln|x| + C\right) = \frac{d}{dx}\ln\left(x^2\right)^{1/2} = \frac{1}{2}\frac{d}{dx}\ln\left(x^2\right)$$

The chain rule implies that

$$\frac{d}{dx}\left(\ln|x| + C\right) = \frac{1}{2}\frac{d}{dx}\ln\left(x^2\right) = \frac{1}{2}\frac{2x}{x^2} = \frac{1}{x}$$

thus confirming that (4.37) is true.

EXAMPLE 6 Evaluate $\int \frac{x^2+1}{x}dx$

Solution: We separate the integrand into two fractions:

$$\int \frac{x^2+1}{x}dx = \int \left(\frac{x^2}{x} + \frac{1}{x}\right) dx = \int \left(x + \frac{1}{x}\right) dx$$

We then evaluate using both the first *and* the second rules:

$$\begin{aligned} \int \frac{x^2+1}{x}dx &= \int \left(x + \frac{1}{x}\right) dx \\ &= \int x\,dx + \int \frac{1}{x}dx \\ &= \frac{x^2}{2} + \ln|x| + C \end{aligned}$$

We check our work by applying the derivative to the result:

$$\frac{d}{dx}\left(\frac{x^2}{2} + \ln|x| + C\right) = \frac{1}{2}\frac{d}{dx}x^2 + \frac{d}{dx}\ln|x| = x + \frac{1}{x} = \frac{x^2+1}{x}$$

Rules for evaluating antiderivatives of the exponential, sine and cosine functions follow from their derivatives. Indeed, if k is a nonzero constant, then

$\int e^{kx}dx = \frac{1}{k}e^{kx} + C$ because $\frac{d}{dx}\left(\frac{1}{k}e^{kx} + C\right) = e^{kx}$

$\int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + C$ because $\frac{d}{dx}\left(\frac{1}{k}\sin(kx) + C\right) = \cos(kx)$

$\int \sin(kx)\,dx = \frac{-1}{k}\cos(kx) + C$ because $\frac{d}{dx}\left(\frac{-1}{k}\cos(kx) + C\right) = \sin(kx)$

EXAMPLE 7 Evaluate $\int e^{2x}dx$.

Solution: The constant is $k=2$, so that

$$\int e^{2x}dx = \frac{1}{2}e^{2x} + C$$

Moreover, we can check our work with the derivative:

$$\frac{d}{dx}\left(\frac{1}{2}e^{2x}+C\right)=\frac{1}{2}e^{2x}\frac{d}{dx}2x=\frac{1}{2}\cdot 2e^{2x}=e^{2x}$$

EXAMPLE 8 Evaluate

$$\int \frac{dx}{\sqrt{e^x}}$$

Solution: Since $\sqrt{e^x}=e^{x/2}$ and $\frac{1}{e^x}=e^{-x}$, we have $k=-1/2$ and

$$\int \frac{dx}{\sqrt{e^x}}=\int e^{-x/2}dx=\frac{1}{-1/2}e^{-x/2}+C=-2e^{-x/2}+C$$

To check our work, we use the derivative:

$$\frac{d}{dx}\left(-2e^{-x/2}+C\right)=-2e^{-x/2}\left(\frac{-1}{2}\right)=e^{-x/2}=\frac{1}{\sqrt{e^x}}$$

EXAMPLE 9 Evaluate $\int \sin(2x)\,dx$ and $\int \cos(\pi x)\,dx$

Solution: In the first antiderivative, $k=2$ so that

$$\int \sin(2x)\,dx=\frac{-1}{2}\cos(2x)+C$$

In the second, $k=\pi$ so that

$$\int \cos(\pi x)=\frac{1}{\pi}\sin(\pi x)+C \tag{4.38}$$

Check your Reading *Verify (4.38) by showing that the derivative of* $\frac{1}{\pi}\sin(\pi x)+C$ *is* $\cos(\pi x)$.

The Remaining Trigonometric Functions

Antiderivative rules for the remaining trigonometric functions are given below:

$\int \sec^2(x)\,dx=\tan(x)+C$	$\int \sec(x)\tan(x)\,dx=\sec(x)+C$
$\int \csc^2(x)\,dx=-\cot(x)+C$	$\int \csc(x)\cot(x)\,dx=-\csc(x)+C$

They can be verified by differentiation.

EXAMPLE 10 Evaluate $\int (\sec x+\tan x)^2\,dx$

Solution: We begin by expanding the integrand:

$$\int (\sec x+\tan x)^2\,dx=\int \left[\sec^2(x)+2\sec(x)\tan(x)+\tan^2(x)\right]dx$$

Now we use the identity $\tan^2(x)=\sec^2(x)-1$ to write this as

$$\begin{aligned}\int (\sec x+\tan x)^2\,dx &= \int \left[\sec^2(x)+2\sec(x)\tan(x)+\sec^2(x)-1\right]dx\\ &= \int \left[2\sec^2(x)+2\sec(x)\tan(x)-1\right]dx\end{aligned}$$

Breaking the antiderivative into three antiderivatives yields

$$\begin{aligned}\int (\sec x + \tan x)^2 \, dx &= 2\int \sec^2(x) \, dx + 2\int \sec(x)\tan(x)\, dx - \int 1 dx \\ &= 2\tan(x) + 2\sec(x) - x + C\end{aligned}$$

which can be verified with differentiation.

Exercises:

Evaluate the following. In 1-20, check your work by differentiating:

1. $\int x^5 dx$
2. $\int x^5 dx$
3. $\int x^{2.14} dx$
4. $\int x^{\pi} dx$
5. $\int e^{3x} dx$
6. $\int e^{-2x} dx$
7. $\int \sin(3x)\, dx$
8. $\int \cos\left(\sqrt{2}x\right) dx$
9. $\int \left(x^3 + 2x^2 + 4x - 2\right) dx$
10. $\int \left(x^3 + 2x - 1\right) dx$
11. $\int (x + 2\sqrt{x})\, dx$
12. $\int (x + \sqrt{x})^2 \, dx$
13. $\displaystyle\int \frac{dx}{\sqrt{x}}$
14. $\displaystyle\int \frac{\sqrt{x}+1}{\sqrt{x}} dx$
15. $\displaystyle\int \frac{x^2 + x + 1}{x^2} dx$
16. $\displaystyle\int \frac{x^2 - 1}{x + 1} dx$
17. $\displaystyle\int \left(1 + \frac{2}{x}\right)^2 dx$
18. $\displaystyle\int \frac{1}{\sqrt{x}}\left(1 + \frac{2}{\sqrt{x}}\right) dx$
19. $\int \cos(t)\, dt$

20 $\int \cos(t)\tan(t)\, dt$

21. $\int (t + \sin(2t))\, dt$
22. $\int (e^x + \sin(3x))\, dx$
23. $\int (\sin(x) + \cos(x))^2 \, dx$
24. $\int (\sin(x) - \cos(x))^2 \, dx$
25. $\displaystyle\int \frac{\sec^2(z)}{1 + \tan^2(z)} dz$
26. $\displaystyle\int \frac{1 - \tan^2(x)}{1 + \tan^2(x)} dx$
27. $\int \left(2\cos^2(x) - 1\right) dt$
28. $\int \left(1 - 2\sin^2(t)\right) dt$
29. $\int \left(\sqrt{2}\cos(x) - \sec(x)\right)^2 dx$
30. $\int (\sec t - \tan t)^2 \, dt$
31. $\int (e^x + 1)^2 \, dx$
32. $\int (e^x + e^{-x})^2 \, dx$
33. $\displaystyle\int \frac{e^x + e^{-x}}{e^x} dx$
34. $\displaystyle\int \frac{e^x + e^{-x}}{e^x} dx$

35. Verify the third antiderivative rule by showing that

$$\frac{d}{dx}\left(\frac{1}{k}e^{kx} + C\right) = e^{kx}$$

36. Verify the fourth antiderivative rule by showing that

$$\frac{d}{dx}\left(\frac{1}{k}\sin(kx) + C\right) = \cos(kx)$$

37. Verify the fifth antiderivative rule by showing that

$$\frac{d}{dx}\left(-\frac{1}{k}\cos(kx)+C\right)=\sin(kx)$$

38. Compute the derivative of $F(x)=-\cot(x)$, and use it to justify

$$\int \csc^2(x)\,dx=-\cot(x)+C$$

39. Only one of the following can be evaluated in closed form. Which one is it?

(a) $\int e^{-x}\left(xe^x+1\right)\,dx$
(b) $\int e^x\sec(x)\,dx$
(c) $\int e^{-x^2}dx$

40. Only one of the following can be evaluated in closed form. Which one is it?

(a) $\int\sqrt{1+\cos(x)}\,dx$
(b) $\int\sqrt{1+\cos(2x)}\,dx$
(c) $\int\sqrt{1+\cos(3x)}\,dx$

41. An automobile travels at a constant speed of 60 miles per hour for one hour. At the beginning of the hour, the odometer reads 10,345.7 miles.

(a) Let $r(t)$ be the odometer reading at time t. Use simple logic to define $r(t)$ explicitly as a function of t.
(b) Explain why $r(t)=\int v(t)\,dt$, where $v(t)$ is the speed of the automobile.
(c) Evaluate the antiderivative in (b), and show that it produces the same function as in (a).

42. An automobile's speed is increasing by 5 miles per second. Initially, its speedometer reads 0 m.p.h. and its odometer reads 25,347.3 miles.

(a) Explain why the speedometer reading at time t is given by

$$v(t)=5t$$

What will the speedometer reading be after 10 seconds?
(b) Let $r(t)$ denote the speedometer reading at time t. Explain why

$$r(t)=\int v(t)\,dt$$

(c) Evaluate the antiderivative in (b). What will the odometer reading be after 10 seconds?

4.6 The Fundamental Theorem

Evaluation Form of the Fundamental Theorem

The fundamental theorem was discovered independently by both Gottfried Liebniz and Sir Isaac Newton at the end of the seventeenth century. In this section, we see that the theorem they discovered unified what had been since antiquity two separate pursuits in mathematics—the study of tangent lines and the computation of areas.

Suppose that $F(x)$ is any antiderivative of a continuous function $f(x)$ over $[a,b]$. Since $F'(x) = f(x)$, the antiderivative $F(x)$ is continuous on $[a,b]$ and differentiable on (a,b). Thus, for each $h > 0$, let us let $\{x_j, c_j\}$ be an h-fine tagged partition of $[a,b]$ where the tags c_j satisfy the Mean Value theorem for $F(x)$. That is, on each $[x_{j-1}, x_j]$, we have

$$F(x_j) - F(x_{j-1}) = F'(c_j)(x_j - x_{j-1})$$

Since $F'(c_j) = f(c_j)$ and $x_j - x_{j-1} = \Delta x_j$, solving for $F(x_j)$ leads to

$$F(x_j) = F(x_{j-1}) + f(c_j)\Delta x_j \tag{4.39}$$

Beginning with $x_n = b$, the formula (4.39) implies that

$$F(x_n) = F(x_{n-1}) + f(c_n)\Delta x_n$$

Since $F(x_{n-1}) = F(x_{n-2}) + f(c_{n-1})\Delta x_{n-1}$, this in turn implies that

$$F(x_n) = F(x_{n-2}) + f(c_{n-1})\Delta x_{n-1} + f(c_n)\Delta x_n$$

and if we continue this proces for n steps, we obtain

$$\begin{aligned} F(x_n) &= F(x_{n-1}) + f(c_n)\Delta x_n \\ &= F(x_{n-2}) + f(c_{n-1})\Delta x_{n-1} + f(c_n)\Delta x_n \\ &= \qquad \vdots \qquad \vdots \\ &= F(x_0) + f(c_1)\Delta x_1 + \ldots + f(c_{n-1})\Delta x_{n-1} + f(c_n)\Delta x_n \end{aligned}$$

However, $F(x_n) = F(b)$ and $F(x_0) = F(a)$, so that we have

$$F(b) - F(a) = f(c_1)\Delta x_1 + \ldots + f(c_n)\Delta x_n$$

Application of the limit as h approaches 0 then yields

$$F(b) - F(a) = \lim_{h\to 0} \sum_{j=1}^{n} f(c_j)\Delta x_j$$

Since the limit of the Riemann sum converges to $\int_a^b f(x)\,dx$, the result is the *evaluation* form of the *fundamental theorem of calculus*:

> **Fundamental Theorem:** If $F(x)$ is *any* antiderivative of a continuous function $f(x)$ on $[a,b]$, then
>
> $$\int_a^b f(x)\,dx = F(b) - F(a) \tag{4.40}$$

A more rigorous proof is included as an exercise.

EXAMPLE 1 Estimate $\int_0^1 x^2dx$ with the midpoint method, and then compare to the result obtained with the Fundmental theorem.

Solution: The midpoint approximation applied to $\int_0^1 x^2dx$ is of the form

j	x_{j-1}	x_j	t_j	$f(t_j)$	Δx_j	$f(t_j)\Delta x_j$
1	0.0	0.2	0.1	$(0.1)^2 = 0.01$	0.2	$0.01 \cdot 0.2 = 0.002$
2	0.2	0.4	0.3	$(0.3)^2 = 0.09$	0.2	$0.09 \cdot 0.2 = 0.018$
3	0.4	0.6	0.5	$(0.5)^2 = 0.25$	0.2	$0.25 \cdot 0.2 = 0.050$
4	0.6	0.8	0.7	$(0.7)^2 = 0.49$	0.2	$0.49 \cdot 0.2 = 0.098$
5	0.8	1.0	0.9	$(0.9)^2 = 0.81$	0.2	$0.81 \cdot 0.2 = 0.162$
						$\int_0^1 x^2dx \approx 0.33$

Alternatively, if we let $F(x) = \int x^2dx$, then

$$F(x) = \int x^2dx = \frac{x^3}{3} + C$$

Thus, over $[0,1]$ the difference $F(b) - F(a)$ is

$$F(1) - F(0) = \left(\frac{1}{3} + C\right) - \left(\frac{0}{3} + C\right) = \frac{1}{3}$$

so that the fundamental theorem implies an exact value of

$$\int_0^1 x^2dx = F(1) - F(0) = \frac{1}{3}$$

Check your Reading *Evaluate* $\int_0^2 x^2dx$.

Area and a New Notation

To facilitate the use of the fundamental theorem (4.40), let us define the new notation

$$F(x)|_a^b = F(b) - F(a)$$

As a result, the evaluation theorem (4.40) can be written as

$$\int_a^b f(x)\,dx = \int f(x)\,dx\,\Big|_a^b \tag{4.41}$$

where $\int f(x)\,dx$ is the familiar antiderivative of $f(x)$ studied in the last chapter. Moreover, (4.41) gives us an algorithm for evaluating definite integrals—evaluate the antiderivative $\int f(x)\,dx$ and then compute the difference implied by $|_a^b$.

EXAMPLE 2 Evaluate $\int_1^3 x^3\,dx$ using the fundamental theorem

Solution: We use (4.41) and the fact that $\frac{x^4}{4}$ is an antiderivative of x^3:

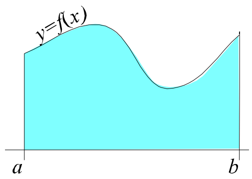
$$\int_1^3 x^3\,dx = \int x^3dx\Big|_1^3 = \frac{x^4}{4}\Big|_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = 20$$

We use the symbol $\int$ for both antiderivatives and definite integrals because it allows us to omit the "antiderivative" step when calculating definite integrals with the Evaluation theorem. That is, example 2 could have read

$$\int_1^3 x^3\,dx = \left.\frac{x^4}{4}\right|_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = 20$$

Recall that if $f(x) \geq 0$ for all x in $[a,b]$, then the area under $y = f(x)$ over $[a,b]$ is

$$Area = \int_a^b f(x)\,dx$$



Thus, if $\int f(x)\,dx$ can be determined in closed form, then the area of the region under $y = f(x)$ and over $[a,b]$ can be determined exactly with the fundamental theorem.

EXAMPLE 3 Find the area of the region under the graph of $f(x) = 2x$ and over $[0,3]$.

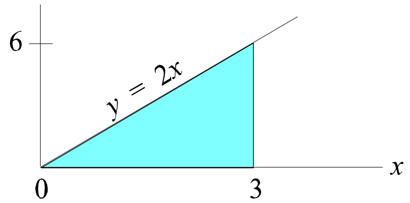
Solution: Since $f(x) = 2x$ is non-negative over $[0,3]$, the region under the graph of $f(x) = 2x$ and over $[0,3]$ is

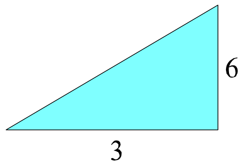
$$A = \int_0^3 2x\,dx$$

As a result, the fundamental theorem implies that

$$\int_0^3 2x\,dx = x^2\big|_0^3 = 3^2 - 0^2 = 9$$

To confirm the result in example 3, let us notice that the region under $f(x) = 2x$ over the interval $[0,3]$ is a triangle with height 6 and width 3,



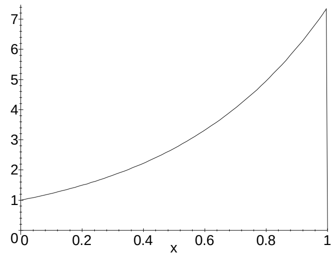


Thus, it has an area of $\frac{1}{2}(6)(3) = 9$.

EXAMPLE 4 Find the area of the region under $y = e^{2x}$ over the interval $[0,1]$.

Solution: Since $e^{2x} > 0$ over $[0, 1]$, the area of the region is

$$Area = \int_0^1 e^{2x} dx$$



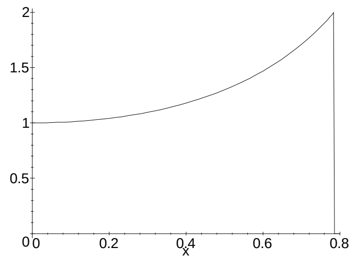
To find the area, we use the basic form $\int e^{kx} dx = \frac{1}{k} e^{kt} + C$ to obtain

$$Area = \int_0^1 e^{2x} dx = \frac{1}{2} e^{2x} \Big|_0^1 = \frac{1}{2} e^2 - \frac{1}{2} e^0 = \frac{1}{2} \left(e^2 - 1 \right)$$

EXAMPLE 5 Find the area of the region under $y = \sec^2(x)$ over $\left[0, \frac{\pi}{4}\right]$:

Solution: Since $\sec^2(x) \geq 0$ on $\left[0, \frac{\pi}{4}\right]$, the area of the region is

$$Area = \int_0^{\pi/4} \sec^2(x)\, dx$$



To do so, we use the basic form $\int \sec^2(x)\, dx = \tan(x) + C$:

$$Area = \int_0^{\pi/4} \sec^2(x)\, dx = \tan(x)|_0^{\pi/4} = \tan\left(\frac{\pi}{4}\right) - \tan(0) = 1$$

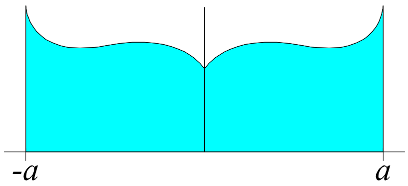
Check your Reading *Why can we drop the "+C" when we use the fundamental theorem? (Hint: read the statement of the fundamental theorem again)*

Even Symmetry and the Fundamental Theorem

The use of symmetry can simplify the computation of area. In particular, a function $f(x)$ is said to be *even* if $f(-x) = f(x)$. For example, $f(x) = x^2$ is even since

$$f(-x) = (-x)^2 = x^2 = f(x)$$

The graph of an even function is symmetric about the y-axis.



6-1: Graph of an even function

Thus, if $f(x)$ is positive, then the area under $y = f(x)$ over $[-a, a]$ is twice the area under $y = f(x)$ over $[0, a]$. That is, if $f(x)$ is even, then

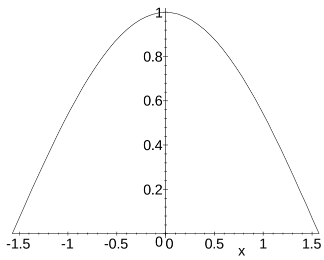
$$\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx \tag{4.42}$$

Moreover, it can be shown that (4.42) holds in general (see the exercises).

EXAMPLE 6 Find the area of the region under the curve $y = \cos(x)$ over the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Solution: To do so, we must evaluate the integral

$$Area = \int_{-\pi/2}^{\pi/2} \cos(x)\, dx$$



Since the region is symmetric about the y-axis—that is, since $f(x) = \cos(x)$ is an even function—the area of the entire region is double that of the region over $\left[0, \frac{\pi}{2}\right]$:

$$Area = 2\int_{0}^{\pi/2} \cos(x)\, dx = 2\sin(x)|_{0}^{\pi/2} = 2\sin\left(\frac{\pi}{2}\right) - 2\sin(0) = 2$$

EXAMPLE 7 Evaluate the integral $\int_{-2}^{2} |x|\, dx$

Solution: Notice that we cannot directly evaluate the integral

$$\int_{-2}^{2} |x|\, dx$$

However, the integrand $f(x) = |x|$ is even since

$$f(-x) = |-x| = |x| = f(x)$$

As a result, (4.42) implies that

$$\int_{-2}^{2} |x|\, dx = 2\int_{0}^{2} |x|\, dx$$

Since $|x| = x$ when $x \geq 0$, we thus have

$$\int_{-2}^{2} |x|\, dx = 2\int_{0}^{2} x\, dx = 2\left(\left. \frac{x^2}{2} \right|_0^2 \right) = 4 \qquad (4.43)$$

Check your Reading *Is $f(x) = \sec(x)$ an even function?*

Definite Integrals of Odd Functions

A function $f(x)$ is said to be *odd* if $f(-x) = -f(x)$ for all x in *dom* (f). For example, $f(x) = x^3$ is odd since

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

Likewise, $f(x) = \sin(x)$ is odd since $\sin(-x) = -\sin(x)$.

Suppose now that $F(x)$ is an antiderivative of a continuous odd function $f(x)$. Then the derivative of $F(-x)$ is

$$\frac{d}{dx}F(-x) = F'(-x)\ \frac{d}{dx}(-x) = -f(-x)$$

Since $f(-x) = -f(x)$, this means that $\frac{d}{dx}F(-x) = f(x)$. However, $f(x) = \frac{d}{dx}F(x)$, so that

$$\frac{d}{dx}F(-x) = \frac{d}{dx}F(x)$$

Thus, there is a constant C such that $F(-x) = F(x) + C$, but because $x = 0$ implies that $F(0) = F(0) + C$, it follows that $C = 0$.

Consequently, we have shown that if $f(x)$ is a continuous odd function and $F(x)$ is an antiderivative of $f(x)$, then

$$F(-x) = F(x)$$

That is, $F(x)$ is an *even* function. Consequently,

$$\int_{-a}^{a} f(x)\, dx = F(a) - F(-a) = 0$$

since $F(-a) = F(a)$. Thus, if $f(x)$ is an odd function, then

$$\int_{-a}^{a} f(x)\, dx = 0 \qquad (4.44)$$

The identity (4.44) is very useful because it holds even when the antiderivative of $f(x)$ cannot be evaluated in closed form.

EXAMPLE 8 Evaluate $\int_{-100}^{100} x^3 \sin(x^2)\, dx$

Solution: If $f(x) = x^3 \sin(x^2)$, then

$$f(-x) = (-x)^3 \sin\left((-x)^2\right) = -x^3 \sin(x^2) = -f(x)$$

Thus, $f(x)$ is an odd function and (4.44) implies that

$$\int_{-100}^{100} x^3 \sin(x^2)\, dx = 0$$

even though $\int x^3 \sin(x^2)\, dx$ cannot be evaluated in closed form.

Exercises:

Evaluate the following definite integrals. You may want to use symmetry and/or the fact that the definite integral is the area under a curve when the integrand is positive.

1. $\int_0^1 3x^2 dx$
2. $\int_0^2 4x^3 dx$
3. $\int_1^3 \left(2x^3 - 6x\right) dx$
4. $\int_{-1}^2 5x^5 dx$
5. $\int_0^2 \sin(\pi x) dx$
6. $\int_{-\pi}^{\pi} \cos\left(3x\right) dx$
7. $\int_1^5 \frac{3}{x^2} dx$
8. $\int_1^5 \frac{1}{x} dx$
9. $\int_{-1}^1 |x| \, dx$
10. $\displaystyle\int_1^9 \frac{dx}{\sqrt{x}}$
11. $\int_0^{\ln(2)} e^{3x} dx$
12. $\int_{0.5}^{1.5} e^{7x} dx$

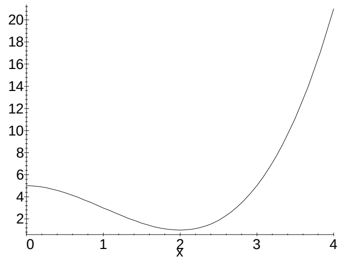
13. $\displaystyle\int_{-1}^1 \cos\left(\frac{\pi x}{2}\right) dx$
14. $\int_{-2}^2 \left(4x^2 - \cos\left(\pi x\right)\right) dx$
15. $\int_0^1 (e^x + e^{-x}) dx$
16. $\int_0^1 (e^x + e^{-x})^2 dx$
17. $\displaystyle\int_{-1}^1 \sin\left(x^3\right) dx$
18. $\int_0^{\pi/4} [\cos^2(x) - \sin^2(x)] dx$
19. $\displaystyle\int_1^3 \frac{x^2+1}{x} dx$
20. $\displaystyle\int_1^4 \frac{\sqrt{x} + x^{3/2}}{\sqrt{x}} dx$
21. $\int_0^{\pi} 2\sin\left(x\right)\cos\left(x\right) dx$
22. $\int_{-\pi/4}^{\pi/4} 2x \sec^2(x) dx$
23. $\int_{-\pi/4}^{\pi/4} \sin\left(x\right) \sec^2(x) dx$
24. $\int_0^{\pi/4} \sin^2\left(x\right) \sec^2(x) dx$

Find the area of each region described below using the evaluation theorem.

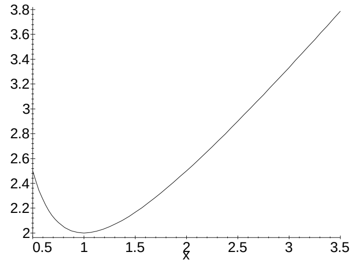
25. under $y = x$ over $[0, 1]$
26. under $y = x^2$ over $[0, 2]$
27. under $y = \sin\left(x\right)$ over $[0, \pi]$
28. under $y = e^{3x}$ over $[0, 1]$
29. under $y = |x|$ over $[0, 1]$
30. under $y = |x|$ over $[-1, 1]$

On the graph of the integrand below each integral, shade the area represented by *the definite integral. Then find the area of the shaded region.*

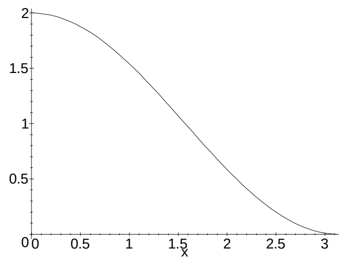
31. $\int_1^{3.5} \left(x^3 - 3x^2 + 5\right)\, dx$



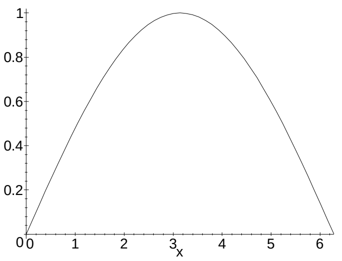
32. $\displaystyle\int_0^{2.5} \left(x + \frac{1}{x}\right)\, dx$



33. $\int_0^{\pi} (\cos(x) + 1)\, dx$



34. $\int_0^{2\pi} \sin(x/2)\, dx$



35. In this exercise, we consider the indefinite integral

$$F(x) = \int_0^x e^{-t} dt$$

(a) What is $F(x)$ in closed form?
(b) What is $F'(x)$? Is this what you expected?

36. In this exercise, we consider the indefinite integral

$$F(x) = \int_0^{\sqrt{x}} t dt$$

(a) What is $F(x)$ in closed form?
(b) What is $F'(x)$? Is this what you expected?

37. In this exercise, we examine the integral

$$\int_{-1}^{1} x^3 dx$$

(a) **Numerical:** Complete the midpoint approximation below

j	x_{j-1}	x_j	t_j	$f(t_j)$	Δx_j	$f(t_j)\Delta x_j$
1	-1.0	-0.5				
2	-0.5	0.0				
3	0.0	0.5				
4	0.5	1.0				

$\int_{-1}^{1} x^3 dx \approx$???

(b) Evaluate $\int_{-1}^{1} x^3 dx$ using the fundamental theorem. How close is the actual value to the approximation in (a)?

38. In this exercise, we examine the integral

$$\int_0^{0.5} \sqrt{x} dx$$

(a) **Numerical:** Complete the midpoint approximation below

j	x_{j-1}	x_j	t_j	$f(t_j)$	Δx_j	$f(t_j)\Delta x_j$
1	0	0.1				
2	0.1	0.2				
3	0.2	0.3				
4	0.3	0.4				
5	0.4	0.5				
					$\int_0^{0.5} \sqrt{x} dx \approx$	???

(b) Evaluate $\int_0^{0.5} \sqrt{x} dx$ using the fundamental theorem. How close is the actual value to the approximation in (a)?

39. In this exercise, we estimate the value of the definite integral

$$\int_0^1 e^{-x^2} dx \tag{4.45}$$

(a) Graph $f(x) = 1 - x^2 + \frac{1}{2}x^4$ and $g(x) = e^{-x^2}$ over the interval $[0, 1]$. Which is bigger?

(b) Estimate (4.45) by evaluating the integral $\int_0^1 \left(1 - x^2 + \frac{1}{2}x^4\right) dx$. Is the estimate above or below the actual value?

40. In this exercise, we estimate

$$\int_0^{\pi} \frac{\sin(x)}{x} dx \tag{4.46}$$

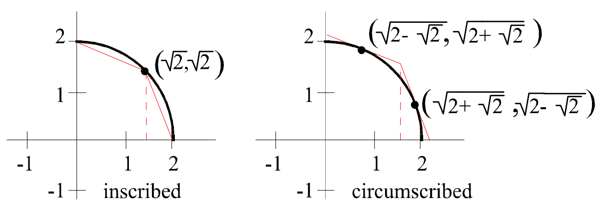
(a) Graph $p(x) = \cos(x/2)$, $q(x) = 1 - x/\pi$ and

$$r(x) = \frac{\sin(x)}{x}$$

over the interval $[0, \pi]$. Arrange them in ascending order.

(b) Evaluate $\int_0^{\pi} \cos(x/2)\, dx$ and $\int_0^{\pi} (1 - x/\pi)\, dx$ to produce a lower and an upper estimate of (4.46).

41. Write to Learn: The ancient Greek mathematician Archimedes estimated π by using inscribed and circumscribed polygons to approximate circles and then computing their areas. In a short essay, revisit Archimedes' work by explaining why the area of the region under $y = \sqrt{4 - x^2}$ and over $[0, 2]$ is π. Then find the areas of the inscribed and circumscribed approximations shown below:



6-2: Approximation of the area of a quarter circle

You will need to use a property of the tangent line to a circle in the circumscribed case.

42. **Write to Learn:** Write a short essay which uses the following steps to present a more rigorous proof of the fundamental theorem (i.e., one that allows tags to be chosen arbitrarily). Suppose that $f(x)$ is continuous over $[a, b]$ and that $F(x)$ is an antiderivative of $f(x)$. Let $\varepsilon > 0$ and suppose that $\{x_j, t_j\}$ is an h-fine partition of $[a, b]$ where h is small enough that

$$|f(x) - f(t_j)| < \frac{\varepsilon}{b-a}$$

for all x in $[x_{j-1}, x_j]$, and suppose this holds for each $j = 1, \ldots, n$.

(a) For each $j = 1, \ldots, n$, let c_j be the point in (x_{j-1}, x_j) for which

$$F(x_j) - F(x_{j-1}) = f(c_j)(x_j - x_{j-1})$$

Mimic the derivation at the beginning of the section to show that

$$F(b) - F(a) = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \ldots + f(c_n)\Delta x_n$$

(b) Show that $F(b) - F(a) - (f(t_1)\Delta x_1 + f(t_2)\Delta x_2 + \ldots + f(t_n)\Delta x_n)$ simplifies to

$$[f(c_1) - f(t_1)]\Delta x_1 + [f(c_2) - f(t_2)]\Delta x_2 + \ldots + [f(c_n) - f(t_n)]\Delta x_n$$

and then use the triangle inequality to show that

$$|F(b) - F(a) - [f(t_1)\Delta x_1 + f(t_2)\Delta x_2 + \ldots + f(t_n)\Delta x_n]| < \varepsilon$$

(c) Use (b) to explain why

$$F(b) - F(a) = \lim_{h \to 0} [f(t_1)\Delta x_1 + f(t_2)\Delta x_2 + \ldots + f(t_n)\Delta x_n]$$

and then explain what the limit represents.

4.7 Substitution

Substitution

Memorize these rules. We will use them often.

We currently have the following basic formulas, where $n \neq 1$ and $k \neq 0$.

1. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$
2. $\int \dfrac{1}{x} dx = \int x^{-1} dx = \ln|x| + C$
3. $\int e^{kx} dx = \frac{1}{k} e^{kx} + C$
4. $\int \cos(kx)\, dx = \frac{1}{k}\sin(kx) + C$
5. $\int \sin(kx)\, dx = -\frac{1}{k}\cos(kx) + C$
6. $\int \sec^2(x)\, dx = \tan(x) + C$
7. $\int \sec(x)\tan(x)\, dx = \sec(x) + C$
8. $\int \csc^2(x)\, dx = -\cot(x) + C$
9. $\int \csc(x)\cot(x)\, dx = -\csc(x) + C$

In addition, we also develop *techniques of integration* to transform a given antiderivative into one of these nine forms. In this section, we introduce the most important of these techniques—the method of substitution.

If $F(x)$ is an antiderivative of $f(x)$ and if $g(x)$ is differentiable, then

$$\frac{d}{dx}F[g(x)] = F'[g(x)]\,g'(x) = f[g(x)]\,g'(x) \tag{4.47}$$

The *method of substitution* is the antiderivative form of (4.47), which is

$$\int f(g(x))\,g'(x)\,dx = F(g(x)) + C \tag{4.48}$$

However, when we use (4.48), we do so by employing the following algorithm:

i. Let $u = g(x)$
ii. Form the differential $du = g'(x)\,dx$ by computing $\frac{du}{dx} = g'(x)$
iii. Replace $g(x)$ by u and replace $g'(x)\,dx$ by du
iv. Evaluate the antiderivative $\int f(u)\,du$ in the variable u
v. Replace u by $g(x)$ in the result of the antidifferentiation

EXAMPLE 1 Evaluate the antiderivative

$$\int \cos\left(x^2\right)\,2xdx \tag{4.49}$$

Solution: First, we let $u = x^2$. Second, we differentiate to obtain $\frac{du}{dx} = 2x$, which yields our differential

$$du = 2xdx$$

Third, we substitute $u = x^2$ and $du = 2xdx$ into (4.49).

$$\int \cos\left(x^2\right)\,2xdx = \int \cos(u)\,du$$

The resulting antiderivative is in one of the basic forms, so fourthly, we evaluate

$$\int \cos(u)\,du = \sin(u) + C$$

and finally, we replace u by x^2 to obtain the final result:

$$\int \cos\left(x^2\right)\,2xdx = \sin\left(x^2\right) + C$$

To check our work, we apply the derivative:

$$\frac{d}{dx}\left(\sin\left(x^2\right) + C\right) = \cos\left(x^2\right)\frac{d}{dx}x^2 = \cos\left(x^2\right)\,2x$$

Notice that the chain rule was used in checking our work. This is because substitution is equivalent to "running the chain rule in reverse."

Check your Reading *Use the substitution $u = x^2$ to evaluate*

$$\int e^{x^2}\,2xdx$$

Substitutions with a Constant Multiplier

In many problems, we must replace a differential of the form $g'(x)\,dx$ by a differential of the form kdu, where k is a constant. In doing so, however, it is important to keep the x terms with their differential dx and the u terms with du.

EXAMPLE 2 Evaluate

$$\int \sec^2\left(x^2\right)\ xdx \tag{4.50}$$

Solution: We let $u = x^2$. Since the derivative of x^2 is $2x$, the differential is $du = 2xdx$. We then divide both sides of the differential by 2 to isolate the expression xdx:

$$\frac{1}{2}du = xdx$$

We replace x^2 by u and xdx by $\frac{1}{2}du$ so that (4.50) becomes

$$\int \sec^2\left(x^2\right) xdx = \int \sec^2(u)\,\frac{1}{2}du = \frac{1}{2}\tan(u) + C$$

Converting back to the original x variable yields the final result

$$\int \sec^2\left(x^2\right)\ xdx = \frac{1}{2}\tan\left(x^2\right) + C$$

Now we check our work with the derivative:

$$\frac{d}{dx}\left(\frac{1}{2}\tan\left(x^2\right) + C\right) = \frac{1}{2}\sec^2\left(x^2\right)\frac{d}{dx}x^2 = \frac{1}{2}\sec\left(x^2\right)\ 2x = \sec^2\left(x^2\right)\ x$$

Changing roots into fractional powers often makes substitution more accessible.

When evaluating antiderivatives which involve radicals, it is easy to become "paralyzed by the notation." To overcome this paralysis, we often convert radicals into numerical exponents.

EXAMPLE 3 Evaluate

$$\int \frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}}dx \tag{4.51}$$

Solution: Although (4.51) may be intimidating, converting the roots to fractional powers leads to a more accessible problem:

$$\int \frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}}dx = \int \sin\left(x^{1/2}\right)\ x^{-1/2}dx$$

If we let $u = x^{1/2}$, then $du = \frac{1}{2}x^{-1/2}dx$, which is the same as $2du = x^{-1/2}dx$. Thus, (4.51) becomes

$$\int \sin\left(x^{1/2}\right)\ x^{-1/2}dx = \int \sin(u)\ 2du = -2\cos(u) + C$$

after which replacing u by $x^{1/2}$ yields

$$\int \frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}}dx = -2\cos\left(\sqrt{x}\right) + C \tag{4.52}$$

Application of the derivative confirms our result.

Check your Reading *Where did the negative sign in (4.52) come from?*

Guidelines for choosing u

The key to the substitution algorithm is to choose a u which also allows the substitution of du. Moreover, there are some guidelines which may aid in the selection of a such a substitution. To begin with, when an integrand contains a fraction, u is often all or some part of the denominator.

EXAMPLE 4 Evaluate $\int \frac{dx}{(x+2)^2}$

Solution: Our guideline motivates us to let $u = x+2$. Since $du = dx$, we have

$$\int \frac{dx}{(x+2)^2} = \int \frac{du}{u^2} = \int u^{-2} du = \frac{u^{-1}}{-1} + C = \frac{-1}{x+2} + C$$

which should now be checked with a derivative.

Indeed, since many applications involve percentage rates of change, it is not uncommon for u to be the entire denominator and thus for the antiderivative to involve logarithms.

EXAMPLE 5 Evaluate the antiderivative

$$\int \frac{\sin(x)}{2-\cos(x)} dx \tag{4.53}$$

Solution: Our guideline motivates us to $u = 2 - \cos(x)$. As a result, $du = \sin(x)\, dx$ and (4.53) is transformed into

$$\begin{aligned} \int \frac{\sin(x)\, dx}{2-\cos(x)} &= \int \frac{du}{u} \\ &= \ln|u| + C \\ &= \ln|2-\cos(x)| + C \end{aligned}$$

In addition, there is often more than one choice of u which will simplify the antiderivative to a basic form. For example, let's evaluate $\int \tan(x)\sec^2(x)\, dx$ using the substitution $u = \tan(x)$ and also the substitution $u = \sec(x)$.

EXAMPLE 6 Evaluate $\int \tan(x)\sec^2(x)\, dx$ using $u = \tan(x)$.

Solution: If $u = \tan(x)$, then $du = \sec^2(x)\, dx$. Thus

$$\int \tan(x)\ \sec^2(x)\, dx = \int u\, du = \frac{u^2}{2} + C$$

which implies that

$$\int \tan(x)\ \sec^2(x)\,dx = \frac{1}{2}\tan^2(x) + C \tag{4.54}$$

EXAMPLE 7 Evaluate $\int \tan(x)\sec^2(x)\,dx$ using $u = \sec(x)$.

Solution: To do so, we rewrite $\int \tan(x)\sec^2(x)\,dx$ in the form

$$\int \tan(x)\sec^2(x)\,dx = \int \sec(x)\ \sec(x)\tan(x)\,dx$$

so that $u = \sec(x)$, $du = \sec(x)\tan(x)\,dx$ results in

$$\begin{aligned}\int \tan(x)\sec^2(x)\,dx &= \int \sec(x)\ \ \sec(x)\tan(x)\,dx \\ &= \int u du \\ &= \frac{u^2}{2} + C_1\end{aligned}$$

As a result, we obtain

$$\int \tan(x)\sec^2(x)\,dx = \frac{1}{2}\sec^2(x) + C_1 \tag{4.55}$$

The results in (4.55) and (4.54) may appear to be different, but in reality, they are the same because $\sec^2(x)$ and $\tan^2(x)$ differ by a constant.

Check your Reading *What trigonometric identity tells us that* $\sec^2(x)$ *and* $\tan^2(x)$ *differ by a constant?*

Additional Considerations

Often we must manipulate both the integrand of the antiderivative and u itself in order to effect a substitution. Moreover, it is often only through trial and error that some manipulation of the integrand leads to a antiderivative that can be evaluated.

For example, there is no obvious method for evaluating $\int \sec(x)\,dx$. Instead, a great deal of trial and error may be necessary before we realize that we must multiply inside the integrand by the ratio of $\sec(x) + \tan(x)$ to itself:

$$\int \sec(x)\,dx = \int \frac{\sec(x)[\sec(x) + \tan(x)]}{\sec(x) + \tan(x)}dx = \int \frac{\sec(x)\tan(x) + \sec^2(x)}{\sec(x) + \tan(x)}dx$$

If $u = \sec(x) + \tan(x)$, then $du = \left(\sec x \tan x + \sec^2 x\right)dx$, so that

$$\begin{aligned}\int \sec(x)\,dx &= \int \frac{\sec(x)\tan(x) + \sec^2(x)}{\sec(x) + \tan(x)}dx \\ &= \int \frac{du}{u} \\ &= \ln|u| + C\end{aligned}$$

Since $u = \sec(x) + \tan(x)$, this leads to a rather frequently occuring rule:

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C \tag{4.56}$$

There are also times when algebraic manipulations of the integrand are required.

EXAMPLE 8 Evaluate $\displaystyle\int \frac{x^3}{x^2+4}dx$

Solution: We let $u = x^2 + 4$ so that $\frac{1}{2}du = xdx$. We also write $x^3 = x^2 \cdot x$ and use the fact that $x^2 = u - 4$ to obtain

$$\int \frac{x^2 \cdot xdx}{x^2+4} = \int \frac{(u-4)\,du}{u} = \int \left(\frac{u}{u} - \frac{4}{u}\right) du = u - 4\ln|u| + C$$

As a result, we have

$$\int \frac{x^3}{x^2+4}dx = \left(x^2+4\right) - 4\ln\left|x^2+4\right| + C$$

However, regardless of how powerful a method substitution is, there remain many antiderivatives which **cannot be evaluated** in closed form using any method—that is, which cannot be expressed as a finite combination of elementary functions. For example, the antiderivative $\int e^{-x^2}dx$ is foundational to the study of statistics, but it cannot be evaluated in closed form.

Many antiderivatives cannot be evaluated in closed form.

EXAMPLE 9 Explain why $\int e^{-x^2}dx$ cannot be evaluated with the substitution $u = x^2$.

Solution: The substitution $u = x^2$ cannot be used because it leads to $\frac{1}{2}du = xdx$, and there is no factor x to place with the dx in

$$\int e^{-x^2}dx$$

In fact, it can be shown that no technique, no computer algebra system, no mathematical genius, nor anything else will ever be able to reduce $\int e^{-x^2}dx$ to a finite combination of elementary functions.

Exercises:

Evaluate using the given substitution. Check your work by differentiating.

1. $\int \cos\left(x^3\right) x^2dx$, $u = x^3$
2. $\int \sin\left(e^{2x}\right) e^{2x}dx$, $u = e^{2x}$
3. $\int e^{x^2}xdx$, $u = x^2$
4. $\int \frac{\sin\left(e^{-x}\right)}{e^x}dx$, $u = e^{-x}$

Evaluate and check your work. You may need to use (4.56).

5.	$\int xe^{x^2}dx$	6.	$\int x\sin\left(x^2\right)\,dx$
7.	$\int e^{\sin(x)}\cos(x)\,dx$	8.	$\int e^{\tan(x)}\sec^2(x)\,dx$
9.	$\int\sqrt{3x+2}dx$	10.	$\int\sqrt[3]{2x+3}dx$
11.	$\displaystyle\int\frac{dx}{2x+1}$	12.	$\displaystyle\int\frac{dx}{\sqrt{2x+1}}$
13.	$\int\cos\left(e^{2x}\right)\,e^{2x}dx$	14.	$\int\sin(3x+1)\,dx$
15.	$\int\cos(x)\sin(x)\,\,dx$	16.	$\int\cos^2(x)\sin(x)\,dx$
17.	$\int\tan(x)\,\,dx$	18.	$\int\cot(x)\,\,dx$
19.	$\displaystyle\int\frac{xdx}{x^2+4}$	20.	$\displaystyle\int\frac{x^2dx}{x^3+4}$
21.	$\displaystyle\int\frac{\sin(x)\,\,dx}{2-\cos(x)}$	22.	$\displaystyle\int\frac{\cos(x)\,dx}{\sin(x)+4}$
23.	$\displaystyle\int\frac{\sec^2(x)}{\tan(x)+1}dx$	24.	$\displaystyle\int\frac{\sec^2(x)}{\sqrt{\tan(x)+1}}dx$
25.	$\displaystyle\int\frac{\sin(\sqrt{x})\,dx}{\sqrt{x}}$	26.	$\displaystyle\int\frac{dx}{x\ln(x)}$
27.	$\displaystyle\int\frac{\cos(x)\,dx}{1-\sin^2(x)}$	28.	$\displaystyle\int\csc(x)\tan(x)\,dx$
29.	$\int(\sec x+1)^2\,dx$	30.	$\int(\sec x-1)^2\,dx$

Choose u and then perform whatever algebraic manipulations are necessary to allow a substitution and to evaluate the antiderivative.

31.	$\int\frac{x^3dx}{x^2+1}$	32.	$\int\frac{x^3dx}{\sqrt{x^2+1}}$
33.	$\int x^3\sqrt{x^2+1}dx$	34.	$\int\frac{x\left(x^2+1\right)\,dx}{x^2}$

35. Only one of the following can be evaluated in closed form. Use substitution to determine which it is.

(*a*) $\displaystyle\int\frac{\ln\left(x^2\right)}{\ln\left(x^2\right)+2}\frac{dx}{x}$ (*b*) $\displaystyle\int\frac{\ln\left(x^3\right)}{\ln\left(x^3\right)+3}\frac{dx}{x^2}$ (*c*) $\displaystyle\int\frac{\ln\left(x^4\right)}{\ln\left(x^4\right)+4}\frac{dx}{x^3}$

36. Only one of the following can be evaluated in closed form. Use substitution to determine which it is.

(*a*) $\displaystyle\int\sin\left(x^2\right)\,x^{-1}dx$ (*b*) $\displaystyle\int\tan\left(x^2\right)\,x^{-1}dx$ (*c*) $\displaystyle\int\ln\left(x^2\right)\,x^{-1}dx$

37. **Computer Algebra System:** Determine the answers to exercises 35 and 36 by attempting to evaluate each with a computer algebra system.

38. In this exercise, we explore the antiderivative

$$\int\csc^2(x)\,\cot(x)\,\,dx \tag{4.57}$$

(a) Evaluate (4.57) using the substitution $u = \cot(x)$
(b) Evaluate (4.57) by writing it as

$$\int \csc(x)\ \csc(x)\cot(x)\,dx$$

and using the substitution $u = \csc(x)$.
(c) Why are the results in (a) and (b) the same, even though they do not look the same?

39. In this exercise, we explore the antiderivatives of

$$\int 2\sin(x)\cos(x)\,dx \tag{4.58}$$

(a) Evaluate (4.58) using the substitution $u = \sin(x)$
(b) Evaluate (4.58) using the substitution $u = \cos(x)$
(c) Evaluate (4.58) by simplifying to

$$\int \sin(2x)\,dx$$

and integrating.
(d) Why are the results in (a), (b) and (c) all the same, even though they do not appear to be the same?

40. Use the substitution $u = \sqrt{x}$ to evaluate the antiderivative

$$\int \frac{dx}{x+\sqrt{x}}$$

4.8 Substitution in Definite Integrals

Substitution in Definite Integrals: Method 1

There are two approaches to using substitution to evaluate definite integrals. One approach is to evaluate the antiderivative first and then substitute afterwards, while the other approach is to change the limits when making the substitution. Although the second approaches has some advantages over the first, either method can be used in many applications.

The first approach will be referred to as "Method 1", in which the fundamental theorem is used to write a definite integral in terms of an antiderivative.

$$\int_a^b f[g(x)]\ g'(x)\,dx = \int f[g(x)]\ g'(x)\,dx\Big|_{x=a}^{x=b}$$

The substitution $u = g(x)$, $du = g'(x)\,dx$ is subsequently applied to the antiderivative, which is then evaluated. Once we replace u with $g(x)$, the limits are then substituted to complete the evaluation:

$$\begin{aligned}\int_a^b f[g(x)]\ g'(x)\,dx &= \int f(u)\,du\Big|_{x=a}^{x=b} \\ &= F(u)|_{x=a}^{x=b} \\ &= F(g(x))|_{x=a}^{x=b} \\ &= F(g(b)) - F(g(a))\end{aligned}$$

Often the substitution is made directly into the definite integral, which unfortunately may lead to confusion if we are not careful to remember that the limits are to be used only after reducing the result to a function of x.

EXAMPLE 1 Use substitution to evaluate

$$\int_0^{\sqrt{\pi}} \sin\left(x^2\right)\ 2xdx$$

Solution: We let $u = x^2$ so that $du = 2xdx$. As a result, we obtain

$$\begin{aligned}
\int_0^{\sqrt{\pi}} \sin\left(x^2\right)\ 2xdx &= \int \sin\left(x^2\right)\ 2xdx\bigg|_{x=0}^{x=\sqrt{\pi}} \\
&= \int \sin(u)\ du\bigg|_{x=0}^{x=\sqrt{\pi}} \\
&= -\cos(u)|_{x=0}^{x=\sqrt{\pi}}
\end{aligned}$$

We replace u by x^2 and then evaluate the difference:

$$\begin{aligned}
\int_0^{\sqrt{\pi}} \sin\left(x^2\right)\ 2xdx &= -\cos\left(x^2\right)|_{x=0}^{x=\sqrt{\pi}} \\
&= -\cos\left(\left(\sqrt{\pi}\right)^2\right) - -\cos(0) \\
&= -\cos(\pi) + \cos(0) \\
&= -1 + 1 \\
&= 2
\end{aligned}$$

EXAMPLE 2 Evaluate

$$\int_0^1 \frac{xdx}{\sqrt{4 - x^2}}$$

Solution: To begin with, we let $u = 4 - x^2$. The differential then becomes

$$du = -2xdx \qquad \Longrightarrow \qquad \frac{-1}{2}du = xdx$$

Substitution thus leads to

$$\begin{aligned}
\int_0^1 \frac{xdx}{\sqrt{4 - x^2}} &= \int \frac{\frac{-1}{2}du}{\sqrt{u}}\bigg|_{x=0}^{x=1} \\
&= -\frac{1}{2}\int u^{-1/2}du\bigg|_{x=0}^{x=1} \\
&= \frac{-1}{2}\frac{u^{1/2}}{1/2}\bigg|_{x=0}^{x=1}
\end{aligned}$$

Simplifying and replacing u by $4 - x^2$ completes the evaluation:

$$\begin{aligned}
\int_0^1 \frac{xdx}{\sqrt{4 - x^2}} &= -\sqrt{u}\ |_{x=0}^{x=1} \\
&= -\sqrt{4 - x^2}\bigg|_{x=0}^{x=1} \\
&= -\sqrt{4 - 1} + \sqrt{4 - 0} \\
&= -\sqrt{3} + 2
\end{aligned}$$

Check your Reading *Why is du negative in example 2?*

Substitution in Definite Integrals: Method 2

The second approach will be referred to as "Method 2" and requires the limits of integration also be changed when the variable is changed. In particular, if $F(x)$ is an antiderivative of $f(x)$, then

$$\frac{d}{dx}F(g(x)) = f(g(x))\,g'(x)$$

Thus, the fundamental theorem says that

$$\int_a^b f[g(x)]\; g'(x)\,dx = F(g(b)) - F(g(a))$$

However, the fundamental theorem also says that

$$\int_{g(a)}^{g(b)} f(u)\,du = F(g(b)) - F(g(a))$$

Combining the two yields the identity

$$\int_a^b f[g(x)]\; g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \tag{4.59}$$

That is, using the substitution $u = g(x)$ and $u = g'(x)\,dx$ in a definite integral requires the limits of integration be changed to $g(a)$ and $g(b)$, respectively.

EXAMPLE 3 Use substitution to evaluate

$$\int_0^{\sqrt{\pi}} \sin\left(x^2\right)\; 2xdx \tag{4.60}$$

Solution: We let $u = x^2$ so that $du = 2xdx$. In addition, we use $u(x) = x^2$ to change the limits of integration into

$$u(0) = 0^2 \qquad\qquad u\left(\sqrt{\pi}\right) = \left(\sqrt{\pi}\right)^2 = \pi$$

When substituting in a definite integral, change the limits of integration.

As a result, (4.60) is transformed into

$$\int_0^{\sqrt{\pi}} \sin\left(x^2\right)\; 2xdx = \int_0^{\pi} \sin(u)\,du$$

The resulting integral $\int_0^{\pi} \sin(u)\,du$ can be evaluated with the fundamental theorem:

$$\begin{aligned}\int_0^{\pi} \sin(u)\,du &= -\cos(u)|_0^{\pi} \\ &= -\cos(\pi) - -\cos(0) \\ &= 2\end{aligned}$$

This in turn implies that (4.60) is given by

$$\int_0^{\sqrt{\pi}} \sin\left(x^2\right)\; 2xdx = 2$$

EXAMPLE 4 Evaluate

$$\int_0^1 \frac{xdx}{\sqrt{4-x^2}}$$

Solution: To begin with, we let $u = 4 - x^2$. The differential then becomes

$$du = -2xdx \qquad \Longrightarrow \qquad \frac{-1}{2}du = xdx$$

Before substitution, we change the limits of integration:

$$u(0) = 4 - 0^2 = 4, \qquad u(1) = 4 - 1^2 = 3$$

The result is

$$\int_0^1 \frac{xdx}{\sqrt{4-x^2}} = \int_4^3 \frac{-\frac{1}{2}du}{\sqrt{u}} = -\frac{1}{2}\int_4^3 u^{-1/2}du$$

We can now use the property of integrals $-\int_b^a f(x)\,dx = \int_a^b f(x)\,dx$ to obtain

$$-\frac{1}{2}\int_4^3 u^{-1/2}du = \frac{1}{2}\int_3^4 u^{-1/2}du = \frac{1}{2}\frac{u^{1/2}}{1/2}\bigg|_3^4$$

Notice that we do not go back to the variable x, but instead simplify and evaluate:

$$\int_0^1 \frac{xdx}{\sqrt{4-x^2}} = \frac{1}{2}\int_3^4 u^{-1/2}du = u^{1/2}\Big|_3^4 = 2 - \sqrt{3}$$

EXAMPLE 5 Evaluate $\int_0^{\pi/4} \tan(x)\,dx$

Solution: We first write the tangent function in terms of sines and cosines,

$$\int_0^{\pi/4} \tan(x)\,dx = \int_0^{\pi/4} \frac{\sin(x)\,dx}{\cos(x)}$$

We then let $u = \cos(x)$. As a result, $du = -\sin(x)\,dx$, which is the same as $-du = \sin(x)\,dx$. The new limits of integration are

$$u(0) = \cos(0) = 1, \qquad u\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

As a result, substitution yields

$$\int_0^{\pi/4} \frac{\sin(x)\,dx}{\cos(x)} = -\int_1^{\sqrt{2}/2} \frac{du}{u} = \int_{\sqrt{2}/2}^1 \frac{du}{u} = \ln|u|\,\Big|_{\sqrt{2}/2}^1$$

The properties of the natural logarithm result in

$$\int_0^{\pi/4} \tan(x)\,dx = \ln|1| - \ln\left|\frac{\sqrt{2}}{2}\right| = \frac{1}{2}\ln(2)$$

Check your Reading *What replaces the question marks in the calculation below?*

$$\int_0^{\sqrt{\pi}} \cos\left(x^2\right)\ 2x dx = \int_{???}^{???} \cos(u)\, du$$

Substitution and Approximation

Even when an integral cannot be evaluated in closed form, the method of substitution can be used to *transform* a given definite integral. Often this leads to integrals which are easier to approximate numerically, or in some cases, to integrals that can be evaluated using other means.

EXAMPLE 6 Evaluate

$$\int_0^{\pi} \frac{\cos^3(x)\sin(x)}{\cos^2(x)+1} dx$$

Solution: If we let $u = \cos(x)$, then $du = -\sin(x)\, dx$,

$$u(0) = \cos(0) = 1, \quad \textit{and} \quad u(\pi) = \cos(\pi) = -1$$

Substituting into the integral yields

$$\int_0^{\pi} \frac{\cos^3(x)\sin(x)\, dx}{\cos^2(x)+1} = -\int_1^{-1} \frac{u^3 du}{u^2+1} = \int_{-1}^{1} \frac{u^3 du}{u^2+1}$$

Notice now that $f(u) = \frac{u^3}{u^2+1}$ is an odd function, and recall that if $f(x)$ is odd, then $\int_{-a}^{a} f(x)\, dx = 0$. Thus, we have

$$\int_0^{\pi} \frac{\cos^3(x)\sin(x)\, dx}{\cos^2(x)+1} = \int_{-1}^{1} \frac{u^3 du}{u^2+1} = 0$$

WARNING: Attempting to evaluate this integral numerically may lead to either a crash or an infinite loop.

Because of the iterative nature of most techniques for numerical integration, attempting to numerically approximate some integrals may lead to incorrect results or may even cause the calculator or computer to crash. For example, we cannot evaluate the definite integral

$$\int_0^{\pi} e^{\cos^2(99x)}\ \sin(99x)\, dx \tag{4.61}$$

and the rapid oscillation of the sine and cosine functions corrupts numerical approximations. (**Do not attempt to numerically integrate (4.61) until after it has been transformed!**).

As a result, substitution is often used to transform definite integrals before numerical methods are applied. In particular, by letting u be the "bad part" of an integrand, we can often remove undesirable features of an integral and thus produce an integral that can be approximated numerically.

EXAMPLE 7 Transform $\int_0^{\pi} e^{\cos^2(99x)}\ \sin(99x)\, dx$ with the substitution $u = \cos(99x)$, and *only then* estimate the result numerically.

Solution: If we let $u = \cos(99x)$ so that $-\frac{1}{99}du = \sin(99x)\,dx$, then the new limits of integration are

$$u(0) = \cos(0) = 1 \qquad\qquad u(\pi) = \cos(99\pi) = -1$$

As a result, (4.61) is transformed into

$$\int_0^{\pi} e^{\cos^2(99x)} \sin(99x)\,dx = -\frac{1}{99}\int_1^{-1} e^{u^2}du$$

The resulting integral no longer includes rapidly oscillating terms. Thus, we can produce a numerical estimate of the transformed integral,

$$\frac{1}{99}\int_{-1}^{1} e^{u^2}du = 0.0295485$$

from which we conclude that $\int_0^{\pi} e^{\cos^2(99x)} \sin(99x)\,dx = 0.0295485$ to seven decimal places of accuracy.

Check your Reading *What is the frequency of* $u = \cos(99x)$?

OPTIONAL: The Pullback Method

The converse of (4.59) says that if we are given an integral of the form $\int_{g(a)}^{g(b)} f(x)\,dx$, then we can let $x = g(u)$, so that $dx = g'(u)\,du$ and

$$\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f(g(u))\,g'(u)\,du$$

The process of letting $x = g(u)$, computing $dx = g'(u)\,du$, changing the limits, and substituting is called the *pullback method,* which is useful both theoretically and in applications.

EXAMPLE 8 Use the pullback $x = u^2$ to evaluate the definite integral

$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$$

Solution: If $x = u^2$, then $dx = 2udu$. Moreover, when $x = 1$, then $u^2 = 1$, so that $u = \pm 1$. To avoid complications, let us choose the positive root $u = 1$. Likewise, when $x = 4$, then $u^2 = 4$, and $u = 2$. Substituting into the definite integral yields

$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \int_1^2 \frac{e^{\sqrt{u^2}}}{\sqrt{u^2}}2udu = \int_1^2 \frac{e^u}{u}2udu = 2\int_1^2 e^u du$$

Evaluating the last integral is straightforward:

$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = 2\int_1^2 e^u du = 2\, e^u\big|_0^2 = 2\left(e^2 - 1\right)$$

When a choice for u is not obvious, we can still employ the pullback method by simply *letting u be everything.* We then change the limits, solve for u as a function of x, compute dx, and substitute. [2]

EXAMPLE 9 Evaluate

$$\int_0^9 \sqrt{1+\sqrt{x}}dx$$

Solution: A choice of substitution is not obvious, so we let u be the entire integrand:

$$u = \sqrt{1+\sqrt{x}}$$

It then follows that $x = 0$ implies that $u = \sqrt{1+\sqrt{0}} = 1$ and when $x = 1$, then $x = \sqrt{1+\sqrt{9}} = \sqrt{1+3} = 2$.

We now solve for x. To begin with, $u^2 = 1+\sqrt{x}$, so $\sqrt{x} = u^2 - 1$ and

$$x = \left(u^2-1\right)^2 = u^4 - 2u^2 + 1$$

As a result, $dx = \left(4u^3 - 4u\right) du$, so that

$$\begin{aligned}
\int_0^9 \sqrt{1+\sqrt{x}}dx &= \int_1^2 u\ \left(4u^3-4u\right) du \\
&= \int_1^2 \left(4u^4 - 4u^2\right) du \\
&= \frac{4u^5}{5} - \frac{4u^3}{3}\bigg|_1^2 \\
&= \left(\frac{4\,(2)^5}{5} - \frac{4\,(2)^3}{3}\right) - \left(\frac{4}{5} - \frac{4}{3}\right) \\
&= \frac{232}{15}
\end{aligned}$$

Exercises:

Evaluate the following using a substitution: Check your work numerically with a

[2] Here and in other calculus related courses, I often refer to the pullback method as the *calculus hammer,* since like a hammer it can be used to pound away at just about any integral.

graphing calculator.

1. $\int_0^3 \sqrt{x+1}dx$ 2. $\int_0^5 \sqrt{3x+1}dx$

3. $\int_0^1 xe^{-x^2}dx$ 4. $\int_0^{\sqrt[3]{\pi}} x^2\sin\left(x^3\right)dx$

5. $\int_0^3 \frac{xdx}{\sqrt{9+x^2}}$ 6. $\int_0^1 \frac{xdx}{\sqrt{4-x^2}}$

7. $\int_0^1 \frac{e^x-e^{-x}}{e^x+e^{-x}}d$ 8. $\int_0^1 \frac{e^x-1}{e^x-x}dx$

9. $\int_{-\pi}^{\pi} e^{\sin(x)}\cos(x)\,dx$ 10. $\int_{-\pi}^{\pi} e^{\cos(x)}\sin(x)\,dx$

11. $\int_0^{\pi/4} \frac{\sin(x)}{1+\cos(x)}dx$ 12. $\int_{-2}^{2} \frac{x^3}{x^2+4}dx$

13. $\int_0^1 x\sin\left(\pi x^2\right)dx$ 14. $\int_0^1 x\cos\left(\pi x^2\right)dx$

15. $\int_0^{\pi} \sin^3(\cos(x))\sin(x)\,dxx$ 16. $\int_{1/e}^{e} \frac{\sin(\sin(\ln x))}{x}dx$

17. $\int_{1/e}^{e} \frac{\sin^3(\ln|x|)}{x}dx$ 18. $\int_0^1 \frac{zdz}{\sqrt{2-z^2}}$

19. $\int_1^2 \frac{\ln(x)}{x}dx$ 20. $\int_1^2 \frac{\ln(\sqrt{x})}{x}dx$

21. $\int_0^1 e^x\sqrt{e^x+1}dx$ 22. $\int_1^2 x^{2x}(\ln(x)+1)\,dx$

Evaluate the following using the given pullback.

23. $\int_0^1 \sqrt[3]{\sqrt{x}+1}dx, \quad x=\left(u^3-1\right)^2$ 24. $\int_0^1 \frac{x}{\sqrt{x}+1}dx, \quad x=(u-1)^2$

25. $\int_0^{\ln(2)} \frac{e^{2x}dx}{\sqrt{1+e^x}}, \quad x=\ln(u)$ 26. $\int_1^2 \frac{\sqrt{\sqrt{x}+1}}{\sqrt{x}}dx, \quad x=\left(u^2-1\right)^2$

27. Use the *pullback method* with $u = everything$ to evaluate

$$\int_0^1 \frac{dx}{\sqrt{x}+1}$$

28. Use the *pullback method* with $u = everything$ to evaluate

$$\int_0^1 \sqrt{1+\sqrt{1+\sqrt{x}}}dx$$

29. For $l>0$, suppose that $f(2l-x)=f(x)$. Show that

$$\int_0^{2l} f(x)\,dx = 2\int_0^l f(x)\,dx$$

What does a graph of $f(x)$ look like? What is significant about $f(l-x) = f(x)$? (Hint: write

$$\int_0^{2l} f(x)\,dx = \int_0^{l} f(x)\,dx + \int_l^{2l} f(x)\,dx$$

30. Suppose that $a > 0$ and that $f(x)$ is a function for which $f\left(\frac{a}{x}\right) = f(x)$. Show that

$$\int_1^a f(x)\,dx = \int_1^a \frac{af(x)}{x^2}dx$$

by transforming the integral on the left using $x = a/u$.

Numerical: *Use a substitution to remove the oscillations from the integrands of the definite integrals below, and then approximate the resulting integral with a calculator. WARNING: Your calculator or computer may crash or enter an infinite loop if you attempt to approximate the untransformed integral.*

31. $\displaystyle\int_0^{\pi} \frac{\sin(15x)\,dx}{\sqrt{1-0.1\cos^4(15x)}}$

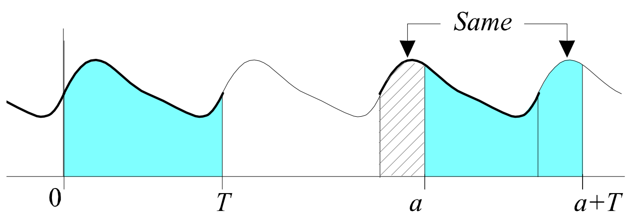
32. $\displaystyle\int_0^{\pi/2} \ln\left(1+\sin^2(15x)\right)\cos(15x)$

33. $\displaystyle\int_0^{\pi} \sin(99x)\sqrt{1+\cos^3(99x)}dx$

34. $\displaystyle\int_0^{\pi} \frac{\sin(99x)\,dx}{\sqrt{1-0.5\cos^4(15x)}}$

35. $\displaystyle\int_0^{\pi} \cos(75x)\cos(\sin(75x))\,dx$

36. * Figure 8-1 shows a function $f(x)$ that is *periodic* with a period of T, which is to say that $f(x+T) = f(x)$ for all real numbers x.



8-1: Periodic with period T

Explain why the figure suggests the identity

$$\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx \tag{4.62}$$

However, a picture is not a proof. The proof of (4.62) uses the identity

$$\int_a^{a+T} f(x)\,dx = \int_a^T f(x)\,dx + \int_T^{a+T} f(x)\,dx$$

and the substitution $u = x - T$. Can you prove (4.62)?

4.9 Integration by Parts

Integrating the Product Rule

In the next chapter, we will consider several applications of the definite integral. However, several of the integrals which arise in those applications will require a technique of integration known as *integration by parts.* In this section, we shall see that integration by parts, like substitution, is important both as a computational and a theoretical tool.

Recall that the product rule is of the form

$$\frac{d}{dx}\left[f(x)\,g(x)\right] = f'(x)\,g(x) + f(x)\,g'(x)$$

Applying the antiderivative operator to both sides yields

$$\int \frac{d}{dx}\left[f(x)\,g(x)\right]dx = \int f'(x)\,g(x)\,dx + \int f(x)\,g'(x)\,dx$$

which simplifies to

$$f(x)\,g(x) = \int f'(x)\,g(x)\,dx + \int f(x)\,g'(x)\,dx$$

Solving for the second antiderivative yields

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int g(x)\,f'(x)\,dx \qquad (4.63)$$

We convert (4.63) into an algorithm by letting $u = f(x)$ and $dv = g'(x)\,dx$. We then differentiate u to obtain

$$du = f'(x)\,dx$$

and integrate dv to obtain

$$v = \int g'(x)\,dx = g(x)$$

As a result, (4.63) is of the form

$$\int u dv = uv - \int v du \qquad (4.64)$$

This method for evaluating antiderivatives is called *integration by parts.*

EXAMPLE 1 Evaluate $\int x\cos(x)\,dx$.

Solution: We let $u = x$ and let $dv = \cos(x)\,dx$. As a result, $du = dx$ and

$$v = \int \cos(x)\,dx = \sin(x)$$

where the "$+C$" is omitted for simplicity (see exercise 31). Application of (4.64) then yields

$$\begin{array}{cccccccc} \int & u & dv & = & uv & - & \int & v & du \\ \int & x & \cos(x)\,dx & = & x\sin(x) & - & \int & \sin(x) & dx \end{array}$$

The remaining antiderivative is one of the basic forms, so that

$$\begin{aligned}\int x\cos(x)\,dx &= x\sin(x) - \int \sin(x)\,dx \\ &= x\sin(x) + \cos(x) + C\end{aligned}$$

To aid in the use of integration by parts, we organize the u, dv information into a table. For example 1, we would obtain the table

$$\begin{array}{cc} u = x & dv = \cos(x)\,dx \\ du = dx & v = \sin(x) \end{array}$$

The expression uv is the product along the diagonal in this table, and the expression vdu is the product along the bottom row.

$$\begin{array}{ccc} u = x & & dv = \cos(x)\,dx \\ & \searrow^{uv} & \\ du = dx & \underset{vdu}{\rightarrow} & v = \sin(x) \end{array}$$

After reading $uv = x\sin(x)$ from the diagonal and $vdu = \sin(x)\,dx$ from the bottom row, we get

$$\begin{array}{cccccccc} \int & u & dv & = & uv & - & \int & v & du \\ \int & x & \cos(x)\,dx & = & x\sin(x) & - & \int & \sin(x) & dx \end{array}$$

from which we get our final answer

$$\begin{aligned}\int x\cos(x)\,dx &= x\sin(x) - \int \sin(x)\,dx \\ &= x\sin(x) + \cos(x) + C \qquad (4.65)\end{aligned}$$

Check your Reading *Check the computation in (4.65) by evaluating the derivative*

$$\frac{d}{dx}\left(x\sin(x) + \cos(x)\right)$$

Guidelines for Integration by Parts

Choose dv so that it can be integrated.

There are two guidelines to remember when applying integration by parts. First, you must be able to find the antiderivative of the factor you designate as dv. Typical choices for dv include functions of the form x^n, e^{kx}, $\sin(kx)$, $\cos(kx)$ and $\sec^2(x)$.

The second guideline is that if one of the factors is of the form x^n where n is a positive integer, then we let $u = x^n$. However, we use the second guideline **only if the first guideline is already satisfied!** In particular, **avoid** letting dv be a function such as $\ln(x)$ or $\sqrt{x^2+4}$, if possible.

EXAMPLE 2 Use integration by parts to evaluate

$$\int x\sec^2(x)\,dx$$

Solution: The first guideline is satisfied by letting dv be either x or $\sec^2(x)$. The second guideline then leads to $u = x$ and the following u, dv information:

$$\begin{array}{cc} u = x & dv = \sec^2(x)\,dx \\ du = dx & v = \tan(x) \end{array}$$

As a result, integration by parts leads to

$$\begin{array}{ccccccccc} \int & u & dv & = & u\ \ v & - & \int & v & du \\ \int & x & \sec^2(x)\,dx & = & x\tan(x) & - & \int & \tan(x) & dx \end{array}$$

Since $\tan(x) = \frac{\sin(x)}{\cos(x)}$, we finish with the substitution $w = \cos(x)$, $dw = -\sin(x)\,dx$, which yields

$$\begin{aligned} \int x\sec^2(x)\,dx &= x\tan(x) - \int \frac{\sin(x)\,dx}{\cos(x)} \\ &= x\tan(x) + \int \frac{dw}{w} \\ &= x\tan(x) + \ln|w| + C \\ &= x\tan(x) + \ln|\cos(x)| + C \end{aligned}$$

EXAMPLE 3 Evaluate $\int x^2 \ln(x)\,dx$

Solution: The first guideline leads us to let $dv = x^2dx$ since $\ln(x)$ is not the integrand of a basic form. Thus, we have

$$\begin{array}{cc} u = \ln(x) & dv = x^2dx \\ du = \dfrac{1}{x}dx & v = \dfrac{x^3}{3} \end{array}$$

so that

$$\begin{array}{ccccccccc} \int & u & dv & = & u\ \ v & - & \int & v & du \\ \int & \ln(x) & x^2dx & = & \dfrac{x^3}{3}\ln(x) & - & \int & \dfrac{x^3}{3} & \dfrac{1}{x}dx \\ & & & = & \dfrac{x^3}{3}\ln(x) & - & \dfrac{1}{3}\int & x^2 & dx \\ & & & = & \dfrac{x^3}{3}\ln(x) & - & \dfrac{1}{3} & \dfrac{x^3}{3} & +C \end{array}$$

The result is

$$\int x^2\ln(x)\,dx = \frac{x^3}{3}\ln(x) - \frac{x^3}{9} + C$$

When integration by parts is applied to a definite integral, then the limits of integration are included in both of the integration by parts difference.

EXAMPLE 4 Evaluate $\int_1^e \ln(x)\,dx$

Sometimes we let $dv = dx$.

Solution: To evaluate $\int_1^e \ln(x)\,dx$, we let $dv = dx$. With the u, dv information

$$\begin{array}{ll} u = \ln(x) & dv = dx \\ du = \dfrac{1}{x}\,dx & v = x \end{array}$$

the integration by parts algorithm $\int udv = uv - \int vdu$ yields

$$\begin{aligned}\int_1^e \ln(x)\,dx &= x\ln(x)|_1^e - \int_1^e \frac{x}{x}dx \\ &= x\ln(x)|_1^e - \int_1^e 1dx \\ &= x\ln(x)|_1^e - x|_1^e \\ &= e\ln(e) - 1\ln(1) - (e-1) \\ &= 1\end{aligned}$$

Check your Reading *Why is there not a "+C" in example 4?*

Repeated Integration by Parts

Often integration by parts must be used more than once. Indeed, antiderivatives of the form $\int x^n e^{kx}dx$, $\int x^n \cos(kx)\,dx$, and $\int x^n \sin(kx)\,dx$ require n integration by parts when n is a positive integer.

EXAMPLE 5 Evaluate $\int_0^\pi x^2 \cos(x)\,dx$

Solution: We let $u = x^2$ and $dv = \cos(x)\,dx$, so that

$$\begin{array}{ll} u = x^2 & dv = \cos(x)\,dx \\ du = 2x\,dx & v = \sin(x) \end{array}$$

which results in

$$\begin{aligned}\int_0^\pi x^2\cos(x)\,dx &= x^2\sin(x)\Big|_0^\pi - 2\int_0^\pi x\sin(x)\,dx \\ &= \pi^2\sin(\pi) - 0^2\sin(0) - 2\int_0^\pi x\sin(x)\,dx \\ &= -2\int_0^\pi x\sin(x)\,dx\end{aligned}$$

Now we must apply integration by parts to the antiderivative $\int_0^\pi x\sin(x)\,dx$ by letting $u = x$ and $dv = \sin(x)\,dx$. The u, dv information thus becomes

$$\begin{array}{ll} u = x & dv = \sin(x)\,dx \\ du = dx & v = -\cos(x) \end{array}$$

so that $\int_0^{\pi} x^2 \cos(x)\,dx$ becomes

$$\begin{aligned}\int_0^{\pi} x^2 \cos(x)\,dx &= -2\left[-x\cos(x)|_0^{\pi} + \int_0^{\pi} \cos(x)\,dx\right] \\ &= -2\left[-x\cos(x)|_0^{\pi} - \sin(x)|_0^{\pi}\right] \\ &= -2\left[-\pi\cos(\pi) - 0\cos(0) - \sin(\pi) + \sin(0)\right] \\ &= -2\left[-\pi(-1) - 0 - 0 - 0\right] \\ &= -2\pi\end{aligned}$$

EXAMPLE 6 Evaluate $\int [\ln x]^2\,dx$.

Solution: We let $u = [\ln x]^2$ and $dv = dx$, so that

$$\begin{array}{cc} u = [\ln x]^2 & dv = dx \\ du = 2\ln(x)\,\frac{1}{x}dx & v = x \end{array}$$

which results in

$$\begin{aligned}\int [\ln x]^2\,dx &= x[\ln x]^2 - \int x\left(2\ln(x)\,\frac{1}{x}\right)dx \\ &= x[\ln x]^2 - 2\int \ln x dx\end{aligned}$$

We now let $u = \ln x$ and $dv = dx$, so that

$$\begin{array}{cc} u = \ln x & dv = dx \\ du = \frac{1}{x}dx & v = x \end{array}$$

which results in

$$\begin{aligned}\int [\ln x]^2\,dx &= x[\ln x]^2 - 2\left[x\ln(x) - \int x\frac{1}{x}dx\right] \\ &= x[\ln x]^2 - 2x\ln(x) + \int dx \\ &= x[\ln x]^2 - 2x\ln(x) + x + C\end{aligned}$$

Check your Reading *Why did we add the "+C" in the last calculation?*

Tabular Integration

Fortunately, we can streamline multiple applications of integration by parts by extending the u, dv information into a longer table. In particular, we construct a table with two columns, the first being a column of derivatives and the second being a column of antiderivatives. We then multiply along the diagonals, alternating signs as we go:

$$\begin{array}{ccc} u & & dv \\ \hline f(x) & & g'(x) \\ & \searrow^{+} & \\ f'(x) & & \int g'(x)\,dx \\ & \searrow^{-} & \\ f''(x) & & \iint g'(x)\,dxdx \\ \vdots & \searrow^{+} & \\ & & \iiint g'(x)\,dxdxdx \\ & & \vdots \end{array}$$

This method for successive integration by parts is called *tabular integration.*

EXAMPLE 7 Use tabular integration to evaluate

$$\int x^2 \cos(x)\,dx$$

Solution: To do so, we place x^2 at the top of the first column and $\cos(x)$ at the top of the second column. The first column is then filled with derivatives of x^2, and the second column is filled with antiderivatives of $\cos(x)$:

$$\begin{array}{ccc} u & & dv \\ \hline x^2 & & \cos(x) \\ & \searrow^{+} & \\ 2x & & \sin(x) \\ & \searrow^{-} & \\ 2 & & -\cos(x) \\ & \searrow^{+} & \\ 0 & & -\sin(x) \end{array}$$

Once 0 appears in the first column, all successive column entries will also be 0. Thus, we can stop once we reach 0. Forming products along diagonals with alternating signs yields

	1st diag		2nd diag		3rd diag	constant
$\int x^2\cos(x)\,dx =$	$x^2\sin(x)$	$-$	$-2x\cos(x)$	$+$	$2(-\sin(x))$	$+C$

which gives us the final answer

$$\int x^2 \cos(x)\,dx = x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C$$

Exercises:

Evaluate the following using integration by parts. Either tabular integration or repeated integration by parts may be used in exercises 21 through 28.

1. $\int xe^x dx$	2. $\int xe^{-x} dx$	3. $\int_0^1 xe^{2x} dx$
4. $\int_0^1 xe^{\pi x} dx$	5. $\int \ln(2x)\,dx$	6. $\int \ln\left(x^3\right) dx$

7.	$\int x\sin(2x)\,dx$	8.	$\int x\sin(x)\cos(x)\,dx$
9.	$\int x\cos(\pi x)\,dx$	10.	$\int x\csc^2(x)\,dx$
11.	$\int (3-x)\,e^x dx$	12.	$\int x\cos(2x)\,dx$
13.	$\int_1^2 x^2\ln(x)\,dx$	14.	$\int x^3\ln(x)\,dx$
15.	$\int x^4\ln(x)\,dx$	16.	$\int_1^2 \sqrt{x}\ln(x)\,dx$
17.	$\displaystyle\int \frac{x+1}{e^x}dx$	18.	$\displaystyle\int_0^{\pi} \frac{x}{\sec(x)}dx$
19.	$\int_1^2 (\ln x)^2\,dx$	20.	$\displaystyle\int \frac{\ln(x)}{\sqrt{x}}dx$
21.	$\int t^2e^tdt$	22.	$\int t^2e^{-2t}dt$
23.	$\int x^2\sin(x)\,dx$	24.	$\int x^2\sin(2x)\,dx$
25.	$\int x^3e^{2x}dx$	26.	$\int x^4e^{-x}dx$
27.	$\int \left(x^2+2x\right)e^xdx$	28.	$\int \left(x^3+x\right)\sin(x)\,dx$
29.	$\int x\tan(x)\sec^2(x)\,dx$	30.	$\int 2x\left(1-2x^2\right)e^{x^2}dx$

31. When integrating dv, we usually neglect to include the "$+C$" term. In this exercise, we examine what happens if we add a constant onto v.

(a) Evaluate $\int xe^xdx$ using integration by parts with

$$\begin{aligned} u &= x & dv &= e^xdx \\ du &= dx & v &= e^x + C \end{aligned}$$

What happens to the "$+C$" term?

(b) Evaluate $\int \ln(x+1)\,dx$ using the u, dv information

$$\begin{aligned} u &= \ln(x+1) & dv &= dx \\ du &= \frac{1}{x+1}dx & v &= x+1 \end{aligned}$$

(c) Evaluate $\int \ln(x+3)\,dx$ using integration by parts with $v = x+3$.

32. Use integration by parts to integrate $\int 2x\ln(x+1)\,dx$ when $v = x^2-1$.

33. Use integration by parts to show that

$$\int x^ne^xdx = x^ne^x - n\int x^{n-1}e^xdx$$

34. Use integration by parts to show that

$$\int (\ln x)^n\,dx = x(\ln x)^{n-1} - n\int (\ln x)^{n-1}\,dx$$

35. Since the antiderivative and derivative operators cancel each other, we have

$$\int f'(x)\,dx = f(x) + C$$

Use this to do the following:

(a) Show that

$$f(x) = xf'(x) - \int xf''(x)\,dx$$

by applying integration by parts to $\int f'(x)\,dx$.

(b) Show further that

$$f(x) = xf'(x) - \frac{x^2}{2}f''(x) + \int \frac{x^2}{2}f''(x)\,dx$$

36. Assume that $g(x)$ is differentiable and simplify the following expressions until there are no antiderivative operators remaining.

(a) $\displaystyle\int xg(x)\,dx + \frac{1}{2}\int x^2 g'(x)\,dx$

(b) $\displaystyle\int e^x g(x)\,dx + \int e^x g'(x)\,dx$

(c) $\displaystyle\int \cos(x)\,g(x)\,dx - \int \sin(x)\,g'(x)\,dx$

37. The first Fourier (For-ee-ay) cosine coefficient of a function $f(x)$ is defined to be

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(x)\,dx$$

Find the first Fourier cosine coefficients of the functions below:

(a) $f(x) = x$ (b) $f(x) = x^2$ (c) $f(x) = \sin(x)$

38. The second Fourier sine coefficient of a function $f(x)$ is defined to be

$$b_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(2x)\,dx$$

Find the first Fourier cosine coefficients of the functions below:

(a) $f(x) = x$ (b) $f(x) = x^2$ (c) $f(x) = \sin(x)$

Evaluate the following by applying a substitution and then by applying integration by parts.

39. $\int 2x^3 \sin\left(x^2\right)dx$ (Hint: write it as $\int x^2 \sin\left(x^2\right)\ 2x dx$)

40. $\int e^{2x}\cos(e^x)\,dx$ (Hint: write it as $\int e^x \cos(e^x)\ e^x dx$)

41. $\int e^{\sqrt{x}}dx$ (Hint: write it as $\int \sqrt{x}\,e^{\sqrt{x}}\,\dfrac{dx}{\sqrt{x}}$)

42. $\int \sin(\sqrt{x})\,dx$

43. $\int \ln(x)\ \sin(\ln(x))\ \dfrac{dx}{x}$

Self Test

A variety of questions are asked in a variety of ways in the problems below. Answer as many of the questions below as possible before looking at the answers in the back of the book.

1. Answer each statement as true or false. If the statement is false, then state why or give a counterexample.

   (a) Given a partition of the interval $[a, b]$, $a = x_0 < x_1 < ... < x_n = b$, the tag t_3 is a number in the subinterval $[x_2, x_3]$.

   (b) A nonconstant simple function must be continuous everywhere.

   (c) If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx$ is an area.

   (d) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du = \int_a^b f(\#)\,d\#$

   (e) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

   (f) If $F(x) = \int_a^x f(t)dt$, then $F(a) = 0$.

   (g) If $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$.

   (h) A continuous function has exactly one antiderivative.

   (i) If $F(x)$ is an antiderivative of $f(x)$ and k is a constant, then $kF(x)$ is an antiderivative of $kf(x)$.

   (j) When an antiderivative is found one can check their work by differentiating the result.

   (k) $\int_a^b f(x)\,dx = f(b) - f(a)$

   (l) The method of u-substitution is derived from the product rule of differentiation..

   (m) An antiderivative in closed form can be found for any combination of elementary functions.

In 2-5, assume that $f(x)$ and $g(x)$ are integrable on $[1, 5]$ and that

$$\int_1^5 f(x)dx = 7, \qquad \int_3^5 f(x)dx = 4, \qquad \textit{and} \qquad \int_1^5 (f(x) + g(x))\,dx = 2$$

Evaluate the following integrals

2. $\int_1^5 2f(x)dx =$ _____ (a) 2 (b) 7 (c) 12 (d) 14

3. $\int_1^3 f(x)dx =$ _____ (a) 1 (b) 2 (c) 3 (d) 4

4. $\int_1^5 g(x)dx =$ _____ (a) -5 (b) -1 (c) 1 (d) 5

5. $\int_1^5 (3f(x) + 2g(x))\,dx =$ _____ (a) 11 (b) 12 (c) 13 (d) 15

6. Which of the simple function approximations of $f(x) = e^x$ over $[0, 1]$ is always below the graph of f?

(a)	left endpoint	(b)	midpoint
(c)	right endpoint	(d)	trapezoidal

7. Which of the following is the derivative of $F(x) = \int_0^{x^2} \sin\left(t^2\right) dt$?

(a)	$\cos\left(x^4\right)$	(b)	$\sin\left(x^4\right)$	(c)	$-\cos\left(x^4\right)$	(d)	$x^2 \sin\left(x^4\right)$
(e)	$2x\cos\left(x^4\right)$	(f)	$2x\sin\left(x^4\right)$	(g)	$2x\cos\left(2x^2\right)$	(h)	$x^2\sin(2x)$

8. If k is a positive integer, then

$$\int_0^{\pi} \cos(kx)\,dx =$$

(a) $-\pi$ (b) 0 (c) 1 (d) π

9. If $k > 0$, then which of the following is the same as

$$\int_0^1 \sqrt{x+k}\,dx$$

(a) $\int_k^{1+k} \sqrt{u}\,du$ (b) $\int_0^1 \sqrt{u}\,du$ (c) $\frac{1}{k}\int_0^1 \sqrt{u}\,du$ (d) $\frac{1}{k}\int_0^{1+k} \sqrt{u}\,du$

10. Integration by parts applied to

$$\int_1^4 \ln(x)\,dx$$

would require the following:

(a) $dv = \ln(x)\,dx$ (b) $dv = dx$ (c) $u = x$ (d) $u = dv$

11. Evaluate $\int (\tan(\theta) + \sec(\theta))^2\,d\theta$

(a) $2\tan\theta - 2\sec(\theta) - \theta + C$ (b) $2\tan\theta + 2\sec(\theta) + \theta + C$

(c) $2\tan\theta - 2\sec(\theta) + \theta + C$ (d) $2\tan\theta + 2\sec(\theta) - \theta + C$

12. Which of the following is the result of applying integration by parts to

$$\int x\sec^2(x)\,dx$$

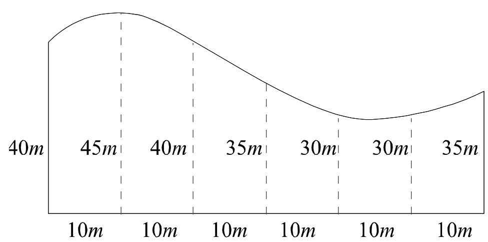
for $u = x$ and $dv = \sec^2(x)\,dx$?

(a) $x\tan(x) - \int \sec^2(x)\,dx$ (b) $x\sec^2(x) - \int \tan(x)\,dx$
(c) $x\tan(x) - \int \tan(x)\,dx$ (d) $x\sec^2(x) - \int \sec^2(x)\,dx$

13. Construct the right endpoint approximation (Riemann Sum) for $\int_0^2 \left(4 - x^2\right) dx$ using the partition $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1.0$, $x_3 = 1.5$, $x_4 = 2.0$

14. Construct the midpoint approximation (Riemann Sum) to $\int_1^3 \left(2x^2 - 1\right) dx$ using the partition $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2.0$, $x_3 = 2.5$, $x_4 = 3.0$

15. Estimate the area in square meters using the trapezoidal approximation.



16. Evaluate $\int_0^1 \sqrt{2-x^2}\, dx$ using basic geometry and properties of the definite integral.

17. Use monotonicity and concavity to sketch the graph of

$$F(x) = \int_0^x \frac{t^2-4}{t^4+9} dt$$

given that is has a horizontal asymptote of $y = 0$.

18. Evaluate the following using whichever technique or formula is necessary. Check by differentiating your solution.

(a) $\int e^{10x} dx$ (b) $\int \frac{x^2+1}{x} dx$ (c) $\int \left(\sec^2 x + x^3\right) dx$

(d) $\int 5\sin(x)\cos(x) dx$ (e) $\int \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$ (f) $\int \frac{e^{\ln x}}{x} dx$

(g) $\int \frac{3x}{x+1} dx$ (h) $\int x\cos(x)\, dx$ (i) $\int \cos^2(x)\sin(x) dx$

(j) $\int (2x+1)\ln(x)\, dx$ (k) $\int \cos^2(x)\sin(x) dx$ (l) $\int \frac{x^3+3x}{x^2+1} dx$

19. What is the area under the curve $y = x^2 + |x|$ over $[-1, 2]$?

20. Use a substitution to evaluate the integral

$$\int_0^{\ln(2)} \frac{e^x}{e^x+1} dx$$

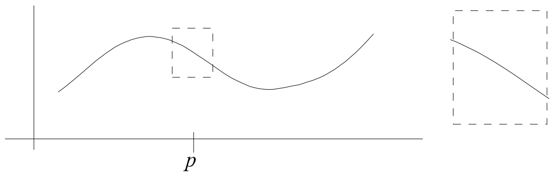
21. **Write to Learn:** In a couple of sentences, explain how you would use the definition of the integral and partitions with midpoint approximations to evaluate the integral

$$\int_0^5 x dx$$

The Next Step... Fractal Interpolation

Why do we have to resort to simple functions and h-fine partitions to define definite integrals? Why can't we simply say that definite integrals measure areas? Our next step is to show that area under a curve is a rather sophisticated concept in its own right, and in fact, defining the concept of area is as involved as defining a definite integral.

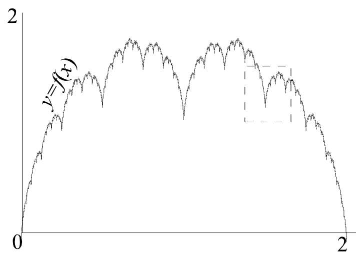
The key is that differentiable functions—the functions that have dominated our study of calculus to this point—are "nice" functions because small sections of their graphs are practically the same as straight lines.



NS-1: Zooming leads to straighter and straighter line

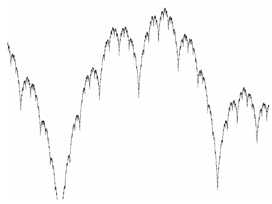
However, there exist continuous functions that are not differentiable at any point. That is, there are functions which have no breaks, no jumps and no vertical asymptotes, and no part of their graph even resembles a straight line.

For example, the curve shown below is the graph of a continuous function



NS-2: Not a differentiable function

However, zooming does not result in the graph becoming more and more like a straight line. To illustrate, let us zoom into the "box" shown in the figure above:



NS-3: Zoom resembles the original

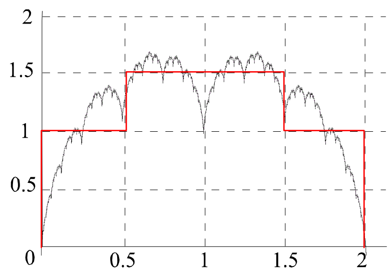
The result looks a great deal like what we started with, a property which is known as *self-similarity*. In fact, every zoom looks like the graph we started with, and as a result, the function never appears to be "practically the same" as a straight line. Thus, the function is continuous for all x in $[0,2]$, but it is not differentiable for any x in $[0,2]$.

Because the function is continuous, however, it must be integrable. That is, we can compute the area under the curve $y = f(x)$ regardless of how strange it looks, and in fact, it can be shown that

$$\int_0^2 f(x)\,dx = 2.5$$

However, dealing with such complicated functions requires a sophisticated definition of the integral. The founders of calculus may have only considered "ordinary" curves, but today's scientists work with today's functions. Thus, they require a definition of the integral which can handle functions like the one above and many others even more difficult to comprehend.

Write to Learn Below is a simple function approximation of $f(x)$. Approximate $\int_0^2 f(x)\,dx$ using the given simple function over $[0,2]$.

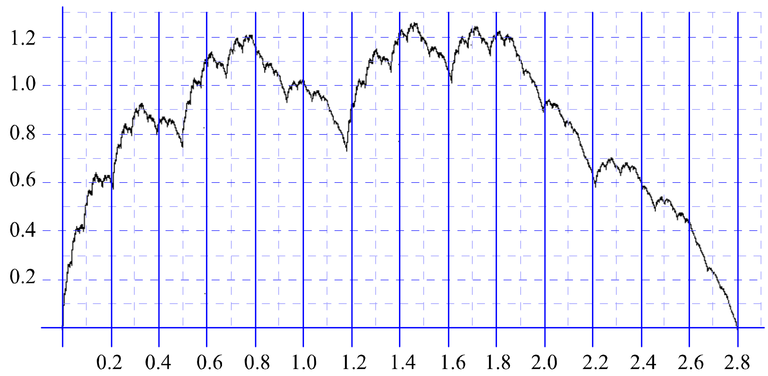


NS-4: Approximate the area of the region under the fractal interpolation function

In a paragraph with complete sentences, explain why you think the simple function approximation of $\int_0^2 f(x)\,dx$ is so accurate?

Write to Learn The function above is an example of a *fractal interpolation function.* Go to the library or to the internet to research fractals, and then write a paper reporting the results of your research.

Group Learning: Shown below is the graph of a continuous function $g(x)$:



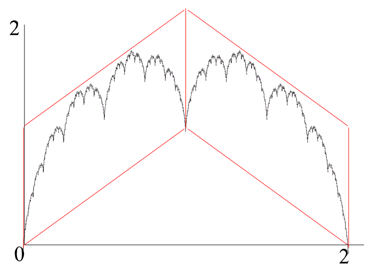
NS-5: Another Fractal Interpolation Function

Have each member of the group construct a simple function approximation of $g(x)$ and then use their simple function approximation to estimate $\int_0^{2.8} g(x)\,dx$. The group should then meet to compare results and agree on a final estimate of $\int_0^{2.8} g(x)\,dx$. Members should report their own individual results when presenting the estimate.

Advanced Contexts

No doubt, there must have been some curiosity piqued when it was mentioned that $\int_0^2 f(x)\,dx = 2.5$. We conclude with a sketch of how the integral of the *fractal interpolation function* $f(x)$ is calculated. The key is that in spite of its rugged appearance, the function $f(x)$ actually has a great deal of symmetry.

For example, each "half" of $f(x)$ is actually $f(x)$ itself re-scaled, shifted and transformed to fit in the red parallelograms shown below.



NS-6: Bounding Boxes of Fractal

In particular, we can use the red boxes and ideas from a course called *linear algebra* to show that

$$\int_0^2 f(x)\,dx = \frac{1}{2}\int_0^2 \left(\frac{1}{2}x + \frac{3}{5}f(x)\right)dx + \frac{1}{2}\int_0^2 \left(\frac{-1}{2}x + \frac{3}{5}f(x) + 1\right)dx \quad (4.66)$$

Simplifying (4.66) leads to

$$\int_0^2 f(x)\,dx = \frac{3}{5}\int_0^2 f(x)\,dx + \frac{1}{2}\int_0^2 dx$$

so that

$$\frac{2}{5}\int_0^2 f(x)\,dx = 1$$

and finally

$$\int_0^2 f(x)\,dx = \frac{5}{2} = 2.5$$

The exercises below illustrate how (4.66) is derived.

1. * Here we examine the transformation

$$u = \frac{1}{2}x, \qquad v = \frac{1}{2}x + \frac{3}{5}y$$

which maps the point (x, y) to the point (u, v).

(a) Show that if (x, y) is any point in the rectangle with vertices $(0, 0)$, $(2, 0)$, $(2, 1)$ and $(0, 1)$, then (u, v) is in the red parallelogram with a vertex at $(0, 0)$. (Hint: show that vertices of the rectangle map to vertices of the parallelogram)

(b) Show that if $y = f(x)$, then we must have

$$v = u + \frac{3}{5}f(2u)$$

(c) It follows that

$$\int_0^1 f(x)\,dx = \int_0^1 v\,du = \int_0^1 \left(u + \frac{3}{5}f(2u)\right) du$$

Now show that the substitution $u = \frac{1}{2}x$, $du = \frac{1}{2}dx$, $u(0) = 0$, $u(2) = 1$ transforms

$$\frac{1}{2}\int_0^2 \left(\frac{1}{2}x + \frac{3}{5}f(x)\right) dx$$

into the integral $\int_0^1 v\,du$.

2. * Here we examine the transformation

$$p = \frac{1}{2}x + 1, \qquad q = \frac{-1}{2}x + \frac{3}{5}y + 1$$

which maps the point (x, y) to the point (p, q)

(a) Show that if (x, y) is any point in the rectangle with vertices $(0, 0)$, $(2, 0)$, $(2, 1)$ and $(0, 1)$, then (p, q) is in the red parallelogram with a vertex at $(2, 0)$. (Hint: show that vertices of the rectangle map to vertices of the parallelogram)

(b) Show that if $y = f(x)$, then we must have

$$q = \frac{-1}{2}(2p - 2) + \frac{3}{5}f(2p - 2)$$

(c) It follows that

$$\int_1^2 f(x)\,dx = \int_1^2 q dp = \int_1^2 \left(\frac{-1}{2}(2p-2) + \frac{3}{5} f(2p-2) + 1 \right) dp$$

Now show that the substitution $p = \frac{1}{2}x + 1$, $dp = \frac{1}{2}dx$, $p(0) = 1$, $p(2) = 2$ transforms

$$\frac{1}{2}\int_0^2 \left(\frac{-1}{2}x + \frac{3}{5} f(x) + 1 \right) dx$$

into the integral $\int_0^1 q dp$.

3. Divide $\int_0^2 f(x)\,dx$ into two integrals, and then use the results from the last two exercises to show that (4.66) is true.

5. APPLICATIONS OF THE INTEGRAL

Definite integrals are not used solely to study areas. They are used throughout mathematics and the sciences with a host of different interpretations. Indeed, such interpretations range from the concrete study of geometric quantities to the relatively abstract study of integral transforms.

In this chapter, we explore several applications of the integral, many of which are connected to the study of geometry. We begin by further exploring the concept of area, and then we move on to the concepts of volume, centroid, and arclength. In each of these applications, the definition of the integral plays a key role in deriving the formulas that are most important to our study.

However, our ultimate goal is one of the most important applications in modern science and mathematics—the use of calculus in the study of *probability.* To reach this goal, we have to introduce two additional concepts related to the integral—the concept of an *improper integral* and the technique of integration known as *integration by parts.*

By the end of the chapter, we feel that there will be little doubt as to the importance of both the integral and the definition of the integral. Indeed, the applications offered here are but a sampling from a vast arena the integral's uses. We only hope that this chapter affords a glimpse into how the integral can be and is used throughout science, engineering, mathematics, and technology.

5.1 Area Between Two Curves

If $f(x)$ is integrable over $[a,b]$, then its definite integral over $[a,b]$ is defined

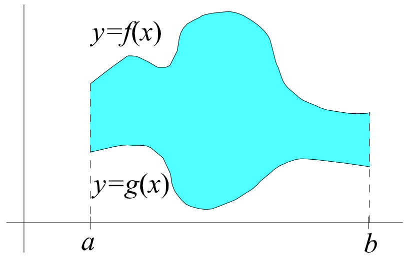
$$\int_a^b f(x)\,dx = \lim_{h\to 0} \sum_{j=1}^{n} f(t_j)\,\Delta x_j \tag{5.1}$$

where the limit is over h-fine partitions of $[a,b]$. In this section, we use (5.1) to extend the integral to a tool for finding areas of regions bound between two curves.

Area of a Type I Region

To begin with, a *type I region* is a region with boundaries of the form $x=a$, $x=b$, $y=f(x)$ and $y=g(x)$, where $f(x)\geq g(x)$ for all x in $[a,b]$ and where $f(x)$ and

$g(x)$ are continuous on $[a,b]$.

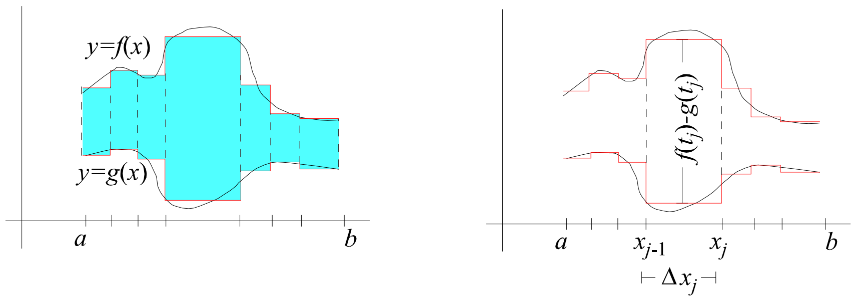


1-1: A Type I Region

To find the area of a type I region, let $h>0$ and let $\{x_j,t_j\}_{j=1}^n$ be an h-fine partition of $[a,b]$. Then the simple functions

$$s_h(x)=\begin{cases} f(t_1) & if \quad x_0\le x<x_1 \\ f(t_2) & if \quad x_1\le x<x_2 \\ \vdots & \qquad \vdots \\ f(t_n) & if \quad x_{n-1}\le x<x_n \end{cases}, \qquad r_h(x)=\begin{cases} g(t_1) & if \quad x_0\le x<x_1 \\ g(t_2) & if \quad x_1\le x<x_2 \\ \vdots & \qquad \vdots \\ g(t_n) & if \quad x_{n-1}\le x<x_n \end{cases}$$

converge to f and g, respectively, as h approaches 0. Thus, the area of the region between the simple functions approximates the area of the region between $y=f(x)$ and $y=g(x)$ respectively.



1-2: Simple function approximation of f and g

Moreover, the region between $s_h(x)$ and $r_h(x)$ is a collection of rectangles with widths Δx_j and heights $f(t_j)-g(t_j)$. The total area of the rectangles is

$$area=[f(t_1)-g(t_1)]\,\Delta x_1+\ldots+[f(t_n)-g(t_n)]\,\Delta x_n$$

and if we let h approach 0, the area of the rectangles converges to the area between the curves:

$$area=\lim_{h\to 0}\sum_{j=1}^{n}[f(t_j)-g(t_j)]\,\Delta x_j$$

By definition, this last limit converges to $\int_a^b (f(x)-g(x))\,dx$.

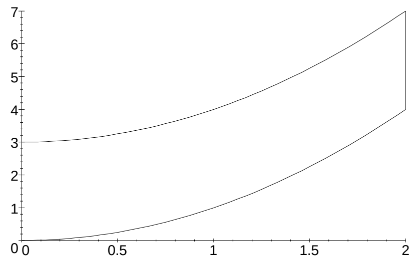
Theorem 1.1. If $f(x)$ and $g(x)$ are integrable on $[a,b]$ with $f(x)\ge g(x)$ for all x in $[a,b]$, then the region between $y=f(x)$ and $y=g(x)$ over $[a,b]$ has an area of

$$A=\int_a^b (f(x)-g(x))\,dx \tag{5.2}$$

That is, (5.2) is the formula for the area of a type I region.

EXAMPLE 1 Find the area of the region between $f(x) = x^2 + 3$ and $g(x) = x^2$ over $[0, 2]$.

Solution: Since $x^2 + 3 \geq x^2$ over $[0, 2]$, the region bound by $x = 0$, $x = 2$, $y = x^2$, and $y = x^2 + 3$ is a type I region:



1-3: Region between $y = x^2$ and $y = x^2 + 3$ over $[0, 2]$

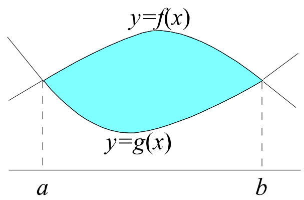
Thus, theorem 1.1 implies that the area of the region is

$$A = \int_0^2 \left(x^2 + 3 - x^2\right) dx = \int_0^2 3dx = 6$$

Check your Reading *What is the area of the region between $y = x + 2$ and $y = x$ over $[0, 2]$?*

Identifying Endpoints and Subdivisions of Regions

When an interval $[a, b]$ is not given, the graphs of the functions must intersect in at least two points in order to bound a type I region.

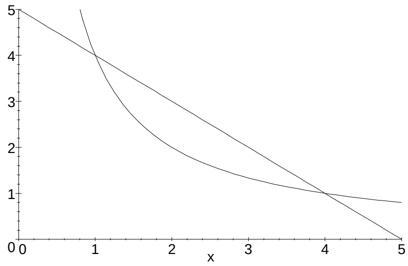


1-4: Limits of integration may be x-coordinates of intersections

To determine the interval $[a, b]$, we must set $f(x) = g(x)$ and solve for x.

EXAMPLE 2 Find the area of the region between the curves $y = 5 - x$ and $y = 4/x$.

Solution: We begin by graphing the two curves to determine if they intersect.



1-5: Region between $y = 5 - x$ and $y = 4/x$

The two graphs intersect when $5 - x = 4/x$. Multiplying both sides by x yields

$$x(5-x) = x\left(\frac{4}{x}\right) \qquad \Longrightarrow \qquad 5x - x^2 = 4$$

The result can be transformed into $x^2 - 5x + 4 = 0$, which factors into

$$(x-1)(x-4) = 0$$

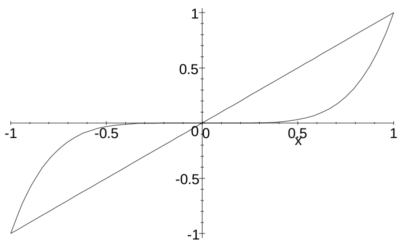
and which in turn has solutions of $x = 1$ and $x = 4$. The graph reveals that $5 - x \geq 4/x$ when x is in $[1, 4]$, so that the area of the region enclosed by $y = 5 - x$ and $y = 4/x$ is

$$area = \int_1^4 \left(5 - x - \frac{4}{x}\right) dx = 5x - \frac{x^2}{2} - 4\ln(x)\bigg|_1^4 = \frac{15}{2} - 8\ln 2$$

If $y = f(x)$ and $y = g(x)$ intersect more than twice, then we find the area of the region between $y = f(x)$ and $y = g(x)$ over each interval implied by the points of intersection. That is, the area of the region between $y = f(x)$ and $y = g(x)$ is the sum of the areas of the individual subregions.

EXAMPLE 3 Find the area of the region between the curves $y = x^5$ and $y = x$.

Solution: Since $x^5 = x$ has solutions $x = -1, 0, 1$, the total region is the union of the regions over $[-1, 0]$ and over $[0, 1]$, respectively.



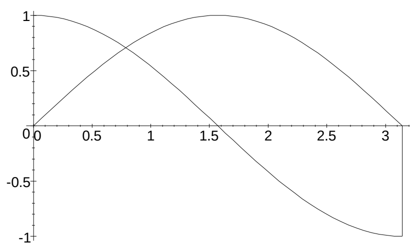
1-6: Region between $y = x$ and $y = x^5$

Notice that $x^5 \geq x$ when x is in $[-1, 0]$, but that $x \geq x^5$ when x is in $[0, 1]$. As a result, the area between the two curves is

$$\begin{aligned} area &= \int_{-1}^{0} \left(x^5 - x\right) dx + \int_{0}^{1} \left(x - x^5\right) dx \\ &= \frac{1}{3} + \frac{1}{3} \\ &= \frac{2}{3} \end{aligned}$$

EXAMPLE 4 Find the area of the region bound between $y = \sin(x)$ and $y = \cos(x)$ over $[0, \pi]$.

Solution: The graph below reveals that $\sin(x) = \cos(x)$ for some x in $[0, \pi]$.



1-7: Region between $y = \sin(x)$ and $y = \cos(x)$ over $[0, \pi]$

To determine the point of intersection, we solve the following:

$$\begin{aligned} \sin(x) &= \cos(x) \\ \tan(x) &= 1 \\ x &= \frac{\pi}{4} \end{aligned}$$

Since $\cos(x) \geq \sin(x)$ when x is in $\left[0, \frac{\pi}{4}\right]$ and $\sin(x) \geq \cos(x)$ when x is in $\left[\frac{\pi}{4}, \pi\right]$, the area of the region enclosed by the two functions is

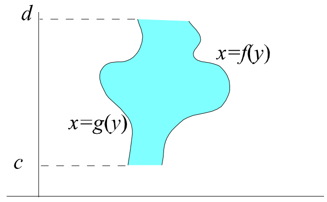
$$\begin{aligned} area &= \int_{0}^{\pi/4} (\cos(x) - \sin(x))\, dx + \int_{\pi/4}^{\pi} (\sin(x) - \cos(x))\, dx \\ &= \sqrt{2} - 1 + \sqrt{2} + 1 \\ &= 2\sqrt{2} \end{aligned}$$

Check your Reading *Why can't we compute the area between $y = x^5$ and $y = x$ with the integral*

$$\int_{-1}^{1} \left(x^5 - x\right) dx$$

Type II Regions

A *type II region* is a region with boundaries of the form $y = c$, $y = d$, $x = f(y)$ and $x = g(y)$, where $f(y) \geq g(y)$ for all x in $[c, d]$ and where f and g are continuous on $[c, d]$.



1-8: A type II region

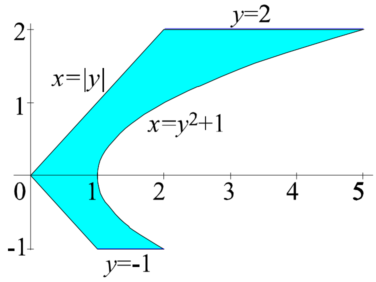
It follows that the area of a type II region is

$$A = \int_c^d [f(y) - g(y)]\, dy$$

That is, if x is considered a function of y, we simply interchange the roles of the two variables.

EXAMPLE 5 Find the area of the type II region between $x = y^2 + 1$ and $x = |y|$ over $[-1, 2]$.

Solution: The graphs of $f(y) = y^2 + 1$ and $g(y) = |y|$ reveal that $f(y) \geq g(y)$ over $[-1, 2]$.



1-9: Region between $x = y^2 + 1$ and $x = |y|$ over $[-1, 2]$.

The definition of the absolute value function says that

$$|y| = \begin{cases} y & if \quad y \geq 0 \\ -y & if \quad y < 0 \end{cases}$$

Thus, the area is computed using integrals over $[-1, 0]$ and $[0, 2]$, respectively.

$$\begin{aligned} area &= \int_{-1}^{0} \left(y^2 + 1 - |y|\right) dy + \int_0^2 \left(y^2 + 1 - |y|\right) dy \\ &= \int_{-1}^{0} \left(y^2 + 1 - -y\right) dy + \int_0^2 \left(y^2 + 1 - y\right) dy \\ &= \frac{5}{6} + \frac{8}{3} \\ &= \frac{7}{2} \end{aligned}$$

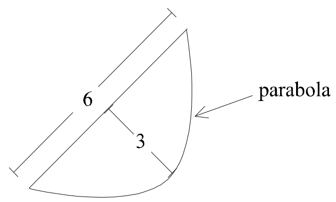
Check your Reading *Where did we use the definition of $|y|$ in the computation above?*

Creating a Coordinate System for a Given Region

When a region is given without benefit of a coordinate system, we must define our own coordinate system and then find representations for the curves in that coordinate system. In particular, let us consider regions bounded by parabolas using the fact that a parabola with nonzero roots r_1, r_2 and with a y-intercept of b has an equation of

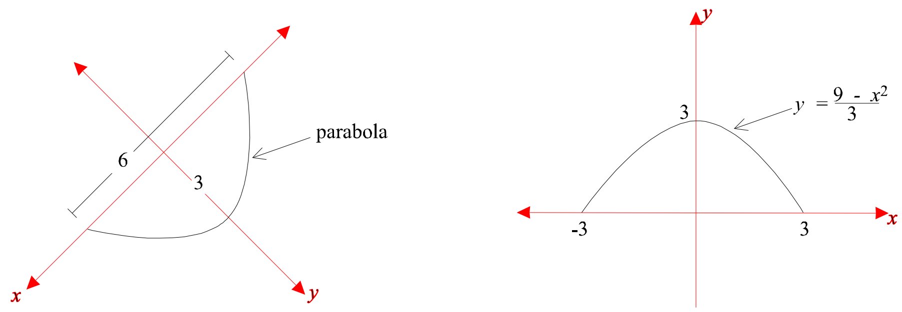
$$y = \frac{b}{r_1 r_2}(x - r_1)(x - r_2) \tag{5.3}$$

EXAMPLE 6 Find the area of the region shown below:



1-10: Parabola for example 6

Solution: We choose the x-axis to be concurrent with—i.e., on top of—the straight segment of length 6, and we choose the y-axis to run through the vertex perpendicular to the x-axis.



1-11: Parabola placed in xy-coordinate system

Orienting the x and y axes as is customary yields the diagram on the right above. The length of the "base" of the region is 6, which corresponds to 3 units to either side of the y-axis. Thus, we have a parabola with a y-intercept of 3 and two x-intercepts of 3 and -3. Its equation is

$$\begin{aligned} y &= \frac{3}{(-3)(3)}(x-3)(x--3) \\ &= \frac{-1}{3}(x-3)(x+3) \\ &= \frac{-1}{3}\left(x^2-9\right) \end{aligned}$$

which simplifies to $y = \frac{1}{3}\left(9 - x^2\right)$. As a result, the area of the region is

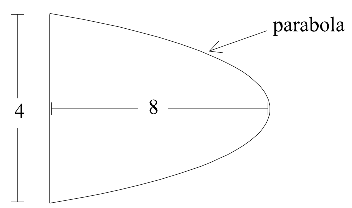
$$A = \frac{1}{3}\int_{-3}^{3}\left(9 - x^2\right)dx = 12$$

Exercises:

Find the area of the region between the two given curves. Some of these will be used again in section 6-2.

1. $y = x,\ y = x^2$
2. $y = 2x + 3,\ y = x^2$
3. $y = 4 - x^2,\ y = 3x$
4. $y = 2x^2 - x^3,\ y = x$
5. $y = 2 - x^2,\ y = x^2$
6. $y = 2x^2,\ y = x^2 + 1$
7. $y = x,\ \ y = x^4$
8. $y = x^4 + 1,\ y = 2x^2$
9. $y = x^3,\ y = x$
10. $y = -x^3,\ y = -x$
11. $y = x^3,\ y = x^2 + 2x$
12. $y = x^4,\ y = 5x^2 - 4$
13. $y = |x|,\ \ y = x^2$
14. $y = |x|,\ y = 2 - x^2$
15. $x = y,\ x = y^2$
16. $x = 2y + 3,\ x = y^2$
17. $x = 4 - y^2,\ x = 3y$
18. $x = 2y^2 - y^3,\ x = y$
19. $x = 2|y|,\ x = y + 3$
20. $x = y + |y|,\ x = y + 1$
21. $y = \sin(x),\ y = \sin(2x)$, over $[0, \pi]$
22. $y = \cos^2(x),\ y = \sin^2(x)$, over $[0, \pi]$
23. $y = \cos(\pi x),\ y = 0$ over $[0, 1]$
24. $y = \sin(\pi x),\ y = 1$ over $[0, 1]$
25. $y = \sec^2(x),\ y = \tan^2(x)$, over $\left[0, \frac{\pi}{4}\right]$
26. $y = 2\cos^2(x),\ y = 1$, over $[0, \pi]$
27. $y = x\sin\left(\pi x^2\right),\ y = x$, over $[0, 1]$
28. $y = 2\ln(x)$ and $y = \ln(x + 2)$
29. $y = e^{2x},\ y = 5e^x - 4$
30. $y = e^x + 4e^{-x},\ y = 5$
31. $y = e^{-x},\ y = e^{-2x}$, over $[0, \ln(2)]$
32. $y = x^{-3},\ y = x^{-2}$, over $[1, \infty)$

33. Let's find the area of the region shown in the figure below.



1-12: Exercise 33

(a) To begin with, define a coordinate system in which y can be written as a function of x.

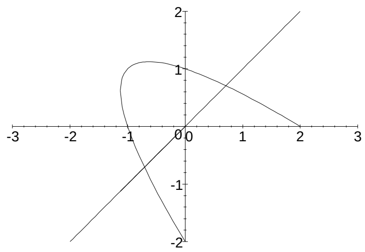
(b) Find the equation of the parabola in this coordinate system using (5.3).

(c) Find the area using a definite integral.

34. What is the area of the region between the line $y = x$ and the parabola

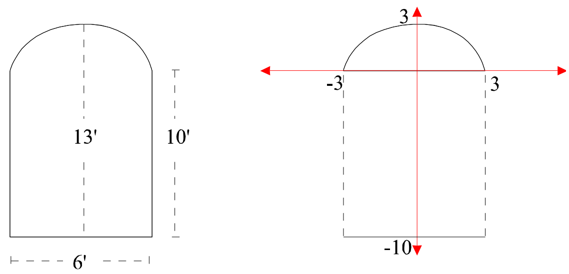
$$x^2 + 2xy + y^2 - x + y = 2$$

The graph of the region is shown below:



1-13: Exercise 34

35. A certain archway is to be filled with a steel-reinforced concrete slab which is to be $\frac{1}{2}$ foot thick.



1-14: Exercise 35

Let's determine how many cubic feet of concrete will be necessary for the job assuming the arc is a parabola

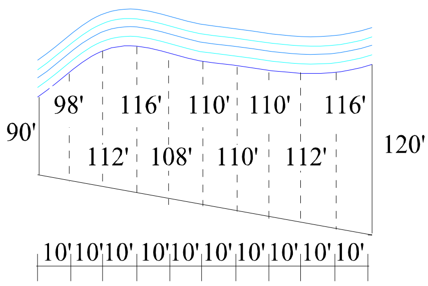
(a) Use the coordinate system shown on the right and (5.3) to define a function whose graph is the arc shown in the picture.

(b) Find the area between the two curves.

(c) Estimate the amount of concrete necessary for the job.

36. Repeat exercise 35 assuming the arc the arc of a circle. Which archway requires more concrete?

37. John buys an irregularly shaped lot bounded on one side by a creek.



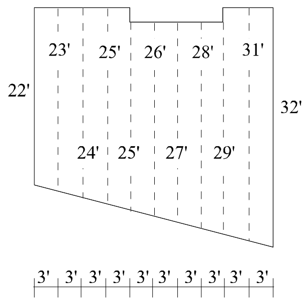
1-15: Exercise 37

He measures distances at 10 foot intervals across the lot.

(a) If the boundary near the stream is the graph of $f(x)$ and the diagonal boundary opposite the stream is the graph of $g(x)$, write an expression for the area of the lot assuming that $x = 0$ corresponds to the left vertical boundary in the plot above.

(b) Use the trapezoidal rule to estimate the area integral in (a). What is the approximate area of the lot in acres? (Note: 1 acre is equal to 43,560 square feet)

38. Below are the dimensions of an irregularly shaped room.

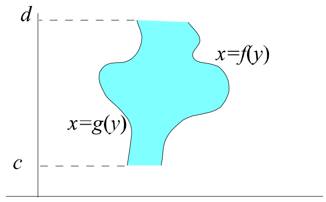


1-16: Exercise 38

(a) Describe the area of the room as a definite integral of the difference of two functions.

(b) Use the trapezoidal rule to compute the integral in (a). Explain why this yields the exact area of the room.

39. **Write to Learn:** Write a short essay which explains why the area of the region



1-17: Exercise 39

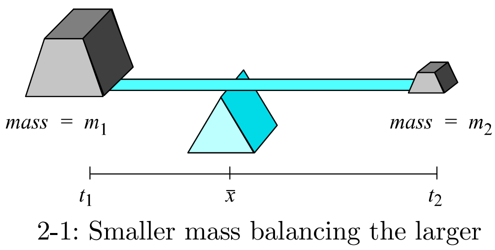
is given by $Area = \int_c^d [f(y) - g(y)]\, dy$. (Hint: mimic the simple function argument leading up to theorem 1.1).

5.2 Centroids

The x coordinate of the Centroid of a Region

In this section, we use definite integrals to define the center or *centroid* of a region in the xy-plane. We do so by extending a simple notion center of mass to a more general notion of *centroid* of a region.

To begin with, the principle of the lever says that a mass of m_2 can balance a larger mass $m_1$if the larger mass is closer to the fulcrum:



2-1: Smaller mass balancing the larger

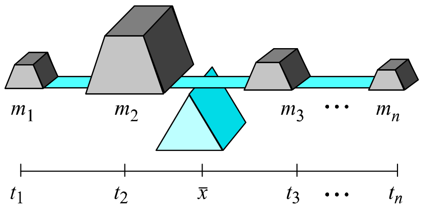
Indeed, suppose the mass m_1 is located at a point t_1, the mass m_2 is located at t_2, and the fulcrum is located at $\bar{x}$. Then the lever balances if the products of the mass and the distance from the fulcrum are the same for both masses:[1]

$$m_1(\bar{x} - t_1) = m_2(t_2 - \bar{x}) \tag{5.4}$$

If we now solve for $\bar{x}$, the result is the *center of mass* of the two masses:

$$\bar{x} = \frac{m_1 t_1 + m_2 t_2}{m_1 + m_2} \tag{5.5}$$

In general, if n masses $m_1, \ldots, m_n$ are located at points $t_1, \ldots t_n$ on the real line,



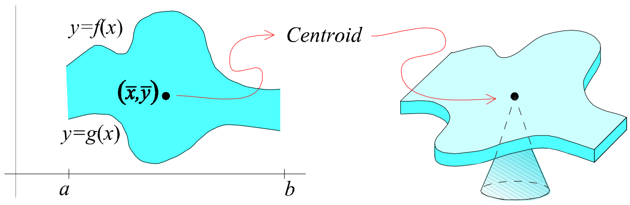
2-2: System of masses balancing each other

then the *center of mass* of the collection of mass-points is

$$\bar{x} = \frac{m_1 t_1 + m_2 t_2 + \ldots + m_2 t_2}{m_1 + m_2 + \ldots + m_n} \tag{5.6}$$

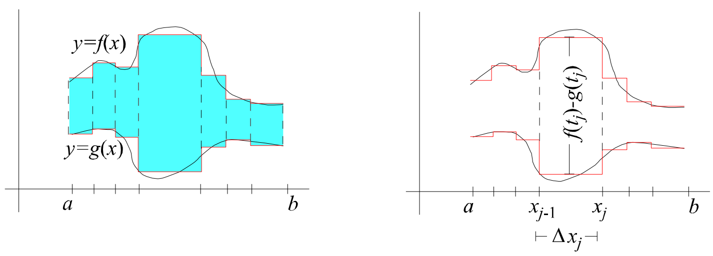
A laminate is a uniform solid which represents a region in the plane.

Let's use (5.6) to define the center of mass of a *laminate* of a type I region in the plane, where a *laminate* of a region is a solid in the shape of the region with a uniform thickness and a uniform mass-density. Moreover, the *centroid* of the region is defined to be the point $(\bar{x}, \bar{y})$ corresponding to the center of mass of the laminate.



2-3: A region and its laminate

To begin with, we approximate both f and g by simple functions over h-fine partitions for each h close to 0.

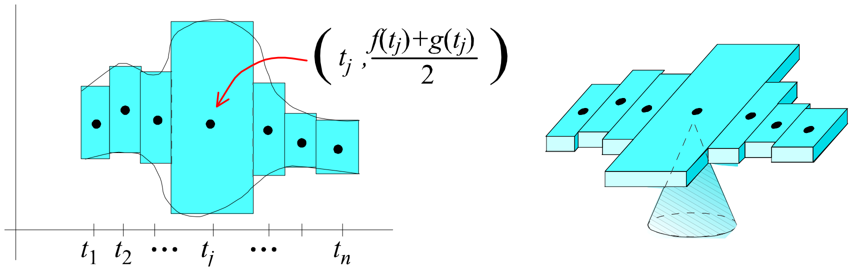


2-4: Simple Function Approximation

Moreover, suppose that the tags t_j are the midpoints of the intervals. Then the result is a collection of n rectangles, where the j^{th} rectangle is centered at

[1] More specifically, the lever balances if the individual torques are equal, where the torque on mass m_j is the product of the gravitational force $-m_j g$ and the distance from the fulcrum.

$\left(t_j, \frac{f(t_j)+g(t_j)}{2}\right)$:



2-5: Rectangular lamina approximation

Their laminates form a system of masses $m_1, \ldots, m_n$. If ρ is the mass of the laminate per unit area of the region, then $m_j = \rho A_j$, where A_j is the area of the j^{th} rectangle. Since $A_j = [f(t_j) - g(t_j)]\,\Delta x_j$, we have

$$m_j = \rho\left[s_\varepsilon(t_j) - r_\varepsilon(t_j)\right]\Delta x_j$$

Thus, if $\bar{x}$ denotes the x-coordinate of the centroid of the *lamina*, then (5.6) implies that

$$\bar{x} \approx \frac{\rho t_1\left[f(t_1) - g(t_1)\right]\Delta x_1 + \ldots + \rho t_n\left[f(t_n) - g(t_n)\right]\Delta x_n}{\rho\left[f(t_1) - g(t_1)\right]\Delta x_1 + \ldots + \rho\left[f(t_n) - g(t_n)\right]\Delta x_n}$$

We now let h approach 0:

$$\bar{x} = \frac{\lim_{h\to 0}\sum_{j=1}^{n}\rho t_j\left[f(t_j) - g(t_j)\right]\Delta x_j}{\lim_{h\to 0}\sum_{j=1}^{n}\rho\left[f(t_j) - g(t_j)\right]\Delta x_j}$$

The denominator is the area of the region. Thus,

$$\bar{x} = \frac{1}{A}\int_a^b x\left[f(x) - g(x)\right]dx \tag{5.7}$$

where $A = \int_a^b \left[f(x) - g(x)\right]dx$ is the area of the region.

Check your Reading *Solve for $\bar{x}$ in (5.4). The result should be (5.5).*

The y coordinate of the Centroid

To find $\bar{y}$, we need only compute the products of the y-coordinate of the centers of the rectangles and the masses of the laminates of the rectangles:

$$\begin{aligned}\left(\frac{f(t_j) + g(t_j)}{2}\right)A_j &= \frac{1}{2}\left(f(t_j) + g(t_j)\right)\left(f(t_j) - g(t_j)\right)\Delta x_j \\ &= \frac{1}{2}\left[f^2(t_j) - g^2(t_j)\right]\Delta x_j\end{aligned}$$

Since $m_j = \rho A_j$, (5.6) implies that

$$\bar{y} = \frac{\rho\left[f^2(t_1) - g^2(t_1)\right]\Delta x_1 + \ldots + \rho\left[f^2(t_n) - g^2(t_n)\right]\Delta x_n}{\rho\left[f(t_1) - g(t_1)\right]\Delta x_1 + \ldots + \rho\left[f(t_n) - g(t_n)\right]\Delta x_n}$$

so that in the limit as h approaches 0 we have

$$\bar{y} = \frac{1}{2A}\int_a^b \left(\left[f(x)\right]^2 - \left[g(x)\right]^2\right)dx \tag{5.8}$$

Let us state these results as a definition.

Definition 3.1: If $f(x) \geq g(x)$ over $[a,b]$, then the centroid of the region between $y = f(x)$ and $y = g(x)$ over $[a,b]$ is defined to be

$$\bar{x} = \frac{1}{A}\int_a^b x\left[f(x) - g(x)\right]dx, \qquad \bar{y} = \frac{1}{2A}\int_a^b \left([f(x)]^2 - [g(x)]^2\right)dx$$

where A denotes the area of the region.

In particular, $(\bar{x}, \bar{y})$ is the center of mass of a laminate of the region.

EXAMPLE 1 Find the centroid of the region between $y = 2 - x$ and $y = 0$ over $[0, 2]$.

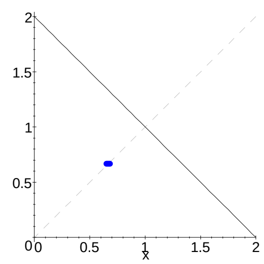
Solution: To begin with, the area of the region is

$$A = \int_0^2 [(2-x) - 0]\,dx = \int_0^2 (2-x)\,dx = \left(2x - \frac{x^2}{2}\right)\Big|_0^2 = 2$$

As a result, the formulas in definition 1.1 become

$$\begin{aligned}\bar{x} &= \frac{1}{2}\int_0^2 x[(2-x) - 0]\,dx = \frac{1}{2}\int_0^2 \left(2x - x^2\right)dx = \frac{2}{3}\\ \bar{y} &= \frac{1}{2\cdot 2}\int_0^2 \left((2-x)^2 - 0^2\right)dx = \frac{1}{4}\int_0^2 \left(4 - 4x + x^2\right)dx = \frac{2}{3}\end{aligned}$$

Since the region is actually a triangle, we have shown that its centroid is $\left(\frac{2}{3}, \frac{2}{3}\right)$.



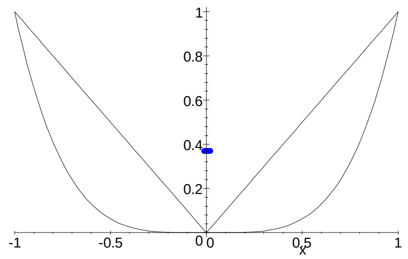
2-6: Symmetry about $y = x$

Check your Reading *Check the computation above by evaluating* $\frac{1}{2}\int_0^2 \left(2x - x^2\right)dx$

Using Symmetry to Simplify Calculation of the Centroid

In example 1, symmetry about the line $y = x$ implies that $\bar{y} = \bar{x}$, so that we actually needed to compute only one of the integrals in example 1. Indeed, symmetry often reduces the amount of computation required in finding $(\bar{x}, \bar{y})$.

EXAMPLE 2 Find $(\bar{x}, \bar{y})$ for the region between $y = |x|$ and $y = x^4$.



2-7: Symmetry about $x = 0$

If the region is symmetric with respect to the y-axis, then $\bar{x} = 0$.

Solution: By symmetry, we can conclude that $\bar{x} = 0$ so that we need only find $\bar{y}$. To do so, we notice that $x^4 = |x|$ when $x = -1, 0, 1$. Thus, the area of the region is

$$\begin{aligned} A &= \int_{-1}^{0} \left(|x| - x^4\right) dx + \int_{0}^{1} \left(|x| - x^4\right) dx \\ &= \int_{-1}^{0} \left(-x - x^4\right) dx + \int_{0}^{1} \left(x - x^4\right) dx \\ &= \frac{3}{10} + \frac{3}{10} \end{aligned}$$

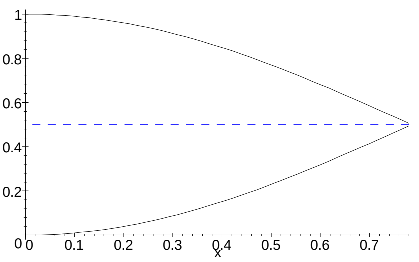
This reduces to $A = \frac{3}{5}$, so that $\bar{y}$ is

$$\bar{y} = \frac{1}{2 \cdot \frac{3}{5}} \int_{-1}^{1} \left((|x|)^2 - \left(x^4\right)^2 \right) dx = \frac{5}{6} \int_{-1}^{1} \left(x^2 - x^8\right) dx = \frac{10}{27}$$

The centroid of a region need not be in the region.

Thus, the centroid of the region is $\left(0, \frac{10}{27}\right)$, which is actually outside of the region.

EXAMPLE 3 Find the centroid of the region between $y = \cos^2(x)$ and $y = \sin^2(x)$ over $\left[0, \frac{\pi}{4}\right]$.



2-8: Symmetry about $y = 0.5$

Solution: Symmetry about the line $y = 0.5$ implies that $\bar{y} = 0.5$. Thus, we need only find $\bar{x}$. The area of the region is

$$A = \int_{0}^{\pi/4} \left(\cos^2(x) - \sin^2(x)\right) dx$$

However, $\cos^2(x) - \sin^2(x) = \cos(2x)$, so that

$$A = \int_{0}^{\pi/4} \cos(2x)\, dx = \frac{1}{2} \sin(2x)\Big|_{0}^{\pi/4} = \frac{1}{2} \sin\left(\frac{2\pi}{4}\right) - \frac{1}{2} \sin(0) = \frac{1}{2}$$

Thus, the x-coordinate of the centroid is

$$\bar{x} = \frac{1}{A}\int_0^{\pi/4} x\left[\cos^2(x) - \sin^2(x)\right]dx = \frac{1}{1/2}\int_0^{\pi/4} x\cos(2x)\,dx$$

Tabular integration with $u = x$ and $dv = \cos(2x)\,dx$ thus yields

u		dv
x		$\cos(2x)$
	$\searrow^{+}$	
1		$\frac{1}{2}\sin(2x)$
	$\searrow^{-}$	
0		$\frac{-1}{4}\cos(2x)$

$$2\int_0^{\pi/4} x\cos(2x)\,dx = 2\left[\frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x)\right]\Big|_0^{\pi/4}$$

As a result, the x-coordinate of the centroid is

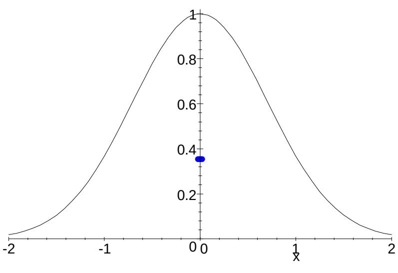
$$\begin{aligned} \bar{x} &= 2\left(\frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x)\Big|_0^{\pi/4}\right) \\ &= 2\left(\frac{1}{2}\cdot\frac{\pi}{4}\cdot 1 + \frac{1}{4}\cdot 0\right) - 2\left(0 + \frac{1}{4}\cdot 1\right) \\ &= \frac{\pi}{4} - \frac{1}{2} \end{aligned}$$

Check your Reading *Mark the centroid of the region in figure 6-14.*

Numerical Integration and Centroids

Finally, the formulas for $\bar{x}$ and $\bar{y}$ can lead to integrals which cannot be evaluated in closed form and thus must be estimated numerically.

EXAMPLE 4 Find the centroid of the region under $y = e^{-x^2}$ over the interval $[-2, 2]$.



2-9: Centroid on y-axis

Solution: The area of the region is estimated numerically to be

$$A = \int_{-2}^{2} e^{-x^2}dx = 1.76416$$

to 5 decimal places. Since the region is symmetric about the y-axis, we must have $\bar{x} = 0$. Thus, we need only find $\bar{y}$, which is

$$\bar{y} = \frac{1}{2A}\int_{-2}^{2}\left[\left(e^{-x^2}\right)^2 - (0)^2\right]dx$$

$$= \frac{1}{2(1.76416)}\int_{-2}^{2} e^{-2x^2}\,dx$$
$$= 0.35519$$

to five decimal places. Thus, the centroid is $(0, 0.35519)$

Exercises:

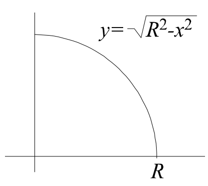
Find the centroids of the regions described below. Whenever possible, use symmetry to reduce the amount of calculation. You found the areas of the regions in 1-12 in section 6-1.

1. $y = x,\ y = x^2$
2. $y = 2x + 3,\ y = x^2$
3. $y = 4 - x^2,\ y = 3x$
4. $y = 2x^2 - x^3,\ y = x$
5. $y = 2 - x^2,\ y = x^2$
6. $y = 2x^2,\ y = x^2 + 1$
7. $y = x,\ \ y = x^4$
8. $y = x^4 + 1,\ y = 2x^2$
9. $y = x^3,\ y = x$
10. $y = -x^3,\ y = -x$
11. $y = x^3,\ y = x^2 + 2x$
12. $y = x^4,\ y = 5x^2 - 4$
13. $y = |x|,\ \ y = x^2$
14. $y = |x|,\ y = 2 - x^2$
15. $y = \cos^2(x),\ y = \sin^2(x)$ over $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$
16. $y = \sec^2(x),\quad y = \tan^2(x)$ over $\left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$
17. $y = \cos(x) + 1,\ y = \sin(x)$ over $[0, \pi]$
18. $y = e^x + e^{-x},\ y = 2$ over $[-\ln(2), \ln(2)]$
19. $y = e^{2x},\quad y = 3e^x - 2$
20. $y = e^{-2x},\quad y = 5e^{-x} - 4$ over $[-2, 2]$
21. $y = \sqrt{4 - x^2},\ y = 0$
22. $y = \sqrt{2 - x^2},\ y = 0$
23. $y = \sqrt{2 - x^2},\ y = |x|$
24. $y = \sqrt{2x - x^2}$

Estimate the centroids of the regions by estimating the integrals in (5.7) and (5.8).

25. $y = \cos\left(x^2\right)$ over $\left[0, \sqrt{\pi/2}\right]$
26. $y = \sin\left(x^2\right)$ over $[0, \sqrt{\pi}]$
27. $y = \sqrt{1 - x^2}$ over $[0, 1]$
28. $y = \sqrt{1 - x^2}$ over $[-1, 1]$
29. $y = \ln(3x - 2),\ \ y = 2\ln(x)$
30. $y = e^{x^2},\ y = e^{5x-4}$

31. In this exercise, we determine the centroid of a quarter circle of radius R.



2-10: Exercise 31

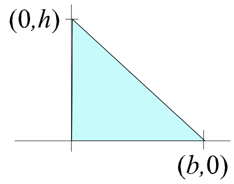
(a) Find the area of the quarter circle without using integrals.
(b) Find $\bar{x}$ as a function of R.

(c) Find $\bar{y}$ as a function of R.
(d) Why would you expect that $\bar{y} = \bar{x}$?

32. In this example, we consider the centroid of the region between $f(x) = 2 - x^2$ and $g(x) = x^2$ over the interval $[-1, 1]$.

(a) Find the area A of the region.
(b) Find the centroid $(\bar{x}, \bar{y})$ of the region.
(c) Graph $f(x) = 2 - x^2$ and $g(x) = x^2$ over the interval $[-1, 1]$. What point would you label as the centroid?

33. **Write to Learn:** Write a short essay in which you discuss the centroid concept and then find the centroid of the triangle with coordinates $(0, h)$ and $(b, 0)$.

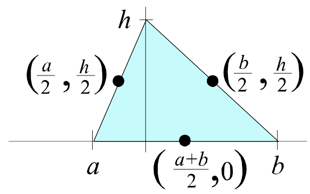


2-11: Exercise 33

34. ***Write to Learn:** In a short essay, define the triangle with coordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3) to be the region between two curves. Then show that the centroid of the triangle is given by the average of the coordinates.

35. It can be shown that the triangle under the graph of

$$f(x) = \begin{cases} h - \frac{h}{a}x & if \quad a \le x < 0 \\ h - \frac{h}{b}x & if \quad x \le 0 \le b \end{cases}$$

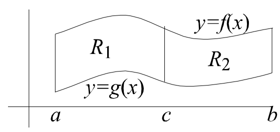


and over $[a, b]$ has a centroid of $\left(\frac{a+b}{3}, \frac{h}{3}\right)$. Use this to answer the following:

(a) Find the equation of the line between the points $(a, 0)$ and the midpoint $\left(\frac{b}{2}, \frac{h}{2}\right)$ of the side opposite $(a, 0)$. Show that the centroid $(\bar{x}, \bar{y})$ is on this line.
(b) Find the equation of the line between the points $(b, 0)$ and the midpoint $\left(\frac{a}{2}, \frac{h}{2}\right)$, and the equation of the line between $(0, h)$ and the midpoint $\left(\frac{a+b}{2}, 0\right)$. Where do these two lines intersect?

36. **Try it out!** Mark off a rectangular grid on a section of cardboard. Locate three non-collinear points on the grid and cut out the implied triangle. Find the average of the coordinates and locate the centroid on the triangle cutout. Show that the triangle cutout balances when supported by a pencil eraser beneath its centroid. Then examine the midpoint relationships discussed in exercise 31.

37. Let R_1 be the region between $y = f(x)$ and $y = g(x)$ over $[a, c]$, and let R_2 be the region between $y = f(x)$ and $y = g(x)$ over $[c, b]$.



Show that if R_1 has centroid $(\bar{x}_1, \bar{y}_1)$ and area A_1 and that if R_2 has centroid $(\bar{x}_2, \bar{y}_2)$ and area A_2, then the centroid $(\bar{x}, \bar{y})$ of the combined region $R_1 \cup R_2$ is

$$\bar{x} = \frac{\bar{x}_1 A_1 + \bar{x}_2 A_2}{A_1 + A_2}, \qquad \bar{y} = \frac{\bar{y}_1 A_1 + \bar{y}_2 A_2}{A_1 + A_2}$$

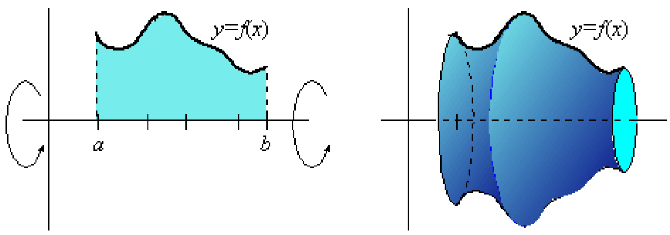
38. Draw a rectangle with one of its sides on the x-axis. Show that $(\bar{x}, \bar{y})$ is at the center of the rectangle.

5.3 Volumes of Solids of Revolution

Revolution of a Region about the x-axis

The definite integral can also be used to determine the volume of a solid obtained by revolving the region between $y = f(x)$ and $y = g(x)$ over $[a, b]$ around a fixed axis. Such a solid is called a *solid of revolution.*

We begin by finding the volume V of the solid generated by revolving the region under a the graph of a continuous function $f(x)$ over the interval $[a, b]$.

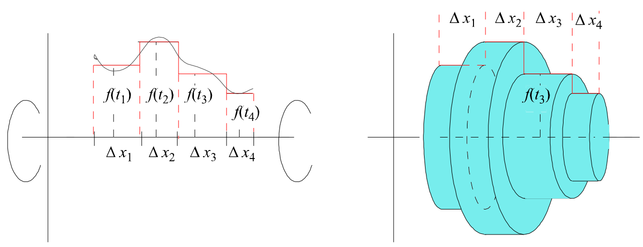


3-1: A Solid of Revolution

Let $\{x_j, t_j\}_{j=1}^{n}$ be an h-fine tagged partition of $[a, b]$ and let

$$s(x) = \begin{cases} f(t_1) & if \quad x_0 \le x < x_1 \\ f(t_2) & if \quad x_1 \le x < x_2 \\ \vdots & \qquad \vdots \\ f(t_n) & if \quad x_{n-1} \le x < x_n \end{cases}$$

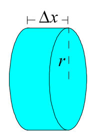
be a simple function approximation of $f(x)$. The solid obtained by revolving the region below the graph of $f(x)$ about the x-axis is approximately the same as the solid obtained by revolving the region below the graph of $s(x)$ about the x-axis.



3-2: Simple function approximation

Since the volume of a circular disk of radius r and width Δx is

$$Volume = \pi r^2 \, \Delta x$$



the volume of the solid of revolution is approximately

$$V \approx \pi \left[f(t_1) \right]^2 \Delta x_1 + \pi \left[f(t_2) \right]^2 \Delta x_2 + \ldots + \pi \left[f(t_n) \right]^2 \Delta x_n$$

If we now let h approach 0, then

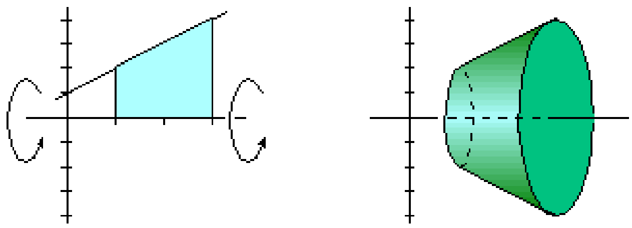
$$V = \lim_{h \to 0} \sum_{j=1}^{n} \pi \left[f(t_j) \right]^2 \Delta x_j$$

so that the volume V of the solid obtained by revolving the region under $f(x)$ over $[a, b]$ around the x-axis is

$$V = \pi \int_a^b \left[f(x) \right]^2 dx \qquad (5.9)$$

when the limit exists.

EXAMPLE 1 Find the volume of the solid which results from revolving the region under $y = x + 1$ and over $[1, 3]$ about the x-axis.



3-3: Solution of revolution of region under $y = x + 1$ over $[1, 3]$

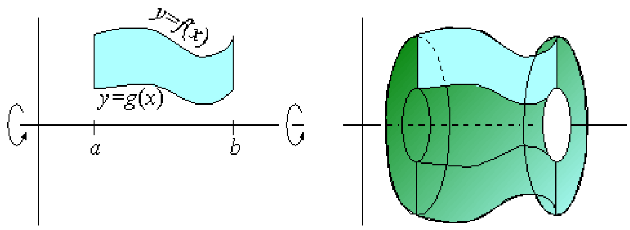
Solution: According to (5.10), the volume of the resulting solid of revolution is

$$\begin{aligned} V &= \pi \int_1^3 (x+1)^2 \, dx \qquad (5.10) \\ &= \pi \int_1^3 \left(x^2 + 2x + 1 \right) dx \\ &= \pi \left(\frac{x^3}{3} + x^2 + x \right) \Big|_1^3 \\ &= \frac{56\pi}{3} \end{aligned}$$

Check your Reading *Explain why $V = \pi r^2 \Delta x$ is the formula for the volume of a circular disk.*

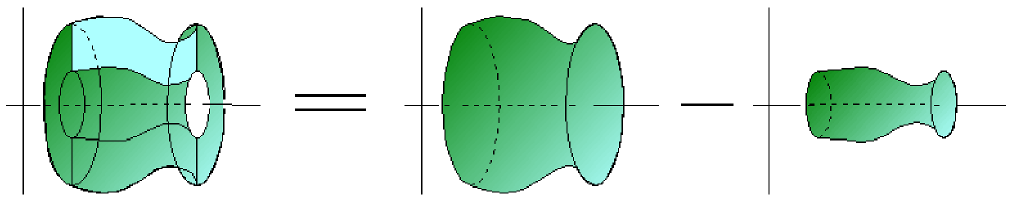
Volume of a Solid of Revolution of a Type I Region

The formula (5.9) can be extended to a formula for the volume of the solid of revolution of a type I region. In particular, suppose that $f(x) \geq g(x)$ over an interval $[a, b]$. To find the volume of the solid obtained by revolving the region between $f(x)$ and $g(x)$ around the x-axis,



3-4: Solid of revolution about x-axis of Type I region

we first notice that the solid can also be constructed by removing the solid obtained by revolving the region under $g(x)$ from the solid obtained by revolving the region under $f(x)$.


3-5: Original is larger solid with smaller solid removed

Thus, the volume of the solid obtained from revolving the region between the graphs of $f(x)$ and $g(x)$ over the interval $[a, b]$ is

$$V = \int_a^b \pi [f(x)]^2 \, dx - \int_a^b \pi [g(x)]^2 \, dx$$

which simplifies to yield the following theorem:

Theorem 3.1: The volume of the solid of revolution of a type I region about the x-axis is given by

$$V = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2 \right) dx \tag{5.11}$$

Notice that if $g(x) = 0$ in (5.11), then theorem 3.1 reduces to (5.9).

EXAMPLE 2 Find the volume of the solid obtained by revolving the region between $y = \sqrt{x}$ and $y = x$ around the x-axis.

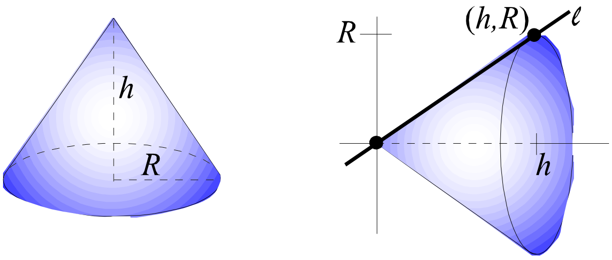
Solution: To do so, we first find that the two curves intersect when $x = 0$ and $x = 1$. Thus, (5.11) becomes

$$\begin{aligned} V &= \pi \int_0^1 \left(\left(\sqrt{x}\right)^2 - (x)^2 \right) dx \\ &= \pi \int_0^1 \left(x - x^2 \right) dx \\ &= \frac{\pi}{6} \end{aligned}$$

Theorem 3.1 can be used to obtain many of the volume formulas of classical geometry, including volumes of cones, frustums, spheres and ellipsoids.

EXAMPLE 3 Let's develop a formula for the volume of a cone with a height of h and with a base of radius R.

Solution: First, we place the cone in a coordinate system so that it is the solid obtained from revolving the region under the line ℓ and over $[0,h]$.



3-6: Volume of a right circular cone

Since ℓ has a slope of $\frac{R}{h}$ and passes through the origin, it is the graph of

$$L(x) = \frac{R}{h}x$$

As a result, the cone is the solid of revolution about the x-axis generated by the region over $[0,h]$ which is between $y = L(x)$ and $y = 0$. Thus, (5.11) yields a volume for the cone of

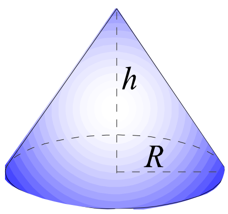
$$V = \pi \int_0^h \left[\left(\frac{R}{h}x \right)^2 - 0^2 \right] dx$$

Evaluating the integral yields the formula

$$V = \frac{\pi R^2}{h^2} \int_0^h x^2 dx = \frac{\pi R^2}{h^2} \left(\left. \frac{x^3}{3} \right|_0^h \right) = \frac{\pi R^2}{h^2} \frac{h^3}{3}$$

Thus, the volume of a cone with a height of h and with a base of radius R is

$$V = \frac{1}{3}\pi R^2 h$$



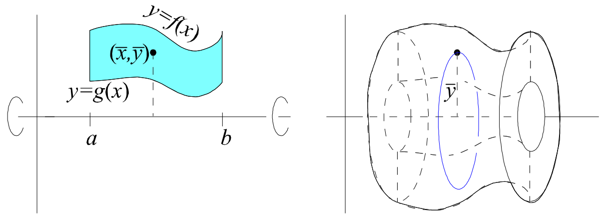
Check your Reading *What is the volume of a cone of with height $h = 9$ inches and with a base of radius $R = 4$ inches?*

A Theorem of Pappus

The integral in (5.11) should be familiar from the last section. Indeed, if we multiply and divide by $2A$, where A is the area of the region, then (5.11) becomes

$$V = \pi \cdot 2A \cdot \frac{1}{2A} \int_a^b \left([f(x)]^2 - [g(x)]^2 \right) dx$$

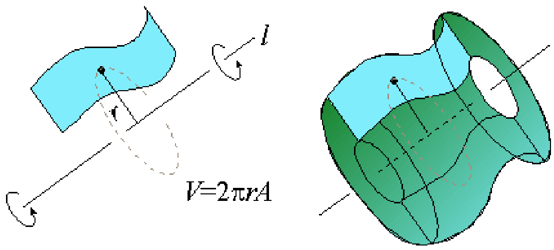
which simplifies to $V = 2\pi\bar{y}A$, where $\bar{y}$ is the y-coordinate of the centroid of the region. Moreover, $2\pi\bar{y}$ is the *circumference* of the path traced out by $\bar{y}$ as the region is revolved about the x-axis:



3-7: Centroid traces out a circle of radius $\bar{y}$

Thus, we have proven a theorem which was first stated in about 320 A.D. by Pappus of Alexandria:

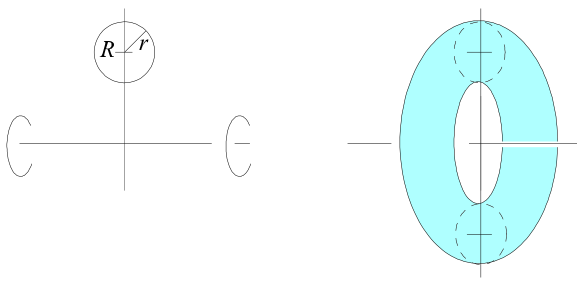
> **Pappus' Theorem:** If a region with area A is revolved around a line l and if r is the distance from the line l to the centroid of the region, then the volume of the solid of revolution is
>
> $$V = \textit{Circumference} \times \textit{Area} = 2\pi r A$$



3-8: Theorem of Pappus

The theorem of Pappus can often be used to find volumes of solids without having to resort to integrals.

EXAMPLE 4 Derive the formula for the volume of the torus obtained by revolving a circle of radius r centered at $(0, R)$ about the x-axis, where $0 < r < R$.



3-9: A Torus is a Circle revolved about the x-axis

Solution: By symmetry, the centroid of the circle is also its center, which is $(0, R)$. Moreover, the circumference of the path of the centroid is $2\pi R$. Since the area of the circle is πr^2, the theorem of Pappus implies that the Volume of the torus is

$$V = 2\pi R \cdot \pi r^2 = 2\pi^2 r^2 R$$

Check your Reading *What is the volume of a torus with inner radius $r = 2$ inches and outer radius $R = 5$ inches?*

Revolution about the y-axis

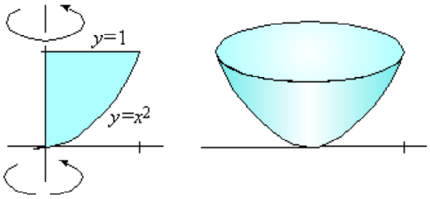
An immediate application of Pappus' theorem is that if a region is rotated about the y-axis, then its volume is $V = 2\pi \bar{x} A$, which can be written

$$V = 2\pi A \cdot \frac{1}{A} \int_a^b x \left[f(x) - g(x) \right] dx$$

Thus, the volume of the solid obtained by revolving the region between $y = f(x)$ and $y = g(x)$ over $[a, b]$ about the y-axis is

$$V = 2\pi \int_a^b x \left[f(x) - g(x) \right] dx \qquad (5.12)$$

EXAMPLE 5 Find the volume of the solid of revolution obtained by revolving the region in the first quadrant between $y = x^2$ and $y = 1$ about the y-axis.



3-10: Revolution about the y-axis

Solution: The formula (5.12) tells us that the volume of this "cereal bowl" type solid is

$$\begin{aligned} V &= 2\pi \int_0^1 x \left[1 - x^2 \right] dx \\ &= 2\pi \int_0^1 \left(x - x^3 \right) dx \\ &= 2\pi \left(\frac{x^2}{2} - \frac{x^4}{4} \right) \Big|_0^1 \\ &= \frac{\pi}{2} \end{aligned}$$

Exercises:

Sketch the given solid of revolution, and then find its volume.

1. under $f(x) = x$ over $[0,3]$ revolved about the x-axis

2. under $f(x) = 3x + 2$ over $[0,1]$ revolved about the x-axis

3. under $f(x) = \sqrt{x}$ over $[0,1]$ revolved about the x-axis

4. under $f(x) = \sqrt{x+1}$ over $[0,3]$ revolved about the x-axis

5. under $f(x) = e^x$ over $[0, \ln 2]$ revolved about the x-axis

6. under $f(x) = \sqrt{\sin(x)}$ over $[0,\pi]$ revolved about the x-axis

7. between $y = x$ and $y = x^2$ revolved about the x-axis

8. between $y = \sqrt{x}$ and $y = x^2$ revolved about the x-axis

9. between $y = |x|$ and $y = x^4$ revolved about the x-axis

10. between $y = e^{2x} - 1$ and $y = 2e^x$ revolved about the x-axis

11. between $y = \cos(x)$ and $y = \sin(x)$ over $\left[0, \frac{\pi}{4}\right]$, about the x-axis

12. between $y = \sqrt{2}\cos(x)$ and $y = 1$ over $\left[0, \frac{\pi}{4}\right]$, about the x-axis

13. between $y = \sec(x)$ and $y = \tan(x)$ over $\left[0, \frac{\pi}{4}\right]$, about the x-axis

14. between $y = \csc(x)$ and $y = \cot(x)$ over $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$, about the x-axis

15. between $y = x$ and $y = 1$ over $[0,1]$ revolved about the y-axis

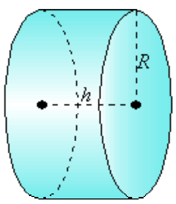
16. under $y = \sqrt{1-x^2}$ and over $[0,1]$ revolved about the y-axis

17. between $y = \cos(x)$ and $y = \sin(x)$ over $\left[0, \frac{\pi}{4}\right]$, about the y-axis

18. between $y = \cos^2(x)$ and $y = \sin^2(x)$ over $\left[0, \frac{\pi}{4}\right]$, about the y-axis

Define a coordinate system in which the given solid is a solid of revolution, and then use (5.9) or (5.11) to determine its volume.

19.



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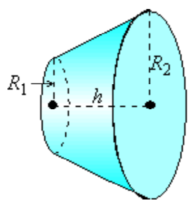

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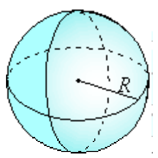

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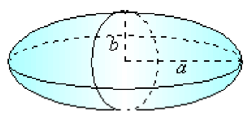

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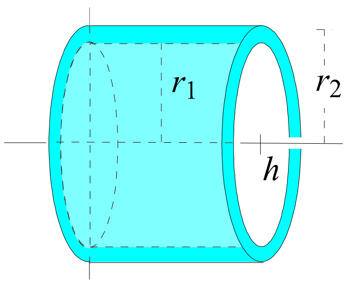

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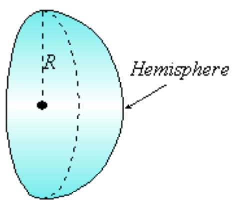
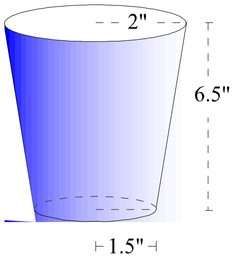


25. A certain drinking cup from a fast food restaurant is in the shape of the *frustum* of a right circular cone.

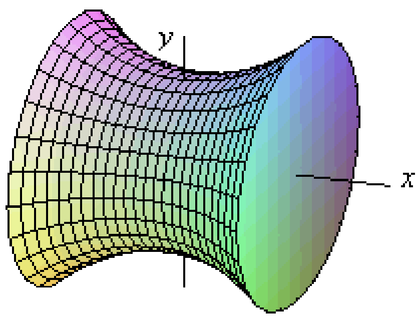


3-11: Drinking cup in exercise 25

If the radius of the bottom is 1.5", the radius of the top is 2" and the height of the cup is 6.5", what is the volume of the cup in cubic inches? How many fluid ounces is that? (1 cubic inch = 0.554 fluid ounces)

26. How many fluid ounces are in the cup in exercise 25 when it is filled to half its height?

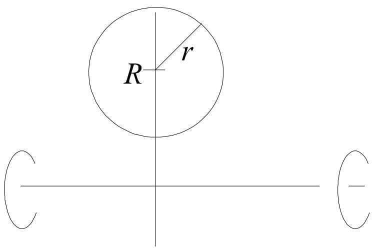
27. The surface of revolution obtained by revolving the graph of $y = e^x + e^{-x}$ about the x-axis is called a *catenoid.*



3-12: Catenoid in exercise 27

What is the volume of the catenoid obtained by revolving $y = e^x + e^{-x}$ over $[-\ln 2, \ln 2]$ about the x-axis?

28. A torus can be generated by revolving a circle of radius r centered at $(0, R)$ about the x-axis, where $r < R$.



3-13: Torus as solid of revolution about x-axis

(a) The equation of the circle is

$$x^2 + (y - R)^2 = r^2$$

Solve for y to obtain a representation of the circle as the region between the graph of two functions.

(b) Use (5.11) to find the volume of a torus with inner radius r and outer radius R.

29. A certain soft drink bottle is approximated by the solid generated by revolving the region over $[0, 9.75"]$ and under the graph of

$$c(x) = \begin{cases} 0.5x + 1 & if \quad 0.0 \le x < 0.5 \\ 1.25 & if \quad 0.5 \le x < 5.0 \\ 2.1875 - 0.1875x & if \quad 5.0 \le x < 9.0 \\ 0.5 & if \quad 9.0 \le x \le 9.75 \end{cases}$$

(a) Graph $c(x)$ over the interval $[0, 9.75]$ and then use the graph to sketch the bottle itself. (Note: to get an accurate representation, you will need equal scaling on both axes).

(b) Use (5.9) to find the volume of the soft drink bottle in cubic inches. How many cubic ounces is that? (1 cubic inch = 0.554 fluid ounces)

30. A certain small mixing bowl is the same as the solid generated by revolving the region between $y = 3.75''$ and

$$y = 0.00035577\, x^{8.7}$$

over $[0, 2.9]$ around the y-axis. Use (5.12) to determine the volume of the mixing bowl. How many fluid ounces will the mixing bowl hold? (1 cubic inch = 0.554 fluid ounces)

31. A certain soft drink bottle is approximated by the solid generated by revolving the region over $[0, 7.4]$ and under the graph of

$$c(x) = \begin{cases} \begin{array}{c} 0.71 + 0.76x - 0.78x^2 + 0.32x^3 \\ -0.053x^4 + 0.003x^5 \end{array} & if \quad 0.0 \le x < 7.05 \\ -109.34 + 30.56x - 2.12x^2 & if \quad 7.05 \le x < 7.4 \end{cases}$$

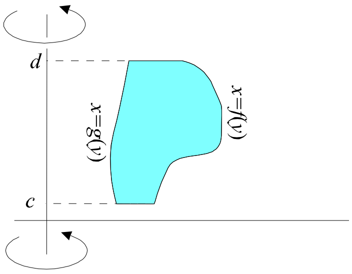
(a) Graph $c(x)$ over the interval $[0, 7.4]$ and then use the graph to sketch the bottle itself. (Note: to get an accurate representation, you will need equal scaling on both axes).

(b) Use (5.9) to find the volume of the soft drink bottle in cubic inches. How many cubic ounces is that? (1 cubic inch = 0.554 fluid ounces)

32. **Try it out!** Find a container that is a solid of revolution and project its silhouette onto a piece of paper. Use the silhouette to determine a region which can be revolved around the x-axis to obtain the solid represented by the container. Then find its volume and compare to the actual volume of the container.

33. A solid can be obtained by revolving the region between the curves $x = f(y)$ and $x = g(y)$ over $[c, d]$ around the y-axis. Use the definition of the integral to explain why its volume is

$$V = \pi \int_c^d \left([f(y)]^2 - [g(y)]^2 \right) dy$$



(5.13)

34. In this exercise, the volume of the solid obtained by revolving the region between $y = \ln(x)$ and $y = 1$ over $[1, e]$ is determined in two different ways.

(a) Use (5.11) to find the volume of the solid directly.

(b) Describe the region between $y = \ln(x)$ and $y = 1$ over $[1, e]$ as the region is described in (5.13).

(c) Find the volume of the solid obtained by revolving the region in (b) about the y-axis using the formula in (5.13).

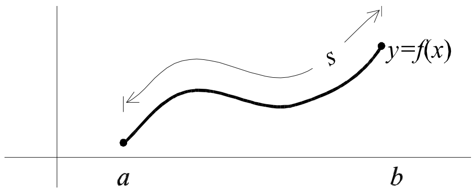
Use (5.13) to find the volume of the solid obtained by revolving the given Type II region around the y-axis.

35. $x = y$, $x = 0$, over $[0, 1]$
36. $x = y^2$, $x = 0$, over $[0, 1]$
37. $x = 2 - y$, $x = y$, over $[0, 1]$
38. $x = e^y$, $x = 1$, over $[0, 1]$
39. $x = y$, $x = y^2$
40. $x = 2 - y$, $x = \sqrt{4 - y^2}$

5.4 Arclength

Derivation of the Arclength Integral

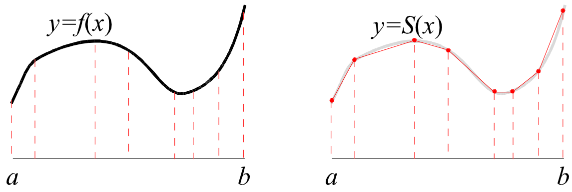
In this section, we see that integrals are also used to define the length of the graph of a function over a given interval. In particular, if $f(x)$ is piecewise differentiable over $[a, b]$, then the *arclength* of the graph of $f(x)$ is the distance s from one end of the graph of $f(x)$ over $[a, b]$ to the other end.



4-1: s is the length of the graph of $f(x)$ over $[a, b]$

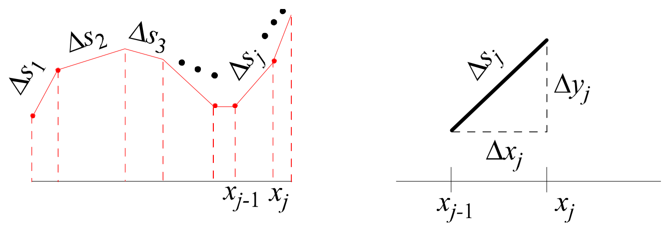
Let's develop a method for measuring the arclength of a function over a given interval.

To begin with, for each $h > 0$, let $\{x_j, t_j\}$ be an h-fine tagged partition of $[a, b]$, and let $y_j = f(x_j)$. Let us then let $S(x)$ be the piecewise linear function whose graph connects the point (x_{j-1}, y_{j-1}) to the point (x_j, y_j). We say that $S(x)$ is a *polygonal approximation* of $f(x)$.



4-2: $S(x)$ is a polygonal approximation of $f(x)$

Also, let us let $\Delta s_1, \ldots, \Delta s_n$ denote the lengths of the individual sections of the graph of $S(x)$,



4-3: Length of a segment in a polygonal approximation

Since a run of length Δx_j results in a rise of distance $\Delta y_j = f(x_j) - f(x_{j-1})$, the Pythagorean theorem implies that

$$\Delta s_j = \sqrt{\Delta x_j^2 + \Delta y_j^2}$$

Thus, if s_{app} denotes the length of the graph of the polygonal approximation $y = S(x)$, then $s_{app} = \sum_{j=1}^{n} \Delta s_j$, which in turn implies that

$$s_{app} = \sum_{j=1}^{n} \sqrt{\Delta x_j^2 + \Delta y_j^2} \tag{5.14}$$

However, the *Mean Value theorem* says that we can choose the tags t_j such that $\Delta y_j = f'(t_j)\,\Delta x_j$. As a result, (5.14) implies that

$$s_{app} = \sum_{j=1}^{n} \sqrt{\Delta x_j^2 + [f'(t_j)\,\Delta x_j]^2} = \sum_{j=1}^{n} \sqrt{1 + [f'(t_j)]^2}\,\Delta x_j$$

The result is a Riemann sum. Thus, in the limit as h approaches 0, we have

$$s = \lim_{h\to 0} s_{app} = \lim_{h\to 0} \sum_{j=1}^{n} \sqrt{1 + [f'(t_j)]^2}\,\Delta x_j$$

where the limit is over h-fine partitions of $[a, b]$. This leads us to the following definition:

Definition 4.1: If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then the length of the curve $y = f(x)$ over the interval $[a, b]$ is given by

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx \tag{5.15}$$

It should be noted that there are curves whose lengths cannot be defined (such as the fractal interpolation function mentioned in section 2.4 and again in the *Next Step* at the end of the last chapter). Indeed, our motivation for definition 4.1 depended heavily on our use of the Mean Value theorem.

EXAMPLE 1 Find the arclength of curve $y = \frac{3}{4}x$ over the interval $[0, 4]$.

Solution: If $f(x) = \frac{3}{4}x$, then $f'(x) = \frac{3}{4}$. Thus, the arclength is

$$s = \int_0^4 \sqrt{1 + \left[\frac{3}{4}\right]^2}\, dx$$

Since the integrand is constant, the fundamental theorem implies that

$$s = 4\sqrt{1 + \left[\frac{3}{4}\right]^2} = 4\sqrt{1 + \frac{9}{16}} = 4\sqrt{\frac{25}{16}} = 5$$

Check your Reading *How is the graph of $f(x) = \frac{3}{4}x$ over $[0, 4]$ related to a 3-4-5 triangle?*

Substitution and Identities

When the arclength integral (5.15) can be evaluated with the fundamental theorem, it often requires either a substitution, an identity or both.

EXAMPLE 2 Find the arclength of $y = x^{3/2}$ over $[0, 4]$.

Solution: Since $y' = \frac{3}{2}x^{1/2}$, the arclength is

$$s = \int_0^4 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\, dx = \int_0^4 \sqrt{1 + \frac{9}{4}x}\, dx$$

Let's let $u = 1 + \frac{9}{4}x$, $du = \frac{9}{4}dx$, $u(0) = 1$, and $u(4) = 10$. Then

$$\begin{aligned} s &= \frac{4}{9}\int_1^{10} \sqrt{u}du \\ &= \frac{4}{9}\int_1^{10} u^{1/2}du \\ &= \frac{4}{9}\frac{u^{3/2}}{3/2}\Big|_1^{10} \\ &= \frac{2}{3}\left[(10)^{3/2} - 1\right] \end{aligned}$$

Indeed, identities such as

$$1 + \tan^2(x) = \sec^2(x)$$

are often used to eliminate the square root in (5.15). Moreover, arclength formulas may involve the antiderivative rule

$$\int \sec(x)\, dx = \ln|\sec x + \tan x| + C \tag{5.16}$$

which was derived on page 337.

Indeed, Mercator used essentially the same formula (5.16) in 1569—fifty years before Napier invented logarithms and a century before Newton and Leibniz discovered the Calculus—in constructing his map of the world now known as a Mercator projection.

EXAMPLE 3 Find the length of the curve $y = \ln|\sec(x)|$ over $\left[0, \frac{\pi}{4}\right]$.

Solution: To do so, we first compute the derivative,

$$f'(x) = \frac{d}{dx}\ln|\sec(x)| = \frac{\frac{d}{dx}\sec(x)}{\sec(x)} = \frac{\sec(x)\tan(x)}{\sec(x)} = \tan(x)$$

after which we substitute into (5.15):

$$s = \int_0^{\pi/4} \sqrt{1 + (\tan x)^2}\, dx$$

The identity $\sec^2(x) = 1 + \tan^2(x)$ thus allows us to write the arclength as

$$s = \int_0^{\pi/4} \sqrt{\sec^2(x)}\, dx = \int_0^{\pi/4} \sec(x)\, dx$$

Thus, the fundamental theorem implies that

$$\begin{aligned} s &= \ln|\sec x + \tan x|\,\Big|_0^{\pi/4} \\ &= \ln\left|\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right| - \ln|\sec(0) + \tan(0)| \\ &= \ln\left|\sqrt{2} + 1\right| - \ln|1 + 0| \\ &= \ln\left(\sqrt{2} + 1\right) \end{aligned}$$

Check your Reading *What happened to* $\ln|1+0|$ *in the next to last line in example 3?*

Perfect Squares

In example 3, the quantity $1 + [f'(x)]^2$ reduced to a perfect square, thus eliminating the square root. In many of the exercises, simplifying $1 + [f'(x)]^2$ to a perfect square is a key step in evaluating the integral.

EXAMPLE 4 Find the arclength of $y = e^{x/2} + e^{-x/2}$over $[0, 1]$.

Solution: If $f(x) = e^{x/2} + e^{-x/2}$, then $f'(x) = \frac{1}{2}e^{x/2} - \frac{1}{2}e^{-x/2}$, so that

$$\begin{aligned} 1 + [f'(x)]^2 &= 1 + \left[\frac{1}{2}e^{x/2} - \frac{1}{2}e^{-x/2}\right]^2 \\ &= 1 + \left(\frac{1}{2}e^{x/2}\right)^2 - 2 \cdot \frac{1}{2}e^{x/2} \cdot \frac{1}{2}e^{-x/2} + \left(\frac{1}{2}e^{-x/2}\right)^2 \\ &= 1 + \left(\frac{1}{2}e^{x/2}\right)^2 - \frac{1}{2} + \left(\frac{1}{2}e^{-x/2}\right)^2 \end{aligned}$$

since $e^{x/2} \cdot e^{-x/2} = 1$. This can then be simplified to

$$1 + [f'(x)]^2 = \left(\frac{1}{2}e^{x/2}\right)^2 + \frac{1}{2} + \left(\frac{1}{2}e^{-x/2}\right)^2$$

Since $\left[\frac{1}{2}e^{x/2} - \frac{1}{2}e^{-x/2}\right]^2 = \left(\frac{1}{2}e^{x/2}\right)^2 - \frac{1}{2} + \left(\frac{1}{2}e^{-x/2}\right)^2$, it follows that

$$1 + [f'(x)]^2 = \left(\frac{1}{2}e^{x/2}\right)^2 + \frac{1}{2} + \left(\frac{1}{2}e^{-x/2}\right)^2 = \left[\frac{1}{2}e^{x/2} + \frac{1}{2}e^{-x/2}\right]^2$$

Thus, the length of the graph of $f(x)$ over $[0, 1]$ is

$$\begin{aligned} s &= \int_0^1 \sqrt{\left[\frac{1}{2}e^{x/2} + \frac{1}{2}e^{-x/2}\right]^2}\, dx \\ &= \int_0^1 \left(\frac{1}{2}e^{x/2} + \frac{1}{2}e^{-x/2}\right) dx \\ &= e^{x/2} - e^{-x/2}\Big|_0^1 \\ &= e^{1/2} - e^{-1/2} \end{aligned}$$

EXAMPLE 5 Find the arclength of $y = x^3 + \frac{1}{12x}$ over $[1, 2]$

Solution: If $f(x) = x^3 + \frac{1}{12x}$, then $f'(x) = 3x^2 - \frac{1}{12x^2}$, so that

$$\begin{aligned} 1 + [f'(x)]^2 &= 1 + \left(3x^2 - \frac{1}{12x^2}\right)^2 \\ &= 1 + 9x^4 - \frac{1}{2} + \frac{1}{144x^4} \\ &= 9x^4 + \frac{1}{2} + \frac{1}{144x^4} \end{aligned}$$

If $\left(3x^2 - \frac{1}{12x^2}\right)^2 = 9x^4 - \frac{1}{2} + \frac{1}{144x^4}$, then similarly $\left(3x^2 + \frac{1}{12x^2}\right)^2 = 9x^4 + \frac{1}{2} + \frac{1}{144x^4}$, which implies that

$$1 + [f'(x)]^2 = \left(3x^2 + \frac{1}{12x^2}\right)^2$$

Thus, the length of the curve over $[1,2]$ is

$$\begin{aligned} s &= \int_1^2 \sqrt{\left(3x^2 + \frac{1}{12x^2}\right)^2}\,dx \\ &= \int_1^2 \left(3x^2 + \frac{1}{12x^2}\right) dx \\ &= x^3 - \frac{1}{12x}\Big|_1^2 \\ &= 7\frac{1}{24} \end{aligned}$$

Check your Reading *Why was it important to obtain perfect squares in examples 4 and 5?*

Numerical Approximation of Arclength

In spite of the effort we expend in evaluating arclength integrals in closed form, it is not uncommon in applications to obtain arclength integrals that cannot be evaluated in closed form. Such integrals often must instead be estimated numerically.

EXAMPLE 6 Estimate numerically the length of the graph of $f(x) = x^2$ over $[0,1]$.

Solution: Since $f'(x) = 2x$, we have

$$1 + [f'(x)]^2 = 1 + 4x^2$$

As a result, the arclength integral is

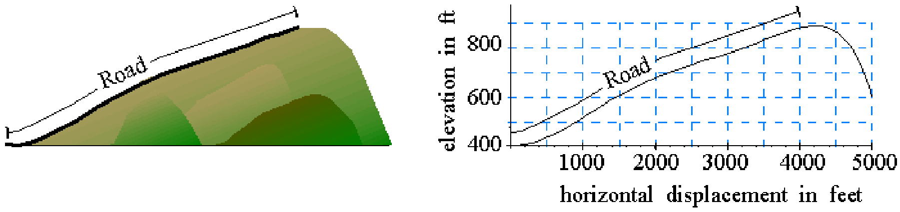
$$s = \int_0^1 \sqrt{1+4x^2}\,dx$$

Although there are techniques for evaluating this integral, none of the techniques we have acquired thus far will work. Thus, we use a calculator or computer to estimate the arclength integral numerically, which results in

$$s = \int_0^1 \sqrt{1+4x^2}\,dx = 1.4798$$

In many applications, the function defining the curve is defined only by a set of data. In such cases, the polygonal approximation itself can be used to estimate the length of the curve.

EXAMPLE 7 Suppose a road is to be built to the top of a ridge. A profile of the ridge is placed on a grid where vertical elevations in feet above sea level are measured at regular horizontal displacements.



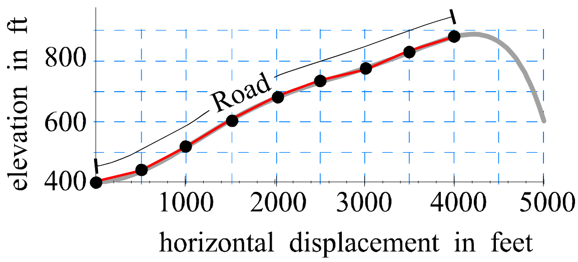
4-8: Road to the top of a ridge

How long is the road?

Solution: To begin with, the elevations at horizontal distances are measured:

$x = horizontal$	0	500	1000	1500	2000	2500	3000	3500	4000
$y = elevation$	400	450	525	600	675	725	775	825	875

A polygonal approximation to the curve is constructed using the data set above.



Let's now use (5.14) to compute the length of this polygonal approximation.

In particular, let's use a table in which we construct Δx_j, Δy_j, and Δs_j.

j	Δx_j	Δy_j	$\Delta s_j = \sqrt{\Delta x_j^2 + \Delta y_j^2}$	
1	500	50	$\sqrt{500^2 + 50^2}$	$= 502.49$
2	500	75	$\sqrt{500^2 + 75^2}$	$= 505.59$
3	500	75	$\sqrt{500^2 + 75^2}$	$= 505.59$
4	500	75	$\sqrt{500^2 + 75^2}$	$= 505.59$
5	500	50	$\sqrt{500^2 + 50^2}$	$= 502.49$
6	500	50	$\sqrt{500^2 + 50^2}$	$= 502.49$
7	500	50	$\sqrt{500^2 + 50^2}$	$= 502.49$
8	500	50	$\sqrt{500^2 + 50^2}$	$= 502.49$
			$\sum_{j=1}^{8} \Delta s_j$	$= 4029.22$

The sum in the last column thus gives us a good approximation of the arclength

$$s_{app} = \sum_{j=1}^{8} \Delta s_j = 4029.22 \; feet$$

This is about $s \approx 4029.22/5380 = 0.75$ miles.

Exercises:

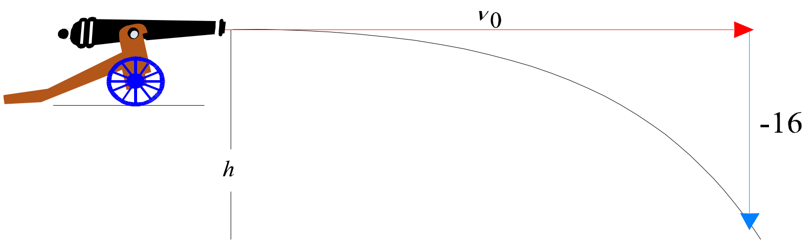
Find the length of the graph of the function over the given interval. Look for opportunities to use identities, substitution or completing the square.

1. $f(x) = 2x + 1$ over $[0, 1]$
2. $f(x) = 3x + 2$ over $[0, 1]$
3. $f(x) = 5$ over $[2, 3]$
4. $f(x) = -2$ over $[3, 4]$
5. $f(x) = |x|$ over $[-1, 1]$
6. $f(x) = x - |x|$ over $[-1, 1]$
7. $f(x) = \ln|\cos x|$ over $\left[0, \frac{\pi}{4}\right]$
8. $f(x) = \ln|\csc(x)|$ over $\left[0, \frac{\pi}{2}\right]$
9. $f(x) = \frac{2}{3}x^{3/2}$ over $[0, 1]$
10. $f(x) = \frac{3}{2}x^{2/3}$ over $[0, 1]$
11. $f(x) = \frac{2}{3}(x-1)^{3/2}$ over $[1, 2]$
12. $f(x) = \frac{2}{3}(x+1)^{3/2}$ over $[-1, 0]$
13. $f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln(x)$ over $[1, 2]$
14. $f(x) = \frac{1}{3}x^{3/2} - x^{1/2}$ over $[0, 1]$
15. $f(x) = e^x + \frac{1}{\sqrt{2}}e^{-x}$ over $[0, \ln(2)]$
16. $f(x) = \frac{1}{4}e^{4x} + \frac{1}{16}e^{-4x}$ over $[0, 1]$
17. $f(t) = \ln\left(\dfrac{e^t - 1}{e^t + 1}\right)$ over $[0, 1]$
18. $f(t) = \frac{1}{2}\ln|\sin(2x)|$ over $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$

Set up the arclength integrals for each of the following and then approximate using numerical integration.

19. $f(x) = \frac{1}{2}x^2$ over $[0, 1]$
20. $f(x) = \frac{1}{3}x^3$ over $[0, 1]$
21. $f(x) = e^x$ over $[0, 1]$
22. $f(x) = \ln(x)$ over $[1, e]$
23. $f(x) = \sin(x)$ over $[0, \pi]$
24. $f(x) = \cos(x)$ over $[0, \pi]$
25. $f(x) = \tan(x)$ over $\left[0, \frac{\pi}{4}\right]$
26. $f(x) = \sec(x)$ over $\left[0, \frac{\pi}{4}\right]$

27. If a cannonball is fired from a height h with an initial velocity of v_0 feet per second,



4-9: Trajectory of a Cannonball

then the trajectory of the cannonball is along the curve

$$y = h - \frac{16}{v_0^2}x^2$$

Set up the arclength integral for the distance traveled by the cannonball during the first second of flight (i.e., over $[0, v_0]$.) Estimate the length of the trajectory when $v_0 = 100$ feet per second and $h = 6$ feet.

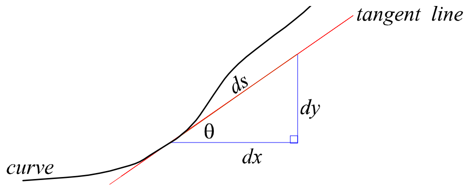
28. A projectile fired from a height of 100 feet with a velocity of 400 feet per second follows the path that is the graph of

$$y = 100 - \frac{x^2}{10000} = 0$$

Show that the projectile strikes the ground after traveling 1000 feet horizontally. Then the length of the graph of y over $[0, 1000]$ to determine the length, in feet, of the trajectory of the projectile.

29. For each x, the *angle of inclination* of $y = f(x)$ is an angle $\theta_f(x)$ for which

$$\frac{dy}{dx} = \tan(\theta)$$

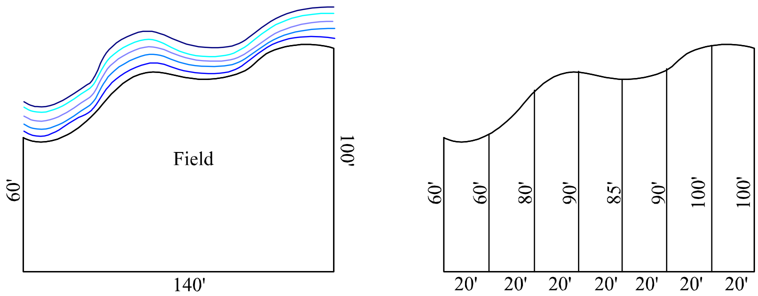


Show that the length of a curve $y = f(x)$ over $[a, b]$ is given by

$$s = \int_a^b \sec[\theta_f(x)]\, dx \tag{5.17}$$

30. Show that if $y = \ln|\cos(x)|$, then the angle of inclination satisfies $\theta(x) = x$ (see exercise 31). Then use (5.17) to find the length of $y = \ln|\cos(x)|$ over $\left[0, \frac{\pi}{4}\right]$.

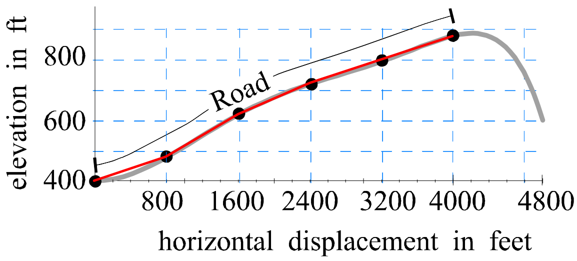
31. A river bounds a field, as shown below left. A surveyor measures distances at right angles to the opposite boundary of the field, as shown below right.



4-11: Measurements of a field

Use a polygonal approximation to estimate the length of the boundary between the field and the river.

32. Use the following alternative grid to estimate the length of the road in example 7:



Why should we expect that our approximations are independent of the underlying grid?

33. For $a > 0$, consider the family of curves

$$y = \frac{1}{a}e^x + \frac{a}{4}e^{-x}$$

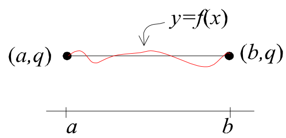
Find the arclength of these curves over $[-1, 1]$ as a function of a. What value of a leads to the shortest curve?

34. Consider the family of curves

$$y = \frac{x^{a+1}}{2a+2} + \frac{x^{-a+1}}{2a-2}$$

for $a > 1$. Find the arclength of these curves over $[1, e]$ as a function of a. Does the arclength increase or decrease as a increases?

35. Write to Learn: Consider points (a, q) and (b, q) which are on the same horizontal line.



4-13: Shortest distance between two points

Write a short essay in which you explain why

$$\int_a^b 1dx \leq \int_a^b \sqrt{1 + [f'(x)]^2}dx$$

for every function $f(x)$ over $[a, b]$ with $f(a) = f(b) = q$. Then use the inequality to prove that the shortest distance between the two points is a straight line.

36. Write to Learn: Suppose that $f'(x) > 0$ over $[a, b]$ is so large that

$$1 + [f'(x)]^2 \approx [f'(x)]^2$$

In a short essay, explain why the length of the graph of $f(x)$ over $[a, b]$ is practically the same as $f(b) - f(a)$.

5.5 Additional Applications

The Mean Value Theorem for Integrals

The definition of the definite integral is one of the most frequently occuring concepts in mathematics and its applications. In this section, we present three additional applications chosen so as to be models for students to mimic when developing still more applications of the integral in the exercises.

For example, let's derive the Mean Value theorem for integrals. For each $h > 0$, let's suppose that $\{x_j, t_j\}$ is an h-fine *regular* partition of an interval $[a, b]$. That is, each subinterval has a width of

$$\Delta x = \frac{b-a}{n} \tag{5.18}$$

where n is the number of subintervals. The *average* value of a function $f(x)$ over the partition $\{x_j, t_j\}$ is a number f_h which satisfies

$$f_h = \frac{f(t_1) + f(t_2) + \ldots + f(t_n)}{n}$$

If we now multiply the numerator and denominator by $b-a$, then (5.18) implies that

$$\begin{aligned} f_h &= \frac{f(t_1)+f(t_2)+\ldots+f(t_n)}{b-a}\left(\frac{b-a}{n}\right) \\ &= \frac{[f(t_1)+f(t_2)+\ldots+f(t_n)]\,\Delta x}{b-a} \\ &= \frac{f(t_1)\,\Delta x+f(t_2)\,\Delta x+\ldots+f(t_n)\,\Delta x}{b-a} \end{aligned}$$

The average value f_{ave} of $f(x)$ over $[a,b]$ is then defined to be the limit as h approaches 0 of the averages f_h over the h-fine partitions, which implies that

$$f_{ave} = \lim_{h\to 0} f_h = \frac{1}{b-a}\lim_{h\to 0}\sum_{j=1}^{n} f(t_j)\,\Delta x$$

If we further assume that $f(x)$ is continuous on $[a,b]$, then the limit converges to $\int_a^b f(x)\,dx$. Thus, we define the *average value* of $f(x)$ on $[a,b]$ to be

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx \tag{5.19}$$

However, the fundamental theorem implies that for any antiderivative $F(x)$ of $f(x)$ over $[a,b]$, we must have

$$f_{ave} = \frac{F(b)-F(a)}{b-a}$$

Since $F'(x)=f(x)$ is continuous on $[a,b]$, the Mean Value theorem for $F(x)$ implies that there is a number c in (a,b) such that

$$f_{ave} = \frac{F(b)-F(a)}{b-a} = F'(c) = f(c)$$

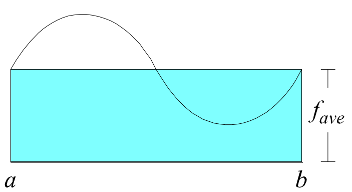
This leads us to the following:

Mean Value Theorem for Integrals: If $f(x)$ is continuous on $[a,b]$, then there is a number c in (a,b) such that

$$f(c) = f_{ave}$$

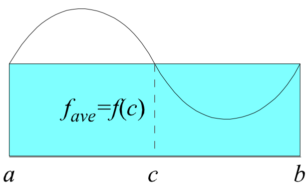
where f_{ave} is the average value of $f(x)$ over $[a,b]$.

Geometrically, if $f(x)\geq 0$ on $[a,b]$, then f_{ave} is the height of the rectangle whose area over $[a,b]$ is the same as the area under $y=f(x)$ over $[a,b]$.



5-1: $\int_a^b f(x)\,dx$ = Area under Curve $= f_{ave}(b-a)$

The Mean Value Theorem for integrals simply says that there is some number c such that the height of the rectangle is equal to the function evaluated at c.



5-2: Mean Value Theorem for Integrals

EXAMPLE 1 Find f_{ave} and the number c in $[0,3]$ for which $f(c) = f_{ave}$ when $f(x) = x^2$.

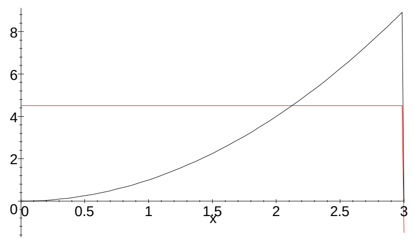
Solution: The average value of the function is

$$f_{ave} = \frac{1}{3-0}\int_0^3 x^2 dx = \frac{9}{3} = 3$$

The equation $f(c) = f_{ave}$ implies that $c^2 = 3$, which yields $c = \sqrt{3}$. To verify the calculation, notice that

$$f_{ave}(b-a) = 3(3-0) = 9$$

which is the same as the area under the curve.



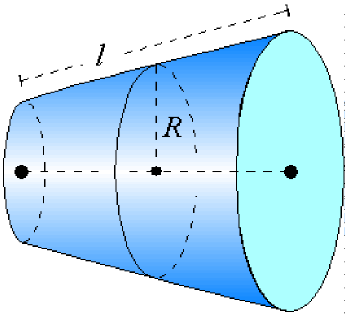
5-3: The area under $y = x^2$ over $[0,3]$ is the same as the area of the rectangle.

Check your Reading *What is the average value of a constant function?*

Surface Area of Solids of Revolution

An application similar to the derivation of the arclength is that of determining the area S of the boundary surface of the solid obtained by revolving the graph of $f(x)$ over $[a,b]$ around the x-axis. To do so, we use the fact that the surface area of the frustum of a cone is

$$S = 2\pi RL$$

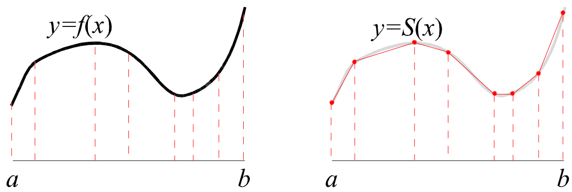


where R is the radius of the cross-section that bisects the axis of the frustum and L is the slant distance.

To begin with, for each $h > 0$, let $\{x_j, t_j\}$ be a tagged partition of $[a, b]$ where the tag t_j in the j^{th} subinterval $[x_{j-1}, x_j]$ is the point where $f(x)$ satisfies the *Mean Value theorem:*

$$f(x_j) - f(x_{j-1}) = f'(t_j)(x_j - x_{j-1})$$

Let us also let $S(x)$ be the piecewise linear function whose graph connects the point (x_{j-1}, y_{j-1}) to the point (x_j, y_j). We say that $S(x)$ is a *polygonal approximation* of $f(x)$.

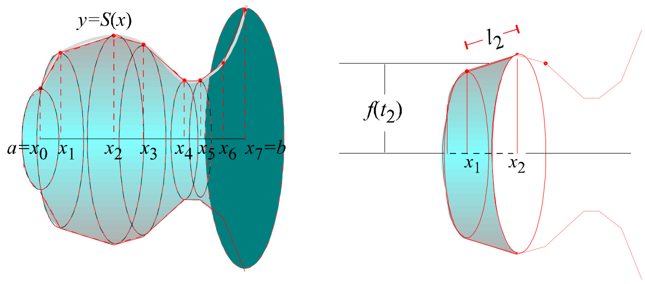


5-4: Polygonal Approximations

As in the derivation of the arclength formula, recall that the length l_j of the j^{th} section of $y = S(x)$ is given by

$$l_j = \sqrt{1 + [f'(t_j)]}\Delta x_j$$

Let us now revolve both $y = S(x)$ and $y = f(x)$ about the x-axis. The surface generated by $y = S(x)$ is an approximation to the surface generated by $y = f(x)$.



5-5: Polygonal Approximation of Surface Area

In addition, each section of the surface generated by revolving $y = S(x)$ about the x-axis is the frustum of a cone.

The radius of the j^{th} frustum is $f(t_j)$ and the slant length is l_j. Thus, the surface area of the j^{th} section is

$$Surface\ Area\ of\ j^{th}\ section = 2\pi RL = f(t_j)\, l_j$$

Since $l_j = \sqrt{1 + [f'(t_j)]^2}\ \Delta x_j$ for each $j = 1, \ldots, n$, the total surface area S of the solid of revolution is approximately

$$S \approx 2\pi f(t_1)\sqrt{1 + [f'(t_1)]}\Delta x_1 + \ldots + 2\pi f(t_n)\sqrt{1 + [f'(t_n)]}\Delta x_n$$

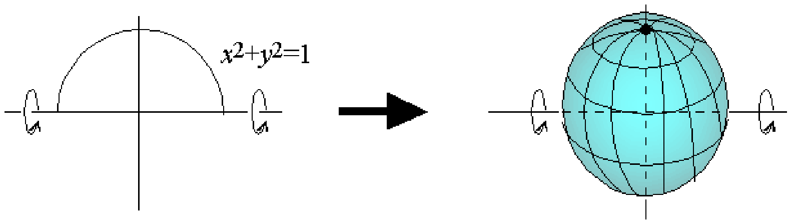
In the limit as h approaches 0, the Riemann sum approximation converges to the surface area, so that

$$S = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}dx$$

is the surface area of the solid of revolution obtained by revolving $y = f(x)$ over $[a, b]$ about the x-axis.

EXAMPLE 2 What is the surface area of the unit sphere?

Solution: The unit sphere can be obtain by revolving the curve $y = \sqrt{1-x^2}$ over $[-1,1]$ about the x-axis.



5-5: Surface area of the unit sphere

However, if $f(x) = \sqrt{1-x^2}$, then $f'(x) = -x/\sqrt{1-x^2}$ and

$$\begin{aligned} f(x)\sqrt{1+[f'(x)]^2} &= \left(\sqrt{1-x^2}\right)\sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2} \\ &= \sqrt{(1-x^2)\left(1+\frac{x^2}{1-x^2}\right)} \\ &= \sqrt{1-x^2+x^2} \\ &= 1 \end{aligned}$$

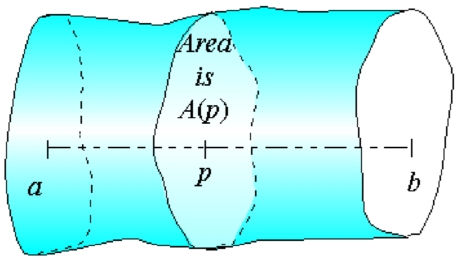
Thus, the surface area of the unit sphere is

$$s = 2\pi\int_{-1}^{1} f(x)\sqrt{1+[f'(x)]^2}dx = 2\pi\int_{-1}^{1} 1dx = 4\pi$$

Check your Reading *How is $x^2+y^2=1$ related to $y=\sqrt{1-x^2}$?*

Volumes by Slicing

If a solid is placed in a coordinate system, then the *slice* at a point p of the solid is a cross-section of the solid perpendicular to the x-axis at the given value of p.



5-6: Slice at $x = p$ perpendicular to x-axis.

Let us also let $A(p)$ denote the area of the slice at p. Our immediate goal is to develop an integral formula for using the areas of the slices to determine the volume V of the solid.

For $h > 0$, let us let $\{x_j, t_j\}_{j=1}^n$ be an h-fine tagged partition of $[a,b]$ and let us let

$$s(x) = \begin{cases} A(t_1) & if \quad x_0 \le x < x_1 \\ A(t_2) & if \quad x_1 \le x < x_2 \\ \vdots & \qquad \vdots \\ A(t_n) & if \quad x_{n-1} \le x < x_n \end{cases}$$

be the corresponding simple function approximation of the area function $A(x)$. Then $s(x)$ is the area function of a solid which is a collection of "disks," where each disk is a lamina of the region in the slice.

It follows that the j^{th} disk has an volume of $A(t_j)\,\Delta x_j$, and as a result, the volume V of the original solid is approximated by the sum of the volumes of the "disks" in the simple function approximation:

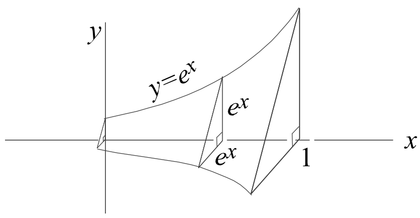
$$V \approx A(t_1)\,\Delta x_1 + A(t_2)\,\Delta x_2 + \ldots + A(t_n)\,\Delta x_n$$

In the limit as h approaches 0, the Riemann sum approximation for V converges to a definite integral. Thus,

$$V = \int_a^b A(x)\,dx \tag{5.20}$$

where $A(x)$ is the area of the slice at x.

EXAMPLE 3 What is the volume of the following solid if each slice is an isosceles right triangle with sides measured in feet?



5-7: Each slice is an iscosceles right triangle

Solution: The base and height of a triangular slice are both e^x. Thus, the area of the slice at x in $[0,1]$ is

$$A(x) = \frac{1}{2}e^x \cdot e^x = \frac{1}{2}e^{2x}$$

The volume by slicing formula (5.20) implies that

$$\begin{aligned} V &= \int_0^1 A(x)\,dx \\ &= \int_0^1 \frac{1}{2}e^{2x}dx \\ &= \frac{1}{4}e^2 - \frac{1}{4} \end{aligned}$$

Thus, the volume of the solid is

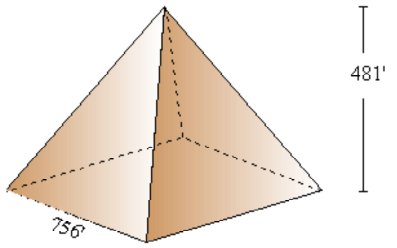
$$V = \frac{1}{4}e^2 - \frac{1}{4} = 1.597\ ft^3$$

Check your Reading *What are the units for $A(x)$ in example 3?*

The Great Pyramid in Egypt

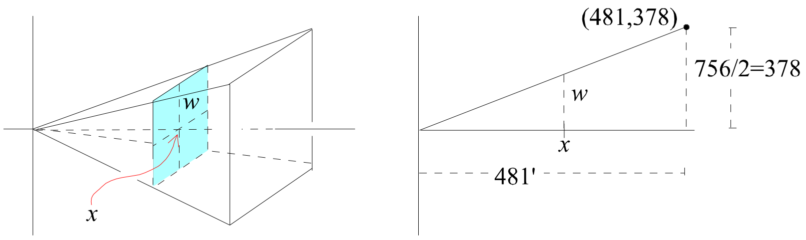
There are many solids which are not solids of revolution but that are nonetheless important in applications. Moreover, there is often not a preferred coordinate system for these solids, which means that we must define the coordinate systems. In doing so, we will want to make sure we orient the solid in our coordinate system so that the areas of the slices can be easily calculated.

EXAMPLE 4 What is the volume of the *Great Pyramid,* which is a square pyramid that is 481 feet high and 756 feet wide at its base?



5-8: The Great Pyramid

Solution: If we "turn the pyramid on its side" and run an x-axis through its center, then each slice is a square with area $A = 4w^2$, where w is half the length of a side.



5-9: Orienting the Great Pyramid in an xy-coordinate system

However, the side of the pyramid now corresponds to a linear function which passes through $(0, 0)$ and also through the point $(481, 378)$. That is, $w = L(x)$ where

$$L(x) = \frac{378}{481}x$$

so that the area of each slice is

$$A(x) = 4w^2 = 4\left(\frac{378}{481}x\right)^2 = 2.47032\,x^2$$

Thus, (5.20) implies that the volume of the great pyramid of Egypt is

$$V = \int_0^{481} A(x)\,dx = \int_0^{481} 2.47032x^2 dx = 9.164 \times 10^7\ ft^3$$

Exercises:

Find the average value f_{ave} of the function over the given interval, and also find the number c in that interval for which $f(c) = f_{ave}$..

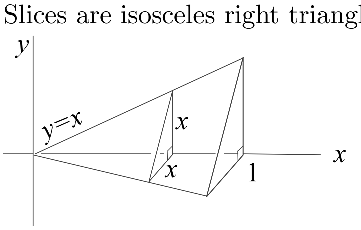
1. $f(x) = x^2$ over $[0, 3]$
2. $f(x) = 3x + 2$ over $[0, 1]$
3. $f(x) = 5$ over $[2, 3]$
4. $f(x) = -2$ over $[3, 4]$
5. $f(x) = |x|$ over $[-1, 1]$
6. $f(x) = x - |x|$ over $[-1, 1]$
7. $f(x) = e^x$ over $[0, \ln(2)]$
8. $f(x) = \ln(x)$ over $[1, e]$

Find the area of the surface obtained by revolving the graph of the given function over the given interval about the x-axis.

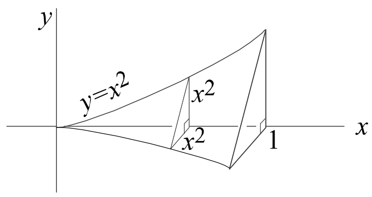
9. $f(x) = 1$ over $[0, 3]$
10. $f(x) = 3x + 2$ over $[0, 1]$
11. $f(x) = x^3$ over $[0, 1]$
12. $f(x) = 2x^3$ over $[3, 4]$
13. $f(x) = 2\sqrt{x}$ over $[0, 3]$
14. $f(x) = x - |x|$ over $[-1, 1]$
15. $f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln(x)$ over $[1, 2]$
16. $f(x) = \frac{1}{3}x^{3/2} - x^{1/2}$ over $[0, 1]$

Determine the area of a slice of the following solids, and then use that to find the volume of the solid.

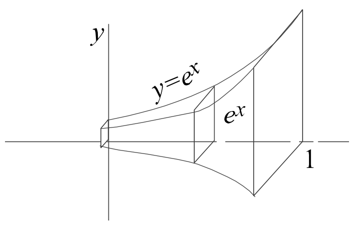
17. Slices are isosceles right triangles



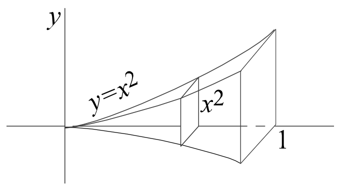
18. Slices are isosceles right triangles



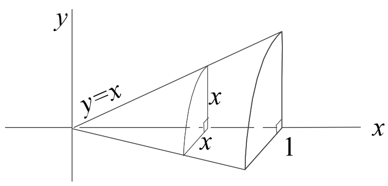
19. Slices are squares



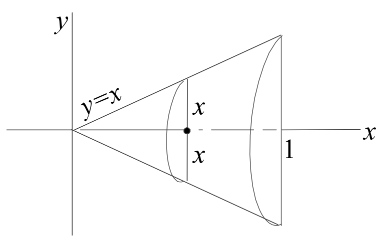
20. Slices are squares



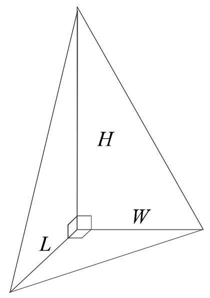
21. Slices are quarter circles



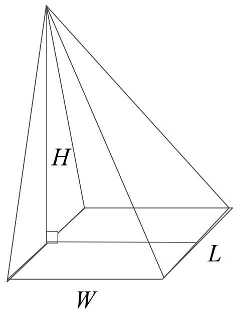
22. Slices are semi-circular



23. Orient the solid in figure 5-11 in a coordinate system and then find its volume using the volumes by slicing formula.



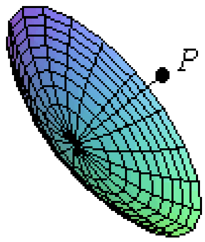
5-11: Exercise 23



5-12: Exercise 24

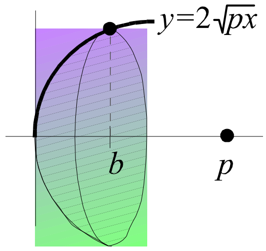
24. Orient the solid in figure 5-12 in a coordinate system and then find its volume using the volumes by slicing formula.

25. A satellite dish is a paraboloid that reflects signals to a focus P.



5-13: Satellite dish is a solid of revolution

A paraboloid with focus at $(p, 0)$ is constructed by revolving $y = 2\sqrt{px}$ about the x-axis.

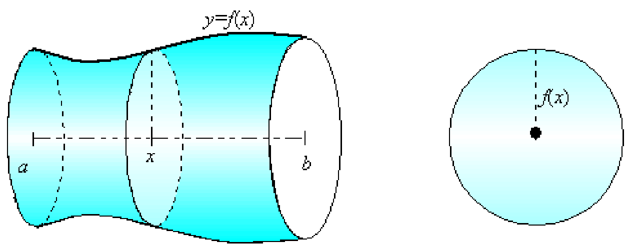


5-14: Paraboloid as a surface of revolution.

What is the surface area of a satellite dish over $[0, b]$, where b is the depth of the dish?

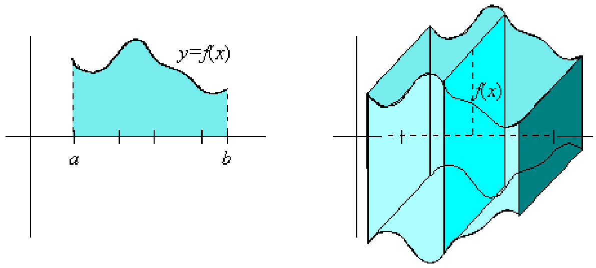
26. Use symmetry to explain why the centroid of a solid of revolution must be on the x-axis. What is the only non-zero coordinate of the centroid and why?

27. Write a short essay explaining how the formula (5.20) for volumes-by-slicing reduces to a formula for the volume of a solid of revolution when each slice is a circle centered on the x-axis whose radius at x is given by $f(x)$.



5-15: Slicing a Solid of Revolution

28. Write a short essay derive the formula for the volume of a solid generated by the region between $y = f(x)$ and $y = g(x)$ over $[a, b]$ if the cross-section is always a square with sides of length $2f(x)$:



5-16: Each cross-section is a square

Write to Learn: In exercises 29-36, you should write a short essay in which you use the definition of the integral to develop an integral formula for the given application.

29. For each $h > 0$, let's suppose that $\{x_j, t_j\}$ is an h-fine *regular* partition of an interval $[a, b]$, so that each subinterval has a width of

$$\Delta x = \frac{b-a}{n}$$

where n is the number of subintervals. The L^1 norm of a function $f(x)$ over the partition $\{x_j, t_j\}$ is a number f_h which satisfies

$$f_h = \frac{|f(t_1)| + |f(t_2)| + \ldots + |f(t_n)|}{n}$$

Use this expression for f_h to define the L^1 norm of a function $f(x)$ over an interval $[a, b]$.

30. For each $h > 0$, let's suppose that $\{x_j, t_j\}$ is an h-fine *regular* partition of an interval $[a, b]$, so that each subinterval has a width of

$$\Delta x = \frac{b-a}{n}$$

where n is the number of subintervals. The *root-mean-square* value of a function $f(x)$ over the partition $\{x_j, t_j\}$ is a number f_h which satisfies

$$(f_h)^2 = \frac{[f(t_1)]^2 + [f(t_2)]^2 + \ldots + [f(t_n)]^2}{n}$$

Use this expression for $(f_h)^2$ to define the *root-mean-square* value of a function $f(x)$ over an interval $[a, b]$.

31. Recall from physics that if an object is moved a given distance along a straight line by a constant force, then the amount of work done is

$$Work = Force \times \ Distance \tag{5.21}$$

(a) Given a nonconstant force $F(x)$, define the simple function

$$s_\varepsilon(x) = \begin{cases} F(t_1) & if \quad a \le x < x_1 \\ F(t_2) & if \quad x_1 \le x < x_2 \\ \vdots & \qquad \vdots \\ F(t_n) & if \quad x_{n-1} \le x < b \end{cases}$$

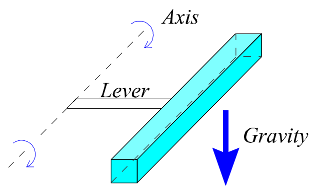
(b) Let W_j be the work done over the j^{th} interval of the partition. Use (5.21) to relate W_j to the j^{th} subinterval of the partition.

(c) Express the total work $W_1 + \ldots + W_n$ as the integral of a simple function.

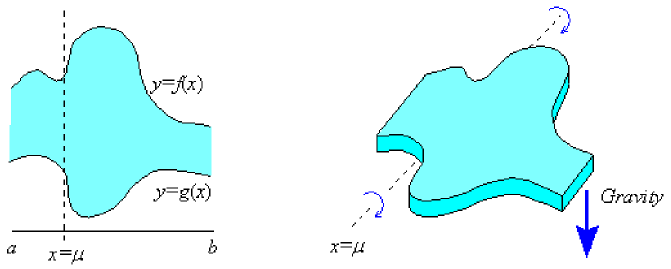
(d) What is the formula for the work done by a nonconstant force $F(x)$ in moving an object across the interval $[a, b]$.

32. *Torque* is the tendency of a force to cause an object to rotate about a given axis. In particular, if the force is gravity, then

$$Torque = Lever \times Gravity$$

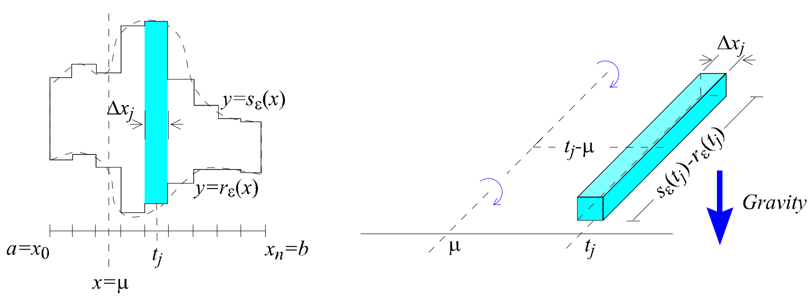


where *Lever* is the length of the lever arm when the rotation is clockwise and is the negative of the length of the lever arm when the rotation is counterclockwise. In this exercise, we derive a integral representation of the torque due to gravity about $x = \mu$ on a laminate of the region between $y = f(x)$ and $y = g(x)$ over $[a, b]$.



5-17: Torque

(a) If $s_\varepsilon(x)$ and $r_\varepsilon(x)$ are the midpoint approximations, of $f(x)$ and $g(x)$ respectively,



5-18: Simple function approximation

Then explain why the mass of the laminate of the j^{th} rectangle is

$$m_j = \rho \left[s_\varepsilon(t_j) - r_\varepsilon(t_j) \right] \Delta x_j$$

where ρ is the mass per unit area of the bar.

(b) Write a formula for the torque τ_j on the laminate of the j^{th} rectangle given that τ_j is the product of the length of the lever arm, $t_j - \mu$, and the mass m_j of the section.

(c) Write the total torque $\tau_1 + \ldots + \tau_n$ as the definite integral of a simple function.

(d) What is the integral representation of the torque τ about the axis $x = \mu$ for the region between $y = f(x)$ and $y = g(x)$ over $[a, b]$?

33. Let $F(t)$ be the rate of blood flowing through the aorta during the time interval $[0, T]$. Suppose also that a dye has been injected into the bloodstream and that its concentration in the aorta is $c(t)$ at time t.

(a) Let $\{x_j, t_j\}_{j=1}^{n}$ be a tagged partition of $[0, T]$. If A_j is the amount of dye in the aorta during the subinterval $[x_{j-1}, x_j]$. Explain why

$$A_j = c(t_j) F(t_j) \Delta x_j$$

(Hint: $F(t_j)$ is in units of volume per unit time. $c(t_j)$ is in units of dye per unit volume. Δx_j is in units of time).

(b) Write the total amount of dye during $[0, T]$ as the definite integral of a simple function. (Hint: the total is $A_1 + \ldots + A_n$).

(c) What is the integral representation of the amount of dye in the aorta during time $[0, T]$ given a rate of blood flow $F(t)$ and a dye concentration at time t of $c(t)$?

34. Given a collection of n rectangles with areas $A_1, \ldots, A_n$ and centroids at

$$(x_1, y_1), \ldots, (x_n, y_n)$$

the centroid $(\bar{x}, \bar{y})$ of the entire collection of rectangles is defined to be

$$\bar{x} = \frac{A_1 x_1 + \ldots + A_n x_n}{A_1 + \ldots + A_n}, \quad \bar{y} = \frac{A_1 y_1 + \ldots + A_n y_n}{A_1 + \ldots + A_n} \tag{5.22}$$

Use (5.22) to derive the centroid formulas in section 2.

35. * Can you derive a "surface area by slicing formula?

36. * Can you use volume by slicing and the definition of the integral to derive Pappus theorem on page 386.

5.6 Improper Integrals

Integrands not Defined at a Point

The integrals that occur in some applications may involve either integrands that are not defined at a point or limits of integration that become infinite. Such integrals are called *improper integrals.* We must pause to consider them before we move on to those applications where they may occur.

An integral in which the integrand is not defined at one or more points is said to be *improper.* Likewise, a definite integral with an infinite limit of integration is also called improper. In this section, we develop methods for evaluating improper integrals of both types.

To begin with, if a function $f(x)$ is not defined at $x = a$, then the *improper integral* of $f(x)$ over $[a, b]$ is defined

$$\int_a^b f(x)\, dx = \lim_{\varepsilon \to a^+} \int_\varepsilon^b f(x)\, dx \tag{5.23}$$

when the limit exists. Likewise, if $f(x)$ is not defined at $x = b$, then the *improper integral* of $f(x)$ over $[a, b]$ is defined

$$\int_a^b f(x)\, dx = \lim_{\varepsilon \to b^-} \int_a^\varepsilon f(x)\, dx$$

when the limit exists.

EXAMPLE 1 Evaluate, if possible, the integral

$$\int_0^1 \frac{dx}{\sqrt{x}}$$

Solution: Since $\frac{1}{\sqrt{x}}$ is not defined at 0, the integral

$$\int_0^1 \frac{dx}{\sqrt{x}}$$

is improper. Thus, (5.23) implies that

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 x^{-1/2} dx = \lim_{\varepsilon \to 0^+} \left. \frac{x^{1/2}}{1/2} \right|_\varepsilon^1 = \lim_{\varepsilon \to 0^+} \left(2 - 2\varepsilon^{1/2} \right)$$

Since the square root function is continuous from the right at $x = 0$, we have

$$\int_0^1 \frac{dx}{\sqrt{x}} = 2$$

EXAMPLE 2 Evaluate, if possible, the integral $\int_0^1 \ln(x)\, dx$

Solution: Since $\ln(x)$ is not defined at $x = 0$, the integral

$$\int_0^1 \ln(x)\, dx$$

is improper. Thus, (5.23) implies that

$$\int_0^1 \ln(x)\, dx = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \ln(x)\, dx$$

Integration by parts can be used to show that $\int \ln(x)\, dx = x \ln(x) - x + C$, so that

$$\int_0^1 \ln(x)\, dx = \lim_{\varepsilon \to 0^+} \left. (x \ln x - x) \right|_\varepsilon^1 = \lim_{\varepsilon \to 0^+} \left(1 \ln(1) - 1 - \varepsilon \ln(\varepsilon) + \varepsilon\right)$$

As a result, we have

$$\int_0^1 \ln(x)\, dx = 0 - 1 - \lim_{\varepsilon \to 0^+} \varepsilon \ln(\varepsilon)$$

Using L'Hôpital's rule, the limit can be shown to converge to 0. Thus,

$$\int_0^1 \ln(x)\,dx = -1$$

If $f(x)$ is not defined at a point c in $[a,b]$, then the improper integral of $f(x)$ over $[a,b]$ is defined

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to c^-}\int_a^\varepsilon f(x)\,dx + \lim_{\varepsilon\to c^+}\int_\varepsilon^b f(x)\,dx \tag{5.24}$$

when both limits exist. Moreover, it is important to note that $\int_a^b f(x)\,dx$ may not exist even though $F(b) - F(a)$ does exist for some antiderivative $F(x)$ of $f(x)$.

EXAMPLE 3 Evaluate, if possible, the integral

$$\int_{-1}^1 \frac{1}{x}dx$$

Solution: Since $1/x$ is not defined at $x = 0$, the integral

$$\int_{-1}^1 \frac{1}{x}dx$$

is improper. Thus, (5.24) becomes

$$\begin{aligned}\lim_{\varepsilon\to 0^-}\int_{-1}^\varepsilon \frac{1}{x}dx + \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{1}{x}dx &= \lim_{\varepsilon\to 0^-}(\ln|\varepsilon| - \ln|-1|) + \lim_{\varepsilon\to 0^+}(\ln|1| - \ln|\varepsilon|) \\ &= \infty - \infty\end{aligned}$$

Since $\infty - \infty$ is undefined, the improper integral $\int_{-1}^1 \frac{1}{x}dx$ does not exist.

Check your Reading *What is the problem with the following computation?*

$$\int_{-1}^1 \frac{1}{x}dx \overset{?}{=} \ln|x|\Big|_{-1}^1 \overset{?}{=} \ln|1| - \ln|-1| \overset{?}{=} 0$$

Improper Integrals over Infinite Intervals

Integrals of the form $\int_a^\infty f(x)\,dx$ are also improper and are defined

$$\int_a^\infty f(x)\,dx = \lim_{R\to\infty}\int_a^R f(x)\,dx \tag{5.25}$$

when the limit exists. When the limit in (5.25) exists, then we say that the integral *converges,* and when the limit does not exist, then we say that the integral *diverges.*

EXAMPLE 4 Evaluate, if possible, the integral

$$\int_1^\infty \frac{dx}{x^2}$$

Solution: We first evaluate the indefinite integral

$$\int_1^R \frac{dx}{x^2} = \int_1^R x^{-2}dx = \left.\frac{x^{-1}}{-1}\right|_1^R = 1 - \frac{1}{R}$$

We then apply the limit as R approaches ∞:

$$\int_1^\infty \frac{dx}{x^2} = \lim_{R\to\infty} \int_1^R \frac{dx}{x^2} = \lim_{R\to\infty} \left(1 - \frac{1}{R}\right) = 1$$

Thus, the integral *converges* to 1, which is to say that

$$\int_1^\infty \frac{dx}{x^2} = 1$$

EXAMPLE 5 Evaluate, if possible, the integral

$$\int_0^\infty \frac{2x}{x^2+1}dx$$

Solution: We begin with the integral

$$\int_0^R \frac{2x}{x^2+1}dx$$

which we evaluate using the substitution $u = x^2 + 1$, $du = 2xdx$ and the new limits of integration

$$u(0) = 0^2 + 1 = 1 \qquad\qquad u(R) = R^2 + 1$$

As a result, we have

$$\int_0^R \frac{2x}{x^2+1}dx = \int_1^{R^2+1} \frac{du}{u} = \ln|u||_1^{R^2+1}$$

which can be simplified to

$$\int_0^R \frac{2x}{x^2+1}dx = \ln\left|R^2+1\right| - \ln|1| = \ln\left|R^2+1\right|$$

Applying the limit as R approaches ∞ thus yields

$$\int_0^\infty \frac{2x}{x^2+1}dx = \lim_{R\to\infty} \int_0^R \frac{2x}{x^2+1}dx = \lim_{R\to\infty} \ln\left|R^2+1\right| = \infty$$

since $\ln(x)$ does not have a horizontal asymptote. Thus, the integral

$$\int_0^\infty \frac{2x}{x^2+1}dx$$

diverges. Equivalently, we might say that the integral does not exist.

Integrals of the form $\int_{-\infty}^{a} f(x)\,dx$ are defined by

$$\int_{-\infty}^{a} f(x)\,dx = \lim_{S\to-\infty} \int_{S}^{a} f(x)\,dx$$

when the limit exists. Integrals of the form $\int_{-\infty}^{\infty} f(x)\,dx$ are then defined to be *two separate limits*

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{S\to-\infty} \int_{S}^{a} f(x)\,dx + \lim_{R\to\infty} \int_{a}^{R} f(x)\,dx$$

for any value of a.

EXAMPLE 6 Evaluate

$$\int_{-\infty}^{\infty} \frac{2x}{(x^2+1)^2}dx$$

Solution: If we let $a = 0$, then we obtain

$$\int_{-\infty}^{\infty} \frac{2x}{(x^2+1)^2}dx = \lim_{S\to-\infty} \int_{S}^{0} \frac{2x}{(x^2+1)^2}dx + \lim_{R\to\infty} \int_{0}^{R} \frac{2x}{(x^2+1)^2}dx$$

If we let $u = x^2 + 1$, then $du = 2xdx$, $u(0) = 1$, and $u(S) = S^2 + 1$. Likewise, $u(R) = R^2 + 1$ and we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{2x}{(x^2+1)^2}dx &= \lim_{S\to-\infty} \int_{S^2+1}^{1} \frac{du}{u^2} \quad + \quad \lim_{R\to\infty} \int_{1}^{R^2+1} \frac{du}{u^2} \\
&= \lim_{S\to-\infty} \left(\left. \frac{u^{-1}}{-1} \right|_{S^2+1}^{1} \right) \quad + \quad \lim_{R\to\infty} \left(\left. \frac{u^{-1}}{-1} \right|_{1}^{R^2+1} \right) \\
&= \lim_{S\to-\infty} \left(\frac{1}{S^2+1} - 1 \right) \quad + \quad \lim_{R\to\infty} \left(1 - \frac{1}{R^2+1} \right) \\
&= 0 - 1 \quad + \quad 1 - 0 \\
&= 0
\end{aligned}$$

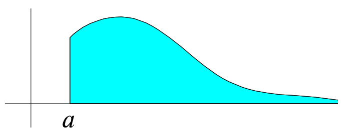
Check your Reading *Does $\int_{1}^{\infty} x^2 dx$ converge or diverge?*

Area of Regions over Infinite Intervals

If $f(x)$ is positive, then the integral

$$\int_{a}^{\infty} f(x)\,dx$$

represents the area under the curve $y = f(x)$ over the interval $[a, \infty)$.



6-1: Area under a curve over $[a, \infty)$

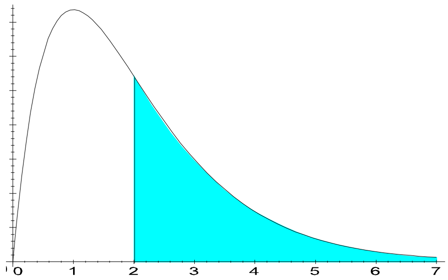
Moreover, we often use a result from in an exercise in chapter three which says that if $p(R)$ is a polynomial and $s > 0$, then

$$\lim_{R\to\infty} p(R)\, e^{-sR} = 0 \tag{5.26}$$

EXAMPLE 7 Find the area under the graph of $f(x) = xe^{-x}$ over the interval $[2, \infty)$.

Solution: To do so, we must evaluate the integral

$$\int_2^\infty xe^{-x}dx$$



We begin with the integral

$$\int_2^R xe^{-x}dx$$

which we evaluate using tabular integration:

u		dv
x		e^{-x}
	$\searrow^{+}$	
1		$-e^{-x}$
	$\searrow^{-}$	
0		e^{-x}

$$\int_2^R xe^{-x}dx = \left(-xe^{-x} - e^{-x}\right)\Big|_2^R$$

To finish the evaluation, we substitute and apply the limit

$$\int_2^\infty xe^{-x}dx = \lim_{R\to\infty}\left(-Re^{-R} - e^{-R}\right) - \left(-2e^{-2} - e^{-2}\right)$$

Applying (5.26) to the limits as R approaches ∞ then leads to

$$\int_2^\infty xe^{-x}dx = -0 - 0 - \left(-2e^{-2} - e^{-2}\right) = 3e^2$$

EXAMPLE 8 Find the area of the region over $[1, \infty)$ and under the graph of

$$f(x) = \frac{1}{x}$$

Solution: To do so, we must evaluate $\int_1^\infty \frac{1}{x}dx$, which leads to

$$\begin{aligned}\int_1^\infty \frac{dx}{x} &= \lim_{R\to\infty}\int_1^R \frac{dx}{x}\\ &= \lim_{R\to\infty} \ln|x|\big|_1^R\\ &= \lim_{R\to\infty}\left(\ln(R) - \ln(1)\right)\end{aligned}$$

Since $\ln(R)$ approaches ∞ as R approaches ∞, the improper integral *diverges.* That is, there is an infinite amount of area under the graph of $f(x) = 1/x$ over the interval $[1, \infty)$.

Check your Reading *How much area is under* $y = x^2$ *over* $[1, \infty)$?

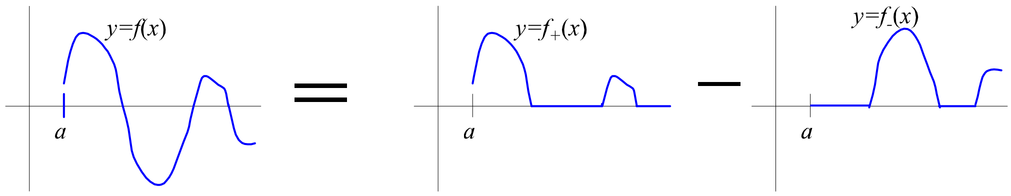
A Comparison Test for Integrals

In many applications, it is as important to know *if* an integral converges as it is to know what it converges to. Thus, we conclude with a theorem for determining if an integral converges without actually computing the integral.

Given $f(x)$ defined on an interval $[a, \infty)$, let us define

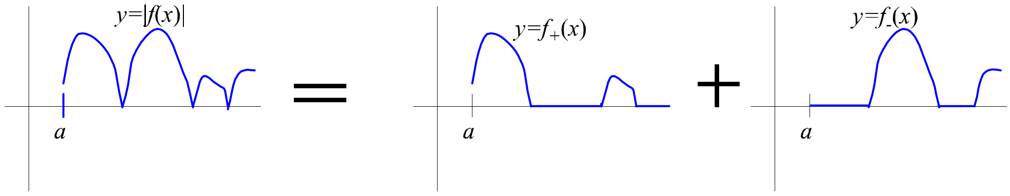
$$f_+(x) = \begin{cases} f(x) & if \quad f(x) \geq 0 \\ 0 & if \quad f(x) < 0 \end{cases} \qquad and \qquad f_+(x) = \begin{cases} -f(x) & if \quad f(x) \leq 0 \\ 0 & if \quad f(x) > 0 \end{cases}$$

It then follows that $f(x) = f_+(x) - f_-(x)$,



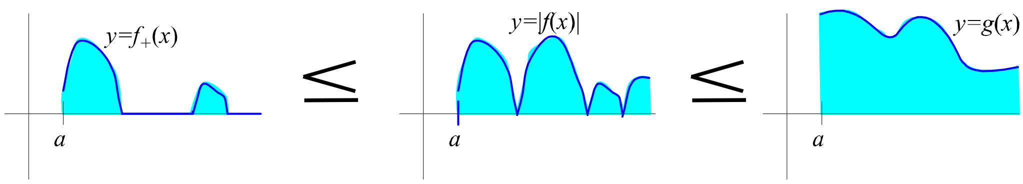
6-2: $f(x)$ is the difference between its positive and negative parts

and it also follows that $|f(x)| = f_+(x) + f_-(x)$.



6-3: $|f(x)|$ is the sum of positive and negative parts

Clearly, $0 \leq f_+(x) \leq |f(x)|$ and also $0 \leq f_-(x) \leq |f(x)|$ on $[a, \infty)$, which means that the region under the graph of $y = f_\pm(x)$ is contained in the region under the graph of $y = |f(x)|$. Moreover, if $g(x)$ is a function for which $|f(x)| \leq g(x)$ on $[a, \infty)$, then the region under $y = |f(x)|$ is contained in the region under $y = g(x)$.



6-4: Area under negative part is bounded by area under $y = g(x)$

As a result, if $\int_a^\infty g(x)\,dx$ converges, then that implies that there is only a finite amount of area under $y = g(x)$ over $[a, \infty)$, thus implying only a finite amount of area under $y = |f(x)|$ over $[a, \infty)$, and further implying that there is only a finite amount of area under both $y = f_+(x)$ and $y = f_-(x)$ over $[a, \infty)$. This discussion motivates the following theorem.

Theorem 7.1: If $|f(x)| \leq g(x)$ for all x in some interval $[a, \infty)$, and if $\int_a^\infty g(x)\,dx$ converges, then both

$$\int_a^\infty |f(x)|\,dx \qquad \textit{and} \qquad \int_a^\infty f(x)\,dx$$

also converge.

This allows us to determine if an improper integral converges without having to actually compute the integral.

EXAMPLE 9 Use theorem 7.2 to determine if

$$\int_1^\infty \frac{\sin(x)}{x^4+1}dx$$

converges or diverges.

Solution: We begin by studying

$$\int_1^\infty \frac{|\sin(x)|\,dx}{x^4+1}$$

However, $|\sin(x)| \leq 1$ for all x, and similarly, $x^4 + 1 \geq x^4$. Thus, we have

$$\frac{1}{x^4+1} \leq \frac{1}{x^4}$$

which implies that

$$\frac{|\sin(x)|}{x^4+1} \leq \frac{1}{x^4}$$

It then follows that

$$\int_1^\infty \frac{|\sin(x)|}{x^4+1}dx \leq \int_1^\infty \frac{1}{x^4}dx = \frac{1}{3}$$

Since $\int_1^\infty \frac{1}{x^4}dx$ converges, theorem 7.2 implies that

$$\int_1^\infty \frac{\sin(x)}{x^4+1}dx \qquad \textit{and} \qquad \int_1^\infty \frac{|\sin(x)|}{x^4+1}dx$$

both converge as well.

Exercises:

Evaluate the following, if they exist. If they do not exist, explain why not.

1. $\displaystyle\int_0^4 \frac{dx}{\sqrt{x}}$
2. $\displaystyle\int_0^2 \frac{dx}{x}$
3. $\displaystyle\int_0^1 \frac{dx}{x^2}$
4. $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x}}dx$
5. $\displaystyle\int_{-2}^3 \frac{dx}{x}$
6. $\displaystyle\int_{-1}^1 \frac{dx}{x^2}$
7. $\displaystyle\int_{-1}^1 \frac{dx}{x^{2/5}}$
8. $\displaystyle\int_{-1}^1 \frac{dx}{x^{7/5}}$
9. $\displaystyle\int_{-1}^1 \frac{dx}{x^{\pi}}$
10. $\displaystyle\int_0^1 \frac{x}{x^2-1}dx$
11. $\displaystyle\int_0^1 \frac{x}{\sqrt{x^2-1}}dx$
12. $\displaystyle\int_0^1 \frac{\cos(\ln x)}{x}dx$

Determine if each of the integrals below converges or diverges. If it converges, compute its value. If it diverges, write "diverges" as the result.

13. $\displaystyle\int_1^{\infty} e^{-x}dx$

14. $\displaystyle\int_0^{\infty} xe^{-x}dx$

15. $\displaystyle\int_1^{\infty} \frac{1}{x^{3/2}}dx$

16. $\displaystyle\int_1^{\infty} \frac{1}{x^2}dx$

17. $\displaystyle\int_0^{\infty} \frac{e^x}{e^x+1}dx$

18. $\displaystyle\int_0^{\infty} \frac{e^{-x}}{e^{-x}+1}dx$

19. $\displaystyle\int_0^{\infty} xe^{-x^2}dx$

20. $\displaystyle\int_{-\infty}^{\infty} xe^{-2x^2}dx$

21. $\displaystyle\int_{-\infty}^{\infty} x^2e^{-|x|}dx$

22. $\displaystyle\int_0^{\infty} x^3e^{-2x}dx$

23. $\displaystyle\int_0^{\infty} x^3e^{-x^2}dx$

24. $\displaystyle\int_0^{\infty} x^5e^{-x^2}dx$

25. Find the area under the graph of

$$f(x) = \frac{x^{-2}}{1+x^{-1}}$$

over the interval $[1,\infty)$.

26. Find the area under the graph of

$$f(x) = e^{-x} + e^{-2x}$$

over the interval $[1,\infty)$.

In exercises 27-30, use the fact that the type II improper integral

$$\mathcal{L}(f) = \int_0^{\infty} f(t)\,e^{-st}dt \tag{5.27}$$

is called the ***Laplace Transform*** *of the function $f(t)$.*

27. Find the Laplace transform of $f(t) = t$. (see (5.27))

28. Find the Laplace transform of $f(t) = t^2$. (see (5.27))

29. Find the Laplace transform of $f(t) = e^{-t}$. (see (5.27))

30. Use integration by parts to show that if $f(x)$ is differentiable, then

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

31. For $x > 1$, the *Gamma Function* is defined

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt$$

(a) What is $\Gamma(1)$?
(b) What is $\Gamma(2)$?
(c) What is $\Gamma(3)$?

32. The gamma function in exercise 33 is important in many applications because it has the two properties

$$\begin{aligned}\Gamma(1) &= 1 \\ \Gamma(x+1) &= x\Gamma(x) \qquad (5.28)\end{aligned}$$

Use the definition above and improper integrals to show that (5.28) is true.

33. The following improper integral converges for all $z > -1$

$$\int_1^{\infty}\left(\frac{1}{t}-\frac{1}{t+z}\right)dt$$

Can you evaluate it?

34. *The following improper integral converges for all $t > 0$:

$$\int_0^{\infty}\frac{e^{-x}-e^{-xt}}{x}dx$$

Can you evaluate it?

35. There are occasions when a substitution transforms an integral which does not exist into an integral which does exist, thus forcing us to be careful when using substitution with improper integrals. For example, let us consider

$$\int_{-\pi}^{\pi}\frac{\sin(x)\,dx}{1-\cos(x)} \qquad (5.29)$$

(a) Use a substitution to show that

$$\int\frac{\sin(x)\,dx}{1-\cos(x)} = \ln|1-\cos(x)|+C$$

(b) Explain why $\ln|1-\cos(x)|+C$ is not continuous over $[-\pi,\pi]$, and thus why (5.29) does not exist.
(c) Show that $u = 1-\cos(x)$ transforms (5.29) into

$$\int_2^2\frac{du}{u}$$

Does this integral exist? If so, what is its value?

36. It is also important that integrals of the form $\int_{-\infty}^{\infty} f(x)\,dx$ be defined as 2 limits and not just one. For example, let us consider

$$\int_{-\infty}^{\infty}4x^3dx$$

(a) Show that $\int_{-\infty}^{0}4x^3dx$ and $\int_0^{\infty}4x^3dx$ both diverge.
(b) Show that for any number R, we have

$$\int_{-R}^{R}4x^3dx = 0$$

(c) Use (a) and (b) to explain why $\int_{-\infty}^{\infty} f(x)\,dx$ should not be defined by

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

37. Use theorem 7.1 to demonstrate the convergence of

$$\int_{0}^{\infty} \frac{\sin(x)\,dx}{e^x + 1}$$

38. Use theorem 7.1 to demonstrate the convergence of

$$\int_{1}^{\infty} \frac{\ln(x)\,dx}{x^4 + 1}$$

39. **Write to Learn:** Explain why if $0 \le g(x) \le |f(x)|$ for all x in $[a,\infty)$ and if $\int_a^\infty g(x)\,dx$ diverges, then so also does $\int_a^\infty |f(x)|\,dx$.

40. *__Write to Learn: (The Birth of Quantum Mechanics)__ In the late 1800's, Max Planck was interested in the total electromagnetic energy per unit volume radiated by a blackbody at a constant absolute temperature T. In particular, he knew that the total energy per unit volume is an improper integral of the form

$$E_{perV} = \int_{0}^{\infty} \rho(\omega)\,d\omega$$

where $\rho(\omega)$ is the *spectral energy density* at a frequency of ω and is defined so that

$$\rho(\omega)\,d\omega = Energy\ over\ frequencies\ between\ \omega\ and\ \ \omega + d\omega$$

Write a short essay in which you consider the following ideas:

(a) **The Ultraviolet Catastrophe:** Classical mechanics using harmonic oscillators implies that

$$\rho(\omega) = \frac{8\pi kT}{c^3}\,\omega^2$$

where c is the speed of light and $k = 1.38 \times 10^{-23} Joules/^\circ K$. Explain why classical mechanics implies that E_{perV} is infinite due to a divergent improper integral.

(b) Planck assumed that energy is radiated in small packets so that he could model the emission of energy with population models and probability theory. This led him to propose that the true form of the spectral energy density is

$$\rho(\omega) = \frac{8\pi h}{c^3}\,\frac{\omega^3}{e^{h\omega/kT} - 1} \tag{5.30}$$

where h is *Planck's constant.* Use linearization of $e^{h\omega/kT}$ at $\omega = 0$ to show that

$$e^{h\omega/kT} \approx 1 + \frac{h\omega}{kT}$$

when ω is close to 0. Show that Planck's law (5.30) matches the classical prediction for frequencies ω that are not too large.

(c) We can use L'hopital's rule to show that

$$\lim_{\omega\to\infty} \frac{\omega^5}{e^{h\omega/kT}-1} = 0 \tag{5.31}$$

Explain why (5.31) means that we can choose R such that if $\omega > R$, then

$$e^{h\omega/kt} - 1 > \omega^5 \tag{5.32}$$

Planck's $\rho(\omega)$ is continuous and bounded on $(0, R)$, so the integral $\int_0^R \rho(\omega)\, d\omega$ exists. Use (5.32) and theorem 7.1 to explain why

$$\int_R^\infty \rho(\omega)\, d\omega \textit{ converges}$$

and thus why Planck's form of $\rho(\omega)$ avoids the ultraviolet catastrophe.

5.7 Geometric Probability

Probability Density Functions

The study of probability begins with the concept of a *random variable,* which is a variable X whose values are possible outcomes of a random process. An *event* is a subset E of the possible outcomes of a random process, and the *probability* of an event is a measure of the likelihood that the value of X is in the event subset E.

For example, suppose a person is selected at random from a given population and that the random variable X is that person's height in feet. An *event* E would be a collection of possible heights for that person. To illustrate, the event $E = [5, 6]$ is the event of the person's height being between 5 and 6 feet. The probability of E is then denoted by $\Pr(E)$ and is the likelihood that the randomly selected person's height is one of the values in the event E. Thus, $\Pr([5, 6])$ represents the probability of the randomly selected person's height being between 5 and 6 feet.

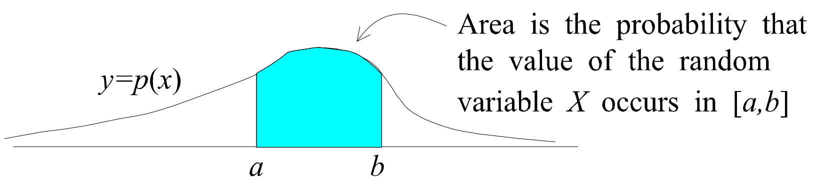
If the event E is an interval, such as $E = [a, b]$, then the probability $P(E)$ may also be denoted by

$$\Pr(X \textit{ in } [a, b]), \qquad \Pr(a \le X \le b), \qquad \textit{or} \qquad P(X \textit{ in } E)$$

Finally, we measure probabilities with integrals. In particular, if X denotes a continuous random variable, then the probability of the value of X being in $[a, b]$ is defined to be

$$P(a \le X \le b) = \int_a^b p(x)\, dx \tag{5.33}$$

where $p(x)$ is a function known as a *probability density* or *p.d.f.* of the continuous random variable X. That is, $P([a, b])$ is the area of the region under the density curve $y = p(x)$ and over $[a, b]$.



7-1: Probability as an Area of a region under a curve

Since there is a 100% chance that the value of X is in $(-\infty, \infty)$, we have

$$\int_{-\infty}^{\infty} p(x)\,dx = 1 \tag{5.34}$$

Finally, probabilities must be positive, which implies that $p(x) \geq 0$ for all x.

EXAMPLE 1 Show that $p(x)$ is a probability density function:

$$p(x) = \begin{cases} 0 & if \quad x < 0 \\ 0.2e^{-0.2x} & if \quad x \geq 0 \end{cases} \tag{5.35}$$

Solution: Since $p(x) \geq 0$ by definition, we need only show that $p(x)$ satisfies (5.34). Since $p(x) = 0$ if $x < 0$, we have

$$\begin{aligned} \int_{-\infty}^{\infty} p(x)\,dx &= \lim_{R\to\infty} \int_0^R 0.2e^{-0.2x}dx \\ &= \lim_{R\to\infty} 0.2 \left(\frac{1}{-0.2} e^{-0.2x} \Big|_0^R \right) \\ &= \lim_{R\to\infty} \left(-e^{-0.2R} + e^0 \right) \\ &= 1 \end{aligned}$$

In statistics, we often study families of probability densities. For example, suppose that at a certain establishment, the random variable T is the time that elapses between one customer being served and the next being served. If r is the average time that elapses between customers, then T is *exponentially distributed* with p.d.f.

$$p(t) = \begin{cases} 0 & if \quad t < 0 \\ \frac{1}{r}e^{-t/r} & if \quad t \geq 0 \end{cases} \tag{5.36}$$

as is shown rigorously in the *Next Step* at the end of this chapter.

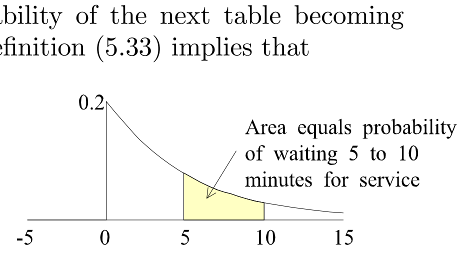
EXAMPLE 2 At a certain restaurant, a table becomes available about once every 5 minutes, on average. If a table opens up at 6:00 p.m., what is the probability of the next table becoming available between 6:05 and 6:10 p.m.?

Solution: Since $r = 5$ minutes, (5.36) for this restaurant is

$$p(t) = \begin{cases} 0 & if \quad t < 0 \\ \frac{1}{5}e^{-t/5} & if \quad t \geq 0 \end{cases} = \begin{cases} 0 & if \quad t < 0 \\ 0.2e^{-0.2t} & if \quad t \geq 0 \end{cases}$$

Since $P(5 \leq T \leq 10)$ is the probability of the next table becoming available between 6:05 and 6:10, definition (5.33) implies that

$$P(5 \leq T \leq 10) = \int_5^{10} p(t)\,dt$$



Since $p(t) = 0.2e^{-0.2t}$ if $t > 0$, we have

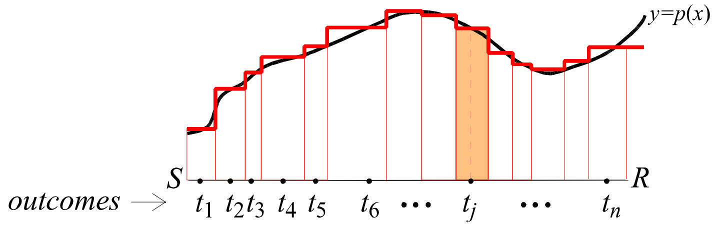
$$P(5 \leq T \leq 10) = \int_5^{10} 0.2e^{-0.2t} dt = \frac{0.2}{-0.2} e^{-0.2t} \bigg|_5^{10} = 0.232544$$

Thus, there is a 23% chance the next table becomes available between 6:05 and 6:10 p.m.

Check your Reading *What does the integral $\int_2^7 0.2e^{-0.2t} dt$ represent in example 2?*

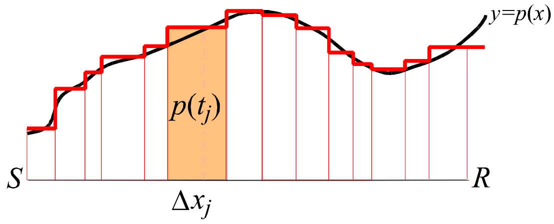
Expected Value and Standard Deviation

Let's suppose that M is a large positive integer, and suppose that M values of a random variable are produced by a random process. Let $[S, R]$ denote an interval containing all M outcomes, and let $\{x_j, t_j\}$ be a h-fine partition of $[S, R]$ that is so fine that the tags t_j can be chosen very close to the M outcomes.



7-2: Simple function approximation

Since $p(t_j)\Delta x_j$ is the approximate area under $y = p(x)$ over $[x_{j-1}, x_j]$, the quantity $p(t_j)\Delta x_j$ is the approximate probability that one of the M outcomes is equal to t_j.



7-3: Probability of $[x_{j-1}, x_j]$

Correspondingly, $p(t_j)\Delta x_j$ is the approximate *percentage* of the M outcomes that are equal to t_j, which is to say that t_j occurs $M\ p(t_j)\Delta x_j$. For example, if $p(t_j)\Delta x_j$ is 0.05 and $M = 100,000$, then approximately 5% of the 100,000 outcomes is equal to t_j, which is to say that t_j occurs $5,000$ times.

The average for the M outcomes is

$$\begin{aligned} average &= \frac{\overbrace{t_1 + \ldots + t_1}^{Mp(t_1)\Delta x_1 \ times} + \overbrace{t_2 + \ldots + t_2}^{Mp(t_2)\Delta x_2 \ times} + \ldots + \overbrace{t_n + \ldots + t_n}^{Mp(t_n)\Delta x_n \ times}}{M \ total \ outcomes} \\ &= \frac{t_1 \cdot Mp(t_1)\Delta x_1 + t_2 \cdot Mp(t_2)\Delta x_2 + \ldots + t_n \cdot Mp(t_n)\Delta x_n}{M} \\ &= t_1 p(t_1)\Delta x_1 + t_2 p(t_2)\Delta x_2 + \ldots + t_n p(t_n)\Delta x_n \end{aligned}$$

As M gets larger and larger, two things must occur. First, the partitions must become finer and finer, thus leading to

$$average = \lim_{h \to 0} \sum_{j=1}^{n} t_j p(t_j)\Delta x_j = \int_S^R x p(x)\, dx$$

Second, the intervals $[S, R]$ containing the outcomes might have to become arbitrarily large, which is to say that S approaches $-\infty$ and R approaches ∞.

$$average = \lim_{S\to-\infty} \lim_{R\to\infty} \int_S^R xp(x)\,dx = \int_{-\infty}^{\infty} xp(x)\,dx$$

The result is also called the mean or *expected value* of the random variable and is denoted by μ, which is the Greek letter "mu." That is, the expected value of the random variable with p.d.f. $p(x)$ is given by

$$\mu = \int_{-\infty}^{\infty} xp(x)\,dx \tag{5.37}$$

when the improper integral exists.

EXAMPLE 3 Show that if r is constant, then r is the expected value of a random variable with p.d.f.

$$p(t) = \begin{cases} 0 & if \quad t < 0 \\ \frac{1}{r}e^{-t/r} & if \quad t \geq 0 \end{cases}$$

Solution: Since $p(t) = 0$ if $t < 0$, the formula (5.37) implies that

$$\mu = \int_0^{\infty} t\,\frac{1}{r}e^{-t/r}dt = \frac{1}{r}\lim_{R\to\infty}\left[\int_0^R te^{-0.2t}dt\right]$$

Integration by parts in the form of tabular integration yields

$$\begin{array}{cccc} u & & dv \\ \hline t & & e^{-t/r} \\ & \searrow^{+} & \\ 1 & & -re^{-t/r} \\ & \searrow^{-} & \\ 0 & & r^2e^{-t/r} \end{array} \qquad \mu = \frac{1}{r}\lim_{R\to\infty}\left(-rte^{-0.2t} - r^2e^{-0.2t}\right)\Big|_0^R$$

from which we get

$$\mu = \frac{1}{r}\lim_{R\to\infty}\left(-rR\,e^{-0.2R} - r^2e^{-0.2R} + r^2\right) = \frac{1}{r}\left(0 + 0 + r^2\right) = r$$

If μ is the expected value of X, then the random variable $(X-\mu)^2$ is the predicted "error" in expecting X to have a value of μ. The expected value of $(X-\mu)^2$ is called the *variance or uncertainty* in X, and is given by

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2\,p(x)\,dx$$

where σ is called the *standard deviation* in the random variable X.

EXAMPLE 4 Find σ for the $p.d.f.$ given in example 2.

Solution: Integration by parts can be used to show that the standard deviation for the p.d.f. $p(t)$ in example 2 is

$$\sigma^2 = \int_0^\infty (t-5)^2\, 0.2e^{-0.2t} dt = 25, \qquad \sigma = \sqrt{25} = 5$$

That is, T has an expected value of $\mu = 5$, give or take an estimated $\sigma = 5$ minutes.

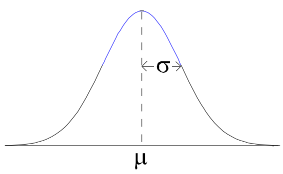
Check your Reading *Should the customer expect to wait longer than 10 minutes?*

Gaussian Distributions

A random variable X is said to be *normally distributed* with mean μ and standard deviation σ has a p.d.f of the form

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2} \tag{5.38}$$

The density (5.38) is called a *Gaussian density function,* and its graph is called a *bell curve.* That is, $p(x)$ has a maximum at μ and inflection points at $\mu+\sigma$ and $\mu-\sigma$.



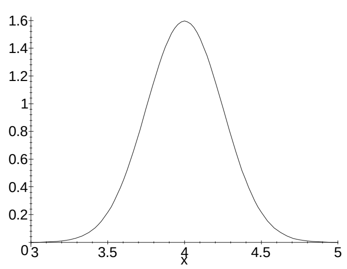
7-4: Gaussian Density

The multiplier $1/\left(\sigma\sqrt{2\pi}\right)$ is included so that (5.38) satisfies (5.34).

EXAMPLE 5 Heights of individual cornstalks in a field of corn are known to be normally distributed with an average of 4 feet and a standard deviation of 3 inches (= 0.25 feet). What is the probability that a corn plant chosen at random is between 4 and 5 feet tall?

Solution: The probability density of the corn plants is

$$p(x) = \frac{1}{0.25\sqrt{2\pi}} e^{-\frac{1}{2}(x-4)^2/(0.25)^2}$$



A numerical integrator yields

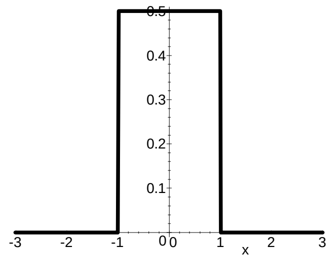
$$P(4 \le X \le 5) = \int_4^5 \frac{1}{0.25\sqrt{2\pi}} e^{-\frac{1}{2}(x-4)^2/(0.25)^2} dx = 0.49997$$

That is, about half the plants are between 4 and 5 feet tall.

Normal distributions occur naturally in the study of *sample means,* where a sample mean is the average over n sample outcomes of a random variable. Specifically, the *central limit theorem* says that if X is a random variable with mean μ and a finite standard deviation σ, then the p.d.f. of the sample mean for a sample of size n is approximately Gaussian with mean μ and standard deviation $\sigma/\sqrt{n}$.

EXAMPLE 6 If the value of a random variable X is chosen at random from $[-1, 1]$, then X is said to be *uniformly distributed* with a p.d.f. of

$$p(x) = \begin{cases} 0.5 & if \ -1 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$



(a) What is the mean and standard deviation of X?

(b) Approximate the p.d.f. of the values of the sample mean with a Gaussian density.

(c) What is the probability that the average of 10 numbers chosen uniformly randomly from $[-1, 1]$ will be between -0.5 and 0.5?

Solution: (a) The expected value of a uniformly distributed X is

$$\mu = \int_{-\infty}^{\infty} xp(x)\, dx = \int_{-1}^{1} x(0.5)\, dx = 0$$

A mean of $\mu = 0$ reflects an equal probability of choosing either a positive or a negative number. Moreover, the standard deviation for X is

$$\sigma^2 = \int_{-1}^{1} 0.5x^2 dx = \frac{1}{3}, \qquad \sigma = \frac{1}{\sqrt{3}}$$

(b) Since $n = 10$, the standard deviation for the sample means is

$$\frac{\sigma}{\sqrt{10}} = 0.1826$$

Thus, the sample mean of 10 samples chosen uniformly is approximated by the Gaussian density

$$p(x) = \frac{1}{(0.1826)\sqrt{2\pi}} e^{-\frac{1}{2}x^2/(0.1826)^2}$$

(c) The probability that the average of 10 samples is in $[-0.5, 0.5]$ is

$$\begin{aligned} P(-0.5 \leq average \leq 0.5) &= \int_{-0.5}^{0.5} \frac{1}{(0.1826)\sqrt{2\pi}} e^{-\frac{1}{2}x^2/(0.1826)^2} dx \\ &= 0.9938 \end{aligned}$$

where the integral is approximated

Check your Reading *A computer generates 1000 sample means of 10 numbers chosen uniformly from* $[-1,1]$. *About how many of those sample means is between* -0.5 *and* 0.5?

Families of Densities

The study of statistics and probability is in many ways the study of individual families of probability densities. We have already encountered two of these families—exponentially distributed random variables with mean r (5.36)and normally distributed random variables with mean μ and standard deviation σ (5.38).

New families of densities are being developed all the time for use in new applications of probability and statistics. Like the two we have already encountered, these families tend to be parametrized by a mean μ, a standard deviation σ, or other relevant parameters.

EXAMPLE 7 The Weibull distribution is used in reliability theory to study the failure rate of devices such as light bulbs and batteries. It has a probability density of

$$p(x) = \begin{cases} n\beta^{-n}x^{n-1}e^{-(x/\beta)^n} & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

where n and β are positive constants. Show that $p(x)$ is a probability density function.

Solution: Clearly, $p(x) \geq 0$ if $x \geq 0$ and $p(x) = 0$ if $x < 0$. Thus,

$$\int_{-\infty}^{\infty} p(x)\,dx = \int_0^{\infty} n\beta^{-n}x^{n-1}e^{-(x/\beta)^n}\,dx = \lim_{R\to\infty}\int_0^R e^{-(x/\beta)^n}\,n\beta^{-n}x^{n-1}dx$$

If we let $u = (x/\beta)^n$, then $du = n\beta^{-n}x^{n-1}$, $u(0) = 0$, and $u(R) = (R/\beta)^n$:

$$\begin{aligned} \int_{-\infty}^{\infty} p(x)\,dx &= \lim_{R\to\infty}\int_0^{(R/\beta)^n} e^{-u}\,du \\ &= \lim_{R\to\infty}\left(-e^{-u}\Big|_0^{(R/\beta)^n}\right) \\ &= \lim_{R\to\infty}\left(1 - e^{-(R/\beta)^n}\right) \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

Exercises:

Use probability densities listed in the section above to compute the following probabilities. Some of these require numerical integration.

1. The probability that a number chosen at random from $[-1,1]$ will be in the interval $[0.2, 0.7]$. (see example 6)

2. The probability of the "next" table becoming available between 6:00 p.m. and 6:05 p.m. (see example 2)

3. The probability of the "next" table becoming available between 6:00 p.m. and 6:10 p.m. . (see example 2)

4. The probability that a stalk of corn is between 3 and 4 feet tall. (see example 5)

5. The probability that a stalk of corn is between 3.5 and 4.5 feet tall. (see example 5)

6. The probability that a stalk of corn is between 30 and 40 feet tall. (see example 5)

Show that the given function $p(x)$ is a probability density. Then find its expected value.

7. $p(x) = \begin{cases} 1 & if\ \ 0 \le x \le 1 \\ 0 & otherwise \end{cases}$

8. $p(x) = \begin{cases} \frac{1}{6} & if\ \ 0 \le x \le 6 \\ 0 & otherwise \end{cases}$

9. $p(x) = \begin{cases} 0 & if \quad x < 0 \\ 0.7e^{-0.7x} & if \quad x \ge 0 \end{cases}$

10. $p(x) = \begin{cases} 0 & if \quad x < 0 \\ \pi e^{-\pi x} & if \quad x \ge 0 \end{cases}$

11. $p(x) = \begin{cases} 0 & if \quad x < 0 \\ xe^{-x} & if \quad x \ge 0 \end{cases}$

12. $p(x) = \begin{cases} 0 & if \quad x < 0 \\ \frac{1}{2}x^2e^{-x} & if \quad x \ge 0 \end{cases}$

13. $p(x) = \begin{cases} 1 - |x| & if \quad -1 \le x \le 1 \\ 0 & otherwise \end{cases}$

14. $p(x) = \begin{cases} 1 - x^2 & if \quad -1 \le x \le 1 \\ 0 & otherwise \end{cases}$

15. $p(x) = \frac{1}{2}e^{-|x|}$

16. $p(x) = e^{-2|x|}$

Find the standard deviation of each of the following:

17. $p(x) = \begin{cases} 1 & if\ \ 0 \le x \le 1 \\ 0 & otherwise \end{cases}$

18. $p(x) = \begin{cases} \frac{1}{6} & if\ \ 0 \le x \le 6 \\ 0 & otherwise \end{cases}$

19. $p(x) = \begin{cases} 0 & if \quad x < 0 \\ 0.7e^{-0.7x} & if \quad x \ge 0 \end{cases}$

20. $p(x) = \begin{cases} 0 & if \quad x < 0 \\ \pi e^{-\pi x} & if \quad x \ge 0 \end{cases}$

21. $p(x) = \begin{cases} \frac{3}{4} - \frac{3}{4}x^2 & if \quad -1 \le x \le 1 \\ 0 & otherwise \end{cases}$

22. $p(x) = \frac{1}{2}e^{-|x|}$

23. Intelligence quotient (IQ) scores are normally distributed with a mean of $\mu = 100$ and a standard deviation of $\sigma = 15$. Use numerical integration to determine the percentage of the population which has an IQ score between 85 and 115.

24. The heights of adult males in the United States are normally distributed with a mean of 69 inches and a standard deviation of 2.8 inches. Use numerical integration to estimate the percentage of the adult male population which is between five feet and six feet tall.

25. In 1999, SAT I Verbal test scores were normally distributed with a mean of 505 and a standard deviation of 111. What is the probability of a student scoring between 600 and 700 on the SAT verbal test?

26. In 1999, SAT I Math test scores were normally distributed with a mean of 511 and a standard deviation of 114. What is the probability of a student scoring between 400 and 600 on the SAT math test?

27. The weight of a peach from a certain orchard is normally distributed with mean 8 ounces and standard deviation of 1 ounce. What is the probability of a peach from that orchard having a weight between 5 ounces and 6 ounces?

28. The antennae lengths of a sample of 32 woodlice were measured and found to have a mean of 4 mm and standard deviation of 2.37 mm. Assuming the antennae lengths are normally distributed, what is the probability of a woodlice antennae being between 3 mm and 5 mm in length.

29. Shown below are 100 sample means, where each sample mean is the average of 8 numbers chosen at random from the interval $[0,1]$.

$$0.30, 0.50, 0.60, 0.46, 0.52, 0.55, 0.54, 0.27, 0.47, 0.50, 0.66, 0.40, 0.76, 0.39, 0.55,$$
$$0.64, 0.43, 0.61, 0.49, 0.80, 0.44, 0.49, 0.50, 0.44, 0.35, 0.36, 0.20, 0.62, 0.63, 0.27,$$
$$0.40, 0.48, 0.43, 0.42, 0.53, 0.46, 0.63, 0.62, 0.40, 0.60, 0.54, 0.39, 0.46, 0.45, 0.30,$$
$$0.37, 0.54, 0.46, 0.42, 0.68, 0.61, 0.63, 0.60, 0.51, 0.59, 0.34, 0.52, 0.60, 0.52, 0.29,$$
$$0.48, 0.47, 0.46, 0.67, 0.60, 0.45, 0.27, 0.73, 0.50, 0.58, 0.41, 0.45, 0.36, 0.39, 0.51,$$
$$0.68, 0.62, 0.32, 0.42, 0.49, 0.54, 0.31, 0.45, 0.38, 0.73, 0.52, 0.34, 0.50, 0.47, 0.33,$$
$$0.67, 0.66, 0.44, 0.47, 0.54, 0.64, 0.68, 0.45, 0.68, 0.67$$

Let's approximate the distribution of the sample means with a Gaussian distribution.

(a) The probability density for choosing a number X at random from $[0,12]$ is

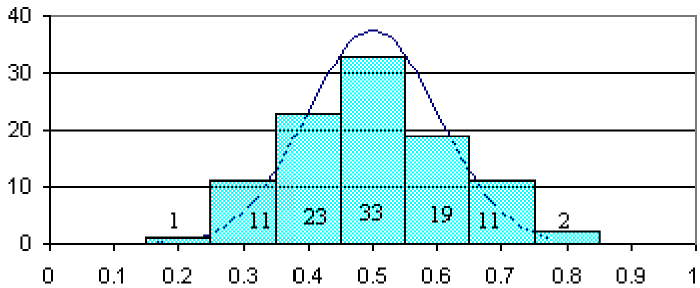
$$p(x) = \begin{cases} 1 & \text{if} \quad 0 \le x \le 1 \\ 0 & \text{if} \quad x < 0 \text{ or } x > 1 \end{cases}$$

What is the mean and standard deviation of X?

(b) The 100 sample means above were placed into a *histogram*, which is a bar graph showing the number of sample mean data points which occur in each of the intervals

$$[0, 0.05), [0.05, 0.15), [0.15, 0.25), [0.25, 0.35), [0.35, 0.45), [0.45, 0.55),$$
$$[0.55, 0.65), [0.65, 0.75), [0.75, 0.85), [0.85, 0.95), [0.95, 1]$$

The result is shown in the image below:



7-5: Histogram approximation of Gaussian

The equation of the bell curve in the image is

$$p(x) = 37.5e^{-(x-\mu)^2/(2\sigma^2)}$$

What are μ and σ?

30. **Write to Learn: Try it out!:** Use a spreadsheet or write a program that chooses 8 numbers at random from $[0,1]$ and then computes their average. Generate 100 such sample means and count the number in each of the subintervals in exercise 29(b). How close is the histogram to the bell curve in 29(b)?

31. The lifetime of a lightbulb is exponentially distributed with an average lifetime of 1000 hours. What is the probability that a lightbulb will last more than 2000 hours?

32. Distribution of incomes are ofetn modeled with the *Pareto Distribution*, which has a density of

$$p(x) = \begin{cases} \dfrac{nb^n}{x^{n+1}} & \text{if} \quad x \geq b \\ 0 & \text{if} \quad x < b \end{cases}$$

where b and n are positive constants.

(a) Show that $p(x)$ is a probability density. What is the mean and standard deviation of a pareto-distributed random variable?

(b) Use (a) to answer the following: Let's suppose that in a certain population, the lowest possible earned income is $b = \$18,000$ and the average income is $b = \$27,000$. What should n be? What is the probability of an income of more than $100,000, assuming a pareto distribution?

33. In the *Bohr model* of the Hydrogen atom an electron orbits a single proton in a circular orbit with radius $r_B = 5.29 \times 10^{-11}$ meters.[2] A different model of the hydrogen atom, called the *wave model*, assumes the electron is in a cloud and its radius can vary. The probability density for the distance r of the electron from the center of the atom is

$$p(r) = \frac{4}{r_B^3} r^2 e^{\frac{-2r}{r_B}}$$

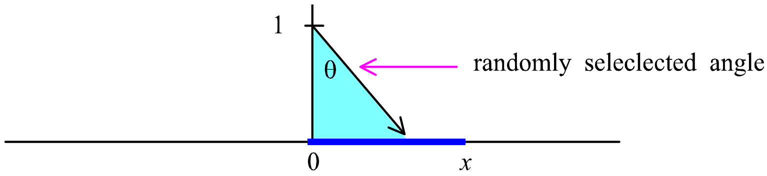
That is, $p(r)(r_2 - r_1)$ is the approximate probability of finding the electron orbiting with a radius between r_1 and r_2 when r_1 and r_2 are close together.

(a) Graph $p(r)$ for r in the domain $\left[0, 2.116 \times 10^{-10}\right]$, and then show that $p(r)$ is maximized at the Bohr radius r_B (that is, the Bohr radius has the highest probability per unit length of being the distance from the proton to the electron in the hydrogen atom.)

(b) Assuming that $p(r) = 0$ if $r < 0$ (negative radii are impossible), what is the expected value of the random variable r with probability density $p(r)$? Why does the expected value not occur at the maximum of the density?

(c) What is the standard deviation of r?

34. Let X be the x-intercept of a line through $(0, 1)$ whose angle θ with the y-axis is chosen at random from between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$.



7-6: θ is selected at random

[2] r_B is the radius at which the centrifugal force of the circular orbit (given by Newton's law) is offset by the electrostatic attraction between the electron and the proton (given by Coulomb's law).

The probability density for X is the *cauchy density*

$$p(x) = \frac{1}{\pi(1+x^2)}$$

Does X have a mean? Does X have a standard deviation? Explain.

35. In this exercise, we show that a random variable with the Gaussian density

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}$$

has an expected value of μ.

(a) Explain the following:

$$\int_{-\infty}^{\infty} xp(x)\,dx = \int_{-\infty}^{\infty} (x-\mu)\,p(x)\,dx + \mu\int_{-\infty}^{\infty} p(x)\,dx$$

(b) Evaluate $\int_{-\infty}^{\infty} (x-\mu)\,p(x)\,dx$ using a substitution.

(c) Combine (a) and (b). What is the expected value of the random variable with density $p(x)$?

36. In this exercise, we show that a random variable with the Gaussian density

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}$$

has a standard deviation of σ.

(a) Use integration by parts with $u = x-\mu$ and $dv = (x-\mu)\,e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}dx$ to evaluate

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2\, e^{-\frac{1}{2}(x-\mu)^2/\sigma^2} dX$$

(b) Use the fact that $p(x)$ is a probability density to simplify the result in (a).

37. *If the temperature of a collection of molecules remains constant, then the speeds of the molecules have a *Maxwell distribution,* which has a density of

$$p(x) = \sqrt{\frac{2}{\pi}}\,\frac{x^2}{s^3} e^{-x^2/(2s^2)}$$

Use the properties of the Gaussian distribution to show that $p(x)$ is a probability density and then to determine its mean and standard deviation.

38. Fisher's F-distribution is used in analysis of variance, population genetics, and most anywhere else where ratios of sums of squares of random variables are considered. The density of the F-distribution is

$$p(x) = C_{m,n} \frac{x^{(m-2)/2}}{\left(1+\frac{m}{n}x\right)^{(m+n)/2}}\ if\ x \geq 0$$

where m, n, and $C_{m,n}$ are positive constants (and $p(x) = 0$ if $x < 0$). What should the constant $C_{m,n}$ be when $m = 2$?

39. **Write to Learn:** The *median* of a probability density is the number M for which

$$\int_{-\infty}^{M} p(x)\,dx = 0.5$$

That is, the median is the *50th percentile* for the distribution. Find the median for the Gaussian and exponential distributions, and then discuss in a short essay how the shape of the probability density is related to the relative positions of the median and mean.

40. The *failure rate,* which is also known as the *hazard rate,* for a random variable X with p.d.f. $p(x)$ is given by

$$h(t) = \frac{p(x)}{\int_x^{\infty} p(x)\,dx}$$

Show that the failure rate for an exponentially distributed random variable is constant, and that the failure rate for a Weibull distribution is constant only when it reduces to an exponential.

Self Test

A variety of questions are asked in a variety of ways in the problems below. Answer as many of the questions below as possible before looking at the answers in the back of the book.

1. Answer each statement as true or false. If the statement is false, then state why or give a counterexample.

   (a) For any two integrable functions f and g on $[a,b]$, the area between the graphs of $f(x)$ and $g(x)$ over $[a,b]$ is $\int_a^b (f(x)-g(x))dx$.

   (b) If an integrable function $f(x) \leq 0$ for all x in $[a,b]$, then the area between the graph of $f(x)$ and the $x-$axis over $[a,b]$ is $\int_b^a f(x)dx$.

   (c) A solid of revolution always has circular cross-sections when taken perpendicular to the axis of revolution.

   (d) $\int_0^3 \sqrt{1+9x^4}dx$ is the arclength of the graph of the quadratic function $f(x)=3x^2$ from $x=0$ to $x=3$.

   (e) The x-coordinate for the centroid of a 30^o-60^o-90^o triangle with hypotenuse of length 2 and adjacent side of length $\sqrt{3}$ is at $\overline{x}=\frac{2}{\sqrt{3}}$ along the adjacent side.

   (f) The center of gravity of an elliptical laminate occurs at one of the foci of the ellipse.

   (g) To generate a right circular cone with height twice its base diameter we can revolve a line with slope 2 about the $x-axis$.

   (h) The mean value theorem for integrals is used to compute the area of a surface of revolution.

   (i) $\int_1^\infty \frac{\sin(x)}{x}dx$ is an improper integral.

   (j) $\int_{-1}^1 \frac{dx}{x-2}$ is an improper integral.

   (k) $\int_0^1 \ln(x)dx$ is an improper integral.

   (l) The expected value of a random variable is equal to the average value of its probability density.

   (m) If $p\,(x)$ is a probability density that vanishes (i.e., is equal to 0) outside of the interval $[0,2]$, then its average value over $[0,2]$ is $\frac{1}{2}$.

2. The area bounded by $y=x^2$ and $y=x+2$ is described by the following integral:

   (a) $\int_{-1}^2 \left(x+2-x^2\right)dx$

   (b) $\int_{-1}^2 \left(x+1-x^2\right)\,dx$

   (c) $\int_{-1}^2 \left(x+2-x^2\right)\,dx$

   (d) $\int_{-1}^2 \left(x^2-x-2\right)\,dx$

3. Find the volume when the region between $y=-x^2+3x$ and the x-axis is rotated about the x-axis.

   (a) 4.5π (b) 6.7π (c) 8.1π (d) 13.5π

4. Which of the following is true of the centroid of the region between $y = 4x - x^2$ and the x-axis.

(a) $\bar{x} = 2$ (b) $\bar{y} = 0$ (c) $\bar{x} = 0$ (d) $\bar{y} = 4$

5. Which of the following definite integrals represents the area of the surface generated by revolving the curve $y = x^3$, $0 \leq x \leq 4$ about the x-axis?

(a) $2\pi \int_0^4 x^3\sqrt{1+3x^2}\,dx$ (b) $\pi \int_0^4 x^3\sqrt{1+9x^4}\,dx$

(c) $2\pi \int_0^4 x^3\sqrt{1+9x^4}\,dx$ (d) $\pi \int_0^4 x^6\,dx$

6. Which of the integrals below represents the volume of the solid obtained by revolving the region bounded by $x = 0$, $x = 4$, $y = 0$ and $y = \sqrt{x^2+4}$ about the y-axis over the interval $[0, 4]$?

(a) $\int_0^4 \sqrt{x^2+4}dx$ (b) $\pi \int_0^4 \sqrt{x^2+4}dx$
(c) $\pi \int_0^4 \left(x^2+4\right) dx$ (d) $2\pi \int_0^4 x\sqrt{x^2+4}\,dx$

7. Find the volume of the solid obtained by revolving the region between $y = |x|$ and $y = \sqrt{2-x^2}$ about the x-axis.

(a) 4π (b) $\frac{8\pi}{3}$ (c) $\frac{14\pi}{3}$ (d) 22π

8. If $F(x) = \int_1^x \sqrt{t^2-1}\,dt$, then what is the length of the graph of $F(x)$ over $[1, 3]$?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5 (f) 6 (g) 7

9. Which of the following is **not** a probability density on the real line?

(a) $p(x) = e^{-x^2/2}$ (b) $p(x) = 1$ (c) $p(x) = \frac{1}{\pi(x^2+1)}$ (d) $p(x) = e^{-|x|}$

10. Find the volume of the solid of revolution formed by revolving the area between $f(x) = ex$ and $g(x) = e^x$ about the $x-$axis.

11. Find the length of the curve $y = x^{3/2}$ from $x = 0$ to $x = 5$.

12. Find the centroid of the semicircular area between the x-axis and $y = \sqrt{4-x^2}$.

13. Find the average value of $f(x) = \lfloor x^2 \rfloor$ over $[0, 2]$ where $\lfloor x \rfloor$ is the *floor function* and is the largest integer less than or equal to x. (For example, $\lfloor 2.3 \rfloor = 2$, $\lfloor 5 \rfloor = 5$, and $\lfloor -1.3 \rfloor = -2$).

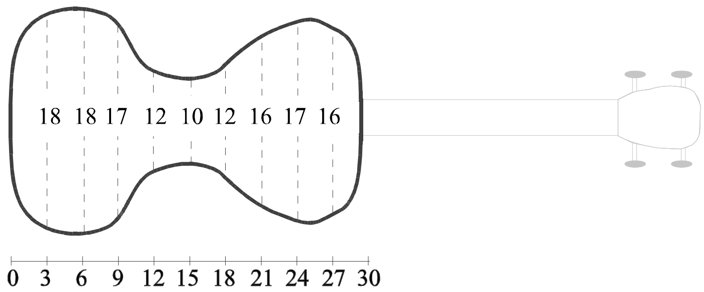
14. Evaluate the improper integral, if it exists.

$$\int_0^1 x^2 \ln(x)\,dx$$

15. Evaluate the improper integral, if it exists.

$$\int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

16. Compute the following for the back panel of a guitar as shown in the figure.



(a) Find a numerical approximation of the area of the panel

(b) Find a numerical approximation of the perimeter of the panel.

(c) Find a numerical approximation of the centroid of the panel.

17. For what value of b is the function

$$p(x) = x^2 e^{-bx}$$

a probability density over $[0, \infty)$. What is its expected value?

18. **Write to Learn:** Find a positive function on $[0, 2]$ whose graph has "infinite length" but that encloses a region with an area of 2. In a short essay, present your function and then explain why the volume of the solid obtained by rotating the region about the x-axis must be infinite. Can you connect it to a theorem of Pappus? Explain.

The Next Step... Conditional Probability

Why are bell curves Gaussian? Why are waiting times exponentially distributed? That is, in general, where do probability densities come from? Our next step is to demonstrate how probability models can be developed from first principles using the concept of *conditional probability.*

Suppose that X is a random variable and suppose that it is known already that all values of X must occur in a subset B. Then given any other subset A, we define

$$\Pr(A \mid B) = \text{probability that } X \text{ has a value in } A \text{ given that the value of } X \text{ is known to be in } B$$

We say that $\Pr(A \mid B)$ is the *conditional probability* of A given that B must occur. Let's derive a formula for $\Pr(A \mid B)$ given that $\Pr(B) \neq 0$.

To begin with, $\Pr(B \mid B)$ is the probability that X has a value in B given that X must have a value in B, which means that

$$\Pr(B \mid B) = 1$$

Similarly, $\Pr(A \mid B)$ should depend only on those values in A that are also in B, which means that $\Pr(A \mid B)$ should depend only on $A \cap B$. These and other considerations lead us to define conditional probability as follows:

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} \tag{5.39}$$

Now that we have introduced the concept of conditional probability, let's look at an example of how it and equation (5.39) are used to construct *probability models* of real world processes. In particular, let's use conditional probability to explore why waiting times are exponentially distributed (that is, why $p(t) = \frac{1}{r}e^{-t/r}$ for $t \geq 0$ is the probability density for waiting times).

Suppose a large number of print jobs have been sent to a printer where one of the print jobs has just started, and suppose that the random variable T denote the amount of time that elapses before the next print job begins. Clearly, the amount of time that elapses before the next print job begins does **not** depend on how long a print job has been waiting, but instead depends on how quickly, on average, the printer can process print jobs.

In particular, suppose each print job requires r seconds, on average. If τ seconds have passed already without a new print job beginning, then there is about an $\frac{h}{r}$ chance of the next print job beginning in the interval $[\tau, \tau+h]$ when h is close to 0. For example, if $r = 10$ seconds and $\tau = 5$ seconds have passed already, then there is only a 1 out of 10 chance that the next print job will begin before 6 seconds have elapsed.

The key here is that intuitive statements like the one above are very often conditional probabilities waiting to be revealed. In the preceding paragraph, we are saying that if we **know already** that $T > \tau$, then the probability that T is in $[\tau, \tau+h]$ is approximately $\frac{h}{r}$. As a conditional probability, we would write that

$$\Pr(T \text{ } in \text{ } [\tau, \tau+h] \mid T > \tau) \approx \frac{h}{r} \tag{5.40}$$

However, equation (5.39) implies that

$$\Pr(T \text{ } in \text{ } [\tau, \tau+h] \mid T > \tau) = \frac{\Pr(T \text{ } in \text{ } [\tau, \tau+h] \cap [\tau, \infty))}{\Pr(T > \tau)}$$

Since $[\tau, \tau+h] \cap [\tau, \infty) = [\tau, \tau+h]$, and since $\Pr(T > \tau) = 1 - \Pr(T \leq \tau)$, this implies that

$$\Pr(T \text{ } in \text{ } [\tau, \tau+h] \mid T > \tau) = \frac{\Pr(T \text{ } in \text{ } [\tau, \tau+h])}{1 - \Pr(T \leq \tau)} \tag{5.41}$$

Combining (5.40) and (5.41) leads to

$$\frac{\Pr(T \text{ } in \text{ } [\tau, \tau+h])}{1 - \Pr(T \leq \tau)} \approx \frac{h}{r}$$

In terms of integrals, $\Pr(T \leq \tau) = \int_0^\tau p(t)\,dt$ and $\Pr(T \text{ } in \text{ } [\tau, \tau+h]) = \int_\tau^{\tau+h} p(t)\,dt$, which leads to

$$\frac{\int_\tau^{\tau+h} p(t)\,dt}{1 - \int_0^\tau p(t)\,dt} \approx \frac{h}{r} \qquad \Longrightarrow \qquad \int_\tau^{\tau+h} p(t)\,dt \approx \frac{h}{r}\left(1 - \int_0^\tau p(t)\,dt\right)$$

Our goal is to determine what $p(t)$ actually is, so we divide by h.

$$\frac{1}{h}\int_\tau^{\tau+h} p(t)\,dt \approx \frac{1}{r}\left(1 - \int_0^\tau p(t)\,dt\right)$$

Since $\int_\tau^{\tau+h} p(t)\,dt = \int_0^{\tau+h} p(t)\,dt - \int_0^\tau p(t)\,dt$, we obtain

$$\frac{\int_0^{\tau+h} p(t)\,dt - \int_0^\tau p(t)\,dt}{h} \approx \frac{1}{r}\left(1 - \int_0^\tau p(t)\,dt\right)$$

As h approaches 0, the approximation becomes exact, which is to say that

$$\lim_{h\to 0} \frac{\int_0^{\tau+h} p(t)\,dt - \int_0^{\tau} p(t)\,dt}{h} = \frac{1}{r}\left(1 - \int_0^{\tau} p(t)\,dt\right) \tag{5.42}$$

However, the limit in (5.42) above is clearly a derivative, so that we obtain

$$\begin{aligned} \frac{d}{d\tau}\int_0^{\tau} p(t)\,dt &= \frac{1}{r}\left(1 - \int_0^{\tau} p(t)\,dt\right) \\ p(\tau) &= \frac{1}{r}\left(1 - \int_0^{\tau} p(t)\,dt\right) \end{aligned}$$

Differentiation with respect to τ again then yields

$$p'(\tau) = -\frac{1}{r}\,p(\tau)$$

and the only solution to the differential equation $p' = \frac{-1}{r}p$ is the exponential function

$$p(t) = Ce^{-t/r}$$

where C is a constant. The fact that $\int_0^{\infty} p(t)\,dt = 1$ can then be used to show that $C = \frac{1}{r}$, so that the probability density for print jobs waiting in a que is

$$p(t) = \frac{1}{r}e^{-t/r} \quad for \;\; t \geq 0$$

However, the most important observation in this example may be that the model itself was derived using conditional probability.

Write to Learn In a short essay, derive the "memoryless" property of the exponential density, which says that for any $a > 0$, we have

$$\Pr(T > s + t \mid T > t) = \Pr(T > s)$$

Can you use the concept of conditional probability to explain what the memoryless property means? Also, be sure to derive the equivalent property

$$\Pr(T > s + t) = \Pr(T > s)\Pr(T > t)$$

Write to Learn A random variable T is said to have a *Weibull distribution* if given that $T > \tau$, the probability that T is in $[\tau, \tau + h]$ is approximately $\lambda\alpha t^{\alpha-1}h$ when h is small, where λ and a are positive constants. Mimic the derivation of the exponential density above to obtain the p.d.f. of a Weibully distributed random variable.

Write to Learn If X is a random variable with p.d.f. of $p(x)$ and if it is known that X is in a subset B of $(-\infty, \infty)$ where $\Pr(B) > 0$, then the *conditional density* of X given B is

$$p_B(x) = \frac{p(x)}{\Pr(B)}$$

Show that $p_B(x)$ for x restricted to the set B is a probability density function, and then explain why if A is a set of possible outcomes for X, then

$$\Pr(A \cap B') = 0$$

where $B' = \{x \text{ in } (-\infty, \infty) \text{ such that } x \text{ is not in } B\}$. Finally, use these properties and the concept of a conditional density to explain why

$$P(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Write to Learn Go to the library and/or search the internet to learn more about conditional probability and its use in modeling real-world processes. Write a short report documenting what you discover in that research.

Group Learning The *hazard rate function* $r(t)$ of a random variable T is defined by

$$r(t) = \lim_{h \to 0} \frac{\Pr(t \le T \le t + h \mid T > t)}{h}$$

It is especially important in determining probabilities such as the probability that a machine will fail in a given time interval, the probability that an employee will retire before a certain age, and so on. In this project, each member of the group should prove a property of the hazard rate function and the results should be presented as a presentation or as a group report.

(a) Show that the hazard rate of an exponentially-distributed random variable is constant.

(b) Show that if $S(t) = \Pr(T > t)$, then

$$r(t) = -\frac{d}{dt} \ln(S(t))$$

(c) Use the result in (b) to show that

$$S(t) = e^{-\int_0^t r(u)du}$$

(d) Show that if a random variable T has a hazard rate of $r(t)$, then the p.d.f. for T is

$$p(t) = r(t)\, e^{-\int_0^t r(u)du}$$

(e) If T is a positive random variable, then its mean μ is given by

$$\mu = \int_0^\infty t f(t)\, dt$$

Use integration by parts with $u = t$ and $dv = f(t)\, dt$ to show that

$$\mu = \int_0^\infty S(t)\, dt$$

and thus, combined with (c), we must have

$$\mu = \int_0^\infty e^{-\int_0^t r(u)du} dt$$

Advanced Contexts:

Although we have been using integrals only to represent probabilities, we can also go the other way and use probability to estimate definite integrals. Methods which use probability to estimate definite integrals and other mathematical quantities are often called *Monte Carlo* methods because they are often based on randomly generated numbers.

In the problems below, our focus will be primarily on estimation methods that use *uniformly distributed random variables*, where a random variable X is said to be *uniformly distributed* across an interval $[a, b]$ if there is an equal probability of X being in any two subintervals of $[a, b]$ with the same length. If X is uniformly randomly distributed across $[a, b]$, then the p.d.f. for X is

$$p(x) = \begin{cases} \frac{1}{b-a} & if\ x\ is\ in\ [a, b] \\ 0 & otherwise \end{cases}$$

For example, if X is uniformly randomly distributed across $[0,1]$, then its p.d.f. if

$$p(x) = \begin{cases} 1 & if\ x\ is\ in\ [0,1] \\ 0 & otherwise \end{cases}$$

We concentrate on uniformly distributed random variables because most computer algebra system random number generators will generate uniformly distributed random numbers.

1. * Suppose that $f(x) \geq 0$ on $[a,b]$ and that there is a number M such that $f(x) \leq M$ for all x in $[a,b]$. Show that if X is uniformly distributed across $[a,b]$ and Y is uniformly distributed across $[0,M]$, then

$$\Pr(Y \leq f(X)) = \frac{1}{M(b-a)} \int_a^b f(x)\,dx$$

   (You might want to draw a picture first)

2. * Use the result in exercise 1 to generate an approximation of

$$\int_0^1 \sin\left(\pi x^2\right) dx$$

   In particular, explain and implement the following pseudo-code:

   i. let count = 0
   ii. For i from 1 to 1000 do
      A. choose X uniformly from $[0,1]$
      B. choose Y uniformly from $[0,1]$
      C. if $Y \leq \sin\left(\pi X^2\right)$ then count = count + 1
   iii. end do
   iv. estimate of integral = count ÷ 1000.

3. * As an alternative, use an argument similar to the last section to show that given a random variable X with *p.d.f.* $p(x)$and a function f, the *expected value* (i.e., the average value) of $f(X)$ is

$$\overline{f(X)} = \int_{-\infty}^{\infty} f(x)\,p(x)\,dx$$

   Also, explain why if X is uniformly randomly distributed over $[a,b]$, then this reduces to

$$\overline{f(X)} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

   If I were to generate 1000 uniformly distributed random numbers $X_1, \ldots, X_{1000}$ from $[a,b]$, then what is the approximate value of

$$\frac{f(X_1) + f(X_2) + \ldots + f(X_{1000})}{1000}$$

6. HYPERBOLIC AND INVERSE FUNCTIONS

Calculus is not only a means of studying functions, but it is also a tool for *creating functions.* This idea will be an important theme in the next four chapters. In fact, this idea is one of the most important aspects of calculus, because it allows us to design functions specifically for the application at hand.

For example, there are processes which are best modeled by arithmetic combinations of exponential functions. Such combinations are called the *hyperbolic functions* because they are closely related to hyperbolas (much like trigonometric functions are related to circle and are thus sometimes referred to as the circular functions).

Moreover, some of the most important functions in mathematics and science are *inverse functions,* where a function $g(x)$ is the inverse of a function $f(x)$ if f and g cancel under composition over some range of inputs. For example, $\sqrt{x}$ is the inverse of x^2 on $(0,\infty)$ since

$$\sqrt{x^2} = x \;\; for\; x > 0 \quad and \qquad \left(\sqrt{x}\right)^2 = x \;\; for\; x > 0$$

Likewise, the exponential and natural logarithm functions are inverses of each other on $(0,\infty)$ because

$$e^{\ln(x)} = \ln\left(e^x\right) = x \; for\; all\; x > 0$$

All of these "new" functions (i.e., they are new to us) have their own unique sets of properties, identities, derivative formulas, and integral forms. In this chapter, we explore some of these new functions both in the role that they play in calculus and in the role that calculus plays in studying them. Subsequent chapters will then continue this idea of calculus as a "function-creation" tool.

6.1 The Hyperbolic Functions

Definition of the Hyperbolic Functions

We begin this chapter by introducing an important new class of functions known as the *hyperbolic functions.* The hyperbolic functions are related to the trigonometric functions and occur frequently in applications.

To begin with, the *hyperbolic cosine* and *hyperbolic sine*, which are called the "cosh" and the "sinch" functions, respectively, are defined

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \qquad \sinh(x) = \frac{e^x - e^{-x}}{2} \tag{6.1}$$

In addition, the *hyperbolic tangent, hyperbolic cotangent, hyperbolic secant and hyperbolic cosecant* are defined respectively

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)},$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}, \qquad \operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

The derivatives of the hyperbolic functions follow from their definitions. Indeed, the definition of $\cosh(x)$ implies that

$$\frac{d}{dx}\cosh(x) = \frac{d}{dx}\frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \sinh(x)$$

Likewise, the definitions of the hyperbolic functions imply the following:

$$\begin{aligned} \frac{d}{dx}\sinh(x) &= \cosh(x), & \frac{d}{dx}\cosh(x) &= \sinh(x) \\ \frac{d}{dx}\tanh(x) &= \operatorname{sech}^2(x), & \frac{d}{dx}\coth(x) &= -\operatorname{csch}^2(x) \\ \frac{d}{dx}\operatorname{sech}(x) &= -\operatorname{sech}(x)\tanh(x), & \frac{d}{dx}\operatorname{csch}(x) &= -\operatorname{csch}(x)\coth(x) \end{aligned}$$

See exercise 31 for details.

EXAMPLE 1 Find $f'(x)$ if $f(x) = \tanh(\ln x)$.

Solution: To do so, we use the chain rule to obtain

$$f'(x) = \frac{d}{dx}\tanh(\text{input}) = \operatorname{sech}^2(\text{input})\frac{d}{dx}\text{input}$$

where the input is the function $\ln(x)$. As a result, we have

$$f'(x) = \operatorname{sech}^2(\ln x)\frac{d}{dx}\ln x = \frac{1}{x}\operatorname{sech}^2(\ln x)$$

EXAMPLE 2 Evaluate $\frac{d}{dx}\left[e^{3x}\cosh(2x)\right]$

Solution: The product rule implies that

$$\begin{aligned} \frac{d}{dx}\left[e^{3x}\cosh(2x)\right] &= \left[\frac{d}{dx}e^{3x}\right]\cosh(2x) + e^{3x}\frac{d}{dx}\cosh(2x) \\ &= e^{3x}\left[\frac{d}{dx}3x\right]\cosh(2x) + e^{3x}\sinh(2x)\left[\frac{d}{dx}2x\right] \\ &= 3e^{3x}\cosh(2x) + 2e^{3x}\sinh(2x) \end{aligned}$$

Check your Reading *How was the chain rule utilized in example 2?*

Properties of the Hyperbolic Functions

Moreover, notice that (6.1) implies that

$$\begin{aligned} \cosh^2(x) - \sinh^2(x) &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x} + 2e^{-x}e^x + e^{-2x}}{4} - \frac{e^{2x} - 2e^{-x}e^x + e^{-2x}}{4} \\ &= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} \end{aligned}$$

since $e^x e^{-x} = 1$. Since the last expression reduces to $\frac{4}{4} = 1$, this leads to

$$\cosh^2(x) - \sinh^2(x) = 1 \tag{6.2}$$

(additional proofs in the exercises). Division throughout by $\cosh^2(x)$ then leads to the identity

$$1 - \tanh^2(x) = \operatorname{sec h}^2(x)$$

The hyperbolic functions have many other properties similar to that of the circular trigonometric functions, including the following:

$$\begin{aligned} \sinh(-x) &= -\sinh(x) \qquad\qquad \cosh(-x) = \cosh(x) && (6.3) \\ \sinh(x+y) &= \sinh(x)\cosh(y) + \cosh(x)\sinh(y) && (6.4) \\ \cosh(x+y) &= \cosh(x)\cosh(y) + \sinh(x)\sinh(y) && (6.5) \end{aligned}$$

Additional properties will be derived in the exercises and examples.

EXAMPLE 3 Show that $\sin(2x) = 2\sinh(x)\cosh(x)$.

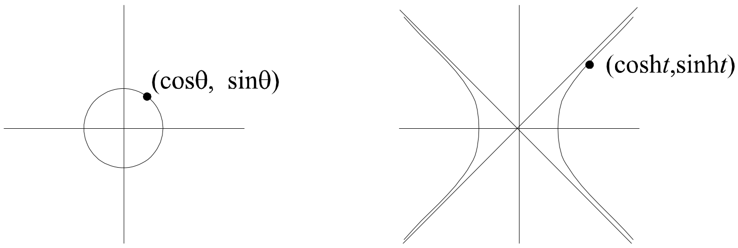
Solution: If we let $x = y$ in (6.4), then

$$\begin{aligned} \sinh(2x) &= \sinh(x+x) \\ &= \sinh(x)\cosh(x) + \sinh(x)\cosh(x) \\ &= 2\sinh(x)\cosh(x) \end{aligned}$$

Graphically, the identity $\cosh^2(t) - \sinh^2(t) = 1$ implies that the point $(\cosh t, \sinh t)$ is on the *hyperbola*

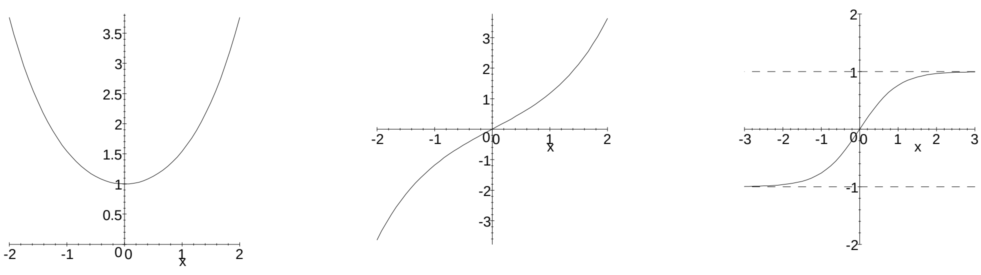
$$x^2 - y^2 = 1$$

which is analogous to the fact that $(\cos\theta, \sin\theta)$ is a point on the unit circle.



1-1: Hyperbolic functions defined by a hyperbola

As a result, $\cosh(x)$ and $\sinh(x)$ both approach ∞ as x becomes arbitrarily large. Since $\tanh(x)$ is the ratio of two functions $\sinh(x)$ and $\cosh(x)$ that are approaching ∞ at the same rate, $\tanh(x)$ approaches 1 as x approaches ∞.



$y = \cosh(x)$ $\qquad$ $y = \sinh(x)$ $\qquad$ $y = \tanh(x)$

1-2: Graphs of three hyperbolic functions

The graph of $\cosh(x)$ should be familiar, since a hanging cable such as power line is modeled by the graph of a hyperbolic cosine, as we will explore at the end of this section and in later sections of the text.

EXAMPLE 4 Find the equation of the tangent line to $y = \tanh(x)$ when $x = 0$.

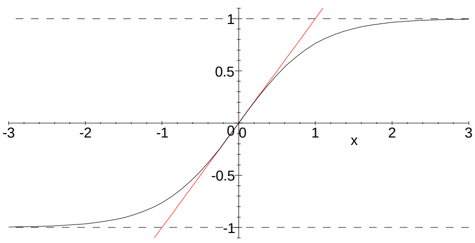
Solution: Since $y' = \operatorname{sech}^2(x)$, the slope of the tangent line at $x = 0$ is

$$y'(0) = \sec\mathrm{h}^2(0) = \frac{1}{\cosh^2(0)} = 1$$

Since $f(0) = \tanh(0) = 0$, the tangent line has a slope of 1 and passes through $(0,0)$:

$$y = 0 + 1(x - 0) = x$$

Thus, $y = x$ is the tangent line to $y = \tanh(x)$ at $(0,0)$.



1-3: The line $y = x$ is tangent to $y = \tanh(x)$ at $(0,0)$.

Check your Reading *What is the slope of the tangent line to $y = \cosh(x)$ at $x = 0$?*

Antiderivatives of the Hyperbolic Functions

Antiderivatives of hyperbolic functions follow from the derivative formulas above. In particular, we have the following:

$$\int \cosh(x)\,dx = \sinh(x) + C \qquad \int \sinh(x)\,dx = \cosh(x) + C$$

$$\int \operatorname{sech}^2(x)\,dx = \tanh(x) + C \qquad \int \operatorname{csch}^2(x)\,dx = -\coth(x) + C$$

$$\int \operatorname{sech}(x)\tanh(x)\,dx = -\operatorname{sech}(x) + C, \qquad \int \operatorname{csch}(x)\coth(x)\,dx = -\operatorname{csch}(x) + C$$

Also, the definition of the hyperbolic functions may be used in evaluating antiderivatives involving hyperbolic functions.

EXAMPLE 5 Evaluate the antiderivative

$$\int x\cosh(x)\,dx$$

Solution: Tabular integration with $u = x$ and $dv = \cosh(x)$ leads to

u		dv
x		$\cosh(x)$
	$\searrow^{+}$	
1		$\sinh(x)$
	$\searrow^{-}$	
0		$\cosh(x)$

$$\int x\cosh(x)\,dx = x\sinh(x) - \cosh(x) + C$$

EXAMPLE 6 Evaluate $\int \tanh(x)\,dx$

Solution: To begin with, we use the definition of the hyperbolic tangent to obtain

$$\int \tanh(x)\,dx = \int \frac{\sinh(x)\,dx}{\cosh(x)}$$

Thus, if $u = \cosh(x)$, then $du = \sinh(x)\,dx$ and

$$\int \tanh(x)\,dx = \int \frac{\sinh(x)\,dx}{\cosh(x)} = \int \frac{du}{u} = \ln|u| + C$$

As a result, the antiderivative is

$$\int \tanh(x)\,dx = \ln|\cosh(x)| + C$$

In addition, the identities

$$\cosh^2(x) = 1 + \sinh^2(x) \qquad \textit{and} \qquad \operatorname{sech}^2(x) = 1 - \tanh^2(x)$$

are also used to simplify antiderivatives.

EXAMPLE 7 Evaluate $\int \cosh^3(x)\,dx$

Solution: Writing $\cosh^3(x) = \cosh^2(x)\cosh(x)$ allows us to use an identity:

$$\begin{aligned}\int \cosh^3(x)\,dx &= \int \cosh^2(x)\cosh(x)\,dx \\ &= \int \left(1 + \sinh^2(x)\right)\cosh(x)\,dx\end{aligned}$$

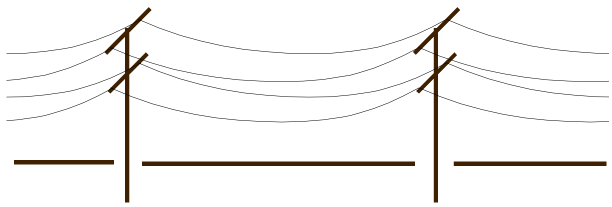
If $u = \sinh(x)$, then $du = \cosh(x)\,dx$ and

$$\begin{aligned}\int \cosh^3(x)\,dx &= \int \left(1 + \sinh^2(x)\right)\cosh(x)\,dx \\ &= \int 1 + u^2 du \\ &= u + \frac{u^3}{3} + C \\ &= \sinh(x) + \frac{1}{3}\sinh^3(x) + C\end{aligned}$$

Check your Reading *What is* $\operatorname{sech}^2(x) + \tanh^2(x)$*?*

Arclength of a Catenary

A hanging power line is an example of a curve called a *catenary,* where a catenary is all or part of the curve $y = \frac{1}{a}\cosh(ax)$ for a constant $a > 0$.



1-4: Power lines imply curves of the form $y = \frac{1}{a}\cosh(ax)$.

Arclengths of catenaries can be computed using the identity

$$\cosh^2(A) = 1 + \sinh^2(A)$$

to simplify under the square root.

EXAMPLE 8 What is the arclength of the catenary $y = \frac{1}{2}\cosh(2x)$ over $[-1, 1]$?

Solution: Since $y' = \frac{1}{2}\sinh(2x)\left(\frac{d}{dx}2x\right) = \frac{1}{2}\sinh(2x)\,2 = \sinh(2x)$, we have

$$1 + [y']^2 = 1 + \sinh^2(2x) = \cosh^2(2x)$$

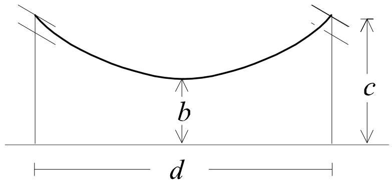
As a result, the arclength integral is

$$s = \int_{-1}^{1}\sqrt{\cosh^2(2x)}dx = 2\int_0^1 \cosh(2x)\,dx$$

since $\cosh(2x)$ is even. Integration then leads to

$$s = \sinh(2x)|_0^1 = \sinh(2)$$

In particular, suppose a cable is suspended between two poles a distance d apart, and suppose that the middle of the cable sags to a height b when it is connected at equal heights c to the two poles:

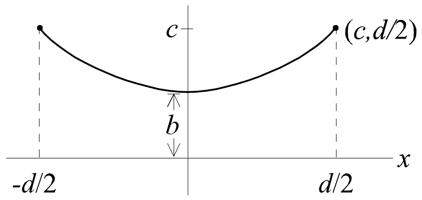


1-5: Parameters for Catenary

At the end of chapter 7 we show that the cable implies the curve

$$y = b + \frac{1}{a}(\cosh(ax) - 1)$$

where a is the solution to the equation $\cosh\left(\frac{1}{2}ad\right) = 1 + a(c-b)$.

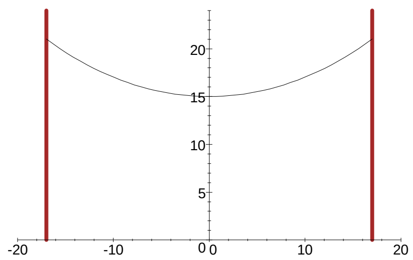


1-6: Catenary with equation $y = b + \frac{1}{a}[\cosh(ax) - 1]$

EXAMPLE 9 A cable hanging between two poles that are 34 feet apart sags to a height of 15 feet in its middle. If it is connected to each pole at a height of 21 feet, then its shape is given by

$$y = 15 + 25(\cosh(0.04x) - 1)$$

How long is the cable?



1-7: Catenary in example 8

Solution: Since $y' = \sinh(0.04x)$, we have

$$1 + [y']^2 = 1 + \sinh^2(0.04x) = \cosh^2(0.04x)$$

As a result, the arclength integral is

$$s = \int_{-17}^{17} \sqrt{\cosh^2(0.04x)}dx = 2\int_0^{17} \cosh(0.04x)\,dx$$

since $\cosh(0.04x)$ is even. Integration then leads to

$$s = \frac{2}{0.04}\sinh(0.04x)\Big|_0^{17} = 50\sinh(0.68) = 36.68\ feet$$

Thus, if the cable was 34 feet long when suspended, then it stretched about 2 feet, 8 inches due to sagging.

Exercises:

Find $f'(x)$ for each of the following: .

1.	$f(x) = \cosh(3x)$	2.	$f(x) = x\sinh(x)$
3.	$f(x) = \cosh\left(x^2\right)$	4.	$f(x) = e^x \operatorname{sech}(x)$
5.	$f(x) = \sinh(2x)$	6.	$f(x) = \sin(x)\sinh(x)$
7.	$f(x) = \sinh\left(x^3\right)$	8.	$f(x) = \cosh^2(x)$
9.	$f(x) = \ln(\cosh(x))$	10.	$f(x) = \sinh^2\left(x^3\right)$

Evaluate the following.

11. $\int \cosh(3x)\,dx$

12. $\int \sinh(\pi x)\,dx$

13. $\int x\text{sech}(x^2)\tanh(x^2)\,dx$

14. $\int x\,\text{csch}^2(x^2)\,dx$

15. $\int x\,\text{sech}^2(x)\,dx$

16. $\int \cosh(3x)\,dx$

17. $\int \cosh(\ln x)\,dx$

18. $\int \tanh(\ln x)\,dx$

Use the definitions in (6.1) to prove the following:

19. $\cosh(-x) = \cosh(x)$

20. $\sinh(-x) = \sinh(x)$

21. $2\sinh(x)\cosh(x) = \sinh(2x)$

22. $\cosh^2(x) + \sinh^2(x) = \cosh(2x)$

23. $\tanh(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

24. $\tanh(x) = \dfrac{e^{2x} - 1}{e^{2x} + 1}$

25. $\frac{d}{dx}\sinh(x) = \cosh(x)$

26. $\text{sech}^2(x) = 1 - \tanh^2(x)$

27. What is the length over $[-\ln(2), \ln(2)]$ of the catenary curve $y = \cosh(x)$.

28. What is the length over $[-10, 10]$ of the catenary curve

$$y = 10 + 20\cosh\left(\frac{x}{20}\right)$$

29. A chain hangs between two poles that are $32\ln(2) \approx 22.18$ feet apart sags to a height of 10 feet in its middle. If either end is at a height of 14 feet, then do the following:

(a) Use figure 1-6 to show that the equation of the catenary formed by the chain is

$$f(x) = 10 + 16\left(\cosh\left(\frac{x}{16}\right) - 1\right)$$

(b) Find the length of the chain using the result in (a).

30. Try it Out! Find a cable connected at equal heights c between two poles that are a distance d apart. Measure the height b of the cable in the middle. Determine either graphically or numerically the approximate value of a where the function

$$E(a) = \cosh\left(\frac{1}{2}ad\right) - 1 - a(c - b)$$

is equal to 0. Substitute this value of a into the equation of the catenary

$$y = b + \frac{1}{a}(\cosh(ax) - 1)$$

and then compute its length over $[-d/2, d/2]$.

31. Let's establish the derivative formulas for the hyperbolic functions.

(a) Use the fact that $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and $\cosh(x) = \frac{e^x + e^{-x}}{2}$ to show that

$$\frac{d}{dx}\sinh(x) = \cosh(x)$$

(b) Use the definitions of the hyperbolic functions, the quotient rule, and the derivatives

$$\frac{d}{dx}\sinh(x) = \cosh(x), \qquad \frac{d}{dx}\cosh(x) = \sinh(x)$$

to show that

$$\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x), \qquad \textit{and} \qquad \frac{d}{dx}\coth(x) = -\operatorname{csch}^2(x)$$

(c) Use the definitions of the hyperbolic functions, the quotient rule, and the derivatives

$$\frac{d}{dx}\sinh(x) = \cosh(x), \qquad \frac{d}{dx}\cosh(x) = \sinh(x)$$

to show that

$$\frac{d}{dx}\operatorname{sech}(x) = -\operatorname{sech}(x)\tanh(x), \quad \textit{and} \quad \frac{d}{dx}\operatorname{csch}(x) = -\operatorname{csch}(x)\coth(x)$$

32. This exercise explores some other identities for the hyperbolic sine and cosine functions:

(a) Use the definitions in (6.1) to show that

$$e^x = \cosh(x) + \sinh(x)$$

(b) Use the definitions in (6.1) to show that

$$e^{-x} = \cosh(x) - \sinh(x)$$

(c) Use the definitions in (6.1) to show that

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$

(d) Use the definitions in (6.1) to show that

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$$

(e) Obtain (6.2) by letting $y = -x$ in the identity in (c).

33. The following exercise explores the graph of sech(x) using

$$\operatorname{sech}(x) = \frac{2}{e^x + e^{-x}}$$

(a) Show that sech(x) has a maximum when $x = 0$.

(b) Show that

$$\lim_{x\to\pm\infty}\operatorname{sech}(x) = 0$$

(c) Explain why sech(x) is always positive.

(d) Use the information in (a)-(c) to sketch the graph of sech(x) . Compare to a graph with a graphing calculator.

34. The following exercise explores the graph of csch(x) using

$$\operatorname{csch}(x) = \frac{2}{e^x - e^{-x}}$$

(a) Explain why $\operatorname{csch}(x)$ has no extrema. Where is the vertical asymptote of $\operatorname{csch}(x)$?
(b) Show that

$$\lim_{x \to \pm\infty} \operatorname{csch}(x) = 0$$

(c) Where is $\operatorname{csch}(x)$ positive, and where is it negative?
(d) Where is $\operatorname{csch}(x)$ concave up, and where is it concave down?
(e) Use the results in (a)-(d) to sketch the graph of $\operatorname{csch}(x)$. Compare to a graph of $\operatorname{csch}(x)$ on a graphing calculator.

35. **Write to Learn:** Write a short essay proving (6.2) by showing that

$$\frac{d}{dx} \cosh^2(x) = \frac{d}{dx} \sinh^2(x)$$

and then showing that $\cosh^2(x)$ differs from $\sinh^2(x)$ by the constant 1. (Hint: consider $x = 0$).

36. Use the definition of $\tanh(x)$ to show that

$$\lim_{x \to \infty} \tanh(x) = 1 \quad \text{and} \quad \lim_{x \to -\infty} \tanh(x) = -1$$

37. Let $y(t) = \cosh(it)$, where i is the imaginary unit (i.e., $i^2 = -1$). Show that $y(t)$ is the solution to the harmonic oscillator

$$y'' + y = 0, \quad y(0) = 1, \quad y'(0) = 0 \tag{6.6}$$

Explain why (6.6) implies that $y(t) = \cos(t)$ and thus that

$$\cosh(it) = \cos(t)$$

38. Show that $u(t) = \sinh(it)$ and $v(t) = i \sin(t)$ are both solutions to

$$y'' + y = 0, \quad y(0) = 0, \quad y'(0) = i$$

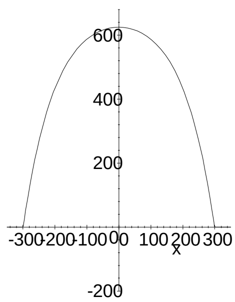
(where $i^2 = -1$). Then explain why this implies that

$$\sinh(it) = i \sin(t)$$

39. The *Gateway Arch* in Saint Louis is an inverted *catenary* with equation

$$y = 693.8597 - 68.7672 \cosh(0.0100333x) \tag{6.7}$$

where y is the height in feet of the center of the arch at x feet from the point immediately below its center.



1-8: The Gateway Arch in Saint Louis is a Hyperbolic Cosine

(a) How high is the arch at its highest point?

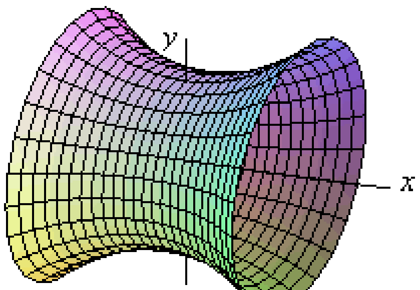
(b) Many people think the Gateway arch is a *parabola.* However, a parabolic shape does not have the structural stability that a catenary has. Graph both the catenary (6.7) and the parabola

$$y = 625.0925\left(1 - \frac{x^2}{299.2261^2}\right)$$

over the interval $[-300, 300]$. How different are the two curves?

(c) How long is the Gateway arch in Saint Louis?

40. The surface of revolution obtained by revolving the graph of $y = \cosh(x)$ about the x-axis is called a *catenoid.*



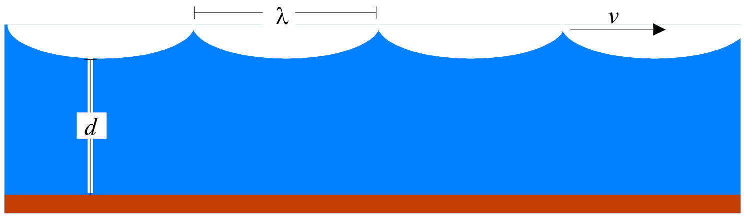
1-9: Catenoid in example 40

It is a type of surface known as a *minimal surface*, in that of all the surfaces connecting the two circles on either end, it is the one with the least surface area. Demonstrate this idea by finding the surface area of the catenoid obtained by revolving $y = \cosh(x)$ over $[-\ln 2, \ln 2]$ about the x-axis, and then show that it is less than the surface area of the cylinder with radius $\cosh(\ln 2)$ and height $2\ln(2)$ (which also connects the two circles in the figure above).

Exercises 41-44 explore the fact that if a traveling wave on the open ocean has a wavelength of λ (pronounced "lambda") and if the ocean has a depth d beneath the surface where the wave is propagating, then the velocity v of the wave is

$$v = \sqrt{\frac{g\lambda}{2\pi}\tanh\left(\frac{2\pi d}{\lambda}\right)}$$

where g is the acceleration due to gravity.



1-10: Ocean wave with velocity v, wavelength λ, and over ocean with depth d

41. Suppose that a wave with wavelength of 300 feet is traveling across a section of the ocean with a depth of 100 feet. How fast is the wave traveling (use $g = 32$ feet/sec^2)? How fast is that in miles per hour? (1 mile = 5280 feet).

42. A Tsunami (i.e., a tidal wave) is considered to have an arbitrarily long wavelength λ as it travels across open ocean, which has an average depth of 13,000 feet. Choose a really big number for λ (e.g., $\lambda = 13,000,000\ feet$), use $g = 32$ feet/sec^2, and then estimate how fast a Tsunami travels across open ocean. How fast is that in miles per hour? (1 mile = 5280 feet)

43. **Shallow-water Waves:** If $d/\lambda < 0.05$, then a wave "feels" the bottom of the ocean. In this case, we usually replace $\tanh\left(\frac{2\pi d}{\lambda}\right)$ by its linearization at 0 (as a function of d). Write a short essay which uses the linearization to explain why the velocity of shallow-water waves depends primarily on the depth of the water and is relatively independent of the wavelength of the wave.

44. **Deep-Water Waves:** The *deep water limit* is the equation given by

$$v = \sqrt{\frac{g\lambda}{2\pi}}$$

In a short essay, explain why we can replace $\tanh\left(\frac{2\pi d}{\lambda}\right)$ by 1 when $\frac{d}{\lambda}$ is sufficiently large, and then show that the result is the deep water limit.[1]

6.2 Inverse Functions

Invertible Functions

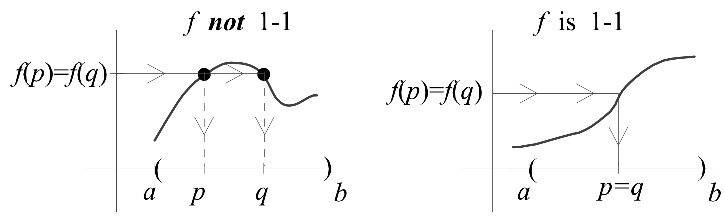
Another important class of functions are the *inverses* of the functions we have previously encountered. For example, we have already seen that the natural logarithm is the *inverse* of the exponential. In this section, we explore the inverse function concept more closely.

A function $f(x)$ is said to be *one-to-one* on an interval (a, b) if the inputs and the outputs are in a *one-to-one* correspondence. Mathematically, we denote *one-to-one* by *1-1,* and we define it to mean that

$$f(p) = f(q) \qquad \Longrightarrow \qquad p = q$$

for all p, q in (a, b).

Graphically, the *horizontal line test* says that a function $f(x)$ is 1-1 on an interval (a, b) if any horizontal line intersects the curve $y = f(x)$ at most once.

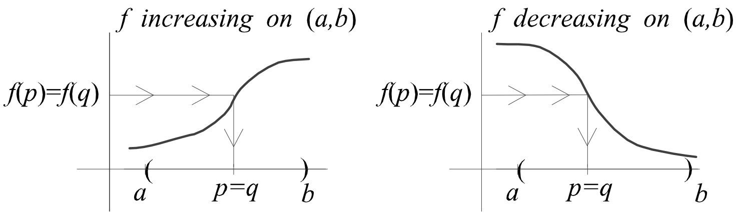


2-1: f is 1-1 if it has only one input for each output

For example, if a continuous function is increasing on (a, b), then it must be

[1] In most applications, "sufficiently large" is assumed to corresond to a ratio of $\frac{d}{\lambda} > 0.5$.

1-1. Likewise, if a continuous function is decreasing on (a,b), then it is *1-1.*

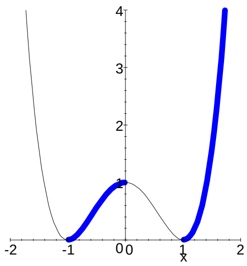


2-2: Monotone Functions are 1-1

Thus, a differentiable function $f(x)$ is 1-1 on (a,b) only if it is monotone on (a,b).

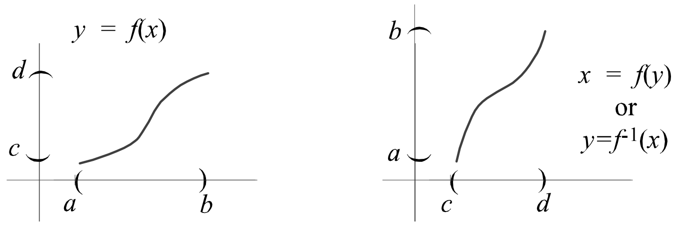
EXAMPLE 1 Find the intervals on which $f(x) = x^4 - 2x^2 + 1$ is 1-1?

Solution: Since $f'(x) = 4x^3 - 4x = 4x\left(x^2 - 1\right)$, we have $f'(x) < 0$ if x is in $(-\infty, -1)$ or if x is in $(0,1)$, but $f'(x) > 0$ on both $(-1,0)$ and $(1,\infty)$. Thus, $f(x) = x^4 - 2x^2 + 1$ is monotone and thus 1-1 on each of the intervals $(-\infty,-1)$, $(-1,0)$, $(0,1)$ and $(1,\infty)$.



2-3: Each section is the graph of a 1-1 function

If $f(x)$ is 1-1 on (a,b) and has a range (c,d), then the *curve* $x = f(y)$ implicitly defines a function f^{-1} with a domain of (c,d) and a range of (a,b).



2-4: Domain of f is range of f^{-1}. Range of f is domain of f^{-1}.

This is **NOT** a quantity raised to a power.

The function f^{-1} is pronounced "eff inverse" and is called the *inverse* of $f(x)$. That is, if $f(x)$ is 1-1 on (a,b), then

$$y = f^{-1}(x) \text{ is the same as } f(y) = x$$

If it is possible to solve for y in $x = f(y)$, then the result is an explicit representation of $y = f^{-1}(x)$.

EXAMPLE 2 Find the inverse explicitly of $f(x) = x^4 - 2x^2 + 1$ over $(1,\infty)$?

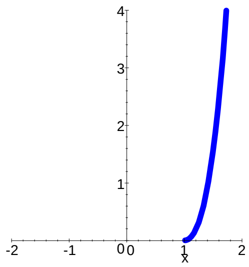
Solution: The inverse is defined implicitly by $x = f(y)$, which is

$$x = y^4 - 2y^2 + 1 \quad for\ y\ in\ (1, \infty)$$

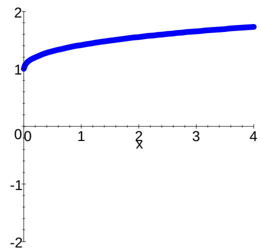
Factoring yields $x = \left(y^2 - 1\right)^2$. Thus, $y > 1$ implies that

$$\begin{aligned} y^2 - 1 &= \sqrt{x} \\ y^2 &= 1 + \sqrt{x} \\ y &= \sqrt{1 + \sqrt{x}} \end{aligned}$$

Since $y = f^{-1}(x)$, we thus have $f^{-1}(x) = \sqrt{1 + \sqrt{x}}$.



2-5a: Graph of f over $(1, \infty)$



2-5b: Graph of $f^{-1}(x) = \sqrt{1 + \sqrt{x}}$

Check your Reading *What does the -1 in the notation f^{-1} indicate?*

Inverse Functions

Since $y = f^{-1}(x)$ implies $x = f(y)$, it follows that $f\left(f^{-1}(x)\right) = f(y) = x$. That is, f and f^{-1} *cancel under composition.* Indeed, if $f(x)$ is 1-1 with a domain (a, b) and a range (c, d), then for each x in (a, b) and each y in (c, d) we have

$$f^{-1}(f(x)) = x \quad and \quad f\left(f^{-1}(y)\right) = y$$

It is for this reason, we say that $f^{-1}(x)$ is the *inverse* of $f(x)$.

EXAMPLE 3 Show that $f(x) = (1 + e^x)^2$ is 1-1 for all x and find its inverse explicitly. Then compute $f^{-1}(f(x))$.

Solution: Since $f'(x) = 2(1 + e^x)e^x > 0$ for all x, f is 1-1 for all x. The inverse function is implicitly defined by $x = f(y)$, which is

$$x = (1 + e^y)^2$$

To find f^{-1} explicitly, we solve for y:

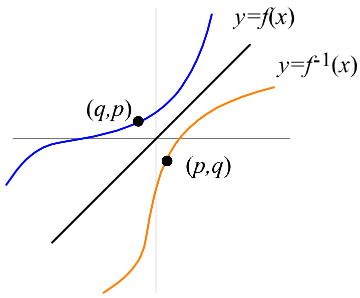
$$\begin{aligned} \sqrt{x} &= 1 + e^y \\ \sqrt{x} - 1 &= e^y \\ y &= \ln\left(\sqrt{x} - 1\right) \end{aligned}$$

Thus, $f^{-1}(x) = \ln(\sqrt{x} - 1)$, and consequently,

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}\left((1+e^x)^2 \right) \\ &= \ln\left(\sqrt{(1+e^x)^2} - 1\right) \\ &= \ln(1+e^x-1) \\ &= \ln(e^x) \\ &= x \end{aligned}$$

That is, $f^{-1}(f(x)) = x$.

Since $y = f^{-1}(x)$ is the same as $x = f(y)$, the *graph* of $f^{-1}(x)$ is the reflection of the graph of $f(x)$ across the line $y = x$. That is, if points (p, q) are mapped to the points (q, p), then the graph of $f(x)$ will be mapped to the graph of $f^{-1}(x)$.



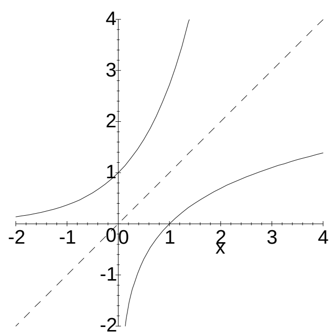
2-6: Graph of f^{-1} is graph of f reflected about $y = x$

EXAMPLE 4 What is the inverse of $f(x) = e^x$?

Solution: In chapter 2, we showed that $x = e^y$ implicitly defines the *natural logarithm,* $\ln(x)$. Indeed, one of the fundamental properties of $\ln(x)$ is that

$$\ln(e^x) = x \qquad \textit{and} \qquad e^{\ln(x)} = x$$

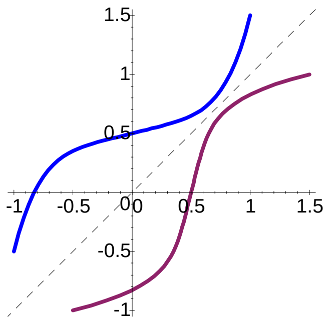
Thus, $f^{-1}(x) = \ln(x)$ is the inverse of $f(x) = e^x$. Moreover, the graph of $y = \ln(x)$ is the graph of $y = e^x$ reflected about the line $y = x$, as is shown in figure



2-7: Graphs of $y = e^x$ and $y = \ln(x)$

EXAMPLE 5 Show that $f(x) = 0.75x^5 + 0.25x + 0.5$ is 1-1 everywhere, and then sketch the graph of its inverse over $[-1, 1.5]$.

Solution: Since $f'(x) = 3.75x^4 + 0.25 > 0$ for all x, the function $f(x)$ is increasing and 1-1 for all x. The graph of $f(x) = 0.75x^5 + 0.25x + 0.5$ over $[-1, 1.5]$ can be produced by a graphing calculator. The graph of $f^{-1}(x)$ is the reflection of the graph of $f(x)$ about the line $y = x$.



2-8: Graph of $f(x)$ is above $y = x$; graph of $f^{-1}(x)$ is below $y = x$.

Check your Reading *In example 4, why not simply solve for y in*

$$x = 0.75y^5 + 0.25y + 0.5$$

Derivatives of Inverse Functions

If $y = f^{-1}(x)$ is differentiable on some interval (a, b), then it is implicitly defined by $f(y) = x$ on that interval. If f is also differentiable, then we can use *implicit differentiation* to find y':

$$\frac{d}{dx} f(y) = \frac{d}{dx} x \quad \Longrightarrow \quad f'(y) \frac{d}{dx} y = 1$$

Division by $f'(y)$ leads to an *autonomous separable differential equation:*

$$\frac{dy}{dx} = \frac{1}{f'(y)} \tag{6.8}$$

That is, the derivative of $f^{-1}(x)$ is implicitly defined by a differential equation.

EXAMPLE 6 What is the derivative of the inverse of the function

$$f(x) = 0.75x^5 + 0.25x + 0.5$$

Solution: In example 5, we saw that $f(x)$ has an inverse for all x. Thus, $x = f(y)$ implies that

$$0.75y^5 + 0.25y + 0.5 = x$$

Implicit differentiation then leads to

$$\begin{aligned}\frac{d}{dx}\left(0.75y^5+0.25y+0.5\right) &= \frac{d}{dx}x\\ 3.75y^4y'+0.25y' &= 1\\ \left(3.75y^4+0.25\right)y' &= 1\end{aligned}$$

Thus, the inverse function $y=f^{-1}(x)$ is a solution to the autonomous separable differential equation

$$y'=\frac{1}{3.75y^4+0.25}$$

Since $y=f^{-1}(x)$, we can rewrite (6.8) in the form

$$\frac{df^{-1}(x)}{dx}=\frac{1}{f'\left[f^{-1}(x)\right]} \tag{6.9}$$

Derivatives of inverse functions are often computed using (6.9).

EXAMPLE 7 Find $\sinh^{-1}(x)$, which is the inverse of $f(x)=\sinh(x)$. What is its derivative?

Solution: Since $\sinh^{-1}(x)$ is implicitly defined by $x=\sinh(y)$, we must solve for y in

$$x=\frac{e^y-e^{-y}}{2}$$

This simplifies to $2x=e^y-e^{-y}$, which upon multiplying by e^y yields

$$\begin{aligned}2xe^y &= \left(e^y\right)^2-e^ye^{-y}\\ 0 &= \left(e^y\right)^2-2xe^y-1\end{aligned}$$

The result is a quadratic in e^y, so we use the quadratic formula:

$$e^y=\frac{2x\pm\sqrt{4x^2+4}}{2}=x\pm\sqrt{x^2+1}$$

Since $e^y>0$ for all y, we choose the positive root and obtain

$$e^y=x+\sqrt{x^2+1} \qquad \Longrightarrow \qquad y=\ln\left(x+\sqrt{x^2+1}\right)$$

Since $y=\sinh^{-1}(x)$, application of the logarithm yields

$$\sinh^{-1}(x)=\ln\left(x+\sqrt{x^2+1}\right)$$

However, we do not compute $\frac{d}{dx}\sinh^{-1}(x)$ using this result, but we instead use (6.9). If $f^{-1}(x)=\sinh^{-1}(x)$, then $f(x)=\sinh(x)$, $f'(x)=\cosh(x)$, and

$$\frac{df^{-1}(x)}{dx}=\frac{1}{f'\left[f^{-1}(x)\right]} \qquad \Longrightarrow \qquad \frac{d\sinh^{-1}(x)}{dx}=\frac{1}{\cosh\left(\sinh^{-1}(x)\right)}$$

Since $\cosh^2(A) - \sinh^2(A) = 1$, we have $\cosh(A) = \sqrt{1+\sinh^2(A)}$, so that if $A = \sinh^{-1}(x)$, then

$$\frac{d\sinh^{-1}(x)}{dx} = \frac{1}{\sqrt{1+\sinh^2\left(\sinh^{-1}(x)\right)}}$$

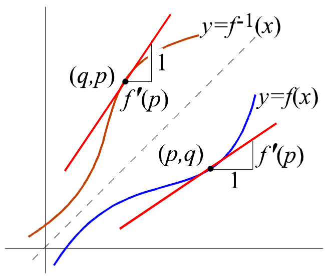
By definition, though, $\sinh^2\left(\sinh^{-1}(x)\right) = \left(\sinh\left(\sinh^{-1}(x)\right)\right)^2 = (x)^2$, so that

$$\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}}$$

Check your Reading *Why did we use the positive root for* $\cosh(A) = \sqrt{1+\sinh^2(A)}$?

Slopes of Tangent Lines to Inverse Functions

The slope of the tangent line to $y = f(x)$ at a point (p, q) is $f'(p)$, which implies that a run of 1 leads to a rise of $f'(p)$.



2-9: The slope of the reflection of a line is the reciprocal of the original slope.

Reflection about the line $y = x$ results in a line tangent to $y = f^{-1}(x)$ at the point (q, p), and sinces rises over runs are reflected to runs over rises, the slope of the line at (q, p) is $1/f'(p)$. However, since $p = f^{-1}(q)$ and since $\left(f^{-1}\right)'(q)$ is the slope of the tangent line to $y = f^{-1}(x)$ at (q, p), we have

$$\left(f^{-1}\right)'(q) = \frac{1}{f'\left[f^{-1}(q)\right]}$$

When $f^{-1}(x)$ cannot be defined explicitly, this method provides a means of determining the slope and equation of the tangent line to $y = f^{-1}(x)$ at the point (q, p).

EXAMPLE 8 Find the equation and tangent line to $y = f^{-1}(x)$ at $(1.5, 1)$ if $f(x) = 0.75x^5 + 0.25x + 0.5$.

Solution: Since $(1.5, 1)$ is on the curve $y = f^{-1}(x)$, we have $f^{-1}(1.5) = 1$ and

$$\left(f^{-1}\right)'(1.5) = \frac{1}{f'\left[f^{-1}(1.5)\right]} = \frac{1}{f'(1)}$$

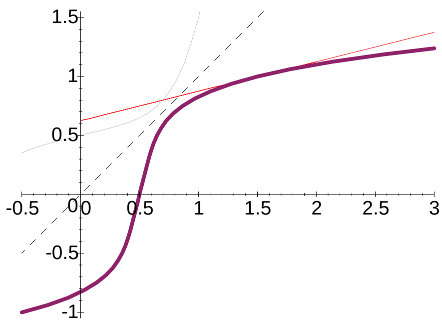
Moreover, $f'(x) = 3.75x^4 + 0.25$, so that

$$\left(f^{-1}\right)'(1.5) = \frac{1}{3.75(1)^4 + 0.25} = \frac{1}{4}$$

Thus, the tangent line is the line through $(1.5, 1)$ with slope $m = \frac{1}{4}$, which is

$$y = 1 + \frac{1}{4}(x - 1.5) = 0.625 + 0.25x$$

as is shown in the figure below:



2-10: Graph of $f(x)$ is above $y = x$; graph of $f^{-1}(x)$ is below $y = x$.

Exercises:

Determine on which intervals the following functions are monotone and hence 1-1.

1. $f(x) = x^4 - 2x^2$
2. $f(x) = 3x^5 - 5x^3$
3. $f(x) = \sin(x)$
4. $f(x) = \sec(x)$
5. $f(x) = e^x - x$
6. $f(x) = \ln(x^2 + 1)$

Find an interval on which $f(x)$ is 1-1, and then find its inverse explicitly by solving for y in terms of x in an equation of the form $f(y) = x$.

7. $f(x) = 3x + 1$
8. $f(x) = 1 - x$
9. $f(x) = x^2$
10. $f(x) = x^3$
11. $f(x) = \frac{1}{x}$
12. $f(x) = x^2 - 3x$
13. $f(x) = \left(1 + \frac{x}{9}\right)^9$
14. $f(x) = \left(1 + \frac{x}{99}\right)^{99}$
15. $f(x) = \ln(x^2 + 1)$
16. $f(x) = \ln(e^x + 1)$
17. $f(x) = \cosh(x)$
18. $f(x) = \text{sech}(x)$
19. $f(x) = \tanh(x)$
20. $f(x) = e^{2x} + e^x$

Show that each of the following functions is 1-1, and then find the differential equation that implicitly defines the derivative of the inverse.

21. $f(x) = x^5 + 3x$
22. $f(x) = x^7 + 3x$
23. $f(x) = 2x - \sin(x)$
24. $f(x) = 4x + \cos(2x)$

Find the derivative of f^{-1} for each of the following functions by employing the rule (6.9). Use algebra and identities to simplify the result.

25. $f(x) = e^x$
26. $f(x) = e^x + 1$
27. $f(x) = \cosh(x)$
28. $f(x) = \tanh(x)$
29. $f(x) = \sqrt{x + 1}$
30. $f(x) = \ln(x)$

31. Find the slope and equation of the tangent line at $(5, 0)$ to $y = f^{-1}(x)$ when $f(x) = x^3 + 3x + 5$. Sketch the graph of $y = f^{-1}(x)$ over $[1, 9]$ by reflecting the graph of $y = f(x)$ over $[-1, 1]$.

32. Find the slope and equation of the tangent line at $(2e, 1)$ to $y = f^{-1}(x)$ when $f(x) = e^x + ex$. Sketch the graph of $f(x)$ over $[-1, 1]$, and use it to sketch the graph of $y = f^{-1}(x)$ when the range is $[-1, 1]$.

33. If a temperature x is given in $°C$, then its equivalent temperature f in $°F$ is a function of x given by

$$f(x) = 32 + \frac{9}{5}x$$

Find the inverse of $f(x)$. What does the inverse of $f(x)$ do?

34. A waiting time random variable T is often exponentially distributed. The *distribution function* for the exponential probability density function is given by

$$P(T) = \int_0^T e^{-t/r} dt$$

where r is a constant representing the *mean waiting time.* Show that P is 1-1, and then compute its inverse. What is represented by $P^{-1}(y)$ if y is a probability in $[0, 1]$.

35. **The Hammer Method:** Show that if $f(x)$ is 1-1 and differentiable, then the substitution $u = f(x)$ leads to

$$\int f(x)\, dx = \int \frac{u du}{f'(f^{-1}(u))}$$

36. Use the hammer method in exercise 35 to evaluate the antiderivative

$$\int \sqrt[3]{1 + \sqrt{x}} dx$$

37. A *function group* is a set G of functions with the following properties:

i. The *identity function* $I(x) = x$ is in G.
ii. If f, g are in G, then their composition $f \circ g$ is also in G.
iii. If f is in G, then f^{-1} is also in G.

Show that the following six functions form a function group.[2]

$$G = \left\{ I(x) = x,\ f(x) = \frac{1}{x},\ g(x) = 1 - x,\quad h(x) = \frac{x-1}{x},\quad p(x) = \frac{1}{1-x},\quad q(x) = \frac{x}{x-1} \right\}$$

38. Show that the functions $f(x) = \frac{1}{x}$, $g(x) = 1 - x$, $q(x) = \frac{x}{x-1}$ and $I(x) = x$ have the unique property that they are their own inverse. Can you find another function that is its own inverse?

39. **Write to Learn:** If $p(x)$ is a probability density function for a random variable X, then the probability that X is in the interval $(-\infty, x)$ is given by the *distribution function,* $P(x)$, which is defined

$$P(x) = \int_{-\infty}^x p(t)\, dt$$

In a short essay, explain why a distribution function is 1-1, why its inverse P^{-1} has a domain of $[0, 1]$, and what $P^{-1}(y)$ represents for a given y in $[0, 1]$.

[2] This function group is known as the *anharmonic group* and is used in the study of cross ratios of linear fractional transformations.

Exercises 40-43 explore the fact that the natural logarithm can be defined by

$$\ln(x) = \int_1^x \frac{dt}{t} \tag{6.10}$$

40. Let's prove that (6.10) is true by showing that the indefinite integral is the inverse of the exponential.

(a) Evaluate the derivative

$$\frac{d}{dx}\int_1^{e^x} \frac{dt}{t}$$

(b) Explain why the result in (a) implies that

$$\int_1^{e^x} \frac{dt}{t} = x + C$$

(c) Let $x = 0$ in (b). What is C?

(d) Explain why (b) with the value of C in (c) implies (6.10).

41. Explain why (6.10) implies that $\ln(1) = 0$. Then show that (6.10) implies that $\ln(x)$ and $\ln(ax)$ have the same derivative and thus that

$$\ln(ax) = \ln(x) + C$$

where $C = \ln(a)$.

42. Show that (6.10) implies that the derivative of $\ln(x^r)$ is $r\ln(x)$. Use this to show that

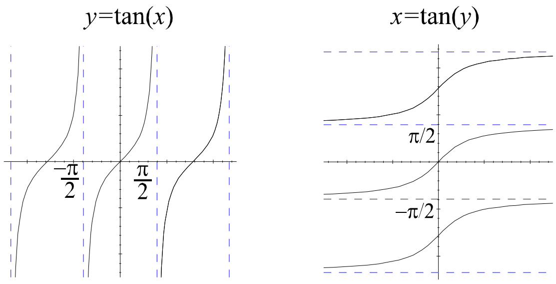
$$\ln(x^r) = r\ln(x)$$

43. Evaluate (6.10) using the fundamental theorem. What is the derivative of (6.10)?

6.3 Inverse Trigonometric Functions

The Inverse Tangent Function

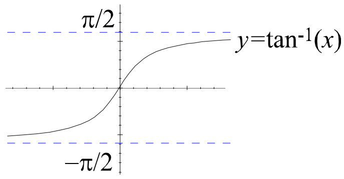
As an immediate application of the ideas in the previous section, notice that the equation $x = \tan(y)$ is not the graph of a function:



3-1: Graph of $x = \tan(y)$

$\tan^{-1}(x)$ is the inverse tangent function.

However, if we restrict y to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then $x = \tan(y)$ implicitly defines the *inverse tangent* function $y = \tan^{-1}(x)$. That is, $y = \tan^{-1}(x)$ is the same as $x = \tan(y)$ for y in the interval $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.



3-2: Graph of $y = \tan^{-1}(x)$

It follows that $y = \tan^{-1}(x)$ is an increasing odd function with horizontal asymptotes

$$\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2} \qquad \textit{and} \qquad \lim_{x\to-\infty} \tan^{-1}(x) = \frac{-\pi}{2}$$

It also follows that when when x is in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, then we have

$$\tan^{-1}(\tan(x)) = x \tag{6.11}$$

In addition, $\tan\left(\tan^{-1}(x)\right) = x$ for all x.

Moreover, the derivative of $y = \tan^{-1}(x)$ can be evaluated by implicitly differentiating $\tan(y) = x$.

$$\begin{aligned} \frac{d}{dx}\tan(y) &= \frac{d}{dx}x \\ \sec^2(y)\,\frac{d}{dx}y &= 1 \\ \frac{d}{dx}\,y &= \frac{1}{\sec^2(y)} \end{aligned}$$

The identity $\sec^2(y) = \tan^2(y) + 1$ then implies that

$$\frac{d}{dx}\,y = \frac{1}{\tan^2(y) + 1}$$

Finally, $\tan(y) = x$ implies that $\tan^2(y) = x^2$. Since $y = \tan^{-1}(x)$, we have

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{x^2 + 1} \tag{6.12}$$

EXAMPLE 1 Evaluate

$$\frac{d}{dx}\tan^{-1}\left(\frac{1}{x}\right)$$

Solution: To do so, we must compute the derivative of $\tan^{-1}(\text{input})$ when the input is x^{-1}. From (6.12), we have

$$\frac{d}{dx}\tan^{-1}(\text{input}) = \frac{1}{(\text{input})^2 + 1}\,\frac{d}{dx}(\text{input})$$

Substituting x^{-1} for the input yields

$$\frac{d}{dx}\tan^{-1}\left(x^{-1}\right) = \frac{1}{(x^{-1})^2 + 1}\,\frac{d}{dx}\left(x^{-1}\right) = \frac{1}{x^{-2} + 1}\left(-x^{-2}\right)$$

which simplifies to

$$\frac{d}{dx}\tan^{-1}\left(x^{-1}\right) = \frac{-1}{\left(x^{-2}+1\right)x^2} = \frac{-1}{1+x^2}$$

EXAMPLE 2 Evaluate the derivative of $f(x) = e^x \tan^{-1}(x)$.

Solution: The product rule implies that

$$\frac{d}{dx}\left(e^x \tan^{-1}(x)\right) = \left(\frac{d}{dx}e^x\right)\tan^{-1}(x) + e^x\left(\frac{d}{dx}\tan^{-1}(x)\right)$$

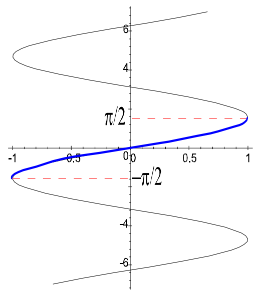
The derivative rule (6.12) then implies that

$$\frac{d}{dx}\left(e^x \tan^{-1}(x)\right) = e^x \tan^{-1}(x) + \frac{e^x}{x^2+1}$$

Check your Reading *What is the derivative of* $y = \tan^{-1}(-x)$?

Definition of the Inverse Sine Function

Similarly, the curve $x = \sin(y)$ is not the graph of y as a function of x. But the **section** of $x = \sin(y)$ where y is in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is a function.



3–3: Graph of $x = \sin(y)$

Thus, we define the *inverse sine function*[3] or $\sin^{-1}(x)$ to be the function defined implicitly by $x = \sin(y)$ when y is in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. It follows that

$$\sin^{-1}(\sin(x)) = x \qquad \text{only if} \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

Moreover, the domain of the inverse sine function is $[-1, 1]$, which implies that

$$\sin\left(\sin^{-1}(x)\right) = x \qquad \text{only if} \qquad -1 \leq x \leq 1$$

To obtain the derivative of the inverse sine function, we recall that $y = \sin^{-1}(x)$ is the same as $\sin(y) = x$. Implicit differentiation implies that

$$\begin{aligned} \frac{d}{dx}\sin(y) &= \frac{d}{dx}x \\ \cos(y)\,y' &= 1 \\ y' &= \frac{1}{\cos(y)} \end{aligned}$$

[3] The inverse sine is also written at times as $\arcsin(x)$ or $\text{asin}(x)$ and is also called the arcsine function. Likewise, $\tan^{-1}(x)$ is also called the arc tangent function and is also sometimes written $\text{atan}(x)$ or $\arctan(x)$.

However, since $\cos(y) = \sqrt{1-\sin^2(y)}$ and since $\sin(y) = x$, this simplifies to

$$y' = \frac{1}{\sqrt{1-\sin^2(y)}} = \frac{1}{\sqrt{1-x^2}}$$

That is, the derivative of $\sin^{-1}(x)$ is

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \tag{6.13}$$

EXAMPLE 3 Evaluate $\frac{d}{dx}\sin^{-1}(e^x)$

Solution: We rewrite the problem as

$$\frac{d}{dx}\sin^{-1}(e^x) = \frac{d}{dx}\sin^{-1}(\text{input})$$

where the input is e^x. The chain rule and (6.13) imply that

$$\frac{d}{dx}\sin^{-1}(\text{input}) = \frac{1}{\sqrt{1-(\text{input})^2}}\,\frac{d}{dx}(\text{input})$$

Substituting e^x for the input yields

$$\frac{d}{dx}\sin^{-1}(e^x) = \frac{1}{\sqrt{1-(e^x)^2}}\,\frac{d}{dx}(e^x) \quad = \frac{e^x}{\sqrt{1-e^{2x}}}$$

EXAMPLE 4 Evaluate $\frac{d}{dx}\left[\, e^{-x}\sin^{-1}(e^x)\,\right]$

Solution: We begin with the product rule:

$$\frac{d}{dx}\left[\, e^{x}\sin^{-1}(e^x)\,\right] = \left(\frac{d}{dx}e^{-x}\right)\sin^{-1}(e^x) + e^{-x}\left[\frac{d}{dx}\sin^{-1}(e^x)\right]$$

The chain rule is then applied to the last term, which yields

$$\begin{aligned}
\frac{d}{dx}e^x\sin^{-1}(e^x) &= -e^{-x}\sin^{-1}(e^x) + e^{-x}\frac{1}{\sqrt{1-(e^x)^2}}\frac{d}{dx}e^x \\
&= -e^{-x}\sin^{-1}(e^x) + e^{-x}\frac{e^x}{\sqrt{1-e^{2x}}} \\
&= -e^{-x}\sin^{-1}(e^x) + \frac{1}{\sqrt{1-e^{2x}}}
\end{aligned}$$

Check your Reading *What is* $\sin^{-1}(\sin(1))$*?*

The derivative rule (6.12) implies the antiderivative rule

$$\int \frac{dx}{1+x^2} = \tan^{-1}(x) + C$$

Moreover, the fundamental theorem says that $\tan^{-1}(x)$ can be defined *explicitly* by an indefinite integral, in that

$$\tan^{-1}(x) = \int_0^x \frac{dt}{t^2+1}$$

Numerical methods for estimating integrals can thus be used to compute $\tan^{-1}(x)$ for given value of x.

However, in applications, we often use the more general form of the antiderivative given by

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \tag{6.14}$$

where a is a constant.

EXAMPLE 5 Evaluate $\displaystyle\int \frac{dx}{x^2+4}$

Solution: Comparison with (6.14) implies that $a^2 = 4$, so that $a = 2$. Thus, (6.14) yields

$$\int \frac{dx}{x^2+4} = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C \tag{6.15}$$

EXAMPLE 6 Evaluate

$$\int \tan^{-1}(x)\,dx$$

Solution: Letting $u = \tan^{-1}(x)$ and $dv = dx$ results in

$$\begin{array}{rlrl} u & = \tan^{-1}(x) & dv & = dx \\ du & = \frac{1}{x^2+1}dx & v & = x \end{array}$$

so that application of integration by parts yields

$$\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \int \frac{x}{x^2+1}dx$$

To evaluate the last integral, we use the substitution $w = x^2+1$, $dw = \frac{1}{2}xdx$ which yields

$$\begin{aligned} \int \tan^{-1}(x)\,dx &= x\tan^{-1}(x) - \frac{1}{2}\int \frac{dw}{w} \\ &= x\tan^{-1}(x) - \frac{1}{2}\ln|w| + C \\ &= x\tan^{-1}(x) - \frac{1}{2}\ln\left|x^2+1\right| + C \end{aligned}$$

Similaly, the inverse sine function satisfies

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + C$$

and the fundamental theorem implies that $\sin^{-1}(x)$ is explictly defined as

$$\sin^{-1}(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}}$$

However, we more often use the more general form of the antiderivative given by

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C \tag{6.16}$$

where a is a constant.

EXAMPLE 7 Evaluate

$$\int \frac{e^x dx}{\sqrt{9-e^{2x}}}$$

Solution: We let $u = e^x$, $du = e^x dx$. Substitution and (6.16) with $a = 3$ yields

$$\int \frac{e^x dx}{\sqrt{9-e^{2x}}} = \int \frac{du}{\sqrt{9-u^2}} = \sin^{-1}\left(\frac{u}{3}\right) + C = \sin^{-1}\left(\frac{e^x}{3}\right) + C$$

EXAMPLE 8 Evaluate

$$\int \frac{\sin^{-1}(x)}{\sqrt{1-x^2}} dx$$

Solution: Notice that if $u = \sin^{-1}(x)$, then $du = \frac{1}{\sqrt{1-x^2}} dx$ and

$$\int \frac{\sin^{-1}(x)}{\sqrt{1-x^2}} dx = \int \sin^{-1}(x) \frac{1}{\sqrt{1-x^2}} dx = \int u du$$

Since $\int u du = \frac{u^2}{2} + C$ and $u = \sin^{-1}(x)$, we have that

$$\int \frac{\sin^{-1}(x)}{\sqrt{1-x^2}} dx = \frac{1}{2}\left[\sin^{-1}(x)\right]^2 + C$$

Check your Reading *Where did the $\frac{1}{2}$ come from in example 8?*

Right Triangles and Inverse Trigonometric Functions

The inverse tangent function maps numbers to angles.

In many applications, the inverse tangent function is written in the form

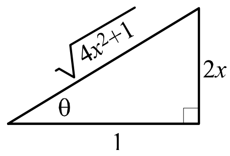
$$\theta = \tan^{-1}(x) \tag{6.17}$$

where θ is an angle. We interpret (6.17) to mean that the *inverse tangent function maps numbers x to angles θ.* Moreover, we often use right triangles to simplify expressions involving $\tan^{-1}(x)$.

EXAMPLE 9 Simplify $\sin\left(\tan^{-1}(2x)\right)$

Solution: To do so, we let $\theta = \tan^{-1}(2x)$. As a result, $\tan(\theta) = 2x$, which implies that

$$\tan(\theta) = \frac{2x}{1} = \frac{opp}{adj}$$



The Pythagorean theorem implies that the hypotenuse has a length of $\sqrt{4x^2+1}$. Thus, simplifying $\sin\left(\tan^{-1}(2x)\right)$ is the same as computing $\sin(\theta)$. That is,

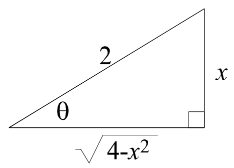
$$\sin\left(\tan^{-1}(2x)\right) = \sin(\theta) = \frac{opp}{hyp} = \frac{2x}{\sqrt{4x^2+1}}$$

We also use right triangles to simplify expressions involving $\sin^{-1}(x)$. For example, to simplify

$$\tan\left(\sin^{-1}\left(\frac{x}{2}\right)\right)$$

we first let $\theta = \sin^{-1}\left(\frac{x}{2}\right)$ so that

$$\sin(\theta) = \frac{x}{2} = \frac{opp}{hyp}$$



Since the lower side is $\sqrt{4-x^2}$ by the Pythagorean theorem, we have

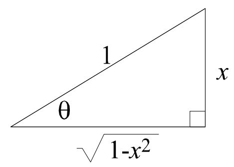
$$\tan\left(\sin^{-1}\left(\frac{x}{2}\right)\right) = \tan(\theta) = \frac{x}{\sqrt{4-x^2}}$$

Moreover, sometimes we must use an identity when simplifying with a right triangle.

EXAMPLE 10 Simplify $\sin\left(2\sin^{-1}(x)\right)$

Solution: We let $\theta = \sin^{-1}(x)$, which yields

$$\sin\theta = \frac{x}{1} = \frac{opp}{hyp}$$



The Pythagorean theorem implies that the lower side has a length of $\sqrt{1-x^2}$. However, simplifying $\sin\left(2\sin^{-1}(x)\right)$ requires the identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. That is,

$$\sin\left(2\sin^{-1}(x)\right) = \sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\,\frac{x}{1}\,\frac{\sqrt{1-x^2}}{1} = 2x\sqrt{1-x^2}$$

Exercises:

Find $f'(x)$ for each of the following and simplify completely:

1. $f(x) = \sin^{-1}(2x)$
2. $f(x) = \tan^{-1}(3x)$
3. $f(x) = \tan^{-1}(x^3)$
4. $f(x) = \sin^{-1}(e^x)$
5. $f(x) = \tan^{-1}(\sinh(x))$
6. $f(x) = \sin^{-1}(\tanh(x))$
7. $f(x) = (1 - x^2)^{1/2} \sin^{-1}(x)$
8. $f(x) = \tan^{-1}\left(\sqrt{x^2 - 1}\right)$
9. $f(x) = \sec\left(\tan^{-1}(x)\right)$
10. $f(x) = \sec\left(\sin^{-1}(x)\right)$
11. $f(x) = \int_0^{\sin^{-1}(x)} \cos(t)\, dt$
12. $f(x) = \int_0^{\tan^{-1}(x)} \tan(t)\, dt$

Evaluate the following:

13. $\int \frac{dx}{x^2 + 16}$
14. $\int \frac{dx}{\sqrt{4 - x^2}}$
15. $\int \frac{\sin(x)\, dx}{\sqrt{9 - \cos^2(x)}}$
16. $\int \frac{\sin(x)\, dx}{\cos^2(x) + 1}$
17. $\int_0^1 \frac{e^x dx}{e^{2x} + 1}$
18. $\int_0^1 \frac{x dx}{\sqrt{3 - x^4}}$
19. $\int x \tan^{-1}(x)\, dx$
20. $\int x \sin^{-1}(x)\ dx$
21. $\int \sin^{-1}(x)\ dx$
22. $\int \frac{\cos(x)\, dx}{\sqrt{\cos(2x)}}$

Simplify the following expressions.

23. $\cos\left(\sin^{-1}(x)\right)$
24. $\sin\left(\cos^{-1}(2x)\right)$
25. $\cos\left(\tan^{-1}(2x)\right)$
26. $\tan\left(\sin^{-1}\left(\sqrt{x}\right)\right)$
27. $\cot\left(\tan^{-1}\left(\frac{1}{x}\right)\right)$
28. $\cos\left(2\sin^{-1}(x)\right)$
29. $\sec\left(\sin^{-1}\left(x^2 + 4\right)\right)$
30. $\csc\left(\tan^{-1}(x + 1)\right)$
31. $\sin\left(2\sin^{-1}(e^x)\right)$
32. $\tan\left(2\tan^{-1}(x)\right)$

33. Use monotonicity and concavity to sketch the graph of $f(x) = x - 5\tan^{-1}(x)$

34. Use monotonicity and concavity to sketch the graph of $f(x) = \sin^{-1}\left(\frac{1}{2}\sin(x)\right)$

35. Estimate the value of the integral

$$\int_0^1 \frac{4\, dt}{t^2 + 1}$$

using a midpoint approximation over the partition

$$0 < 0.2 < 0.4 < 0.6 < 0.8 < 1.0$$

What is the exact value of the integral and how close is the estimate to that value?

36. Estimate the value of the integral

$$6\int_0^{0.5} \frac{dt}{\sqrt{1-t^2}}$$

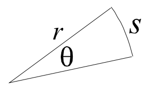
using a midpoint approximation over the partition

$$0 < 0.1 < 0.2 < 0.3 < 0.4 < 0.5$$

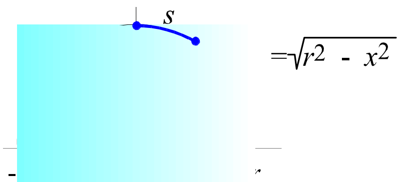
What is the exact value of the integral and how close is the estimate to that value?

37. Show that the length s of the arc of a circle of radius r is

$$s = r\theta$$

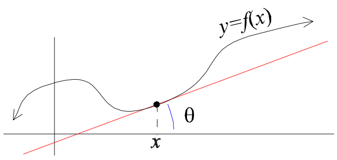


by finding the arclength of the graph of $f(x) = \sqrt{r^2 - x^2}$ over $[0, r\sin(\theta)]$.



3-4: Show that $s = r\theta$

38. The angle θ formed by the tangent line to $f(x)$ at an input of x and the x-axis is called the *angle of inclination* of the graph of $y = f(x)$.



3-5: Angle of inclination

Show that the angle of inclination is $\theta = \tan^{-1}[f'(x)]$.

39. In this exercise, we explicitly define the inverse cosine function $\cot^{-1}(x)$ as the indefinite integral of an elementary function.

(a) Let $y = \cos^{-1}(x)$, and then implicitly differentiate $\cos(y) = x$.

(b) Use the identity $\sin^2(x) = 1 - \cos^2(x)$ and the fact that $\cos(y) = x$ to prove that

$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$$

(c) Use the fundamental theorem of calculus to show that

$$\cos^{-1}(x) = \int_x^1 \frac{dt}{\sqrt{1-t^2}}$$

40. In this exercise, we explicitly define the inverse cotangent function $\cot^{-1}(x)$ as the indefinite integral of an elementary function.

(a) Let $y = \cot^{-1}(x)$, and then implicitly differentiate $\cot(y) = x$.

(b) Use the identity $\csc^2(y) = \cot^2(y) + 1$ and the fact that $\cot(y) = x$ to prove that

$$\frac{d}{dx}\cot^{-1}(x) = \frac{-1}{x^2+1}$$

(c) Use the fundamental theorem to evaluate

$$\int_0^x \frac{-dt}{t^2+1}$$

using the result in (b).

41. In this exercise, we explicitly define the inverse secant function $\sec^{-1}(x)$ as the indefinite integral of an elementary function.

(a) Let $y = \sec^{-1}(x)$, and then implicitly differentiate $\sec(y) = x$.

(b) Use the identity $\tan(y) = \sqrt{\sec^2(y) - 1}$ and the fact that $\sec(y) = x$ to prove that

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}}$$

(c) Use the fundamental theorem to evaluate

$$\int_0^x \frac{dt}{t\sqrt{t^2-1}}$$

42. In this exercise, we explicitly define the inverse cosecant function $\csc^{-1}(x)$ as the indefinite integral of an elementary function.

(a) Let $y = \csc^{-1}(x)$, and then implicitly differentiate $\csc(y) = x$.

(b) Use the identity $\cot(y) = \sqrt{\csc^2(y) - 1}$ and the fact that $\csc(y) = x$ to prove that

$$\frac{d}{dx}\csc^{-1}(x) = \frac{-1}{x\sqrt{x^2-1}}$$

(c) Use the fundamental theorem to evaluate

$$\int_0^x \frac{-dt}{t\sqrt{t^2-1}}$$

using the result in (b).

43. Even though $\sin^{-1}(1) = \frac{\pi}{2}$, it leads to an improper integral. Indeed, direct estimation of the integral

$$\sin^{-1}(1) = \int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

is *unstable,* in that numbers close to 1 do not necessarily lead to accurate approximations of the integral. To illustrate, estimate the three integrals below numerically:

$$(a)\ \int_0^{0.9} \frac{dx}{\sqrt{1-x^2}} \qquad (b)\ \int_0^{0.99} \frac{dx}{\sqrt{1-x^2}} \qquad (c)\ \int_0^{0.999} \frac{dx}{\sqrt{1-x^2}}$$

How close are the results to $\frac{\pi}{2}$?

44. **Computer Algebra Systems:** In spite of the powerful algorithms employed by computer algebra systems, integrals involving inverse trig functions often cannot be evaluated mechanically. And sometimes, they are able to produce a result, but that result is nearly unintelligible. For example, use a computer algebra system to symbolically evaluate

$$\int \frac{\cos(x)\,dx}{\sqrt{\cos(2x)}}$$

45. In this exercise, we derive the identity

$$\tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \tan^{-1}(x)$$

(a) Evaluate and simplify

$$\frac{d}{dx}\tan^{-1}\left(\frac{1}{x}\right)$$

(b) Explain why

$$\tan^{-1}\left(\frac{1}{x}\right) = -\tan^{-1}(x) + C$$

where C is a constant.

(c) Let $x = 1$ in (b) and then solve for C.

46. The *Cauchy probability density function* is defined to be

$$p(x) = \frac{1}{\pi}\frac{r}{x^2 + r^2}$$

where r is constant.

(a) Show that $p(x)$ is a probability function by evaluating

$$\int_{-\infty}^{\infty} p(x)\,dx$$

(b) Find a closed form expression for the probability of the value of a random variable assuming a value in $[a, b]$ if it has a Cauchy probability density.

The exercises below are a review of the trigonometry of the inverse tangent and inverse sine functions, where the inverse sine $\sin^{-1}(x)$ *satisfies*

$$\sin^{-1}(\sin(x)) = x \quad \text{only if} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$\sin\left(\sin^{-1}(x)\right) = x \quad \text{only if} \quad -1 \le x \le 1$$

47. Label each of the following as either true or false: If false, explain why or provide a counter-example.

(a) $\tan\left(\tan^{-1}(1)\right) = 1$ *(b)* $\tan\left(\tan^{-1}(10)\right) = 10$
(c) $\tan^{-1}(\tan(1)) = 1$ *(d)* $\tan^{-1}(\tan(2)) = 2$
(e) $\sin\left(\sin^{-1}\left(\sqrt{2}\right)\right) = \sqrt{2}$ *(f)* $\sin\left(\sin^{-1}\left(\sqrt{3}\right)\right) = \sqrt{3}$
(g) $\sin\left(2\sin^{-1}(0.5)\right) = \sqrt{3}$ *(h)* $\sin\left(3\sin^{-1}(0.5)\right) = 1$

48. Evaluate the following in closed form using radian measure.

(a) $\tan^{-1}(1)$ *(b)* $\tan^{-1}(-1)$ *(c)* $\sin^{-1}\left(-\frac{1}{2}\right)$

(d) $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$ *(e)* $\sin^{-1}(-1)$ *(f)* $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

6.4 Trig Substitutions

Integrals Involving an Expression of the Form $r^2 - x^2$

Substitutions of the form $x = r\sin\theta$ and $x = r\tan(\theta)$ are known as *trig substitutions* and are important because they allow us to use the properties of the trigonometric functions to simplify integrals. Moreover, such substitutions often involve the use of the inverse trigonometric functions.

If $r^2 - x^2$ is in the integrand, let $x = r\sin(\theta)$

To begin with, let us develop a method for evaluating $\int_a^b f(x)\,dx$ when the integrand $f(x)$ contains an expression of the form $r^2 - x^2$. If we let $x = r\sin(\theta)$, then solving for θ results in $\theta = \sin^{-1}(x/r)$. This leads to the substitution

$$\begin{aligned} x &= r\sin(\theta) & \theta(a) &= \sin^{-1}(a/r) \\ dx &= r\cos(\theta)\ d\theta & \theta(b) &= \sin^{-1}(b/r) \end{aligned}$$

which in turn leads to a new integral as is illustrated below:

$$\int_a^b f(x)\,dx = \int_{\theta(a)}^{\theta(b)} f(r\sin\theta)\quad r\cos(\theta)\,d\theta$$

Most importantly, however, the expression $r^2 - x^2$ becomes

$$r^2 - x^2 = r^2 - r^2\sin^2(\theta) = r^2\left(1 - \sin^2(\theta)\right) = r^2\cos^2(\theta)$$

Hopefully, reducing the difference of two squares $r^2 - x^2$ to a single squared expression $r^2\cos^2(\theta)$ will simplify the integral.

EXAMPLE 1 Use a trig substitution to evaluate

$$\int_0^1 \frac{xdx}{\sqrt{4 - x^2}}$$

Solution: The denominator is the square root of $4 - x^2$. Thus, we let $x = 2\sin(\theta)$, which in turn implies that $\theta = \sin^{-1}(x/2)$. This leads to the substitution

$$\begin{aligned} x &= 2\sin(\theta) & \theta(0) &= \sin^{-1}(0/2) = 0 \\ dx &= 2\cos(\theta)\ d\theta & \theta(1) &= \sin^{-1}(1/2) = \pi/6 \end{aligned}$$

Be sure to change the limits of integration when you substitute.

The substitution in turn leads to a new integral:

$$\int_0^1 \frac{x}{\sqrt{4 - x^2}}\,dx = \int_0^{\pi/6} \frac{2\sin\theta}{\sqrt{4 - 4\sin^2\theta}}\,2\cos\theta\,d\theta$$

The expression $4 - 4\sin^2\theta$ factors into $4\left(1 - \sin^2\theta\right)$, thus leading to

$$\int_0^1 \frac{x}{\sqrt{4 - x^2}}\,dx = \int_0^{\pi/6} \frac{2\sin\theta}{\sqrt{4\left(1 - \sin^2\theta\right)}}\,2\cos\theta\,d\theta$$

The denominator is then simplified using the identity $1 - \sin^2\theta = \cos^2(\theta)$:

$$\int_0^1 \frac{x}{\sqrt{4 - x^2}}\,dx = 4\int_0^{\pi/6} \frac{\sin\theta}{\sqrt{4\cos^2\theta}}\cos\theta\,d\theta = \frac{4}{2}\int_0^{\pi/6} \frac{\sin\theta}{\cos\theta}\ \cos\theta\,d\theta$$

Cancellation of the cosine terms thus results in

$$\begin{aligned}\int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx &= 2\int_0^{\pi/6} \sin\theta\,d\theta \\ &= -2\cos\theta|_0^{\pi/6} \\ &= -2\cos\left(\frac{\pi}{6}\right) - -2\cos(0) \\ &= 2-\sqrt{3}\end{aligned}$$

EXAMPLE 2 Evaluate

$$\int_0^{2\sqrt{3}} \frac{dx}{x^2\sqrt{16-x^2}}$$

Solution: The denominator contains the expression $16-x^2$. Thus, we let $x = 4\sin(\theta)$, so that $dx = 4\cos(\theta)\,d\theta$ and $\theta = \sin^{-1}(x/4)$. This leads to the substitution

$$\begin{array}{ll} x = 4\sin(\theta) & \theta(0) = \sin^{-1}(2/4) = \sin^{-1}(1/2) = \pi/6 \\ dx = 4\cos(\theta)\ d\theta & \theta(1) = \sin^{-1}\left(2\sqrt{3}/4\right) = \sin^{-1}\left(\sqrt{3}/2\right) = \pi/3 \end{array}$$

which then yields

$$\int_2^{2\sqrt{3}} \frac{dx}{x^2\sqrt{16-x^2}} = \int_{\pi/6}^{\pi/3} \frac{4\cos(\theta)\,d\theta}{16\sin^2(\theta)\sqrt{16-16\sin^2(\theta)}} = \frac{1}{4}\int_{\pi/6}^{\pi/3} \frac{\cos(\theta)\,d\theta}{\sin^2(\theta)\sqrt{16\cos^2(\theta)}}$$

Since $\sqrt{\cos^2(\theta)} = \cos(\theta)$ for θ in $[\pi/6, \pi/3]$, we have

$$\int_2^{2\sqrt{3}} \frac{dx}{x^2\sqrt{16-x^2}} = \frac{1}{4}\int_{\pi/6}^{\pi/3} \frac{\cos(\theta)\,d\theta}{\sin^2(\theta)\,4\cos(\theta)} = \frac{1}{16}\int_{\pi/6}^{\pi/3} \csc^2(\theta)\,d\theta$$

As a result, the fundamental theorem implies that

$$\int_2^{2\sqrt{3}} \frac{dx}{x^2\sqrt{16-x^2}} = \frac{-1}{16}\cot(\theta)\Big|_{\pi/6}^{\pi/3} = \frac{-1}{16}\cot\left(\frac{\pi}{3}\right) - \frac{-1}{16}\cot\left(\frac{\pi}{6}\right) \tag{6.18}$$

Check your Reading *What is the final result in example 2 (i.e., simplify (6.18))?*

Integrals Involving an Expression of the Form $r^2 + x^2$

If $r^2 + x^2$ is in the integrand, let $x = r\tan(\theta)$

If the integrand of an integral contains an expression of the form $r^2 + x^2$, then we let $x = r\tan(\theta)$, or equivalently, $\theta = \tan^{-1}(x/r)$. This leads to the substitution

$$\begin{array}{ll} x = r\tan(\theta) & \theta(a) = \tan^{-1}(a/r) \\ dx = r\sec^2(\theta)\ d\theta & \theta(b) = \tan^{-1}(b/r) \end{array}$$

which in turn leads to a new integral as is illustrated below:

$$\int_a^b f(x)\,dx = \int_{\theta(a)}^{\theta(b)} f(r\tan\theta)\quad r\sec^2(\theta)\,d\theta$$

Most importantly, however, the expression $r^2 + x^2$ becomes

$$r^2 + x^2 = r^2 + r^2 \tan^2(\theta) = r^2 \left(1 + \tan^2(\theta)\right) = r^2 \sec^2(\theta)$$

Hopefully, reducing the difference of two squares $r^2 + x^2$ to a single squared expression $r^2 \sec^2(\theta)$ will simplify the integral.

EXAMPLE 3 Evaluate

$$\int_0^3 \frac{x\,dx}{\sqrt{9 + x^2}} \tag{6.19}$$

Solution: Since $r = 3$ for $9 + x^2$, we let $x = 3\tan(\theta)$, which yields $dx = 3\sec^2(\theta)\,d\theta$ and $\theta = \tan^{-1}(x/3)$. This leads to the substitution

$$\begin{array}{rl} x = 3\tan(\theta) & \theta(0) = \tan^{-1}(0/3) = 0 \\ dx = 3\sec^2(\theta)\ d\theta & \theta(1) = \tan^{-1}(3/3) = \pi/4 \end{array}$$

The substitution in turn leads to a *new* integral:

$$\int_0^3 \frac{x}{\sqrt{9+x^2}}\,dx = \int_0^{\pi/4} \frac{3\tan(\theta)}{\sqrt{9 + 9\tan^2(\theta)}}\, 3\sec^2(\theta)\,d\theta$$

The expression $9 + 9\tan^2(\theta)$ factors into $9\left(1 + \tan^2\theta\right)$, thus leading to

$$\int_0^3 \frac{x}{\sqrt{9+x^2}}\,dx = \int_0^{\pi/4} \frac{3\tan(\theta)}{\sqrt{9\left(1 + \tan^2\theta\right)}}\, 3\sec^2(\theta)\,d\theta$$

The denominator is then simplified using the identity $1 + \tan^2(\theta) = \sec^2(\theta)$:

$$\int_0^3 \frac{x}{\sqrt{9+x^2}}\,dx = 3\int_0^{\pi/4} \frac{\tan(\theta)}{\sqrt{\sec^2(\theta)}}\, \sec^2(\theta)\,d\theta = 3\int_0^{\pi/4} \frac{\tan(\theta)}{\sec(\theta)} \sec^2(\theta)\,d\theta$$

Cancellation and the fundamental theorem then yield

$$\begin{aligned} \int_0^3 \frac{x}{\sqrt{9+x^2}}\,dx &= 3\int_0^{\pi/4} \sec(\theta)\tan(\theta)\,d\theta \\ &= 3\sec(\theta)\big|_0^{\pi/4} \\ &= 3\sec\left(\frac{\pi}{4}\right) - 3\sec(0) \\ &= 3\sqrt{2} - 3 \end{aligned}$$

Check your Reading *Verify the result by estimating (6.19) using numerical integration.*

Antiderivatives and Trig Substitutions

The following antiderivative forms often occur in trig substitutions.

$$\int \sec(\theta)\,d\theta = \ln|\sec\theta + \tan\theta| + C \tag{6.20}$$

$$\int \cos^2(\theta)\,d\theta = \frac{1}{2}\theta + \frac{1}{2}\sin(\theta)\cos(\theta) + C \tag{6.21}$$

$$\int \sin^2(\theta)\,d\theta = \frac{1}{2}\theta - \frac{1}{2}\sin(\theta)\cos(\theta) + C \tag{6.22}$$

Also, when trig substitutions are used to evaluate antiderivatives in closed form, we frequently must simplify the result using right triangle trigonometry.

EXAMPLE 4 Use a trig substitution to evaluate the antiderivative

$$\int \sqrt{1-x^2}\,dx$$

Solution: If we let $x = \sin(\theta)$, then $dx = \cos(\theta)\,d\theta$ and $\theta = \sin^{-1}(x)$, thus leading to

$$\int \sqrt{1-x^2}dx = \int \sqrt{1-\sin^2(\theta)}\ \cos(\theta)\,d\theta$$

This in turn simplifies to

$$\int \sqrt{1-x^2}dx = \int \sqrt{\cos^2(\theta)}\ \cos(\theta)\,d\theta = \int \cos^2(\theta)\ d\theta$$

As a result, (6.22) leads to

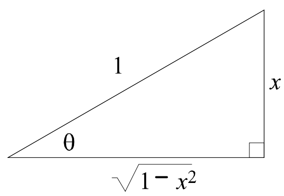
$$\int \sqrt{1-x^2}\ dx = \frac{1}{2}\theta + \frac{1}{2}\sin(\theta)\cos(\theta) + C$$

which after substituting $\theta = \sin^{-1}(x)$ becomes

$$\int \sqrt{1-x^2}\ dx = \frac{1}{2}\sin^{-1}(x) + \frac{1}{2}\sin\left(\sin^{-1}(x)\right)\cos\left(\sin^{-1}(x)\right) + C$$

Since $\sin\left(\sin^{-1}(x)\right) = x$, we need only simplify the expression $\sin\left(\cos^{-1}(x)\right)$. However, $x = \sin(\theta)$ implies that

$$\sin(\theta) = \frac{x}{1} = \frac{opp}{hyp}$$



As a result, $\cos\left(\sin^{-1}(x)\right) = \cos(\theta) = \sqrt{1-x^2}$, so that

$$\int \sqrt{1-x^2}\ dx = \frac{1}{2}\sin^{-1}(x) + \frac{1}{2}x\sqrt{1-x^2} + C$$

EXAMPLE 5 Use a trig substitution to evaluate

$$\int \frac{dx}{\sqrt{9+x^2}}$$

Solution: The denominator is the square root of $9+x^2$, so we let $x = 3\tan(\theta)$, which in turn yields $dx = 3\sec^2(\theta)\,d\theta$ and $\theta = \tan^{-1}(x/3)$. As a result,

$$\begin{aligned}\int \frac{dx}{\sqrt{9+x^2}} &= \int \frac{3\sec^2(\theta)\,d\theta}{\sqrt{9+9\tan^2(\theta)}} \\ &= \int \frac{3\sec^2(\theta)\,d\theta}{\sqrt{9\sec^2(\theta)}} \\ &= \int \frac{3\sec^2(\theta)\,d\theta}{3\sec(\theta)}\end{aligned}$$

Simplifying and applying (6.20) then yields

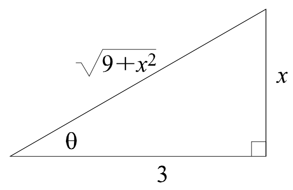
$$\int \frac{dx}{\sqrt{9+x^2}} = \int \sec(\theta)\, d\theta = \ln|\sec(\theta) + \tan(\theta)| + C$$

and since $\theta = \tan^{-1}(x/3)$, this in turn becomes

$$\begin{aligned} \int \frac{dx}{\sqrt{9+x^2}} &= \ln\left|\sec\left(\tan^{-1}\left(\frac{x}{3}\right)\right) + \tan\left(\tan^{-1}\left(\frac{x}{3}\right)\right)\right| + C \\ &= \ln\left|\sec\left(\tan^{-1}\left(\frac{x}{3}\right)\right) + \frac{x}{3}\right| + C \end{aligned}$$

To simplify $\sec\left(\tan^{-1}(x/3)\right)$, we notice that

$$\tan(\theta) = \frac{x}{3}$$



As a result, $\sec\left(\tan^{-1}(x/3)\right) = \sec(\theta) = \left(\sqrt{9+x^2}\right)/3$. Thus,

$$\int \frac{dx}{\sqrt{9+x^2}} = \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C \tag{6.23}$$

Check your Reading *Explain why properties of the logarithm allow us to write (6.23) as*

$$\int \frac{dx}{\sqrt{9+x^2}} = \ln\left|x + \sqrt{9+x^2}\right| + C_1$$

where $C_1 = C - \ln(3)$.

Transforming Elliptic Integrals

The use of trig substitutions may have been inspired by the study of *elliptic integrals,* which are a class of improper integrals that can be transformed into proper integrals using trigonometric substitutions.

EXAMPLE 6 The following improper integral is called a *complete elliptic integral of the first kind:*

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \qquad |k| < 1, \tag{6.24}$$

Transform it into a proper integral using the substitution $x = \sin(\theta)$.

Solution: Since $x = \sin(\theta)$, we have $dx = \cos(\theta)\, d\theta$ and also $\theta = \sin^{-1}(x)$. Thus, the new limits of integration are $\theta = \sin^{-1}(0) = 0$ and $\theta = \sin^{-1}(1) = \pi/2$. As a result, we have

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \int_0^{\pi/2} \frac{\cos(\theta)\, d\theta}{\sqrt{\left(1-\sin^2(\theta)\right)\left(1-k^2\sin^2(\theta)\right)}}$$

The identity $1-\sin^2(\theta)=\cos^2(\theta)$ then yields

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}=\int_0^{\pi/2}\frac{\cos(\theta)\,d\theta}{\sqrt{(\cos^2(\theta))\left(1-k^2\sin^2(\theta)\right)}}$$

Simplifying and canceling thus yields

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}=\int_0^{\pi/2}\frac{\cos(\theta)\,d\theta}{\cos(\theta)\sqrt{1-k^2\sin^2(\theta)}}=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2(\theta)}}$$

As a result, the improper integral (6.24) is transformed into the proper integral

$$\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2(\theta)}}$$

Elliptic integrals occur in a surprising number of applications ranging from the swinging of a pendulum to the motion of a planet.

EXAMPLE 7 If a pendulum with a length l is initially at rest at an angle θ_0, then the amount of time required for the pendulum to complete one swing is

$$T=4\sqrt{\frac{l}{g}}\int_0^1\frac{dx}{\sqrt{(1-k^2x^2)(1-x^2)}} \tag{6.25}$$

where $k=\sqrt{\frac{1}{2}(1-\cos(\theta_0))}$ and where $g=32$ feet per sec^2. For $l=4$ feet and $\theta_0=\frac{\pi}{3}$, transform the integral to a proper integral and then use numerical integration to estimate T.

Solution: We let $x=\sin(\theta)$, $dx=\cos(\theta)\,d\theta$. Thus, $\theta=\sin^{-1}(x)$ so that

$$\theta(0)=\sin^{-1}(0)=0 \quad \textit{and} \quad \theta(1)=\sin^{-1}(1)=\frac{\pi}{2}$$

Substituting into (6.25) yields

$$\begin{aligned} T &= 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{\cos(\theta)\,d\theta}{\sqrt{\left(1-k^2\sin^2(\theta)\right)\left(1-\sin^2(\theta)\right)}} \\ &= 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{\cos(\theta)\,d\theta}{\cos(\theta)\sqrt{1-k^2\sin^2(\theta)}} \\ &= 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2(\theta)}} \end{aligned}$$

The resulting integral is proper and is thus suitable for numerical approximation. Since $l=4$ and $\theta=\frac{\pi}{3}$, it follows that $k=\frac{1}{2}$ and that numerical estimation of the integral leads to

$$T=4\sqrt{\frac{4}{32}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-\left(\frac{1}{4}\right)^2\sin^2(\theta)}}=2.26\ \text{sec}$$

Exercises:

Evaluate the following definite integrals with a trig substitution:

1. $\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx$
2. $\int_0^{1.5} \frac{x}{9-x^2}dx$
3. $\int_0^1 \frac{x}{\sqrt{x^2+3}}\,dx$
4. $\int_0^{2\sqrt{3}} \frac{x^2}{\sqrt{16-x^2}}dx$
5. $\int_0^3 \frac{dx}{x^3+9x}$
6. $\int_0^{\sqrt{3}} \frac{dx}{x\sqrt{x^2+3}}\,dx$
7. $\int_{-1}^1 \frac{x^3}{\sqrt{1+x^2}}\,dx$
8. $\int_{-1}^1 \sqrt{1-x^2}\,dx$
9. $\int_0^1 \frac{dx}{(25-x^2)^{3/2}}$
10. $\int_0^{\sqrt{15}} \frac{dx}{(x^2+5)^{3/2}}$
11. $\int \frac{dx}{\sqrt{x^2+4}}$
12. $\int \frac{dx}{1.44-x^2}$
13. $\int \frac{dx}{x^2\sqrt{4-x^2}}$
14. $\int \frac{dx}{x^2\sqrt{4+x^2}}$
15. $\int \frac{dx}{(x^2+25)^{3/2}}$
16. $\int \frac{dx}{(25-x^2)^{3/2}}$
17. $\int \frac{\sqrt{4-x^2}}{x^2}dx$
18. $\int \frac{1-2x^2}{\sqrt{1-x^2}}\,dx$

Convert the following into proper integrals and then use numerical integration to estimate the result.

19. $\int_0^1 \frac{e^{-x}}{\sqrt{1-x^2}}dx$
20. $\int_0^1 \frac{\sqrt{x}dx}{\sqrt{1-x^2}}$
21. $\int_0^1 \frac{\sin\left(x^2\right)}{\sqrt{1-x^2}}dx$
22. $\int_0^3 \sqrt{\frac{16-x^2}{9-x^2}}dx$
23. $\int_0^2 \frac{dx}{\sqrt{16-x^4}}$
24. $\int_0^3 \frac{dx}{\sqrt{9x-x^3}}$

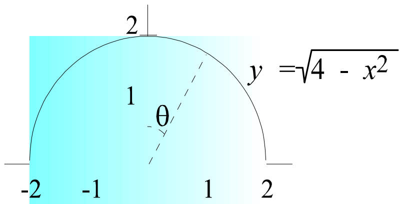
25. Find the area of the region between the upper and lower branches of

$$y^2 - x^2 = 1$$

over the interval $[-1, 1]$.

26. In this exercise, we use two methods to evaluate

$$\int_0^1 \sqrt{4-x^2}dx$$



(a) Evaluate the integral using a trigonometric substitution.

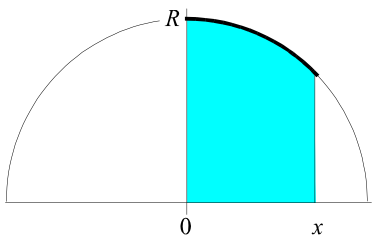
(b) Use the fact that the area of a sector of a circle is $A = \frac{1}{2}r^2\theta$ to find the area of the sector in the shaded region above.

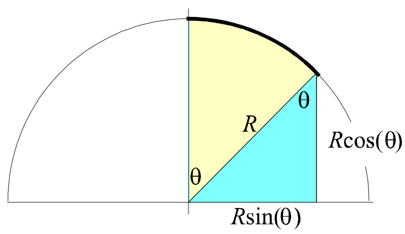
(c) Find the area of the triangle in the shaded region add it to the result in (b). Is it the same as the result in (a)?

27. **Write to Learn:** Write a short essay in which you evaluate

$$\int_0^x \sqrt{R^2-t^2}dt$$

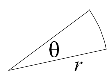
with a trig substitution. Then relate the trig substitution to the diagrams below by finding the sum of the area of the triangle and the area of the sector. (Hint: the sector with angle θ of a circle with radius r has an area of $A = \frac{1}{2}r^2\theta$)



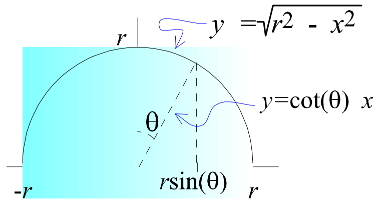


28. **Write to Learn:** Write a short essay that proves that the area of a sector is

$$A = \frac{1}{2}r^2\theta$$



by finding the area of the shaded region below:



29. **Write to Learn:** Write a short essay in which you use the graph of $\theta = \tan^{-1}(x)$ to explain why $x = \infty$ corresponds to $\theta = \pi/2$ and why $x = -\infty$ corresponds to $\theta = -\pi/2$. Then use the substitution

$$\begin{aligned} x &= b\tan(\theta) & \theta(-\infty) &= \frac{-\pi}{2} \\ dx &= b\sec^2(\theta)\,d\theta & \theta(\infty) &= \frac{\pi}{2} \end{aligned}$$

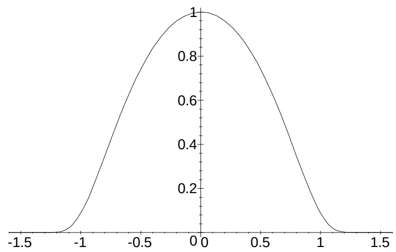
to transform the following improper integral[4]

$$\int_{-\infty}^{\infty} \frac{e^{-x^2}}{x^2+b^2} dx \tag{6.26}$$

30. Use the transformation $x = \tan(\theta)$, $dx = \sec^2(\theta)\, d\theta$ to show that

$$\int_{-\infty}^{\infty} \frac{e^{-x^2}}{x^2+1} dx = \int_{-\pi/2}^{\pi/2} e^{-\tan^2(\theta)}\, d\theta \tag{6.27}$$

(see exercise 31). Although the resulting integral is also improper, the function $e^{-\tan^2(\theta)}$ converges quickly to 0 at both $-\pi/2$ and $\pi/2$.



Use numerical integration to estimate the integrals in (6.27)

31. A *complete elliptic integral of the second kind* is an integral of the form

$$\int_0^a \sqrt{\frac{a^2 - \varepsilon^2 x^2}{a^2 - x^2}}\, dx$$

where $a > 0$ and $0 \le \varepsilon < 1$. Use a trig substitution to transform it into a proper integral.

32. What does the integral in 31 reduce to when $\varepsilon = 0$?

32. A *complete elliptic integral of the third kind* is an integral of the form

$$\int_0^1 \frac{dx}{(1 - nx^2)\sqrt{(1-x^2)(1-k^2x^2)}}$$

where $n < 1$. Use a trig substitution to transform it into a proper integral.

34. Many mechanical problems (e.g., precession of a spinning top, period of an orbit in certain multi-body problems) involve elliptic integrals of the form

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(\alpha + \beta x + \gamma x^2)}}$$

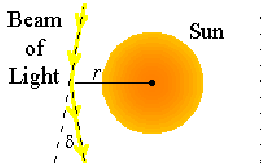
Convert this elliptic integral to a proper integral.

35. According to general relativity (in particular, the Schwartzchild metric), the "bending angle" δ of a beam of light when it is closest to the sun is approximately

$$\delta \approx \frac{2Gm}{c^2 r} \int_0^1 \frac{(1 - v^3)\, dv}{(1-v^2)^{3/2}} \tag{6.28}$$

[4] This integral appears in the study of the *Voight Profile*, which is used in the design of lasers.

where r is the shortest distance from the beam to the center of the sun, $m = 2 \times 10^{33}$ *grams* is the mass of the sun, $c = 3 \times 10^{10} \frac{cm}{\sec}$ is the speed of light in a vacuum, and $G = 6.67 \times 10^{-8} \frac{cm^3}{g \sec^2}$ is the universal gravitational constant.



Use a trigonometric substitution to evaluate the improper integral in (6.28). What is δ (in radians) when $r = 7 \times 10^{10} cm$ (i.e., the light "barely grazes" the surface of the sun)? (Hint: after substituting, use integration by parts with $u = 1 - \sin^3(\theta)$ and $dv = \sec^2(\theta)\, d\theta$).

36. Trig substitutions are often used in creative ways to study elliptic integrals. For example, transform the complete elliptic integral of the first kind

$$\int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}$$

using the substitution $x = \frac{1}{k} \sin(\theta)$.

37. What is the period of a pendulum with a length of $l = 4$ feet when $\theta_0 = \frac{\pi}{2}$?

38. What is the period of a pendulum with a length of $l = 4$ feet when $\theta_0 = 89°$?

39. **Try it out!** Construct a pendulum with a length of 4 feet, and then release it from an initial angle of $\theta_0 = \frac{\pi}{6}$ (or some other relatively small initial angle). Measure the amount of time required for the pendulum to swing back and forth 10 times, and then divide by 10 to obtain an estimate of T. Compare your result to the value of T obtain from (6.25) with the appropriate parameters.

40. **Try it out!** A pendulum can be used to estimate the acceleration due to gravity at your location. In particular, construct a pendulum with a length of 4 feet, and then release it from an initial angle of $\theta_0 = \frac{\pi}{6}$ (or some other relatively small initial angle). Measure the amount of time required for the pendulum to swing back and forth 10 times, and then divide by 10 to obtain an estimate of T. Substitute for T, l, and k in (6.25), transform and estimate the integral, and then solve for g.

6.5 Additional Antiderivatives

Integration Using Trigonometric Identities

Although substitution and integration by parts may be the most important techniques of integration, there are today literally hundreds of techniques of integration and thousands of basic antiderivative formulas. In this section, we explore a few additional techniques and formulas that are representative of the more common approaches to evaluating antiderivatives.

For example, the antiderivative of a square of a sine or a cosine is evaluated using one of the identities

$$\cos^2(A) = \frac{1}{2} + \frac{1}{2}\cos(2A) \tag{6.29}$$

$$\sin^2(A) = \frac{1}{2} - \frac{1}{2}\cos(2A) \tag{6.30}$$

which follow immediately from the identities for $\cos(2A)$ (see the exercises).

EXAMPLE 1 Evaluate $\int \sin^2(3x)\,dx$

Solution: The identity $\sin^2(3x) = \frac{1}{2} - \frac{1}{2}\cos(6x)$ transforms the antiderivative into

$$\begin{aligned}\int \sin^2(3x)\,dx &= \int \left(\frac{1}{2} - \frac{1}{2}\cos(6x)\right)dx \\ &= \frac{1}{2}x - \frac{1}{12}\sin(6x) + C\end{aligned}$$

Analogously, antiderivatives of products of sines and cosines are often evaluated with one of the identities

$$\sin(A)\sin(B) = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B) \tag{6.31}$$

$$\cos(A)\cos(B) = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B) \tag{6.32}$$

$$\sin(A)\cos(B) = \frac{1}{2}\sin(A-B) + \frac{1}{2}\sin(A+B) \tag{6.33}$$

EXAMPLE 2 Evaluate $\int \cos(3x)\cos(5x)\,dx$

Solution: We use (6.32) with $A = 3x$ and $B = 5x$ and the fact that $\cos(-2x) = \cos(2x)$:

$$\begin{aligned}\cos(3x)\cos(5x) &= \frac{1}{2}\cos(3x-5x) + \frac{1}{2}\cos(3x+5x) \\ &= \frac{1}{2}\cos(2x) + \frac{1}{2}\cos(8x)\end{aligned}$$

As a result, we have

$$\begin{aligned}\int \cos(3x)\cos(5x)\,dx &= \frac{1}{2}\int \cos(2x)\,dx + \frac{1}{2}\int \cos(8x)\,dx \\ &= \frac{1}{4}\sin(2x) + \frac{1}{16}\sin(8x) + C\end{aligned}$$

Additionally, it may be appropriate to utilize a *Pythagorean identity:*

$$\begin{aligned}\sin^2(x) &= 1 - \cos^2(x), \qquad \sec^2(x) = 1 + \tan^2(x) \\ \cos^2(x) &= 1 - \sin^2(x), \qquad \csc^2(x) = 1 + \cot^2(x)\end{aligned}$$

EXAMPLE 3 Evaluate $\int \cos^3(x)\,dx$

Solution: We write $\cos^3(x) = \cos^2(x)\cos(x)$ and use the identity $\cos^2(x) = 1 - \sin^2(x)$:

$$\int \cos^2(x)\cos(x)\,dx = \int \left(1 - \sin^2(x)\right)\cos(x)\,dx$$

As a result, we can let $u = \sin(x)$, $du = \cos(x)\,dx$ to obtain

$$\begin{aligned} \int \left(1 - \sin^2(x)\right)\cos(x)\,dx &= \int \left(1 - u^2\right) du \\ &= u - \frac{u^3}{3} + C \end{aligned}$$

Replacing u by $\sin(x)$ thus yields

$$\int \cos^3(x) = \sin(x) - \frac{\sin^3(x)}{3} + C$$

EXAMPLE 4 Evaluate $\int \sin^2(x)\cos^3(x)\,dx$

Solution: To do so, we write $\cos^3(x) = \cos^2(x)\cos(x)$ and use $\cos^2(x) = 1 - \sin^2(x)$:

$$\int \sin^2(x)\cos^3(x)\,dx = \int \sin^2(x)\left(1 - \sin^2(x)\right)\cos(x)\,dx$$

Letting $u = \sin(x)$, $du = \cos(x)\,dx$ then yields

$$\begin{aligned} \int \sin^2(x)\cos^3(x)\,dx &= \int u^2\left(1 - u^2\right) du \\ &= \int \left(u^2 - u^4\right) du \\ &= \frac{u^3}{3} - \frac{u^5}{5} + C \\ &= \frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + C \end{aligned}$$

Check your Reading *Which identity should be applied to* $\int \sin(2x)\sin(7x)\,dx$*?*

Additional Inverse Function Formulas

The inverse hyperbolic functions also result in some important antiderivative formulas. For example, in section 2 we used implicit differentiation to show that

$$\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}}$$

It follows that $\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}(x) + C$, or more generally,

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \tag{6.34}$$

for a positive number a. When combined with the fact that

$$\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$$

this antiderivative formula sometimes occurs in the form

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left(x + \sqrt{x^2+a^2}\right) + C$$

EXAMPLE 5 Evaluate the antiderivative

$$\int \frac{dx}{\sqrt{x^2+4}}$$

Solution: The identity (6.34) implies that

$$\int \frac{dx}{\sqrt{x^2+4}} = \sinh^{-1}\left(\frac{x}{2}\right) + C$$

or equivalently, in logarithmic form we have

$$\int \frac{dx}{\sqrt{x^2+4}} = \ln\left(x + \sqrt{x^2+4}\right) + C$$

The inverse functions of $\cosh(x)$ and $\tanh(x)$ also produce commonly used antiderivative formulas. For example, implicit differentiation can be used to show that

$$\frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}} \qquad \textit{and} \qquad \frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}$$

As a result, we obtain the antiderivative formulas

$$\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1}(x) + C \qquad \textit{and} \qquad \int \frac{dx}{1-x^2} = \tanh^{-1}(x) + C$$

or more generally, for a constant $a > 0$, the formulas

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \qquad \textit{and} \qquad \int \frac{dx}{a^2-x^2} = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + C$$

EXAMPLE 6 Evaluate the antiderivative

$$\int \frac{\cos(x)\,dx}{1-\sin^2(x)}$$

Solution: If we let $u = \sin(x)$, then $du = \cos(x)\,dx$ and

$$\int \frac{\cos(x)\,dx}{1-\sin^2(x)} = \int \frac{du}{1-u^2} = \tanh^{-1}(u) + C$$

Substituting for u using $u = \sin(x)$ thus results in

$$\int \frac{\cos(x)\,dx}{1-\sin^2(x)} = \tanh^{-1}(\sin(x)) + C$$

Notice in example 6 that we can simplify to

$$\int \frac{\cos(x)\, dx}{1-\sin^2(x)} = \int \frac{\cos(x)\, dx}{\cos^2(x)} = \int \frac{1}{\cos(x)} dx = \int \sec(x)\, dx$$

Thus, example 6 yields another form for the antiderivative of the secant function, namely

$$\int \sec(x)\, dx = \tanh^{-1}(\sin(x)) + C$$

However, it can be shown that the result in example 6 simplifies to the usual antiderivative of $\sec(x)$, namely

$$\int \sec(x)\, dx = \ln|\sec(x) + \tan(x)| + C$$

Check your Reading *Which formula would we use to evaluate*

$$\int \frac{e^x}{\sqrt{e^{2x}-1}} dx$$

Integrals Involving Expressions of the Form $x^2 - a^2$

Inverse hyperbolic functions and the identity (6.2) allow us to transform a definite integral whose integrand contains an expression of the form x^2-a^2. In particular, we let $x = a\cosh(t)$ and then use (6.2) in the form

$$\cosh^2(t) - 1 = \sinh^2(t) \tag{6.35}$$

so that to transform a definite integral whose integrand contains an expression of the form $x^2 - a^2$, we use the substitution $x = a\cosh(t)$. That is, we let

$$\begin{aligned} x &= a\cosh(t) \\ dx &= a\sinh(t)\ dt \end{aligned}$$

To simplify an integral involving $x^2 - a^2$, let $x = a\cosh(t)$.

The expression $x^2 - a^2$ then becomes

$$x^2 - a^2 = a^2\cosh^2(t) - a^2 = a^2\left(\cosh^2(t) - 1\right) = a^2\sinh^2(t)$$

and in the final result, we can return to the t variable using the fact that $t = \cosh^{-1}(x/a)$.

EXAMPLE 7 Evaluate the antiderivative

$$\int \frac{dx}{x^2\sqrt{x^2-4}}$$

To do so, we let $x = 2\cosh(t)$, so that $dx = 2\sinh(t)\, dt$ and

$$\int \frac{dx}{x^2\sqrt{x^2-4}} = \int \frac{2\sinh(t)\, dt}{4\cosh^2(t)\sqrt{4\cosh^2(t)-4}} = \int \frac{\sinh(t)\, dt}{4\cosh^2(t)\sqrt{\cosh^2(t)-1}}$$

Simplifying with (6.35) then leads to

$$\int \frac{dx}{x^2\sqrt{x^2-4}} = \int \frac{\sinh(t)\, dt}{4\cosh^2(t)\sinh(t)} = \frac{1}{4}\int \operatorname{sech}^2(t)\, dt$$

since $1/\cosh^2(t) = \text{sech}^2(t)$. However, $\frac{d}{dt}\tanh(t) = \text{sech}^2(t)$ implies that

$$\int \frac{dx}{x^2\sqrt{x^2-4}} = \frac{1}{4}\int \text{sech}^2(t)\,dt = \frac{1}{4}\tanh(t) + C$$

Finally, $x = 2\cosh(t)$ implies that $t = \cosh^{-1}(x/2)$. Thus,

$$\int \frac{dx}{x^2\sqrt{x^2-4}} = \frac{1}{4}\tanh\left[\cosh^{-1}\left(\frac{x}{2}\right)\right] + C$$

Let's conclude by showing that hyperbolic expressions can be simplified in much the same way that trigonometric expressions are simplified using right triangles. In particular, notice that $1 - \tanh^2(x) = \text{sech}^2(x)$ implies that

$$\tanh(x) = \sqrt{1 - \sec\text{h}^2(x)} = \sqrt{1 - \frac{1}{[\cosh(x)]^2}}$$

As a result, we have

$$\frac{1}{4}\tanh\left[\cosh^{-1}\left(\frac{x}{2}\right)\right] = \frac{1}{4}\sqrt{1 - \frac{1}{\left[\cosh\left(\cosh^{-1}(x/2)\right)\right]^2}}$$

Since $\cosh\left(\cosh^{-1}(x/2)\right) = x/2$, we thus have

$$\begin{aligned} \frac{1}{4}\tanh\left[\cosh^{-1}\left(\frac{x}{2}\right)\right] &= \frac{1}{4}\sqrt{1 - \frac{1}{(x/2)^2}} \\ &= \frac{1}{4}\sqrt{\frac{x^2/4 - 1}{x^2/4}} \\ &= \frac{\sqrt{4(x^2/4 - 1)}}{4x} \\ &= \frac{\sqrt{x^2-4}}{4x} \end{aligned}$$

As a result, the result in example 7 can also be expressed as

$$\int \frac{dx}{x^2\sqrt{x^2-4}} = \frac{\sqrt{x^2-4}}{4x} + C$$

Check your Reading *Which formula would we use to evaluate*

$$\int \frac{e^x}{\sqrt{e^{2x}-1}}dx$$

Computer Algebra Systems

A large number of the remaining hundreds of techniques and thousands of formulas have been incorporated into software tools known as *computer algebra systems*, so that those who use computer algebra systems can evaluate many of the myriad antiderivatives which arise in applications.

EXAMPLE 8 Use a computer algebra system to evaluate $\int \sin^4(x)\,dx$

Solution: The computer algebra system *Maple* produces the result

$$\int \sin^4(x)\,dx = \frac{3}{8}x - \frac{1}{4}\sin^3 x\cos x - \frac{3}{8}\cos x\sin x + C$$

EXAMPLE 9 Use a computer algebra system to evaluate

$$\int \sin(x)\sin(2x)\cos(3x)\,dx$$

Solution: The computer algebra system *Maple* produces the result

$$\int \sin(x)\sin(2x)\cos(3x)\,dx = \frac{1}{8}\sin 2x + \frac{1}{16}\sin 4x - \frac{1}{24}\sin 6x - \frac{1}{4}x + C$$

There are, however, two dangers in using a computer algebra system to evaluate an antiderivative. First, there are many antiderivatives that even the most powerful software tools cannot evaluate. For example,

$$\int e^{-x^2}dx, \quad \int \frac{dx}{x^5 + \pi x + 1}, \quad \textit{and} \quad \int \ln(e^x + 1)\,dx$$

cannot be evaluated in closed form by any machine, technique, or person.

Second, the result may not be in the form we might have expected. For example, consider the antiderivative

$$\int \sec^4(x)\,dx \tag{6.36}$$

We can write $\sec^4(x) = \sec^2(x)\sec^2(x)$ and use the identity $\sec^2(x) = 1 + \tan^2(x)$:

$$\int \sec^4(x)\,dx = \int \left(1 + \tan^2(x)\right)\,\sec^2(x)\,dx$$

We then let $u = \tan(x)$, $du = \sec^2(x)\,dx$ to obtain

$$\begin{aligned} \int \sec^4(x)\,dx &= \int \left(1 + u^2\right)du \\ &= u + \frac{u^3}{3} + C \\ &= \tan(x) + \frac{\tan^3(x)}{3} + C \end{aligned} \tag{6.37}$$

However, if we evaluate (6.36) with *Maple*, we obtain

$$\int \sec^4(x)\,dx = \frac{\sin(x)}{3\cos^3 x} + \frac{2\sin x}{3\cos x} + C_1 \tag{6.38}$$

Although (6.36) and (6.38) represent the same family of antiderivatives, they are radically different in appearance.

Exercises:

Evaluate the following by first simplifying with trigonometric identities or by using an inverse hyperbolic form.

1.	$\int \cos^2(x)\,dx$	2.	$\int \cos^2(2x)\,dx$
3.	$\int \sin^2(\pi x)\,dx$	4.	$\int \sin^2\left(\sqrt{2}\;x\right)dx$
5.	$\int \cos(3x)\sin(2x)\,dx$	6.	$\int \cos(2x)\cos(4x)\,dx$
7.	$\int \sin(2x)\sin(x)\,dx$	8.	$\int \cos(\pi x)\sin(ex)\,dx$
9.	$\int \sin^3(x)\,dx$	10.	$\int \cos^2(x)\,dx$
11.	$\int \sin^3(x)\cos^3(x)\,dx$	12.	$\int \cos^5(x)\,dx$
13.	$\displaystyle\int \frac{e^x dx}{\sqrt{e^{2x}+1}}$	14.	$\displaystyle\int \frac{\sin(x)\,dx}{\sqrt{1+\cos^2(x)}}$
15.	$\displaystyle\int \frac{dx}{\sqrt{x}\sqrt{x+1}}$	16.	$\displaystyle\int \frac{\sin(x)\,dx}{\sqrt{\cos(2x)}}$

Evaluate using a substitution of the form $x = a\cosh(t)$.

17.	$\displaystyle\int \frac{dx}{\sqrt{x^2-9}}$	18.	$\displaystyle\int \frac{dx}{x^2-16}$
19.	$\displaystyle\int \frac{x dx}{\sqrt{x^2-9}}$	20.	$\displaystyle\int \frac{x dx}{\sqrt{x^2-1.44}}$
21.	$\displaystyle\int \frac{\sqrt{x^2-9}}{x^2}dx$	22.	$\displaystyle\int \frac{x^2 dx}{\sqrt{x^2-1}}$

Evaluate with a computer algebra system.

23.	$\int \sin^2(3x)\sin^2(11x)\,dx$	24.	$\int \sin^2(3x)\cos^2(2x)\,dx$
25.	$\int \cos^4(x)\,dx$	26.	$\int \sin^6(x)\,dx$
27.	$\int \tan^4(x)\,dx$	28.	$\int \cot^4(x)\,dx$
29.	$\int e^x \sin(2x)\,dx$	30.	$\int e^{3x}\sin(2x)\,dx$

31. Try evaluating the following antiderivative with a computer algebra system:

$$\int \sqrt{\cos^2(x)-\cos^4(x)}dx$$

Then evaluate it by hand using the fact that

$$\begin{aligned}\cos^2(x)-\cos^4(x) &= \cos^2(x)\left(1-\cos^2(x)\right)\\ &= \cos^2(x)\sin^2(x)\end{aligned}$$

32. Try evaluating the following antiderivative with a computer algebra system:

$$\int \sqrt{\sin^2(x)-\sin^4(x)}dx$$

Then evaluate it by hand using the fact that

$$\sin^2(x)-\sin^4(x) = \cos^2(x)\sin^2(x)$$

33. Try evaluating the following antiderivative with a computer algebra system:

$$\int \sqrt{\cos(2x) + \sin^4(x)}\, dx$$

Then evaluate it by hand using the fact that .

$$\cos(2x) + \sin^4(x) = \cos^4(x)$$

34. Try evaluating the following antiderivative with a computer algebra system:

$$\int x^{1/\ln(x)} dx$$

Then evaluate it by hand using the fact that $x^{1/\ln(x)} = e$.

35. In this exercise, we evaluate the antiderivative

$$2\int \cos(x)\cos(x)\, dx \tag{6.39}$$

(a) Evaluate (6.39) using the identity (6.29).
(b) Evaluate (6.39) using the identity (6.32).

36. In this exercise, we evaluate the antiderivative

$$2\int \sin(x)\cos(x)\, dx \tag{6.40}$$

(a) Evaluate (6.40) using the identity $2\sin(x)\cos(x) = \sin(2x)$.
(b) Evaluate (6.40) using the identity (6.33).

37. Evaluate $\int \sin^3(x)\, dx$ using the identity

$$\sin^3(x) = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$$

38. Evaluate $\int \left[\cos^4(x) + \sin^4(x)\right] dx$ using the identity

$$\cos^4(x) + \sin^4(x) = \frac{1}{4}\cos 4x + \frac{3}{4}$$

39. **Write to Learn:** Write a lecture that you would give to a trigonometry class in which you derive (6.31) by applying sum-of-the-angles identities to

$$\frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$$

and in similar fashion you apply (6.32) and (6.33) respectively to

$$\frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B) \qquad \text{and} \qquad \frac{1}{2}\sin(A-B) + \frac{1}{2}\sin(A+B)$$

40. **Write to Learn:** Write a lecture that you would give to a trigonometry class in which you derive (6.29) and (6.30) from (6.31), (6.32), and (6.33).

41. Use implicit differentiation to show that

$$\frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}}$$

What antiderivative formula is implied by this result?

42. Use implicit differentiation to show that

$$\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}$$

What antiderivative formula is implied by this result?

Self Test

A variety of questions are asked in a variety of ways in the problems below. Answer as many of the questions below as possible before looking at the answers in the back of the book.

1. Answer each statement as true or false. If the statement is false, then state why or give a counterexample.

   (a) $\cosh x$ is the same as $\cos(hx)$

   (b) The hyperbolic tangent function has infinitely many vertical asymptotes

   (c) $\sinh(x + 2\pi) = \sinh(x)$

   (d) The derivative of $\cosh(x)$ is $\sinh(x)$

   (e) If $f(x)$ is increasing on (a, b), then $f^{-1}(x)$ exists on $(f(a), f(b))$.

   (f) $\sin^{-1}(\sin(\pi)) = \pi$

   (g) If x is in $\left[0, \frac{\pi}{4}\right]$, then

   $$\sin^{-1}\left(\tan^{-1}(\tan(\sin(x)))\right) = \tan\left(\sin\left(\sin^{-1}\left(\tan^{-1}(x)\right)\right)\right)$$

   (h) The inverse sine is defined for all x.

   (i) The inverse tangent is defined for all x.

   (j) The inverse hyperbolic cosine is defined for all x.

   (k) The derivative of $\sin^{-1}(x)$ is $\cos^{-1}(x)$.

   (l) Evaluating $\int \cos^2(x)\,dx$ requires a trigonometric identity.

2. Which of the following is **not** a critical point of $f(x) = \cosh(\cos(x))$?

   (a) 0 (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) π

3. Evaluate $\int x \sinh(x)\,dx$

   (a) $x\sinh(x) + \cosh(x) + C$ (b) $x\sinh(x) - \cosh(x) + C$
   (c) $x\cosh(x) + \sinh(x) + C$ (d) $x\cosh(x) - \sinh(x) + C$

4. If $f(x) = x^{-1}$, then $f^{-1}(x) =$

   (a) x (b) x^{-1} (c) $-x^{-2}$ (d) $\left(x^{-1}\right)^{-1}$

5. Which of the following has the same derivative as

   $$\tan^{-1}\left(\frac{-1}{x}\right)$$

   (a) $\tan^{-1}(x)$ (b) $\tan^{-1}(-x)$ (c) $\sin^{-1}(x)$ (d) $\sin\left(\cos^{-1}(x)\right)$

6. Which of the following substitutions would **not** eliminate the square root in the integrand of

   $$\int \sqrt{x^2 + 1}dx$$

   (a) $x = \sin(\theta)$ (b) $x = \tan(\theta)$ (c) $x = \sinh(t)$ (d) $x = \cot(\theta)$

7. If $x = \tan(\theta)$, then

$$\int \frac{dx}{x^2\sqrt{x^2+1}}$$

simplifies to

(a) $\displaystyle\int \frac{d\theta}{\tan(\theta)\sec(\theta)}$ (b) $\displaystyle\int \frac{d\theta}{\tan^2(\theta)\sec(\theta)}$

(c) $\displaystyle\int \frac{\sec^2(\theta)\ d\theta}{\tan^2(\theta)[\tan(\theta)+1]}$ (d) $\displaystyle\int \frac{\sec(\theta)\,d\theta}{\tan^2(\theta)}$

8. Simplify the expression $\sin(\tan^{-1}(\frac{1}{x}))$

9. Evaluate the antiderivative $\int x\tan^{-1}(x)dx$

10. Evaluate the antiderivative

$$\int \frac{dx}{2x^2+5}$$

11. Evaluate the antiderivative

$$\int \frac{dx}{\sqrt{4-x^2}}$$

12. Evaluate the antiderivative

$$\int \frac{1}{x\sqrt{1+x^2}}dx$$

13. What is the area of the region bounded by the hyperbola $x^2 - y^2 = 1$ and the line $x = 2$?

14. What is the area of the region bounded by the hyperbola $y^2 - x^2 = 1$ and the line $y = 2$?

15. In an earlier self-test, you were asked to evaluate the definite integral

$$\int_0^1 \sqrt{2-x^2}dx$$

using geometry. Now evaluate this integral using a trigonometric substitution.

16. What is the arclength of the catenary $y = 2\cosh(x/2)$ over $[-\ln 2, \ln 2]$?

17. **Write to Learn:** In a short essay, explain why the substitution $x = r\sinh(t)$, $dx = r\cosh(t)\,dt$ can also be used to transform integrands that contain expressions of the form $x^2 + r^2$. Compare this substitution to the $x = r\tan(\theta)$, $dx = r\sec^2(\theta)\,d\theta$ method by applying both to

$$\int \frac{dx}{x^2\sqrt{x^2+4}}$$

18. **Computer Algebra System:** Show that

$$\int \sqrt{1+\sqrt[3]{1+x}}dx = \frac{6}{7}\left(\sqrt{1+\sqrt[3]{1+x}}\right)^7 - \frac{12}{5}\left(\sqrt{1+\sqrt[3]{1+x}}\right)^5 + 2\left(\sqrt{1+\sqrt[3]{1+x}}\right)^3 + C$$

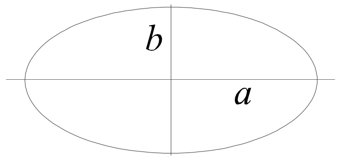
by differentiating the right side and simplifying (preferably with a computer algebra system). Can your computer algebra system evaluate

$$\int \sqrt{1+\sqrt[3]{1+x}}dx?$$

Trigonometric substitutions are frequently employed in the study of elliptic integrals. But what is an elliptic integral? And why are they so important? Our next step is to show that elliptic integrals occur naturally, thus making trigonometric substitutions an important tool in the study of integrals.

In analytic geometry, it is shown that the equation of an ellipse centered at the origin is given by

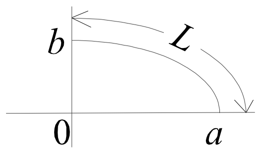
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



where a is the length of the semi-major axis and b is the length of the semi-minor axis. As we will now show, finding the arclength of the ellipse requires us to work with an elliptic integral.

To begin with, notice that by symmetry we need only find the length of the section of the ellipse in the first quadrant and then multiply the result by 4. That is, the distance around the ellipse is $4L$, where

$$L = \int_0^a \sqrt{1 + (y')^2}\, dx$$



To find y', we solve for y^2 to obtain

$$y^2 = b^2 - \frac{b^2}{a^2}x^2 \tag{6.41}$$

Implicit differentiation then yields

$$2y\,y' = -2\frac{b^2}{a^2}x, \qquad y' = -\frac{b^2}{a^2}\frac{x}{y}$$

As a result, the quantity $1 + (y')^2$ becomes

$$1 + (y')^2 = 1 + \frac{b^4}{a^4}\frac{x^2}{y^2}$$

We can now use (6.41) to replace y^2, which yields

$$1 + (y')^2 = 1 + \frac{b^4}{a^4}\frac{x^2}{\left(b^2 - \frac{b^2}{a^2}x^2\right)}$$

Finding a common denominator and simplifying yields

$$1 + (y')^2 = \frac{a^2 - \left(1 - b^2/a^2\right)x^2}{a^2 - x^2}$$

If we now define the *eccentricity* ε of the ellipse to be

$$\varepsilon^2 = 1 - \frac{b^2}{a^2} \tag{6.42}$$

then the quantity $1+(y')^2$ reduces to

$$1+(y')^2 = \frac{a^2-\varepsilon^2x^2}{a^2-x^2}$$

Consequently, the distance around an ellipse is

$$4L = 4\int_0^a \sqrt{1+(y')^2}dx = 4\int_0^a \sqrt{\frac{a^2-\varepsilon^2x^2}{a^2-x^2}}\,dx \tag{6.43}$$

The integral in (6.43) is a *complete elliptic integral of the second kind.* It is improper and cannot be evaluated in closed form except for special choices of the eccentricity ε. Thus, estimating the distance around an ellipse requires us to use a trig substitution to transform (6.43) into a proper integral and then estimate the resulting integral numerically.

Write to Learn What is the distance around an ellipse with a semi-major axis of 2 inches and a semi-minor axis of 1 inch? Use (6.42) to determine the eccentricity of the ellipse and use it to set up the integral (6.43). Then transform the integral using the trig substitution $x = 2\sin(\theta)$, $dx = 2\cos(\theta)\,d\theta$ and estimate the resulting integral numerically. Finally, report your results in a short essay which includes a sketch of the ellipse whose arclength is being estimated.

Write to Learn The earth's orbit is an ellipse with a semi-major axis of $a = 94,511,000$ miles and a semi-minor axis of $b = 91,404,000$ miles. How many miles does the earth travel in one year in its orbit about the sun? Write a short essay describing the method used to estimate the length and the result of that estimation.

Write to Learn What kind of curve is implied by $\varepsilon = 0$? What does (6.43) reduce to when $\varepsilon = 0$? What is its value? Write a short essay which describes the significance of the eccentricity, including why $0 < \varepsilon < 1$ for an ellipse and what type of curve results when $\varepsilon = 0$.

Write to Learn Go to the library and investigate the topic of elliptic integrals. Report the results of your research in a short essay on elliptic integrals.

Group Learning: Each member of the group chooses a planet and then obtains the semi-major axis and the semi-minor axis of the planet's orbit. Each member should then use (6.42) to determine the eccentricity of the orbit, should set up the integral in (6.43), should transform it into a proper integral, and then should estimate the distance the planet travels in one of its years.

Advanced Contexts:

Not only do trigonometric substitutions convert improper elliptic integrals into proper integrals, but they also convert elliptic integrals into a form that allows them to be estimated with one of the most powerful numerical techniques ever discovered. In particular, if a_0 and b_0 are positive numbers, then let

$$\begin{aligned} a_1 &= \frac{a_0+b_0}{2}, & b_1 &= \sqrt{a_0b_0} \\ a_2 &= \frac{a_1+b_1}{2}, & b_2 &= \sqrt{a_1b_1} \\ &\vdots & &\vdots \end{aligned}$$

$$a_n = \frac{a_{n-1}+b_{n-1}}{2}, \qquad b_n = \sqrt{a_{n-1}b_{n-1}}$$

$$\vdots \qquad\qquad \vdots$$

and so on. Then it can be shown that a_n and b_n both approach the same number m as n approaches ∞. That is,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = m$$

The number m is called the *arithmetic-geometric* mean of a_0 and b_0.

For example, if $a_0 = 1$ and $b_0 = 9$, then

$$\begin{aligned}
a_1 &= \frac{1+9}{2} = 5, \qquad b_1 = \sqrt{1\cdot 9} = 3 \\
a_2 &= \frac{3+5}{2} = 4, \qquad b_2 = \sqrt{3\cdot 5} = 3.87298 \\
a_3 &= \frac{4+3.87298}{2} = 3.93649, \qquad b_3 = \sqrt{4\cdot 3.87298} = 3.93598 \\
a_4 &= \frac{3.93649+3.93598}{2} = 3.93623, \qquad b_4 = \sqrt{(3.93649)(3.93598)} = 3.93623
\end{aligned}$$

Thus, to 5 decimal places, the arithmetic-geometric mean of $a_0 = 1$ and $b_0 = 9$ is $m = 3.93623$.

Why are we talking about the arithmetic-geometric mean? It's because about 200 years ago, the mathematician Karl Gauss proved something truly amazing. He showed that a *complete elliptic integral of the first kind,* which is of the form

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

can be computed with an arithmetic-geometric mean. In the problems below, we introduce his method and its relationship to elliptic integrals.

1. Gauss' method is based on the fact that

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2(\theta)+b^2\sin^2(\theta)}} = \int_0^{\pi/2} \frac{d\phi}{\sqrt{\left(\frac{a+b}{2}\right)^2\cos^2(\phi)+\left(\sqrt{ab}\right)^2\sin^2(\phi)}}$$

Use numerical integration to verify this result when $a = 1$ and $b = 9$.

Exercises 2-4 present the derivation of the result in exercise 1, and then exercise 5 shows how it is used to compute a complete elliptic integral of the first kind.

2. * Show that if $2\theta = \phi + \sin^{-1}(\gamma\sin(\phi))$, then

$$2d\theta = \frac{\gamma\cos(\phi)+\sqrt{1-\gamma^2\sin^2(\phi)}}{\sqrt{1-\gamma^2\sin^2(\phi)}}d\phi$$

and also that

$$\sqrt{(\gamma+1)^2 - 4\gamma\sin^2(\theta)} = \gamma\cos(\phi)+\sqrt{1-\gamma^2\sin^2(\phi)} \tag{6.44}$$

Finally, show that if $\gamma = \frac{a-b}{a+b}$, then (6.44) becomes

$$\frac{2}{a+b}\sqrt{a^2\cos^2(\theta)+b^2\sin^2(\theta)} = \gamma\cos(\phi)+\sqrt{1-\gamma^2\sin^2(\phi)}$$

3. * Use the substitution $2\theta = \phi + \sin^{-1}(\gamma \sin(\phi))$ with $\gamma = \frac{a-b}{a+b}$ and its properties from exercise 2 to show that

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2(\theta) + b^2\sin^2(\theta)}} = \int_0^{\pi} \frac{d\phi}{\sqrt{(a+b)^2 - (a-b)^2\sin^2(\phi)}}$$

4. *Use the substitution $\phi = \pi - \theta$ to show that

$$\int_0^{\pi/2} \frac{d\phi}{\sqrt{(a+b)^2 - (a-b)^2\sin^2(\phi)}} = \int_{\pi/2}^{\pi} \frac{d\phi}{\sqrt{(a+b)^2 - (a-b)^2\sin^2(\phi)}}$$

and then use this to explain why

$$\int_0^{\pi} \frac{d\phi}{\sqrt{(a+b)^2 - (a-b)^2\sin^2(\phi)}} = 2\int_0^{\pi/2} \frac{d\phi}{\sqrt{(a+b)^2 - (a-b)^2\sin^2(\phi)}}$$

Finally, show that $\frac{(a-b)^2}{4} = \frac{(a+b)^2}{4} - ab$ and then use it to show that

$$\int_0^{\pi} \frac{d\phi}{\sqrt{(a+b)^2 - (a-b)^2\sin^2(\phi)}} = \int_0^{\pi/2} \frac{d\phi}{\sqrt{\left(\frac{a+b}{2}\right)^2\cos^2(\phi) + \left(\sqrt{ab}\right)^2\sin^2(\phi)}}$$

5. *Combine the steps in 2-4 to show that

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2(\theta) + b^2\sin^2(\theta)}} = \int_0^{\pi/2} \frac{d\phi}{\sqrt{\left(\frac{a+b}{2}\right)^2\cos^2(\phi) + \left(\sqrt{ab}\right)^2\sin^2(\phi)}}$$

Use the substitution $x = \sin(\theta)$ to show that

$$\int_0^{1} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos^2(\theta) + (1-k^2)\sin^2(\theta)}}$$

and then use this to explain why if m is the arithmetic-geometric mean of $a_0 = 1$ and $b_0 = \sqrt{1-k^2}$, then

$$\int_0^{1} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \int_0^{\pi/2} \frac{dx}{\sqrt{m^2\cos^2(\theta) + m^2\sin^2(\theta)}} = \frac{\pi}{2m}$$

7. DIFFERENTIAL EQUATIONS

Calculus was developed in part to be a tool for solving differential equations. In particular, it was used to find *closed form* solutions to differential equations, where a solution is said to be in *closed form* when it is expressed as a finite combination of elementary functions. Hypothetically, a closed form solution tells us all there is to know about the differential equation, and thus about the real-world process which the differential equation models.

Finding closed form solutions to differential equations usually reduces to the problem of evaluating antiderivatives. As a result, the study of differential equations has been the prime motivation for the development of many of the techniques of integration, as well as being an unending source of new functions with exciting new properties.

In this chapter, we will concentrate on finding closed form solutions of *separable differential equation,* which are differential equations of the form

$$y' = \frac{g(x)}{f(y)}$$

In doing so, we will develop several additional techniques of integration, introduce a variety of new functions, and explore a sampling of the myriad real-world applications of differential equations. Hopefully, we will also discover that finding a closed form solution to a differential equation produces a sense of completion, a sense of accomplishment, and a sense of understanding that places it among the most worthwhile of intellectual pursuits.

7.1 Separable Differential Equations

Separating Variables

Antiderivatives are instrumental in finding solutions of differential equations. Thus, we should be and are ready to solve differential equations, beginning with a class of equations known as *separable differential equations.*

Suppose that y is defined implicitly as a function of x by the equation

$$F(y) = G(x) + C \tag{7.1}$$

for some constant C. If F and G are antiderivatives of f and g, respectively, then

$$\begin{aligned} \frac{d}{dx}F(y) &= \frac{d}{dx}G(x) + \frac{d}{dx}C \\ F'(y)\,y' &= G'(x) \\ f(y)\,y' &= g(x) \end{aligned}$$

Solving for y' then yields a *separable differentiable equation*

$$y' = \frac{g(x)}{f(y)} \tag{7.2}$$

To find a solution (7.1) to a separable differential equation (7.2), we simply reverse the process. That is, we must "undo" the differentiation above. First we replace y' by dy/dx:

$$\frac{dy}{dx} = \frac{g(x)}{f(y)} \tag{7.3}$$

Separate

We then **separate** variables in (7.3) to get

$$f(y)\,dy = g(x)\,dx$$

Integrate

after which we **apply the antiderivative** to both sides:

$$\int f(y)\,dy = \int g(x)\,dx$$

If F and G are antiderivatives of f and g, respectively, then as expected

$$F(y) = G(x) + C \tag{7.4}$$

Solve for y (if possible)

for some constant C. If possible, we then **solve for** y in order to transform the *integral curve* (7.4) into an *explicitly* defined solution to the differential equation.

EXAMPLE 1 Find solutions to the separable differential equation $y' = 2xy$

Solution: First, $y' = 2xy$ can be written

$$\frac{dy}{dx} = 2xy$$

Multiplication by dx and division by y then yields

$$\frac{dy}{y} = 2xdx$$

Thus, application of the antiderivative yields

$$\begin{aligned} \int \frac{dy}{y} &= 2\int xdx \\ \ln|y| &= x^2 + C \end{aligned}$$

so that in solving for y we obtain

$$|y| = e^{x^2+C} = e^{x^2}e^C$$

The solution to $|y| = A$ is given by $y = \pm A$.

Moreover, the definition of the absolute value function implies that

$$y = \pm e^C e^{x^2}$$

so that if we define a new constant $P = \pm e^C$, then the solution is

$$y = Pe^{x^2} \tag{7.5}$$

However, it must be pointed out that while separation of variables can produce some of the solutions to a separable differential equation, it does not necessarily produce *all* the solutions to a separable differential equation. For example, see exercises 43 and 44.

Check your Reading *How is $y = Pe^{x^2}$ related to $y' = 2xy$?*

Initial Values and Autonomous Equations

If an initial condition is given, then it can be used to determine the value of the arbitrary constant. However, we determine the value of the arbitrary constant before solving for y.

EXAMPLE 2 Find a solution to the separable initial value problem

$$\frac{dy}{dx} = e^{x-y}, \quad y(0) = 1$$

Solution: Since $e^{x-y} = e^x e^{-y}$, the differential equation can be written as

$$dy = e^x e^{-y} dx$$

which in separated form is

$$e^y dy = e^x dx$$

Integration yields

$$\begin{aligned} \int e^y dy &= \int e^x dx \\ e^y &= e^x + C \end{aligned} \tag{7.6}$$

Before we solve for y, however, let us use the initial condition to determine C. The initial condition says that when $x = 0$, then $y = 1$. As a result,

$$e^1 = e^0 + C \quad \Longrightarrow \quad C = e - 1$$

Correspondingly, the integral curve (7.6) becomes

$$e^y = e^x + e - 1$$

so that

$$y = \ln(e^x + e - 1)$$

An *autonomous differential equation* is a differential equation in which the input variable does not appear explicitly. In such cases, it is common to replace y' by dy/dt since often $y(t)$ is a function of time t.

EXAMPLE 3 Find solutions to $y' = \tan(y)$

Solution: First, we replace y' by $\frac{dy}{dt}$ and separate

$$\begin{aligned} \frac{dy}{dt} &= \tan(y) \\ \frac{dy}{\tan(y)} &= dt \\ \cot(y)\,dy &= dt \end{aligned}$$

Writing $\cot(y)$ as $\cos(y)/\sin(y)$ and integrating yields

$$\int \frac{\cos(y)}{\sin(y)} dy = \int dt$$

We now let $u = \sin(y)$, $du = \cos(y)\,dy$ and evaluate the antiderivatives:

$$\begin{aligned} \int \frac{du}{u} &= \int dt \\ \ln|u| &= t + C \\ \ln|\sin(y)| &= t + C \end{aligned} \quad (7.7)$$

The result (7.7) is an integral curve form of the solution. Another integral curve form is given by

$$\sin(y) = Pe^t$$

where $P = \pm e^C$.

Check your Reading *What function do we obtain if we solve for y in $\sin(y) = Pe^t$?*

Constants of Proportionality

A variable A is said to be *vary directly* with B, or equivalently, to be *directly proportional* to B if there is a constant k such that $A = kB$. Since proportionality occurs frequently in applications, separable equations often appear in the form

$$y' = k\,\frac{g(x)}{f(y)} \quad (7.8)$$

where k is *constant of proportionality.* When separating, it is common to leave the k with the $g(x)$ term, so that a solution to (7.8) is of the form

$$F(y) = k\,G(x) + C$$

where F and G are antiderivatives of f and g, respectively.

EXAMPLE 4 Find solutions to $y' = k\,x\,(y+2)$for a constant $k > 0$.

Solution: To do so, we replace y' by dy/dx to obtain

$$\frac{dy}{dx} = k\,x\,(y+2)$$

after which separation yields

$$\frac{dy}{y+2} = k\,xdx$$

Since k is a constant and can be factored out of an antiderivative, integration yields

$$\int \frac{dy}{y+2} = k \int xdx$$

We let $u = y + 2$, $du = dy$ in the first antiderivative to obtain

$$\begin{aligned} \int \frac{du}{u} &= k \int xdx \\ \ln|u| &= k\frac{x^2}{2} + C \\ \ln|y+2| &= k\frac{x^2}{2} + C \qquad (7.9) \end{aligned}$$

Finally, application of the exponential yields

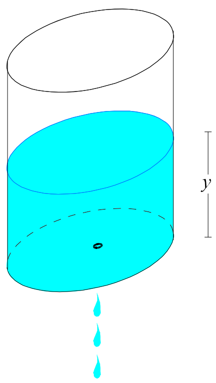
$$y + 2 = \pm e^C e^{kx^2/2} \qquad (7.10)$$

so that if we define a new constant $P = \pm e^C$, the solution becomes

$$y = -2 + Pe^{kx^2/2}$$

Constants of proportionality are common in applications, especially when those applications use differential equations as *mathematical models* of real-world processes.

EXAMPLE 5 A special case of **Torricelli's law** says if a cylinder containing water has a small leak in its bottom, then the rate of change in the height of the water in the cylinder is proportional to the square root of the height.



1-1: Leaking Tank

(a) Translate this law into a separable differential equation.

Use "$-k$" as the constant of proportionality.

(b) Solve the separable differential equation in (a).

Solution: (a) If $y(t)$ dentoes the height of the water in the cylinder at time t, then

rate of change of height	is proportional to	square root of height
$\frac{dy}{dt}$	$= -k$	$\sqrt{y}$

(b) We solve the differential equation in (a) by first separating variables:

$$\frac{dy}{dt} = -k\sqrt{y} \qquad \Longrightarrow \qquad y^{-1/2}dy = -kdt$$

We then integrate to produce

$$\begin{aligned} \int y^{-1/2}dy &= -k\int dt \\ 2y^{1/2} &= -kt + C \end{aligned}$$

after which, we solve for y

$$y = \left(\frac{-kt + C}{2}\right)^2$$

However, since k and C are arbitrary, so also are $\frac{k}{2}$ and $\frac{C}{2}$. Thus, we replace these with the new constants k_1 and C_1:

$$y(t) = (C_1 - k_1 t)^2$$

Similarly, a variable A is said to *vary inversely* with B, or equivalently, to be *inversely proportional* to B if there is a constant k for which $A = \frac{k}{B}$.

For example, it has been shown that a layer of ice on the surface of a lake acts as an insulator and slows the rate of new ice formation, thus causing the rate of ice formation to *vary inversely* with the thickness of the ice. Thus, if $y(t)$ is the thickness of the layer of ice on the surface of a lake at time t, then

$$\frac{dy}{dt} = \frac{k}{y}$$

for some constant $k > 0$.

EXAMPLE 6 Find solutions to the model of ice formation

$$\frac{dy}{dt} = \frac{k}{y}$$

Solution: To solve this differential equation, we separate,

$$ydy = kdt$$

integrate,

$$\begin{aligned} \int ydy &= \int kdt \\ \frac{y^2}{2} &= kt + C \end{aligned}$$

and solve for y:

$$y = \sqrt{2kt + 2C}$$

Check your Reading *How did we go from (7.9) to (7.10)?*

Orthogonal Trajectories

If $G'(x) = g(x)$ and $F'(y) = f(y)$, then the separable differential equation

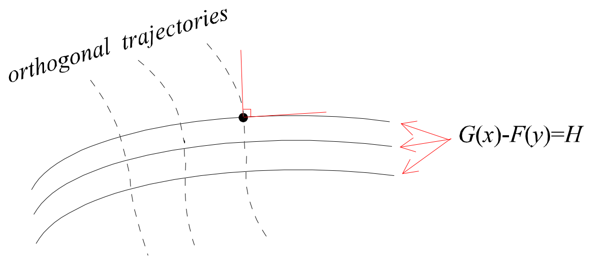
$$\frac{dy}{dx} = \frac{g(x)}{f(y)} \tag{7.11}$$

implicitly defines a *family* of curves of the form $G(x) - F(y) = H$, where H is a constant. That is, for each value of H, we obtain an integral curve solution to (7.11).

Let us now define a *new* differential equation by taking the negative reciprocal of (7.11):

$$\frac{dy}{dx} = \frac{-f(y)}{g(x)} \tag{7.12}$$

When a solution to the *new* differential equation (7.12) intersects curves of the form $G(x) - F(y) = H$, then the tangent lines at the point of intersection to the curves must intersect at *right angles.*



1-2: Orthogonal Trajectories

For this reason, solutions to (7.12) are called *orthogonal trajectories* to the solutions of (7.11)..

EXAMPLE 7 Find the integral curve solutions to the separable equation

$$\frac{dy}{dx} = \frac{-y}{x} \tag{7.13}$$

and then find a family of orthogonal trajectories to (7.13).

Solution: Separation of variables leads to

$$\begin{aligned} \frac{dy}{y} &= -\frac{dx}{x} \\ \int \frac{dy}{y} &= \int -\frac{dx}{x} \\ \ln|y| &= -\ln|x| + C \\ |y| &= e^{-\ln|x|}e^{C} \\ y &= \pm e^{C}\frac{1}{x} \end{aligned}$$

Thus, if we let $k = \pm e^C$, then the solutions are of the form $y = \frac{k}{x}$.

To find orthogonal trajectories to (7.13), we take the negative reciprocal of the right side to obtain the *new* equation

$$\frac{dy}{dx} = \frac{x}{y}$$

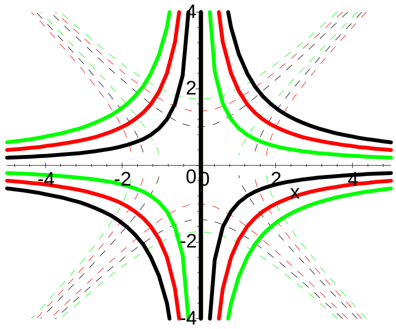
Separation of variables then leads to

$$\begin{aligned} ydy &= xdx \\ \int ydy &= \int xdx \\ \frac{1}{2}y^2 &= \frac{1}{2}x^2 + C \\ y^2 - x^2 &= 2C \end{aligned}$$

If we let $C_1 = 2C$, then we find the the orthogonal trajectories to the curves $y = \frac{k}{x}$ are the hyperbolas of the form

$$y^2 - x^2 = C_1$$

as is shown in the figure below.



1-3: Dashed curves are orthogonal to solid curves

Exercises:

Solve the following separable differential equations. If possible, solve for y so that the solution is defined explicitly as a function of the input variable. (k, μ, and σ are constants)

1.	$y' = y$	2.	$y' = 2y$	3.	$y' = -y$
4.	$y' = -3y$	5.	$y' = y + 2$	6.	$y' = 3 - 0.1y$

7. $\dfrac{dy}{dx} = 4x^3y^2$

8. $\dfrac{dy}{dx} = -x^3y^2$

9. $\dfrac{dy}{dx} = e^{x+y}$

10. $\dfrac{dy}{dx} = e^{x-y}$

11. $\dfrac{dy}{dx} = k\,\dfrac{x}{y}$

12. $\dfrac{dy}{dx} = k\,\dfrac{y}{x}$

13. $\dfrac{dy}{dx} = \dfrac{y^2-1}{xy}$

14. $\dfrac{dy}{dx} = \dfrac{x^3y}{x^4+1}$

15. $\dfrac{dy}{dx} = kx\,(y-1)$

16. $y' = \dfrac{k}{y-1}$

17. $y' = \dfrac{-1}{\sigma}xy$

18. $y' = \dfrac{-1}{\sigma}\,(x-\mu)\,y$

19. $\sin(y)\,\dfrac{dy}{dx} = x$

20. $\sec^2(y)\,y' = 1$

21. $\dfrac{dy}{dx} = \sec(y)\cos(x)$

22. $\dfrac{dy}{dx} = \csc(y)\sin(x)$

23. $\dfrac{dy}{dx} = \cos(x)\cos^2(y)$

24. $\dfrac{dy}{dx} = \sin(x)\sin^2(y)$

25. $\dfrac{dy}{dx} = \cos(y)\cot(y)$

26. $\dfrac{dy}{dx} = \sin(y)\tan(y)$

27. $y' = y+1, \quad y(0) = 1$

28. $y' + 2y = 0, \quad y(0) = 3$

29. $\dfrac{1}{y}\dfrac{dy}{dx} = -2x, \quad y(0) = 1$

30. $\dfrac{dy}{dx} = \dfrac{-x}{y}, \quad y(0) = 1$

31. $y' = \dfrac{\cos(x)}{y}, \quad y(0) = 0$

32. $y' = y\sin(x), \quad y(0) = 1$

33. In this exercise, we consider an object subject only to the force of friction, which leads to the acceleration of the object being proportional to its velocity.

(a) Translate this law into a separable differential equation. Use "$-k$" as the constant of proportionality.

(b) Solve the separable differential equation in (a).

34. In some models, the force of friction leads to the acceleration being proportional to the **square** of the velocity.

(a) Translate this law into a separable differential equation. Use "$-k$" as the constant of proportionality.

(b) Solve the separable differential equation in (a).

35. Newton's law of cooling says that the rate at which an object's temperature is changing is proportional to the difference between the temperature of the surrounding environment, denoted by E, and the temperature of the object.

(a) Translate this law into a separable differential equation.

(b) Solve the separable differential equation in (a).

36. Suppose $P(t)$ is the level of performance at time t of an individual learning a certain skill. If the maximum level of performance in that skill is M, then the rate of change in performance is proportional to the difference between M and the current level of performance.

(a) Translate this law into a separable differential equation.
(b) Solve the separable differential equation in (a).

37. Show that the integral curve solutions to the separable differential equation

$$\frac{dy}{dx} = \frac{-x}{y}$$

form a family of circles centered at the origin. What are the orthogonal trajectories to these solutions?

38. What family of curves is formed by the solutions to

$$\frac{dy}{dx} = \frac{2y}{x}$$

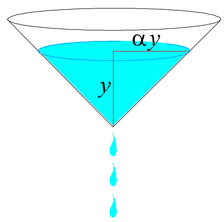
What are the orthogonal trajectories to the solutions?

39. **Torricelli's law** says that if water is draining from a tank, the rate of change of the volume with respect to time is proportional to the square root of the depth of the water. That is, if V is the volume of the water and $y(t)$ is the height of the water at time t, then

$$\frac{dV}{dt} = k\sqrt{y}$$

Use Torricelli's law to derive the differential equation in example 5 for water dripping from a right circular cylinder. (Hint: the volume of a right circular cylinder is $V = \pi r^2 y$, where r is the radius of the cylinder and y is its height).

40. When water is at a height y in a cone, the radius of the surface of the water is αy, where α is the slope of the side of the cone.



1-4: Leaking Cone

Let's suppose that water is draining from the bottom of a conical tank.

(a) Use Torricelli's law to derive the differential equation for the height y of the water at time t. (Hint: volume of a cone with a base of radius r and height h is $V = \frac{1}{3}\pi r^2 y$).
(b) Solve the differential equation in (a).

41. Let a, b be constants and let $y(t)$ be a solution to the *logistic equation*

$$y' = ay + by^2 \tag{7.14}$$

Show that if we let $z(t) = \frac{1}{y(t)}$, then

$$z' = az + b \tag{7.15}$$

What is the solution to (7.15)? What does this imply is the solution to (7.14)?

42. The following equation is not separable:

$$y\frac{dy}{dx} = x - \frac{y^2}{x}$$

Show that the substitution $u = xy$ transforms the equation above into

$$\frac{u}{x^2}\frac{du}{dx} = x$$

and then solve using separation. Replace u by xy and solve for y to find solutions to the original equation.

43. Separation does not necessarily produce all the solutions to a differential equation, as we will see by exploring

$$y' = 2\sqrt{y} \tag{7.16}$$

(a) Use separation of variables to solve (7.16).

(b) Show that $y = 0$ is also a solution to (7.16).

(c) Is there any value of the arbitrary constant in (a) that produces the solution in (b)? That is, is the solution in (b) a result of separation of variables?

44. Does separation of variables produce all possible solutions to the differential equation

$$y' = 3y^{1/3}$$

Explain. (Hint: Consider the steps in exercise 43)

45. Without air resistance, a raindrop which fell from a height of 3000 feet would reach the ground with a velocity of about 300 miles per hour. Ouch! Thus, it is safe to assume that air resistance plays a major role in the motion of a raindrop. In fact, the velocity v of a raindrop with a diameter of 0.003 inches (about the size of a raindrop in a light drizzle) satisfies the following differential equation

$$\begin{array}{ccccc} mass \times accel & = & \text{gravity} & - & \text{air resistance} \\ m\frac{dv}{dt} & = & 32m & - & mcv \end{array}$$

where m is the mass of the raindrop and c is the *coefficient of drag.*[1] (we are assuming that down is the positive direction so that positive velocities correspond to rate of descent).

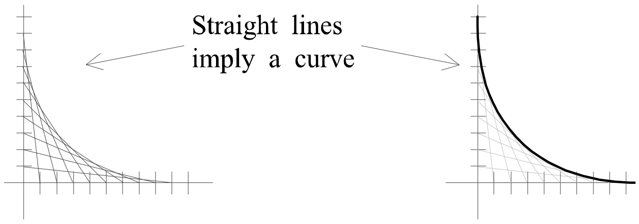
(a) Eliminate m and solve the resulting separable equation when $c = 52.64$ per second. Use an initial velocity of $v(0) = 32$ feet per second.

(b) Integrate $v(t)$ in (a) to obtain $r(t)$, the height of the object at time t. Assume that $r(0) = 3,000$ feet. How long until the raindrop reaches the ground? How fast is it traveling when it hits?

46. **Write to Learn:** "String art" often involves the use of straight lines to imply a curve. For example, a curve through the point $(0, 10)$ is implied by

[1] Adapted from "Raindrops," section 7.1 of *The Calculus Reader, Volume II* by David Smith and Lawrence C. Moore, CD Heath & Co, 1994.

the line segments that connect the points $(0, 10-a)$ on the y-axis to the points $(a, 0)$ on the x-axis, where $a = 1, 2, \ldots, 10$



1-5: String Art

In a short essay, explain why the equations of the lines are of the form

$$y = \left(1 - \frac{10}{a}\right) x + (10 - a)$$

and then show that this set of lines is a solution to the non-separable differential equation

$$xy' - y = \frac{10y'}{1 - y'}$$

Then apply implicit differentiation to obtain a new differential equation which is separable and whose solution is the curve implied by the straight lines. (Hint: y'' factors out of the equation after implicit differentiation).[2]

7.2 Growth and Decay

Percentage Rates of Change

In this section, we explore one of the most important separable differential equations, which is the differential equation of the form

$$y' = ky, \quad y(0) = P$$

In particular, we will see that this equation is instrumental in the study of growth and decay in many applications.

If $y(t)$ is a function of t, then the instaneous *percentage rate of change (%ROC)* of y is the ratio of the rate of change y' to the actual amount y at time t. That is,

$$\%ROC = \frac{\text{instantaneous rate of change of } y}{\text{current value of } y} = \frac{y'}{y}$$

Moreover, if the $\%ROC$ of y is equal to a constant k, then $\frac{y'}{y} = k$, which is the same as $y' = ky$. Thus, all functions with a constant rate of change k must be a solution to a differential equation of the form

$$y' = ky, \qquad y(0) = P \tag{7.17}$$

where P is a constant known as the *initial value* of $y(t)$.

[2] The non-separable differential equation in exercise 45 is a differential equation of the form $xy' - y = g(y')$, which is called a *Clairaut equation.*

Since (7.17) is the same as $\frac{dy}{dx} = ky$, separation of variables leads to

$$\frac{dy}{y} = kdt$$

Integration then leads to

$$\begin{aligned} \ln|y| &= kt + C, \\ |y| &= e^{kt}e^{C} \\ y &= \pm e^{C}e^{kt} \end{aligned}$$

We can then let $P = \pm e^{C}$ to obtain $y = Pe^{kt}$.

Let us now show that $y = Pe^{-kt}$ is the *only* solution to (7.17). If $u(t)$ is also a solution to (7.17) and we let $h(t) = e^{-kt}u(t)$, then

$$h'(t) = e^{-kt}u'(t) - ke^{-kt}u(t)$$

Since $u'(t) = ku(t)$ by assumption, this simplifies to

$$h'(t) = ke^{-kt}u'(t) - ke^{-kt}u(t) = 0$$

Thus, h is constant and since $h(0) = e^{0}u(0) = P$, it must follow that $h(t) = P$ for all t. Thus,

$$e^{-kt}u(t) = P \qquad \Longrightarrow \qquad u(t) = Pe^{kt}$$

That is, $y(t) = Pe^{kt}$ is the *only* solution to (7.17) for k and P fixed.

As a result, exponential functions are exactly those with a *constant percentage rate of change.* If $k > 0$, then y is said to be *growing exponentially* with respect to t, and if $k < 0$, then y is said to be *decaying exponentially*. Moreover, determining k is a key step in constructing models of exponential processes.

EXAMPLE 1 Solve the initial value problem

$$y' = ky, \quad y(0) = 5$$

given that $y(2) = 8$.

Solution: The solution to $y' = ky$ is of the form $y = Pe^{kt}$. Since P is the initial value and $y(0) = 5$, we must have $P = 5$ and consequently,

$$y = 5e^{kt}$$

The given value $y(2) = 8$ implies that $y = 8$ when $t = 2$. Thus,

$$8 = 5e^{2k}$$

Solving for k then yields

$$e^{2k} = \frac{8}{5}, \quad 2k = \ln\left(\frac{8}{5}\right), \quad k = \frac{1}{2}\ln\left(\frac{8}{5}\right) = 0.235$$

Thus, the solution to $y' = ky$, $y(0) = 5$ given that $y(2) = 8$ is

$$y = 5e^{0.235\,t}$$

We often assume that $P \neq 0$, since otherwise the process under consideration would not be changing. This allows us to estimate k even when the value of P is either impossible to estimate or irrelevant to the application.

EXAMPLE 2 What annual interest rate will cause an investment to double in value every five years assuming exponential growth?

Solution: If y denotes the investment's value at time t in years, then we are assuming that

$$y(t) = Pe^{kt} \tag{7.18}$$

for some values of P and k. To find k, we observe that an investment of P doubling in 5 years implies that $y(5) = 2P$. Substituting into (7.18) yields

$$2P = Pe^{k5} \tag{7.19}$$

Assuming $P \neq 0$ allows us to reduce (7.19) to $e^{5k} = 2$. Applying $\ln()$ then yields

$$\begin{aligned} \ln\left(e^{5k}\right) &= \ln(2) \\ 5k &= \ln(2) \end{aligned}$$

and as a result, we get

$$k = \frac{\ln(2)}{5} = 0.13862\,94 = 13.86\,\%$$

Check your Reading *Interpret the result in example 2. What does $k = 13.86\,\%$ represent?*

Using Data Sets to Estimate k

More often than not, an entire set of data is used to estimate k. To do so, however, we must transform the exponential model into a linear model. Applying the natural logarithm to $y = Pe^{kt}$ yields

$$\ln(y) = \ln\left(Pe^{kt}\right) = \ln(P) + \ln\left(e^{kt}\right) = \ln(P) + kt$$

so that letting $C = \ln(P)$ results in

$$\ln(y) = C + kt \tag{7.20}$$

An exponential model is transformed into a linear model with the natural logarithm.

The transformation $Y = \ln(y)$ then converts (7.20) the linear model

$$Y = C + k\,t$$

which can be fit to a data set using a least squares line.

EXAMPLE 3 World population figures since 1950 are reported below, where t is the year and y is the number of people in billions:

$t = year$	$y = population\ in\ billions$
1950	2.520
1960	3.020
1970	3.700
1980	4.450
1990	5.300

Use the data set to estimate the instantaneous percentage rate of growth of the world's population from 1950 to 1990.

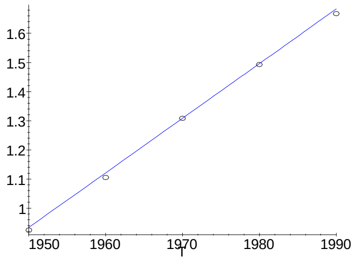
Solution: If we assume that $y(t) = Pe^{kt}$, then $Y = \ln(y)$ produces the new data set given in the last two columns below:

t	y	t	$Y = \ln(y)$
1950	2.520	1950	0.924
1960	3.020	1960	1.105
1970	3.700	1970	1.308
1980	4.450	1980	1.493
1990	5.300	1990	1.668

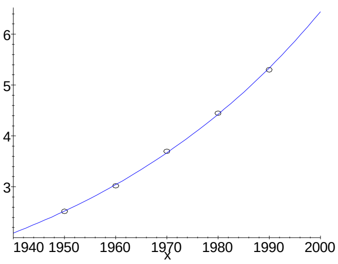
Applying the least squares algorithm to the last two columns yields the least squares line

$$Y = -35.629 + 0.01875t \tag{7.21}$$

from which we see that $k = 0.01875$. Moreover, the correlation coefficient is $r = 0.9997$. Thus, assuming exponential growth, we can safely conclude that the world's population has grown with an instantaneous rate of $k = 1.875\%$ per year since 1950.



8-1: $Y = -35.629 + 0.01875t$



8-2: $y = e^{-35.629+0.01875t}$

EXAMPLE 4 Use the model in example 3 to predict the world's population for the year 2010.

Solution: To do so, we replace Y by $\ln(y)$ in (7.21) to obtain

$$\ln(y) = -35.629 + 0.01875t \tag{7.22}$$

We now let $t = 2010$ and solve for y :

$$\begin{aligned} \ln(y) &= -35.629 + 0.01875(2010) \\ \ln(y) &= 2.0585 \\ y &= e^{2.0585} = 7.834209677 91 \end{aligned}$$

There will be a little more than 7.8 billion people on earth in the year 2010, according to this model.

Check your Reading *What will the world population be in 2020, according to (7.22)?*

Exponential Decay

Processes in which a quantity decays occur frequently in applications. For example, the force of friction for some (but not all) moving objects is proportional to the product of the mass m and the velocity v of the object. That is,

$$Force = -cmv$$

where $c > 0$. Since the acceleration a is the derivative of v, the motion of some moving objects is modeled by

$$mv' = -cmv, \qquad \text{or} \qquad v' = -cv \tag{7.23}$$

That is, the force of friction causes the velocity of some objects to decay exponentially.

EXAMPLE 5 An automobile slams on its brakes while it is going 60 m.p.h.. After one second, it has slowed to 40 m.p.h. Determine the amount of time until it has slowed to 5 m.p.h. if its velocity, $v(t)$, satisfies (7.23).

Solution: The equation $v' = -cv$ has only one solution, namely

$$v(t) = v_0 e^{-ct}$$

where v_0 is the initial velocity of the object. If we let $t = 0$ at the moment the brakes are engaged, then $v_0 = 60$ and

$$v(t) = 60e^{-ct}$$

To determine c, we observe that when $t = 1$ second, then $v(1) = 40$ m.p.h.. As a result,

$$\begin{aligned} v(1) &= 60e^{-c(1)} \\ 40 &= 60e^{-c} \end{aligned}$$

and solving for c in the usual way yields $c = 0.4054651$. Thus, $v(t) = 50e^{-0.4054651t}$ and we must find t such that $v(t) = 5$. That is, we must solve for t in the following:

$$5 = 60e^{-ct}$$

To do so, we divide by 60 and apply the natural logarithm:

$$e^{-ct} = \frac{5}{60}, \quad -ct = \ln\left(\frac{5}{60}\right) = -2.4849$$

Dividing by $-c = -0.4054651$ then yields

$$t = \frac{-2.4849}{-c} = \frac{2.4849}{0.4054651} = 6.1285 \text{ sec}$$

When a plant or animal dies, the carbon-14 in its body begins to decay to carbon-12. It has been determined experimentally that the *half-life* of carbon-14 is 5730 years. That is, if P is the initial amount of carbon-14, then after 5730 years the amount remaining will be $\frac{1}{2}P$. If we assume that the ratio of carbon-14 to carbon-12 in living tissue has not changed over the past 20,000 years, then the age of a sample will follow from the amount of carbon-14 it contains.

EXAMPLE 6 A certain plant sample contains only 65% as much carbon-14 as does a living plant today. How old is it?

Solution: We let y be the amount of carbon-14 in the sample after t years, and let

$$y(t) = Pe^{kt}$$

Since the half-life of carbon-14 is 5730 years, we can assume that $y(5730) = P/2$, so that

$$Pe^{k5730} = \frac{P}{2}$$

In an exponential decay, the percentage rate of change is negative.

The initial amount, P, is not zero and thus $e^{k5730} = \frac{1}{2}$, which leads to

$$k = \frac{1}{5730} \ln\left(\frac{1}{2}\right) = -1.209681 \times 10^{-4}$$

Our goal is to find t such that $y(t) = 0.65P$. Since $y(t) = Pe^{kt}$, we have

$$Pe^{kt} = 0.65P$$

Again, P cancels, and we are left to solve for t (where $k = -1.209681 \times 10^{-4}$):

$$\begin{aligned} e^{kt} &= 0.65 \\ kt &= \ln(0.65) \\ t &= \frac{1}{k}\ln(0.65) = \frac{\ln(0.65)}{-1.209681 \times 10^{-4}} = 3561.13 \textit{ years} \end{aligned}$$

Check your Reading *How old is a sample that contains 50% as much carbon-14 as does a living plant today?*

Modeling with Differential Equations

Differential equations of the form $y' = ky$ occur naturally in many different applications. When they do, we use the fact that the solution to $y' = ky$ must be of the form $y = Pe^{kt}$, where $P = y(0)$.

EXAMPLE 7 The isothermal compressibility κ_T of a substance with a constant temperature that has volume V at pressure p is given by

$$\kappa_T = -\frac{1}{V}\frac{dV}{dp}$$

At $20°C$, the isothermal compressibility of water is 49.6×10^{-6} per atmospheres. If 500 ml of water has a pressure of 1 atmosphere, then what is the volume of the water when the pressure is increased to 2 atmospheres?

Solution: Solving for $\frac{dV}{dp}$ leads to

$$\frac{dV}{dp} = -\kappa_T V$$

which has a solution of $V = Be^{-\kappa_T p}$, where B is a constant. When $p = 1$, then $V = 500$ ml. Thus,

$$500 = Be^{-\kappa_T} \qquad \Longrightarrow \qquad B = 500e^{\kappa_T} = 499.9752$$

As a result, when $p = 2$, then

$$V = 499.9752e^{-\kappa_T 2} = 499.9752e^{-\left(49.6\times10^{-6}\right)2} = 499.926\ ml$$

Thus, doubling the pressure does not cause a great deal of change in the volume of the water.

Let's conclude with an example involving the *concentration* y of a pollutant in a lake, where the concentration is the amount of the pollutant per unit volume. In particular, let's suppose that the pollutant is uniformly distributed in a lake[3] with a constant volume V, and that the source of the pollutant has been stopped. Then Vy is the total amount of pollutant in the lake, and if r denotes the rate at which polluted water is flowing out of the lake, then

$$\begin{aligned}
\text{Pollutant Outflow} &= \frac{\text{amount of pollutant leaving the lake}}{\text{units of time}} \\
&= \frac{\text{amount of pollutant}}{\text{units of volume}} \times \frac{\text{units of volume leaving lake}}{\text{units of time}} \\
&= \qquad y \qquad\qquad \times \qquad\qquad r
\end{aligned}$$

Since the inflow of pollution is zero, we have

$$\begin{array}{ccccc}
\text{Rate of Change of Pollution} & = & \text{Pollution Inflow} & - & \text{Pollution Outflow} \\
\frac{d}{dt}\left[Vy(t)\right] & = & 0 & - & ry
\end{array}$$

Since V is constant, this reduces to $Vy' = -ry$, which leads to

$$y' = -\frac{r}{V}\,y, \qquad y(0) = y_0 \tag{7.24}$$

where y_0 is the initial concentration.

EXAMPLE 8 Lake Erie has a volume of

$$V = 4.58 \times 10^{11} \text{ cubic meters}$$

and a mean outflow rate of about

$$r = 4.80 \times 10^{8} \text{ cubic meters per day}$$

[3] This is hardly a justifiable assumption, but it is one which will allow us to use exponential functions to obtain a good approximation.

How long until the amount of a certain pollutant in the lake decays to 5% of its current concentration?

Solution: To begin with, the solution to (7.24) is

$$y = y_0 \, e^{-kt} \tag{7.25}$$

where $k = r/V$. The percentage rate of change is given by

$$k = \frac{r}{V} = \frac{4.80 \times 10^8 \text{ cubic meters per day}}{4.58 \times 10^{11} \text{ cubic meters}} = 0.001048035 \text{ per day}$$

The current value is y_0, so that we must find t such that $y(t) = 0.05y_0$. Substituting into (7.25) yields

$$y_0 e^{-0.001048035t} = 0.05y_0$$

Canceling y_0 and applying $\ln(x)$ then yields

$$-0.001048035t = \ln(0.05)$$

which translates into about $t = 2858$ years. That is, it will take about 2,858 years until Lake Erie's outflow can reduce the amount of a given pollutant in the lake to 5% of its current concentration.

Exercises

Solve the initial value problems. State whether the equation is modeling exponential growth or decay.

1. $y' = 0.2y$, $y(0) = 3$
2. $y' = -y$, $y(0) = 5$
3. $y' = ky$, $y(0) = 3$, $y(2) = 7$
4. $y' = ky$, $y(0) = e$, $y(2) = 1$
5. $y' = ky$, $y(0) = 5$, $y(2) = 1$
6. $y' = ky$, $y(0) = 1$, $y(2) = e$

7. The doubling time of an exponentially-growing process is the time required to grow from an initial value of P to a value of $2P$. What is the doubling time of the process modeled by

$$y' = 5y, \quad y(0) = 2$$

8. The doubling time of an exponentially-growing process is the time required to grow from an initial value of P to a value of $2P$. What is the doubling time of the process modeled by $y' = 1.09y$?

9. The *half-life* of an exponentially-decaying process is the time required to decay from an initial value of P to a value of $\frac{1}{2}P$. What is the half-life of the process modeled by

$$y' = -3y, \quad y(0) = 7$$

10. The *half-life* of an exponentially-decaying process is the time required to decay from an initial value of P to a value of $\frac{1}{2}P$. What is the half-life of the process modeled by $y' = -1.1y$?

11. Use (7.22) to predict what the population will be in 2050.

12. Use (7.22) to predict when the world population will pass 15 billion.

13. Beautifa National Park once had wolves, but they were hunted out. In 1980, however, park rangers reintroduced wolves into the park by releasing four pairs of wolves. In 2000, park rangers numbered the wolf population at 92 wolves. Assuming exponential population growth, how many wolves will be in the park in 2010? In what year will the wolf population become too large for the park if it is estimated that the park can support no more than 1000 wolves?

14. A textbook that now costs \$90 only cost your parents \$30 when they were in school 25 years ago. Assuming exponential growth, what will that same textbook cost when your children are in college some 25 years from now?

15. If an investment doubles once every 4 years, then how long until it triples, assuming exponential growth?

16. **Write to Learn:** John invests \$1,000 in the stock market at the beginning of a certain year. At the end of that year, his investment is worth \$1,240. Assuming exponential growth, find the growth rate for John's investment. Write an essay explaining why the assumption of exponential growth is a good one and why the growth rate is a better measure of the growth of John's investment than is the simple interest rate of 24% per year.

17. A petri dish which initially contains 100 bacteria has 700 bacteria an hour later.

(a) Assuming the growth is exponential, find the model $y = Pe^{kt}$ for this colony of bacteria.

(b) How many bacteria were in the dish 15 minutes into the experiment?

(c) How long will it take the number of bacteria to double?

(d) How long will it take the number of bacteria to triple?

18. A cell of the bacterium Escherichia coli (E-coli) in a nutrient solution will divide every 15 minutes when the population is sufficiently small. If 10^8 bacteria are necessary for symptoms of the illness to appear and if there is an initial intake of 100 bacteria, then approximately how many hours will elapse before symptoms occur?

19. A rumor spreads exponentially through an arbitrarily large population. If 5 people know the rumor initially, and 25 people have heard the rumor by the end of the first day, then how many people will have heard the rumor by the end of the 7th day?

20. A sphere whose radius is originally R melts according to the law

$$\frac{dV}{dt} = -4\pi k r^3$$

where V is the volume of the sphere and $k = 2\ \mathrm{sec}^{-1}$. Show that $r = R\, e^{-3kt}$ and find the time necessary to reduce the radius to $R/2$. (Hint: $V = \frac{4}{3}\pi r^3$)

21. A certain plant sample contains only 35% as much carbon-14 as does a living plant today. Use carbon-14 dating to determine the age of the plant.

22. When only a small percentage of carbon-14 remains, its use in dating is unstable because small changes in the sample amount lead to large changes in the age estimate. This is illustrated in the problems below.

(a) Suppose that a certain sample contained only 10% as much carbon-14 as does a living plant today. How old is the sample?
(b) Suppose that a certain sample contained only 11% as much carbon-14 as does a living plant today. How old is the sample?
(c) Suppose that a certain sample contained only 12% as much carbon-14 as does a living plant today. How old is the sample?

23. How much carbon-14 will a 6,000 year old sample contain as a percentage of the amount of carbon-14 in living tissue?

24. In a laboratory, a scientist produces Pu-239 (Plutonium-239). To find the half-life the scientist measures the amount present at a particular time, 0.05 grams. One month (0.08333333 years) later she measures 0.04999988 grams. Find the half-life for Pu-239 (note. there will be an error of about 350 years due to lack of precision in the measurements —i.e., the lack of significant digits)

25. Suppose that your grade in this course is based on the time required to master a new topic. Specifically, suppose you begin with a grade of 100% each time a new topic is introduced, and then your grade is reduced by 0.5% per hour until you pass a "mastery exam" on that topic. Assuming exponential growth, how many hours would elapse before you had a grade of 90% for that topic? Before you had a grade of 80%? Before you had a grade of 70%?

26. What annual interest rate will lead to an investment's doubling in value every four years? Every three years? Every two years? Every year?

27. Suppose the amount of a certain drug in the bloodstream decays exponentially, and suppose the drug has a half-life of 4 hours and a minimum of 50 milligrams of the drug needs to be in the blood stream at all times. If an initial dose of 200 milligrams is ingested, when should the second dose be taken in order for the amount of the drug in the blood to remain above the minimum level? How long after the second dose of 200 milligrams should the third dose be ingested to maintain this minimum level?

28. Suppose the amount of a certain painkiller in the bloodstream decays exponentially. It is recommended not to exceed two 500mg tablets of a particular pain killer every six hours. If the minimum level to maintain in the blood in order to have relief is 150 mg and this is maintained by taking the maximum allowable dosage, then what is the half-life of this drug?

29. The United States population recorded in the censuses since 1950 is given in the table below.

t-year	1950	1960	1970	1980	1990
y-population (millions)	150.697	179.323	203.185	226.546	248.710

(a) Transform the data using $Y = \ln(y)$.
(b) Use least-squares to determine the exponential model.
(c) If the current trends continue, approximately what will the U.S. population be in the year 2000? in 2020?
(d) In what year will the population be 1 billion? Do you think this is a reasonable answer? Explain.

30. The yearly price of a certain textbook is given below.

$t =$ year	1989	1990	1991	1992	1993
$y =$ price	\$50.00	\$52.50	\$55.00	\$58.00	\$61.00

Assume that the price of the book increases exponentially.

(a) Transform the data using $Y = \ln(y)$ and use a least squares line to determine k.

(b) Predict the price of the book in the year 2000. In the year 2010.

31. John invests \$1000 in the stock market. Over the course of several months, his investment grows steadily, as the data set below reveals:

$t =$ month	0	1	2	3	4
$y =$ amount	\$1000	\$1017.24	\$1035.70	\$1053.90	\$1072.50

Assume that the investment grows exponentially during those months.

(a) Transform the data using $Y = \ln(y)$ and use a least squares line to estimate k.

(b) Multiply k by 12 to get the annual growth rate. At about what annual percentage rate is John's investment growing?

(c) Assuming his investment continues to grow at the same rate, how much will John's investment be worth at the end of the year? At the end of five years?

32. From early February through the first part of May, 1999, the Dow Jones Industrial Average (DJIA) gained almost 2,000 points. Below is a selection of Dow Jones Industrial Averages for a few selected dates during that time, with y as the index value t days after February 1: [4]

	Feb. 1	Feb. 16	Mar. 1	Mar. 15	Apr. 1	Apr. 15	May 3
$t =$ day	0	15	28	42	58	72	88
$y =$ index	9,345.70	9,291.11	9,324.78	9,958.77	9,832.51	10,462.72	11,014.69

(a) Transform the data using $Y = \ln(y)$ and then graph the transformed data along with its least squares line. Does it appear that the DJIA grew exponentially during that time period?

(b) Multiply k in (a) by 365. Is this a good indicator of the annual percentage rate of growth of the DJIA during this time period, or is it too high or too low?

33. A biologist discovers a new bacteria and desires to know the average period of time between when a bacterium divides and when it divides again. A bacterial culture is mixed until the bacteria can be assumed to be uniformly distributed throughout the mixture, and then the mixture is distributed uniformly to five petri dishes. The five petri dishes are placed in an incubator, and once every hour, a single dish is removed and quickly frozen. The number of bacteria in each dish are counted and the following set of data is produced.

time	2:00	3:00	4:00	5:00	6:00
count (in thousands)	1235	1941	3050	4794	7534

How long does it take for the bacteria population to double? (the doubling time of the population is equivalent to the average period of time between when a bacteria divides and when it divides again.)

[4] From "Dow Jones Averages" at http://averages.dowjones.com/home.html.

34. Repeat exercise 33 when the data set is

time	2:00	3:00	4:00	5:00	6:00
count (in thousands)	1056	2013	3898	7992	16109

35. When N_2O_5 is dissolved in the inert solvent carbon tetrachloride, it decomposes according to the balance equation

$$2N_2O_5 \longrightarrow 4NO_2 + O_2$$

If y denotes the concentration in moles per liter of N_2O_5 at time t in seconds, then

$$\frac{dy}{dt} = -ky$$

where the rate constant is $k = 6.284 \times 10^{-4}$ sec^{-1} at $31.84°C$. How many *hours* does it take for a given amount of N_2O_5 to decompose into half of the original amount?[5]

36. Repeat exercise 35 using the fact that $k = -4.82 \times 10^{-3}$ sec^{-1} at $64°C$.

37. The adiabatic compressibility α of a substance with a constant pressure that has volume V at temperature T is given by

$$\alpha = -\frac{1}{V}\frac{dV}{dT}$$

The adiabatic compressibility of water is 2.1×10^{-4} per degrees Kelvin ($°K$). If 500 ml of water has a temperature of $300°K$, then what is the volume of the water when the temperature is increased to $310°\ K$?

38. Repeat example 7 using the fact that air is 5 orders of magnitude more compressible than water.

39. The air pressure p at an altitude of z feet above sea level satisfies the *hydrostatic equation*

$$\frac{dp}{dz} = -\delta g$$

where δ is the density of the air at height z and g is the acceleration due to gravity. The *Equation of State* for moist air is given by

$$p = \delta R_d T_v$$

where R_d is the gas constant for dry air and T_v is the *virtual temperature* at height z.[6]

(a) Assume that R_d and T_v are constant, and then show that the density of the air δ satisfies

$$\frac{d\delta}{dz} = -k\delta$$

where $k = g/\left(R_d T_v\right)$ is a constant.

[5] From "Chemical Sciences" at http://www.chem.ualberta.ca/~plambeck/che/p102/p02143.htm by James Plambeck.

[6] The virtual temperature T_v is a function of the actual temperature that incorporates the dewpoint of the atmosphere as well.

(b) Solve the equation in (a), transform the following data using $Y = \ln(y)$, use a least squares line to estimateboth k and P, and then solve for δ as a function of z.

$z =$ altitude in meters	0	500	1000	1500	2000	2500	3000
$\delta =$ air density in $\frac{kg}{m^3}$	1.23	1.17	1.11	1.06	1.01	0.957	0.910

40. Suppose we have n moles of an ideal gas whose molecules have an average mass of m. The ideal gas law says that

$$pV = nkT$$

where p is pressure, V is the volume of the gas, k is a constant, and T is the temperature of the gas. Use the hydrostatic equation $p'(z) = -\delta g$ in exercise 39 (where δ is the density of the gas and g is the acceleration due to gravity) and the ideal gas law to obtain the *barometric equation*

$$\frac{dp}{dz} = \frac{-mg}{kT}\,p$$

Assuming that m, g, k, and T are constant, what is the solution to the barometric equation? (Hint: $\delta = \frac{mass}{volume} = \frac{nm}{V}$)

41. The *rocket equation* says that if M is the mass of a rocket and v_e is the velocity of the expanding gasses, then

$$dv = -v_e \frac{dM}{M}$$

where $v(t)$ is the velocity of the rocket.

(a) Transform the equation into

$$\frac{dM}{dv} = \frac{-1}{v_e} M$$

and then find the mass of the rocket as a function of its velocity.

(b) Suppose a single-stage rocket which is initially at rest on the launch-pad attains a velocity of 4.79 miles per second at the point where all of its fuel has been consumed. If the mass of the rocket without any fuel is 1 ton, then how much fuel is consumed in reaching a velocity of 4.79 miles per second?

(c) Suppose the exhaust gas velocity is 1.7 feet per second and the mass of the rocket at a velocity of 4.79 miles per second is 1 ton. What was the mass of the rocket initially?

42. Write to Learn: An automobile's speedometer reading is the same as its odometer reading for all times between 3:00 p.m. and 3:05 p.m. on a certain day. Write a short essay in which you describe the odometer reading $y(t)$ as a function of t and determine the speed of the automobile at 3:05 given that the odometer read 60 miles at 3:00 p.m.

43. Write to Learn: Suppose that a car slams on its brakes while it is going 60 m.p.h., and suppose that it has slowed to 40 m.p.h. after 1 second. Assuming exponential decay of the cars velocity, answer the following:

(a) How long until the car slows to 1 m.p.h.?

(b) How long until the car slows to 0.1 m.p.h.?

(c) How long until the car slows to 0.01 m.p.h.?

(d) Do you believe these results? Write a short essay explaining why an exponential decay model predicts that the car will never come to a complete stop.

Remark 2 *The prediction of a never-stopping automobile in exercise 43 is known as a* mathematical artifact, *which is a mathematical result that does not correspond to a real world phenomenon.*

7.3 Partial Fractions

Integration of Rational Functions

Many separable differential equations require a special method of integration called *the method of partial fractions.* In this section, we introduce this method along with several of the separable differential equations that require its use.

To begin with, let's use algebra to develop a method for evaluating integrals of the form

$$\int \frac{(r\,x+s)\ \ dx}{(x-p)\,(x-q)}$$

To do so, we use the fact that there are numbers A and B such that

$$\frac{r\,x+s}{(x-p)\,(x-q)}=\frac{A}{x-p}+\frac{B}{x-q} \tag{7.26}$$

The expansion on the right of (7.26) is called the *partial fraction expansion* of the rational function on the left.

To determine the partial fraction expansion (7.26), we multiply both sides by the product $(x-p)\,(x-q)$, which results in

$$r\,x+s=A\,(x-q)+B\,(x-p)$$

Expanding the expression on the right yields

$$r\,x+s=(A+B)\,x-Aq-Bp$$

As a result, $A+B=r$ and $-Aq-Bp=s$, from which we solve for A and B.

EXAMPLE 1 Use a partial fraction expansion to evaluate

$$\int \frac{2x+1}{x^2+x-6}dx$$

Solution: To do so, we factor the denominator and write the integrand in the form (7.26).

$$\frac{2x+1}{(x-2)\,(x+3)}=\frac{A}{x-2}+\frac{B}{x+3}$$

Multiplying both sides by the product $(x-1)\,(x-2)$ yields

$$\begin{aligned} 2x+1 &= A\,(x+3)+B\,(x-2) \\ &= Ax+3A+Bx-2B \end{aligned}$$

Grouping terms then yields

$$2x+1=(A+B)\,x \ + \ 3A-2B$$

By inspection, we see that $A+B=2$ and $3A-2B=1$.

Since $A+B=2$ is the same as $A=2-B$, the second equation becomes

$$\begin{aligned} 3\,(2-B)-2B &= 1 \\ 6-3B-2B &= 1 \end{aligned}$$

Thus, $-5B=-5$ or $B=1$, so that $A+B=2$ becomes

$$A+1=2, \qquad A=1$$

and as a result, we have the partial fraction expansion

$$\frac{2x+1}{(x-2)\,(x+3)}=\frac{1}{x-2}+\frac{1}{x+3} \tag{7.27}$$

Finally, the partial fraction expansion is integrated to obtain

$$\begin{aligned} \int \frac{2x+1}{x^2+x-6}dx &= \int \frac{1}{x-2}dx+\int \frac{1}{x+3}dx \\ &= \ln|x-2|+\ln|x+3|+C \end{aligned}$$

A quadratic with complex roots is said to be irreducible over the reals.

If the denominator has a factor that is an *irreducible quadratic*—i.e., a quadratic with complex roots—then a linear term "$Bx+C$" is placed over that irreducible factor in the partial fraction expansion.

EXAMPLE 2 Evaluate the antiderivative

$$\int \frac{x^2+x+4}{x\,(x^2+4)}dx$$

Solution: The integrand can be written as

$$\frac{x^2+x+4}{x\,(x^2+4)}=\frac{A}{x}+\frac{Bx+C}{x^2+4}$$

Multiplying both sides by the product $x\left(x^2+4\right)$ yields

$$\begin{aligned} x^2+x+4 &= A\left(x^2+4\right)+(Bx+C)\,x \\ 1x^2+1x+4 &= (A+B)\,x^2+Cx+4A \end{aligned}$$

Thus, $A+B=1$, $C=1$, and $4A=4$. Consequently, $A=1$ and

$$1+B=1 \qquad \Longrightarrow \qquad B=0$$

so that the partial fraction expansion of the integrand is

$$\frac{x^2+x+4}{x\,(x^2+4)}=\frac{1}{x}+\frac{1}{x^2+4}$$

Substituting into the antiderivative then yields

$$\begin{aligned} \int \frac{x^2+x+4}{x\,(x^2+4)} &= \int \frac{1}{x}dx+\int \frac{1}{x^2+4}dx \\ &= \ln|x|+\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C \end{aligned}$$

Check your Reading *Where did the inverse tangent come from in example 2?.*

Repeated Roots and Computer Algebra Systems

It is also possible that repeated roots or powers of irreducible quadratics occur as factors in the denominator of a rational function. In such cases, each successive positive integer power appears as a term in the partial fraction expansion. For example, if $(x-r)^n$ is a factor of the denominator, then the first n terms of the partial fraction expansion is of the form

$$\frac{q(x)}{(x-r)^n p(x)} = \frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \ldots + \frac{A_n}{(x-r)^n} + \ldots$$

where $p(x)$ and $q(x)$ are polynomials.

EXAMPLE 3 Use a partial fraction expansion to evaluate

$$\int \frac{x+3}{(x-4)^2} dx \tag{7.28}$$

Solution: To do so, we place "A" over an $x-4$ factor and a "B" over $(x-4)^2$:

$$\frac{x+3}{(x-4)^2} = \frac{A}{x-4} + \frac{B}{(x-4)^2}$$

To solve for A and B, we multiply throughout by $(x-4)^2$:

$$x+3 = A(x-4) + B = Ax - 4A + B$$

Thus, $A=1$ and $B-4A=3$, so that $B-4=3$ and $B=7$. It follows that

$$\frac{x+3}{(x-4)^2} = \frac{1}{x-4} + \frac{7}{(x-4)^2}$$

and substituting into the integrand of (7.28) leads to

$$\begin{aligned} \int \frac{x+3}{(x-4)^2} dx &= \int \frac{1}{x-4} dx + \int \frac{7}{(x-4)^2} dx \\ &= \ln|x-4| - \frac{7}{x-4} + C \end{aligned}$$

The denominator of a rational function may possibly contain all 3 types of factors—linear factors, irreducible quadratics, and repeated roots. Fortunately, however, the algebra involved in computing coefficients of partial fractions is ideally suited to the computer. Thus, computer algebra systems are often used to compute partial fraction expansions.

EXAMPLE 4 Use a computer algebra system to aid in the calculation of

$$\int \frac{x^4 + 5x^2 - 2x + 2}{x(x^2+1)(x-1)^3} dx \tag{7.29}$$

Solution: We write the integrand as a partial fraction expansion

$$\frac{x^4+5x^2-2x+2}{x\left(x^2+1\right)\left(x-1\right)^3}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{D}{x-1}+\frac{E}{\left(x-1\right)^2}+\frac{F}{\left(x-1\right)^3}$$

To solve for A, B, C, D, E, and F, we use the computer algebra system *Maple* to obtain

$$\frac{x^4+5x^2-2x+2}{x\left(x^2+1\right)\left(x-1\right)^3} \quad \xrightarrow{\textit{Maple}} \quad -\frac{2}{x}+\frac{x}{x^2+1}+\frac{1}{x-1}+\frac{3}{\left(x-1\right)^3}$$

We then integrate term by term to evaluate (7.29):

$$\begin{aligned}\int\frac{x^2+x+2}{x\left(x-1\right)^3}dx &= -\int\frac{2}{x}dx+\int\frac{x}{x^2+1}dx+\int\frac{1}{x-1}dx+\int\frac{3}{\left(x-1\right)^3}dx \\ &= -2\ln x+\frac{1}{2}\ln\left(x^2+1\right)+\ln\left(x-1\right)-\frac{3}{2\left(x-1\right)^2}+C\end{aligned}$$

Check your Reading *Will your graphing calculator produce partial fraction expansions?*

The Logistic Equation

In 1838, the Belgian mathematician P. F. Verhulst realized that populations cannot become arbitrarily large, but instead grow toward a limiting value N known as the *carrying capacity* or *saturation level* of the population. He then reasoned that if the population y at time t is well below the carrying capacity N, then it grows exponentially with an *intrinsic growth rate* of k. As a result, he introduced the *logistic equation*

$$y'=ky\left(1-\frac{y}{N}\right) \tag{7.30}$$

as a new model of population growth. Notice that if y is very small with respect to N, then $y'\approx ky$.

EXAMPLE 5 Suppose that y denotes the number of trees per acre at time t on a certain tract of land, and suppose that y increases at a rate of 10% per year when the tree population is well below the carrying capacity of $N=1000$ trees per acre. Find and solve the logistic model of the tree growth on that land.

Solution: For y well below $N=1000$, the growth rate is $k=10\%$ per year. Thus, the equation (7.30) becomes

$$\frac{dy}{dt}=0.1y\left(1-\frac{y}{1000}\right) \tag{7.31}$$

To solve this differential equation, we separate variables to obtain

$$\frac{dy}{y\left(1-\frac{y}{1000}\right)}=0.1\,dt$$

Multiplying by $\frac{1000}{1000}$ on the left and then integrating both sides yields

$$\int\frac{1000}{y\left(1000-y\right)}dy=\int 0.1dt \tag{7.32}$$

To evaluate the antiderivative on the left, we must expand the integrand into its partial fraction expansion. Indeed,

$$\frac{1000}{y\,(1000-y)} = \frac{A}{y} + \frac{B}{1000-y} \tag{7.33}$$

from which it follows that $A = B = 1$. Thus, (7.32) becomes

$$\int \left(\frac{1}{y} + \frac{1}{1000-y} \right) dy = \int 0.1dt$$

from which we obtain the integral curve solution

$$\ln|y| - \ln|1000-y| = 0.1t + C$$

Check your Reading *Show that $A = B = 1$ in (7.33). (perhaps with your graphing calculator?)*

Chemical Kinetics

Balance equations in elementary chemistry are often of the form

$$\alpha A + \beta B \longrightarrow \gamma C \tag{7.34}$$

which says that α molecules of type A combine with β molecules of type B to produce γ molecules of type C. For example, the balance equation

$$2H_2 + 0_2 \longrightarrow 2H_20$$

says that 2 molecules of hydrogen combine with 1 molecule of oxygen to form 2 molecules of water.

However, balance equations only quantify the end result of a chemical reaction. They do not explain any of the dynamics of the reaction itself. Instead, *chemical kinetics* is the subfield of chemistry in which the actual changes in a chemical reaction are measured and modeled.

Suppose a container containing molecules of both A and B is heated up (which a chemist would describe as "moving the system away from thermodynamic equilibrium"), causing molecules of A and B combine to form molecules of C as described by (7.34). Let $[A]$ denote the concentration of A at time t—that is, the number of molecules of type A per unit volume at time t—, let $[B]$ denote the concentration of B at time t, and let y denote the concentration of C at time t.[7]

It can then be shown that chemical reactions of the form (7.34) are modeled by a differential equation of the form

$$\frac{dy}{dt} = k[A]^{\mu}[B]^{\nu} \tag{7.35}$$

where $\frac{dy}{dt}$ is the *reaction rate* of the reaction and k is the *reaction rate constant.* The numbers μ and ν are the *orders* of A and B in the reaction, which along with k must be estimated experimentally.

Now suppose that when the reaction begins, the concentration $[A]$ is equal to a and the concentration $[B]$ is equal to b. Since α molecules of A result in γ

[7] We use y instead of $[C]$ in keeping with the notational conventions used in this textbook. However, chemists use $[A], [B]$ and $[C]$ exclusively. Moreover, they measure the number of molecules in *moles,* where one mole is about 6.0221367×10^{23} molecules.

molecules of C, the number of A's that have reacted to form y number of C's at time t is $\alpha/\gamma y$, so that at time t we have

$$[A] = \text{initial number} - \text{number converted to } C = a - \frac{\alpha}{\gamma} y$$

Likewise, β molecules of B react to form γ molecules of C, so that

$$[B] = b - \frac{\beta}{\gamma} y$$

Substituting for $[A]$ and $[B]$ in (7.35) thus yields

$$\frac{dy}{dt} = k \left(a - \frac{\alpha}{\gamma} y \right)^{\mu} \left(b - \frac{\beta}{\gamma} y \right)^{\nu}, \qquad y(0) = 0 \tag{7.36}$$

where the initial condition $y(0) = 0$ is equivalent to assuming that there are no molecules of C initially.

The equation (7.36) is a separable differential equation. Solving it requires a partial fraction expansion. And it or something like it is used to describe every chemical reaction known to man—gasoline combustion, plastic production, ozone depletion, and even the process of hydrogen and oxygen combining to form water.

EXAMPLE 6 Let $y(t)$ be the concentration in moles per liter of water produced by the reaction

$$4HBr + O_2 \longrightarrow 2H_20 + 2Br_2$$

Experiments show that the reaction is first order in both HBr and O_2, so that (7.36) becomes

$$\frac{dy}{dt} = k(a - 2y)\left(b - \frac{1}{2} y \right)$$

Solve this equation when y_0 is the initial concentration and $a \neq 4b$.

Solution: Separation of variables leads to

$$\frac{dy}{(a - 2y)\left(b - \frac{1}{2}y\right)} = kdt$$

Multiplying by $\frac{2}{2}$ and integrating leads to

$$\int \frac{2}{(a - 2y)(2b - y)} dy = \int kdt$$

To evaluate the antiderivative on the left, we must expand the integrand into its partial fraction expansion. Indeed,

$$\frac{1}{(a - 2y)(2b - y)} = \frac{A}{(a - 2y)} + \frac{B}{(2b - y)}$$

from which it follows that

$$\frac{1}{(a - 2y)(2b - y)} = \frac{2}{(4b - a)(a - 2y)} - \frac{1}{(4b - a)(2b - y)}$$

Substituting and finishing the integration leads to

$$\begin{aligned} \int \left(\frac{2}{(4b-a)(a-2y)} - \frac{1}{(4b-a)(2b-y)} \right) dy &= \int k dt \\ \frac{1}{4b-a} \left(\ln(2b-y) - \ln(a-2y) \right) &= kt + C \end{aligned}$$

where C is a constant. Consequently, we obtain the integral curve

$$\ln(2b-y) - \ln(a-2y) = k(4b-a)t + C_1$$

where $C_1 = C(4b-a)$.

Exercises:

Evaluate the following antiderivatives using a partial fraction expansion.

1. $\int \frac{dx}{x^2-x}$ 2. $\int \frac{2dx}{x^2-2x}$ 3. $\int \frac{dx}{x^2-4}$

4. $\int \frac{dx}{x^2-9}$ 5. $\int \frac{3x^2+1}{x^3+x} dx$ 6. $\int \frac{x^2-1}{x^3+x} dx$

7. $\int \frac{2x+3}{x^2-5x+6} dx$ 8. $\int \frac{2x-3}{x^2-5x+6} dx$

9. $\int \frac{6x+5}{6x^2+7x+2} dx$ 10. $\int \frac{19x+14}{15x^2+23x+4} dx$

11. $\int \frac{dx}{x^4-1}$ 12. $\int \frac{xdx}{x^4-1}$

13. $\int \frac{dx}{x(x-1)^2}$ 14. $\int \frac{x+2}{x(x-3)^2} dx$

15. $\int \frac{x^2+2}{x^4+x^2} dx$ 16. $\int \frac{1}{x^6+x^4} dx$

17. $\int \frac{2dx}{x-x^{-1}}$ 18. $\int \frac{xdx}{(x-5)^{10}}$

Solve the following logistic equations.

19. $y' = y(1-y)$ 20. $y' = y(2-y)$

21. $y' = 2y(1-y)$ 22. $y' = 2y(10-y)$

23. $y' = 0.05y\left(1-\frac{y}{500}\right)$ 24. $y' = 0.5y\left(1-\frac{y}{100}\right)$

25. $y' = y\left(1-\frac{y}{N}\right)$ 26. $y' = ky(1-y)$

27. Suppose that at time $t = 0$ there are 4 people in a university of 10,000 who have been told to spread the word that the semester will end one day early, and suppose that one day later there are 50 people who have heard the news. How long will it take for half of the university to hear that the semester will end one day early assuming logistic spread of the information?

28. A virus has infected a city of 50,000 people who have no immunity to the disease. The virus was brought to the town by 100 people and it was found that 1,000 people were infected after 10 weeks. How long will it take for half of the population to become infected assuming logistic spread of the disease?

29. Let's suppose that $y(t)$ is the number of bacteria per unit volume at time t in a nutrient solution with a concentration of $c(t)$ units of nutrients per unit volume. Let's further suppose that $y(t)$ satisfies a differential equation of the form

$$\frac{dy}{dt} = k(c)\, y(t) \tag{7.37}$$

where $k(c)$ is the *reproduction rate* of the bacteria.

(a) Often $k(c)$ is assumed to be proportional to the amount of nutrient available. Use this idea to explain why (7.37) can be written in the form

$$\frac{dy}{dt} = \beta c\, y \tag{7.38}$$

for some constant β.

(b) Suppose that α units of nutrient are consumed in producing one unit of population increase. Use this to explain why

$$c = c_0 - \alpha y$$

at time t, where c_0 is the initial amount of nutrients in the solution.

(c) Substitute for c in (7.38), and then solve the resulting logistic equation when y_0 is the initial amount of bacteria in the solution. What is the limit of the solution as t approaches ∞? What does it represent?

30. **Write to Learn:** If $y(t)$ is the population of a certain renewable resource at time t that is being *harvested* at a constant rate $H > 0$, then $y(t)$ can often be considered to be the solution to the modified logistic equation

$$y' = ky\left(1 - \frac{y}{N}\right) - H \tag{7.39}$$

where $N > 0$ is the carrying capacity of the resource in the event of no harvesting and $k > 0$ is the intrinsic growth rate. In a short essay, explain how (7.39) would be solved using separation of variables and a partial fraction expansion if $4H < kN$.

31. At temperatures above $500°K$, the chemical reaction with balance law

$$NO_2 + CO \longrightarrow CO_2 + NO$$

satisfies the separable differential equation

$$\frac{dy}{dt} = k(a - y)(b - y)$$

where y is the amount of CO_2 at time t produced by the reaction, a is the initial amount of NO_2, b is the initial amount of CO, and k is a constant. Separate variables, integrate with partial fractions, and solve for y assuming that $a \neq b$.

32. What is the solution to the separable equation in exercise 31 when $a = b$?

33. If $y(t)$ denotes the amount of NO_2 at time t produced by the chemical reaction with balance law

$$2NO + O_2 \longrightarrow 2NO_2$$

then y satisfies the separable differential equation

$$\frac{dy}{dt} = k(a-y)^2\left(b - \frac{1}{2}y\right)$$

where k is constant, a is the amount of NO initially, and b is the amount of O_2 initially. Solve the separable equation for y as a function of t assuming that $a \neq 2b$.

34. Experimentally, it can be shown that the reaction rate of the chemical reaction

$$CO + NO_2 \longrightarrow CO_2 + NO$$

is first order in both $[CO]$ and $[NO_2]$. If there are 0.2 moles per liter of CO initially, if there are 0.1 moles per liter of NO_2 initially, and if the reaction rate constant is $k = 0.24$ moles per liter per minute, then what will the concentration of CO_2 in moles per liter be after 5 minutes? after 10 minutes?

35. Write to Learn: Chemists often assume that the concentration of one of the chemicals in a *second order* reaction is so high that it is practically constant throughout the reaction. For example, if $\alpha = 1$ and $\nu = 1$, they might assume that $\beta = k[B]$ is constant, so that (7.36) becomes

$$\frac{dy}{dt} = \beta(a-y) \tag{7.40}$$

The resulting reaction and the equation (7.40) are both called *pseudo-first order chemical reactions.* Write a short essay describing the solution to (7.40) and what it represents. What is the limit as t approaches ∞ of $y(t)$? Where does β occur in the solution of (7.40) and what is significant about where it occurs?

36. Try it Out! With the help of a chemist, set up a second order chemical reaction, collect data in the form of the concentration y of the compound C at time t, and then fit (7.36) to the data to predict k. Each member of the group should present a different stage of the process.

37. We can also use limits to find coefficients of partial fraction expansions. Suppose to begin with that

$$f(x) = \frac{A}{x-r} + g(x)$$

where $g(x)$ is differentiable at $x = r$.

(a) Show that

$$A = (x-r)\,g(x) + (x-r)\,f(x)$$

(b) Evaluate the limit as x approaches r to obtain

$$A = \lim_{x \to r}(x-r)\,f(x)$$

(c) Explain why this method does not work with repeated roots.

38. Let's use exercise 37 to determine constants in the partial fraction expansion

$$\frac{1}{x^2-9} = \frac{A}{x-3} + \frac{B}{x+3}$$

(a) Use L'hopital's rule to evaluate

$$A = \lim_{x\to 3}(x-3)\frac{1}{x^2-9}$$

(b) Use L'hopital's rule to evaluate

$$B = \lim_{x\to -3}(x+3)\frac{1}{x^2-9}$$

(c) Check your work by finding A and B using the method from the section above.

39. Substitution versus Partial Fractions: In this exercise, we see that partial fractions is not necessary for all rational function integrands by considering

$$\int \frac{2x}{x^2-9}dx \tag{7.41}$$

(a) Evaluate (7.41) using the substitution $u = x^2 - 9$, $du = 2xdx$.

(b) Evaluate (7.41) by first finding a partial fraction expansion of the integrand.

(c) Use properties of the logarithm to show that the result in (a) is the same as the result in (b). Which method is easier?

40. Substitution versus Partial Fractions: In this exercise, we see that partial fractions is not necessary for all rational function integrands by considering

$$\int \frac{2x-1}{x^2-x}dx \tag{7.42}$$

(a) Evaluate (7.42) using a substitution.

(b) Evaluate (7.42) by first finding a partial fraction expansion of the integrand.

(c) Use properties of the logarithm to show that the result in (a) is the same as the result in (b). Which method is easier?

41. *If $y(t)$ denotes the amount of H_2 at time t consumed by the chemical reaction $H_2 + Br_2 \to 2HBR$, then y satisfies the rather unusual separable differential equation

$$\frac{dy}{dt} = -k(a-y)(b-y)^{1/2}$$

where a is the initial amount of H_2, $b > a$ is the initial amount of Br_2, and k is a positive constant.

(a) Let $y(t) = b - z(t)^2$ (which is the same as $z(t) = (b - y(t))^{1/2}$) and simplify assuming that $0 < z(t) < \sqrt{b}$ at all times t.

(b) Solve the separable differential equation for $z(t)$ produced in (a).

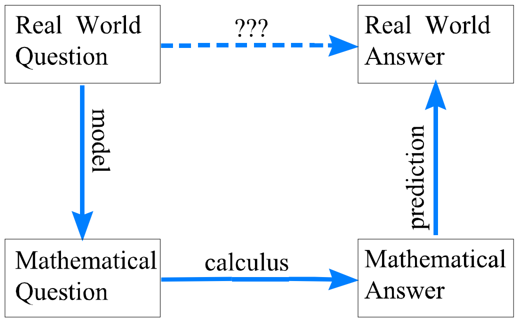
(c) Replace $z(t)$ using $z(t) = (b - y(t))^{1/2}$ and solve for y to obtain the amount of H_2 as a function of time t.

7.4 Mathematical Modeling

Predictions via Mathematical Models

There are many real world questions which cannot be answered with common sense or simple observation. Our five senses cannot tell us if it will rain tomorrow. Simple logic will not tell us if the economy is going to slow down or speed up.

However, a real world question can be translated into a mathematical question by constructing a *mathematical model* of the real world process. *Calculus* can then be used to answer the mathematical question, and the resulting mathematical answer can be used to transform a set of data into a *prediction* of the real world answer.



In this section, *mathematical models* are separable equations of the form

$$y' = k\frac{g(x)}{f(y)}$$

and *calculus* is used to produce an integral curve of the separable equation,

$$F(y) = C + kG(x) \tag{7.43}$$

where k and C are constant. *Prediction* then follows by fitting (7.43) to a set of data with the least squares method.

In particular, $Y = F(y)$ and $X = G(x)$ transforms (7.43) into a linear model

$$Y = C + kX$$

Consequently, if the process being modeled produces a data set of the form

$$(x_1, y_1), \ldots, (x_n, y_n)$$

then letting $X = G(x)$ and $Y = F(y)$ produces a new data set which is suitable for approximation by a least squares line, which in turn can be used to predict values of y given values of x.

EXAMPLE 1 Water in a soup can with a small hole in the bottom has a height y at time t in minutes. Use Toricelli's law (example 5 in section 7-1) and the data set

t minutes	0	0.5	1	1.5	2
y inches	3.375	2.813	2.313	1.875	1.5

to predict how long it will take for the can to drain.

Solution: Toricelli's law says that the rate of change $\frac{dy}{dt}$ is proportional to $\sqrt{y}$, which means that

$$\frac{dy}{dt} = k\sqrt{y}$$

We separate to obtain $y^{-1/2}dy = kdt$ and then we integrate:

$$\begin{aligned} \int y^{-1/2}dy &= k\int dt \\ \frac{y^{1/2}}{1/2} &= kt + C \end{aligned}$$

The result is simplified to the integral curve

$$2\sqrt{y} = kt + C$$

If we now let $Y = 2\sqrt{y}$ and $T = t$, then we obtain the linear model

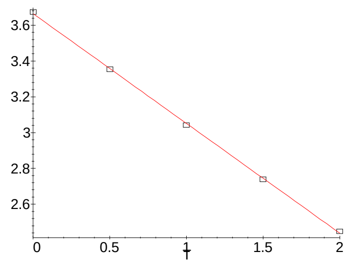
$$Y = kT + C$$

and we transform the original data set into a new T, Y data set:

T	0	0.5	1	1.5	2
Y	3.674	3.354	3.042	2.739	2.449

A computer algebra system is used to determine the least squares line, which is shown in red below along with the (T, Y) data set.

$$Y = -0.613\,T + 3.6646$$



4-2: Least Squares line

Consequently, the data is used to obtain the model

$$2\sqrt{y} = -0.613\,t + 3.6646 \tag{7.44}$$

Moreover, the can is empty when $y = 0$, which leads to

$$-0.613\,t + 3.6646 = 0$$

and which has a solution of $t = 5.978$ minutes.

Check your Reading *Use (7.44) to predict the amount of water in the can after 3 minutes.*

Newton's Law of Cooling

E denotes the temperature of the environment surrounding an object

Newton's law of cooling says that the rate at which an object's temperature is changing is proportional to the difference between the temperature of the surrounding environment, denoted by E, and the temperature of the object. That is, if $y(t)$ is the temperature of the object at time t, then

$$\begin{array}{ccc} \text{rate of change} & \text{is prop to} & \text{difference between } E \text{ and } y \\ \dfrac{dy}{dt} & = \quad k & (E-y) \end{array}$$

Moreover, E is sometimes called the *ambient temperature* of the object.

EXAMPLE 2 A pan of water is heated to $188°F$ and then left to cool in a room which is at a constant temperature of $79°F$. Temperature measurements are made each minute to produce the following set of data:[8]

time in minutes	0	1	2	3	4	5	7	10
Temperature in $°F$	188	184	180	177	174	171	166	159

Let's use Newton's law of cooling to predict the water's temperature after 15 minutes.

Solution: If $y(t)$ is the temperature of the water at time t, then $E = 79°F$ so that

$$\frac{dy}{dt} = k(79-y) \tag{7.45}$$

Separating the y's from the t's leads to

$$\frac{dy}{79-y} = kdt$$

so that integration yields

$$\begin{aligned} \int \frac{dy}{79-y} &= k\int dt \\ -\ln|79-y| &= kt + C \end{aligned} \tag{7.46}$$

If we let $Y = -\ln|79-y|$, then (7.46) is transformed into the linear model

$$Y = kt + C$$

We now use $Y = -\ln|79-y|$ to transform the original temperature data into

t	0	1	2	3	4	5	7	10
Y	-4.691	-4.654	-4.615	-4.585	-4.554	-4.522	-4.466	-4.382

Application of the least squares method yields the least squares line

$$Y = -4.681 + 0.0306\,t$$

Since $Y = -\ln|79-y|$, the integral curve of (7.45) is

$$-\ln|79-y| = -4.681 + 0.0306t \tag{7.47}$$

[8]Using the temperature probe for a TI-89 or even a simple candy thermometer.

Thus, when $t = 15$ minutes, we have

$$-\ln|79 - y| = -4.681 + 0.0306\,(15)$$

Solving for y leads to $y = 147.17°F$. That is, the model predicts that the water will have cooled to $147.17°F$ after fifteen minutes.

It is important to note that a solution to a differential equation model is itself a prediction. That is, modeling the output as a function of the input is equivalent to predicting an output for a given input.

The solution of the differential equation is a prediction of unobserved behaviour.

EXAMPLE 3 Find the function model of cooling in example 2

Solution: In (7.47), we have $-\ln|79 - y| = -4.681 + 0.0306t$, which leads to

$$\ln(y - 79) = 4.681 - 0.0306t$$

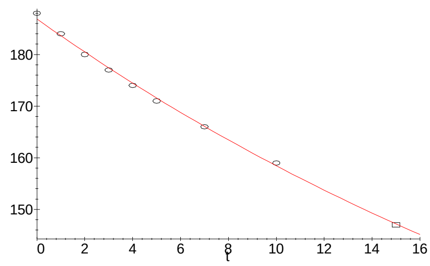
since y is always above room temperature. Application of the exponential then leads to

$$y - 79 = 107.878e^{-0.0306t}$$

Solving for y then yields

$$y(t) = 79 + 107.878\,e^{-0.0306t} \tag{7.48}$$

as is shown in the figure below:



4-3: Curve is a prediction implied by the data

Notice that we can use the curve to estimate that the temperature of the water after fifteen minutes will be about 147° (represented by the box in the graph above).

Check your Reading *Use (7.47) to predict the temperature of the water after $t = 20$ minutes.*

A Simple Model from Epidemiology

The logistic equation can also be used to model an epidemic in which a population with a constant size N is comprised of only two groups—those infected by the disease and those who are susceptible to the disease (i.e., no deaths, no immunity, no incubation, no latency, etc.). If we let y denote the number infected at time t, then $N - y$ represents the number of susceptibles. Moreover, the rate of change of y is proportional to both the number of infecteds and the number of susceptibles. That is,

$$y' = \alpha y (N - y) \tag{7.49}$$

which is a logistic equation with intrinsic growth rate $k = \alpha N$.

EXAMPLE 4 In January of 1978, exactly 763 boys returned to a boy's boarding school in Great Britain, and about a week later, one of the boys developed the flu. The next day, another boy developed the disease, and so on. The following table summarizes the data for the epidemic:[9]

t = time in days	0	1	2	3	4	5	6	7
y = number of boys infected	1	2	4	8	16	31	59	111

Find the intrinsic growth rate of the flu epidemic at this school.

Solution: To find the intrinsic growth rate k, we must solve (7.49) when $N = 763$, transform the data, and then use least squares to estimate α. To begin with, we separate variables to obtain

$$\int \frac{dy}{y(763 - y)} = \int \alpha dt \tag{7.50}$$

To evaluate the antiderivative on the left, we expand the integrand into a partial fraction, the result of which is

$$\frac{1}{y(763 - y)} = \frac{1}{763}\left(\frac{1}{y} - \frac{1}{763 - y}\right)$$

Thus, (7.50) is transformed into

$$\begin{aligned} \frac{1}{763}\int \left(\frac{1}{y} - \frac{1}{763 - y}\right) dy &= \int \alpha dt \\ \ln|y| - \ln|763 - y| &= 763\alpha t + C \end{aligned}$$

after multiplying both sides by 763. Finally, we let $k = 763\alpha$ to obtain

$$\ln\left|\frac{y}{763 - y}\right| = kt + C \tag{7.51}$$

It follows that the transformation

$$Y = \ln\left|\frac{y}{763 - y}\right|, \qquad T = t$$

transforms (7.51) into the linear model

$$Y = kT + C$$

Thus, we transform the data using T, Y to obtain the new data set

T	0	1.0	2.0	3.0	4.0	5.0	6.0	7.0
Y	−6.636	−5.941	−5.246	−4.520	−3.843	−3.162	−2.479	−1.770

[9] Data was originally reported in "Influenza in a Boarding School," *British Medical Journal* March 4, 1978.

The least squares algorithm then results in the fit

$$Y = 0.694\,T - 6.629$$

so that the integral curve solution is

$$\ln\left|\frac{y}{763-y}\right| = 0.694\,t - 6.629 \tag{7.52}$$

Thus, the intrinsic growth rate is $k = 69.4\%$ per day—i.e., the number of students who develop the disease increases by about 70% per day initially.

Often epidemiologists define the *duration* of an epidemic to be twice the time required for half the population to become infected. In example 4, half the total population is $y = 381.5$, so that (7.52) becomes

$$\ln\left|\frac{381.5}{763-381.5}\right| = 0.694\,t - 6.629$$

which has a solution of $t = 9.5$ days. Thus, the duration of the epidemic is 19 days.

Check your Reading *Why do we not define the duration as the amount of time between when the epidemic begins and when all 763 students have become infected?*

Terminal Velocity and the Skydiver

When a skydiver jumps from an airplane, her speed does not increase without bound until she opens her chute. Instead, her speed increases to a *terminal velocity.* This is because her downward velocity v satisfies *Stokes law*,

$$v' = g - kv$$

where g is the acceleration due to gravity and k is an unknown constant which must be determined using least squares.

EXAMPLE 5 A skydiver jumps from a plane flying at a constant altitude of 4,000 feet. She records the number of feet r that she has fallen after t seconds, resulting in the following data set:

t = time in seconds	1	2	3	4
r = feet fallen at time t	15	58	123	208

What is the sky-diver's terminal velocity before she pulls her chute?

Solution: We first solve Stoke's law with $g = 32\ \frac{ft}{\sec^2}$ by separating variables and integrating:

$$\int \frac{dv}{32-kv} = \int dt \qquad \Longrightarrow \qquad \ln|32-kv| = -kt + C$$

Since she was not moving vertically at time $t = 0$, we have $v(0) = 0$, which we substitute into our equation. As a result, $C = \ln(32)$ and solving for v leads to

$$\ln(32 - kv) = -kt + \ln(32) \quad \Longrightarrow \quad v(t) = \frac{32}{k}\left(1 - e^{-kt}\right)$$

We then integrate once more to obtain $r(t)$:

$$r(t) = \int \frac{32}{k}\left(1 - e^{-kt}\right) dt = C_1 + \frac{32}{k}t + \frac{32}{k^2}e^{-kt}$$

and since $r(0) = 0$ (no feet fallen initially), we obtain

$$r(t) = \frac{32}{k}t + \frac{32}{k^2}\left(e^{-kt} - 1\right)$$

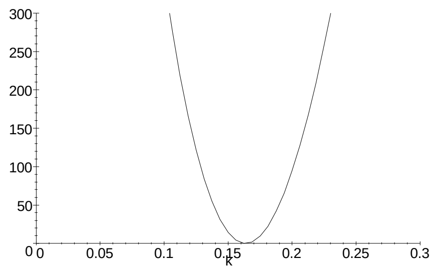
We cannot use a least squares line, but we can form a total squared error function and minimize it. To begin with, the total squared error function is of the form

$$E(k) = (r(t_1) - r_1)^2 + \ldots + (r(t_n) - r_n)^2$$

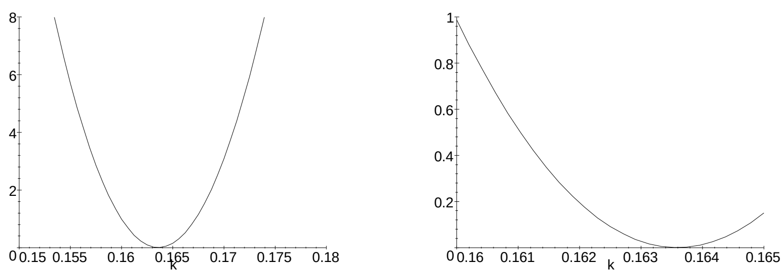
where r_n is the actual number of feet fallen at time t_n. This reduces to

$$\begin{aligned} E(k) &= \left(\frac{32}{k} + \frac{32}{k^2}\left(e^{-k} - 1\right) - 15.16\right)^2 + \left(\frac{32}{k} \cdot 2 + \frac{32}{k^2}\left(e^{-2k} - 1\right) - 57.55\right)^2 \\ &\quad + \left(\frac{32}{k} \cdot 3 + \frac{32}{k^2}\left(e^{-3k} - 1\right) - 123.07\right)^2 + \left(\frac{32}{k} \cdot 4 + \frac{32}{k^2}\left(e^{-4k} - 1\right) - 208.22\right)^2 \end{aligned}$$

and the graph of $E(k)$ is shown in the figure below:



The derivative of $E(k)$ is very involved, and solving $E'(k) = 0$ is impossible to do in closed form. Thus, let us zoom until we have an estimate of the value of k where the minimum occurs:



7-10: Graphs of $E(k)$

It appears that $E(k)$ is minimized when $k \approx 0.1635$. Substituting $k = 0.1635$ into the velocity $v(t)$ then yields

$$v(t) = \frac{32}{0.1635}\left(1 - e^{-0.1635t}\right)$$

The terminal velocity is then defined to be the limit as t approaches ∞ of the velocity function:

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{32}{0.1635}\left(1 - e^{-0.1635t}\right) = 195.718\ \frac{feet}{\text{sec}}$$

Thus, the terminal velocity is approximately 195.718 feet per second, which is approximately 133 miles per hour.

Exercises:

Find an integral curve of the given equation, transform the given data set and use a least squares line to estimate k and C. Then use the model to predict the output for the last input listed.

1. $y' = k$ Data:

t	0	1	2	3	4	5
y	1	2	3	4	5	?

2. $y' = k$ Data:

t	0	1	2	3	4	5
y	3	1	-1	-3	-5	?

3. $y' = ky$ Data:

x	0	1	2	3	4	5
y	1.00	1.10	1.21	1.33	1.46	???

4. $y' = ky$ Data:

t	0	1	2	3	4	5
y	1	2	4	8	16	??

5. $\frac{dy}{dt} = \frac{k}{y}$ Data:

t	0	1	2	3	4	5
y	2	2.449	2.828	3.162	3.464	???

6. $\frac{dy}{dx} = k\frac{x}{y}$ Data:

x	0	1	1.414	1.732	2	3
y	0.099	1.717	2.184	2.646	3.025	???

7. $\frac{dy}{dx} = kxy$ Data:

x	0	1	1.414	1.732	2	3
y	2.65	19.07	118.07	1099.16	9421.98	???

8. $\frac{dy}{dx} = k\frac{y}{x}$ Data:

t	0	1	2	3	4	5
y	0	1	4	9	16	???

9. $\frac{dy}{dt} = k(72 - y)$ Data:

t	0	5	10	15	20	25
y	32	66.445	71.103	71.904	71.989	???

10. $\dfrac{dy}{dx} = -k\,y^2$ Data:

x	1	2	3	4	5
y	1.25	−2.5	−1.67	−1	???

11. $y' = 2k\sqrt{y}$ Data:

t	1	2	3	4	5
y	0.04	0.16	0.36	0.64	???

12. $\dfrac{dy}{dt} = k\cos^2(y)$ Data:

t	0	1	2	3	4
y	1.263	1.379	1.432	1.462	???

13. A glass of water is heated to $180°F$ and then left to cool in a room with a constant temperature of $80°F$. Temperature measurements are made each minute resulting in the following data:

time in minutes	0	1	2	3	4	5	7	10
Temperature in $°F$	180	177	173	170	166	163	159	149

(a) Let $y(t)$ be the temperature of the glass of water at time t. Explain why the mathematical model of the glass of water's cooling is

$$\frac{dy}{dt} = k(80 - y)$$

(b) Find the integral curve of the equation in part (a).

(c) Transform the data using the integral curve in part (b), and then use least squares to estimate k and C.

(d) What will the temperature of the water be after 15 minutes?

14. A glass of water at $136°F$ is placed in a freezer which is at a constant temperature of $28°F$. Regular measurements of its temperature are made, resulting in the data set

time in minutes	0	1	2	3	4	5	7	10
Temperature in $°F$	136	134	130	126	123	120	115	104

(a) Let $y(t)$ be the temperature of the glass of water at time t. What is Newton's law of cooling in this situation?

(b) Find the integral curve of the equation in part (a).

(c) Transform the data using the integral curve in part (b), and then use least squares to estimate k and C.

(d) How long until the glass of water freezes?

15. The soup can on page **??** with the same hole is filled to a different height and again observed, resulting in the data

t minutes	0	0.5	1	1.5	2
y inches	3.063	2.5	2.125	1.75	1.375

How long until the soup can is empty?

16. The hole in the soup can in exercise 17 is enlarged, and then the soup can is again filled and observed, yielding the following data:

t minutes	0	0.25	0.5	0.75	1
y inches	2.688	2.063	1.625	1.25	0.875

How long until the soup can is empty this time?

17. A small hole is punched into the bottom of a large cylindrical bucket, and then the heights are measured every minute, resulting in the data

t minutes	0	1	2	5	10
y inches	6	5.683	5.375	4.5	3.375

How long until the bucket has completely drained?

18. Write to Learn: Try it out! Punch a small hole (but not too small) in the bottom of a soup can with radius r. Insert a ruler so it stands on the bottom of the soup can and fill it with water. Measure the height of the water at 30 second intervals, thus yielding

t minutes	0	0.5	1	1.5	2
y inches	#	#	#	#	#

Then fit the data to the equation

$$\frac{dy}{dt} = k\sqrt{y}$$

and predict the amount of time necessary for the can to completely empty. Report your results in a formal scientific paper.

19. The following data set describes a yeast population y at time t:

t = time in hours	0	10	18	23	34	45	47
y = yeast population	0.37	8.87	10.66	12.50	13.27	12.87	12.7

Assuming logistic growth and that the carrying capacity is $N = 13$, predict the yeast population at 20 hours?

20. Repeat exercise 19 when the carrying capacity is $N = 665$ and we have the data set

t = time in hours	0	1	2	3	4	5
y = yeast population	9.6	18.3	29.0	47.2	71.1	119.1

21. The table below lists the number of people infected per 10,000 population during a 1996 flu outbreak in the Alsace region of France.[10]

t = time in days	0	1	2	3	4	5	6	7
y = number infected per 10,000	2	4	10	22	47	99	196	351

Estimate the intrinsic growth rate k and the duration of the epidemic. (Hint: $N = 10,000$)

22. Write to Learn: Try it out! Present a test subject a list of 50 five digit numbers chosen at random. Let the subject observe the list of numbers for one minute, and then have them write down as many numbers as they can remember in the next minute. Repeat three more times, thus obtaining a set of data of the form

t minutes	0	2	4	6	8
y numbers learned	0	#	#	#	#

Fit the data to the *learning curve* model

$$\frac{dy}{dt} = k(50 - y)$$

Predict how long it will take the subject to learn 25 of the numbers, and then double it as an estimate of how long the subject would need to learn all 50 numbers. Report your results in a formal scientific paper.

[10] Adapted from data supplied by Le Réseau Sentinelles, copyright 1996.

25. Write to Learn: Try it out! Use a thermometer to determine the temperature of a room. Heat a cup of water in a microwave oven for 2.5 minutes. Remove immediately and insert a thermometer into the water. Measure the temperature at 30 second intervals beginning 1 minute after the water has been removed. The result will be a set of data of the form

t minutes	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y °F	#	#	#	#	#	#	#

Fit the data to Newton's law of cooling. Then use the integral curve solution to estimate the initial temperature of the water. Report your results in a formal scientific paper.

26. Write to Learn The Reynolds number of a flow sometimes requires that we model drag by

$$\frac{dv}{dt} = -kv$$

for k constant and sometimes requires that we model drag by

$$\frac{dv}{dt} = -cv^2$$

for c constant where $v(t)$ is the velocity of the flow at time t. Let's determine which type of flow produced this data set:

t	0	1	2	3	4	5	6	7
v	55	38	26	18	12	9	6	4

For each model, solve the differential equation, transform the data set, and fit it to a straight line. Then determine which model produces the best linear fit to the transformed data. Report your results in a short essay.

27. Write to Learn: The amount y in moles of Phenolphthalein in a certain chemical reaction is measured after t seconds, resulting in the data set

t = time in seconds	0	10.5	22.3	35.7	51.1	69.3	91.6
y = concentration in moles	0.0050	0.0045	0.0040	0.0035	0.003	0.0025	0.0020

By (7.34) on page 7.34 of the last section, we know that the form of the chemical reaction is

$$\frac{dy}{dt} = ky^{\mu}$$

where the order μ is either 0, 1, 2, or 3. Solve the differential equation and fit it to the data for each of these values of μ. Which value of μ appears to produce the best fit?

28. Write to Learn: Try it out! Air resistance also plays a role when a feather is falling to the earth, but is it proportional to the feather's velocity, the square of the feather's velocity, or neither? Use a motion detector to take several measurements of a feather's velocity v and acceleration v' as its falls to earth, thus resulting in a data set of the form

t in seconds	#	#	#	#	#
v in feet per second	#	#	#	#	#
v' in feet per second^2	#	#	#	#	#

If air resistance is proportional to velocity, then the velocity satisfies

$$v' = 32 - kv$$

Thus, for air resistance proportional to velocity, the (v, v') data should be well-fit with a linear function whose y-intercept is close to 32. If air resistance is proportional to the *square* of the velocity, then

$$v' = 32 - cv^2$$

If we let $V = v^2$, then this becomes

$$v' = 32 - cV$$

Thus, the (V, v') data should be well-fit with a linear function whose y-intercept is close to 32. Determine which of the data sets (v, v') or (V, v') best fits the balloon data. Report your results in a formal scientific paper.

29. A helium balloon with a volume of V and a mass density of ρ_m has an overall mass of $\rho_m V$ and displaces a air mass of $\rho_{air} V$, where ρ_{air} is the density of the surrounding air. Ignoring drag, the velocity v of a rising helium balloon satisfies

$$\begin{array}{lclcl} \text{mass} \times \text{accel} & = & \text{buoyancy} & - & \text{gravity} \\ \rho_m V \times v' & = & \rho_{air} V\, g & - & \rho_m V\, g \end{array}$$

where $g = 9.8$ m/sec^2 is the acceleration due to gravity. Division by $\rho_m V$ then leads to

$$\frac{dv}{dt} = g\left(\frac{\rho_{air}}{\rho_m} - 1\right) \tag{7.53}$$

where g, ρ_m, and ρ_{air} are assumed constant for small changes in altitude.

(a) Let $r(t)$ denote the altitude at time t of the balloon measured from sea level. Solve (7.53) to show that if a balloon is released from rest from ground level $(r = 0)$, then

$$r(t) = \frac{g}{2}\left(\frac{\rho_{air}}{\rho_m} - 1\right) t^2$$

(b) A simplified approximation[11] to air density as a function of altitude h is

$$\rho_{air}(h) = 1.23 e^{-h/10000}$$

A balloon is released from rest at ground level when $h = 1,000$ feet above sea level, thus yielding

$$\rho_{air} = 1.113\ kg/m^3$$

The height of the balloon is measured at different times t, thus yielding the data set

t = time in seconds	0	1	2	3	4
r = height in meters	0	0.44	1.78	4.01	7.11

Estimate the density of the balloon by finding the value of ρ_m that minimizes the total squared error function

$$\begin{aligned} E(\rho_m) &= \left(0.44 - 4.9\left(\frac{1.113}{\rho_m} - 1\right)1^2\right)^2 + \left(1.78 - 4.9\left(\frac{1.113}{\rho_m} - 1\right)2^2\right)^2 \\ &\quad + \left(4.01 - 4.9\left(\frac{1.113}{\rho_m} - 1\right)3^2\right)^2 + \left(7.11 - 4.9\left(\frac{1.113}{\rho_m} - 1\right)4^2\right)^2 \end{aligned}$$

[11] Air densities actually vary from day to day and hour to hour depending on factors such as temperature, humidity, barometric pressure, and windspeed.

(c) The balloon will rise to an equilibrium height h at which $\rho_m = \rho_{air}(h)$, where $\rho_{air}(h) = 1.23e^{-h/10000}$. What is the equilibrium height for the balloon that produced the data in (b)?

30. **Try it Out!** Obtain a helium-filled balloon and generate time-altitude data by releasing a balloon from rest at ground level and monitoring its position above the earth for the first few seconds.

31. An experienced skydiver increases her terminal velocity by keeping her arms close to her body and tilting her head downward—actions which minimize drag. She records distances fallen r at times t to obtain the following data set:

t = time in seconds	1	2	3	4	5	6	7
r = feet fallen at time t	15	58	127	217	326	453	594

What is the sky-diver's terminal velocity before she pulls her chute?

32. A skydiver leaps from a plane which has a constant altitude of 4000 meters and waits 10 seconds before opening his chute. His velocity v satisfies

$$\begin{array}{ccccc} \text{mass} \times \text{accel} & = & \text{gravity} & - & \text{drag} \\ mv' & = & 9.8m & - & mcv \end{array}$$

where c is the coefficient of drag. He records distances fallen r at times t since leaving the plane to obtain the following data set:

t = time in seconds	1	2	3	4
r = meters fallen at time t	4.7	18.1	39.2	67

What is his velocity when he pulls the chute at 10 seconds after leaving the plane?

33. *A skydiver leaps from an airplane at a constant altitude of 4000 meters and waits 10 seconds before opening his parachute. His velocity v after opening his parachute satisfies the differential equation

$$\begin{array}{ccccc} \text{mass} \times \text{accel} & = & \text{gravity} & - & \text{drag} \\ mv' & = & 9.8m & - & m20v^2 \end{array}$$

where m is his mass. Use exercise 31 to determine how far the skydiver falls in the first 10 seconds and to determine the initial velocity of the skydiver when he opens his chute. Then solve the differential equation for the velocity of the skydiver after he opens his chute, and integrate the result with respect to time. How long does it take the skydiver to reach the ground?

34. ***Write to Learn:** If a raindrop is very small and is approximately spherical with uniform density, then its velocity satisfies the equation

$$\frac{dv}{dt} = 32 - \frac{0.329 \times 10^{-5}}{D^2} v$$

If the velocity of the raindrop is approximately 4 feet per second when it strikes the earth, what is the approximate diameter of the raindrop?

35. * Here is a problem which surfaces from time to time in the popular media. Two men receive equal cups of coffee with equal temperatures at the same time. The first adds cream immediately, but does not drink until ten minutes later. The second waits ten minutes, adds cream, and then drinks. If the temperature of the cream is the same for both, which of the two men drinks the hotter coffee?

36. Group Learning: The ancients used waterclocks and hourglasses to measure time. Here is how they work.

- **Member 1:** Find the volume of the solid of revolution obtained by revolving the region under $y = cx^{1/4}$ and over $[0, h]$ around the x-axis, where c is a constant.
- **Member 2: Torricelli's law** says that if water is draining from a tank and if $h(t)$ is the height of the water at time t, then

$$\frac{dV}{dh}\frac{dh}{dt} = k\sqrt{h} \tag{7.54}$$

  where $V(h)$ is the volume of the water at a height of h. Show that if the tank is in the shape of the solid of revolution obtained by revolving the region under $y = cx^{1/4}$ and over $[0, h]$ around the x-axis, then $\frac{dV}{dh} = c^2\sqrt{h}$.
- **Member 3:** Use Toricelli's law (7.54) and the fact that $\frac{dV}{dh} = c^2\sqrt{h}$ to show that $\frac{dh}{dt}$ is constant.. Why does this imply that $\frac{dh}{dt}$ is a constant function?
- **Member 4:** Construct a tank in the shape of the solid of revolution obtained by revolving the region under $y = cx^{1/4}$ and over $[0, h]$ around the x-axis, punch a hole in its bottom (i.e., at $(0, 0)$), and then mark off equal heights to correspond to equal units of time.

7.5 Second Order Systems

Phase Space

Many mathematical models are constructed using a *system* of autonomous differential equations. In this section, we explore *second order systems*, which are of the form

$$\frac{dx}{dt} = f(x, y), \qquad \frac{dy}{dt} = g(x, y) \tag{7.55}$$

where $f(x, y)$ and $g(x, y)$ are expressions in x and y.

To begin with, an *equilibrium solution* of a system (7.55) is a pair of constant functions $x = A$ and $y = B$ that simultaneously satisfy (7.55). It follows that the equilibria of a system (7.55) are the solutions to the equations

$$f(x, y) = 0 \qquad \textit{and} \qquad g(x, y) = 0$$

Moreover, the chain rule implies that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

which in turn implies that $g(x, y) = \frac{dy}{dx} f(x, y)$. Solving for $\frac{dy}{dx}$ then leads to

$$\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)} \tag{7.56}$$

The set of all integral curves of (7.56) is called the *phase portrait* of the system (7.55). The phase portrait reveals the relationship between x and y in the system (7.55).

EXAMPLE 1 Find the equilibria and the phase portrait of the system

$$\frac{dx}{dt} = 2y, \qquad \frac{dy}{dt} = 1 - 2x \tag{7.57}$$

Solution: The equilibria are the simultaneous solutions to

$$2y = 0 \text{ and } 1 - 2x = 0$$

Thus, $\left(\frac{1}{2}, 0\right)$ is the only equilibrium.

The phase portrait is the set of integral curves of

$$\frac{dy}{dx} = \frac{1 - 2x}{2y} \tag{7.58}$$

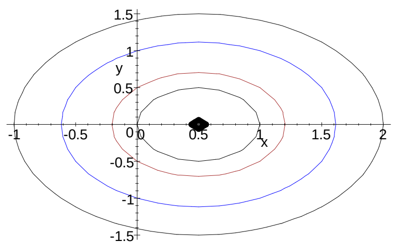
Fortunately, (7.58) is separable, and in fact, separation of variables yields

$$2ydy = (1 - 2x)\, dx$$

Integration leads to $\int 2ydy = \int (1 - 2x)\, dx$, which results in

$$y^2 = x - x^2 + C$$

Thus, the curves $y^2 = x^2 - x + C$ form the *phase portrait* of (7.57).



5-1: Phase Curves for $x' = 2y$, $y' = 1 - 2x$

The phase space curves are closed curves. This indicates that values of $x(t)$ and $y(t)$ repeat at regular intervals, which is to say that $x(t)$ and $y(t)$ are *periodic functions.* Thus, we can conclude that the solutions to (7.57) oscillate about the equilibrium point $(0.5, 1)$.

Check your Reading

Are there solutions to (7.57) in which $x(t)$ is always positive?

Second Order Autonomous Equations

Phase portraits can also be used to study *second order autonomous differential equations*, which are equations of the form

$$\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right) \tag{7.59}$$

where $f(x, x')$ is an expression in x and x'. To do so, we define the variable $y = \frac{dx}{dt}$, so that $\frac{d^2x}{dt^2} = \frac{dy}{dt}$ and (7.59) becomes the system

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = f(x, y)$$

The equilibria must satisfy $y = 0$, $f(x, y) = 0$, which implies that the equilibria are of the form $(p, 0)$ where $f(p, 0) = 0$.

EXAMPLE 2 Find the equilibria and phase portrait of

$$x'' = 1 - x^2$$

Solution: We let $y = x'(t)$, so that $x'' = y'$ implies the following system:

$$x' = y, \qquad y' = 1 - x^2$$

The equilibria are solutions to $y = 0$, $1 - x^2 = 0$, which are $(-1, 0)$ and $(1, 0)$.

The phase portrait is the set of integral curves satisfying

$$\frac{dy}{dx} = \frac{1 - x^2}{y}$$

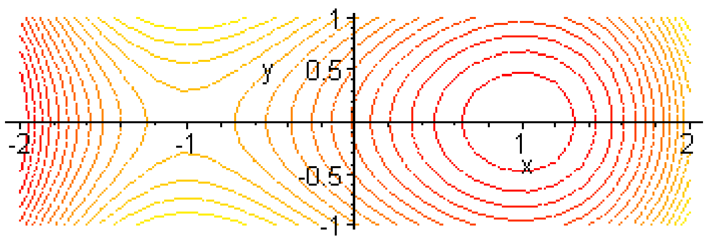
Separation and integration leads to

$$\int y dy = \int (1 - x^2) dx$$

Thus, the phase portrait is the set of curves of the form

$$\frac{y^2}{2} = x - \frac{x^3}{3} + C$$

where C is constant.



5-2: Phase portrait of $x' = y$, $y' = 1 - x^2$

Closed curves in the phase portrait in figure 5-2 implies that there are periodic solutions to $x'' = 1 - x^2$. Let's look at an example in which there are no periodic solutions.

EXAMPLE 3 For $\omega > 0$ constant, find the equilibria and phase portrait of

$$x'' - \omega^2 x = 0$$

Solution: We let $y = x'(t)$, so that $x'' = y'$ implies the following system:

$$x' = y, \qquad y' = \omega^2 x$$

The only equilibrium is $(0,0)$, and the phase portrait is formed by the solutions to

$$\frac{dy}{dx} = \frac{\omega^2 x}{y}$$

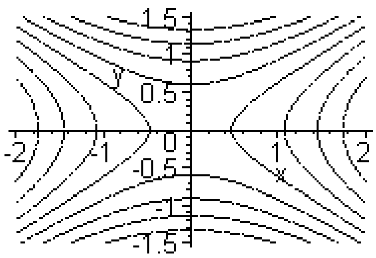
Separation leads to $ydy = \omega^2 x dx$, and integration leads to

$$\begin{aligned} \int ydy &= \omega^2 \int xdx \\ \frac{1}{2}y^2 &= \frac{1}{2}\omega^2 x^2 + H \end{aligned}$$

Multiplication by 2 leads to $y^2 = \omega^2 x^2 + H$ or

$$y^2 - \omega^2 x^2 = H$$

which is a family of *hyperbolas* centered at the origin.



5-3: Phase curves for $x' = y$, $y' = \omega^2 x$

In addition, a phase portrait can be used to find solutions to a second order differential equation. In particular, the substitution $y = x'$ often leads to a separable differential equation that can be solved in closed form.

EXAMPLE 4 Use the phase portrait in example 3 to find solutions to

$$x'' - \omega^2 x = 0$$

Solution: In example 3, we showed that the phase portrait of $x'' - \omega^2 x = 0$ is

$$y^2 - \omega^2 x^2 = H$$

As a result, $y = x'$ implies that $(x')^2 = \omega^2 x^2 + H$ or

$$\frac{dx}{dt} = \sqrt{\omega^2 x^2 + H}$$

Factoring out ω leads to

$$\frac{dx}{dt} = \omega \sqrt{x^2 + \frac{H}{\omega^2}}$$

Separation of variables leads to

$$\int \frac{dx}{\sqrt{x^2 + \frac{H}{\omega^2}}} = \int \omega dt$$

However, formula (6.34) with $a = \sqrt{H}/\omega$ on page 483 implies that

$$\sinh^{-1}\left(\frac{\omega x}{\sqrt{H}}\right) = \omega t + C$$

which yields a final solution of

$$x = \frac{\sqrt{H}}{\omega}\sinh(\omega t + C)$$

In fact, the identity for the hyperbolic sine of a sum implies that the *general solution* to

$$x'' - \omega^2 x = 0$$

is given by

$$x(t) = A\cosh(\omega t) + B\sinh(\omega t)$$

where A and B are constants.

Check your Reading *If* $x(t) = \cosh(t)$, *then what is* x''?

Applications in Biology and Physics

Phase portraits are used frequently in applications. Let's look an example from epidemiology.

EXAMPLE 5 In a population, let S denote the number of people susceptible to a given disease, let I denote the number infected, and let R denoted the number removed from the infecteds. Then the progression of the disease is in the form

$$S \quad \rightarrow \quad I \quad \rightarrow \quad R$$

A simple model of an SIR epidemic model is given by

$$\frac{dS}{dt} = -rSI, \qquad \frac{dI}{dt} = rSI - aI$$

where r is the rate of infection and a is the rate of removal of the infecteds. Find the phase portrait of this simple SIR model.

Solution: The phase portrait is formed by solutions to

$$\frac{dI}{dS} = \frac{rSI - aI}{-rSI} = \frac{a - rS}{rS}$$

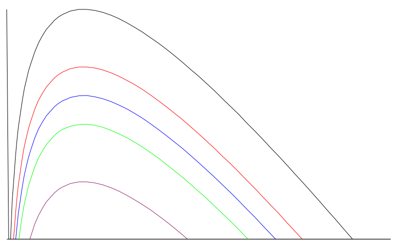
Separation of variables leads to

$$dI = \left(\frac{a}{r}\frac{1}{S} - 1\right)dS$$

and integration leads to

$$\begin{aligned}\int dI &= \int \left(\frac{a}{r}\frac{1}{S} - 1\right) dS \\ I &= \frac{a}{r}\ln|S| - S + C\end{aligned}$$

where C is a constant. The result is a family of curves that can be used to determine what the maximum number of infecteds is for a given epidemic.



5-4: Phase portrait for an SIR epidemic

Now let's look at an example from physics.

EXAMPLE 6 Spring-mass oscillations in the "real world" are not perfect harmonic oscillators, but are instead modeled by *Duffing's equation*

$$x'' + \omega^2 x + \varepsilon x^3 = 0$$

where ε is a measure of the "stiffness" of the spring. Find the phase portrait of Duffing's equation.

Solution: If $y = x'$, then $y' = -\omega^2 x - \varepsilon x^3$ and

$$\frac{dy}{dx} = \frac{-\omega^2 x - \varepsilon x^3}{y}$$

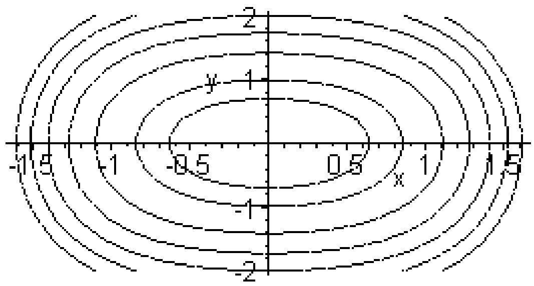
Separation and integration leads to

$$\begin{aligned}\int y dy &= \int \left(-\omega^2 x - \varepsilon x^3\right) dx \\ \frac{1}{2}y^2 &= -\omega^2\frac{x^2}{2} - \varepsilon\frac{x^4}{4} + H\end{aligned}$$

As a result, the phase portrait is given by

$$\frac{\varepsilon x^4}{2} + \omega^2 x^2 + y^2 = 2H$$

which for $\varepsilon > 0$ is a collection of closed curves about the origin.

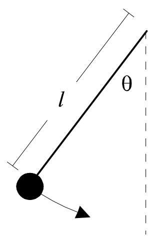


5-5: Phase portrait of Duffing's Equation

Check your Reading *What do the closed curves in the phase portrait of Duffing's equation imply about the solutions?*

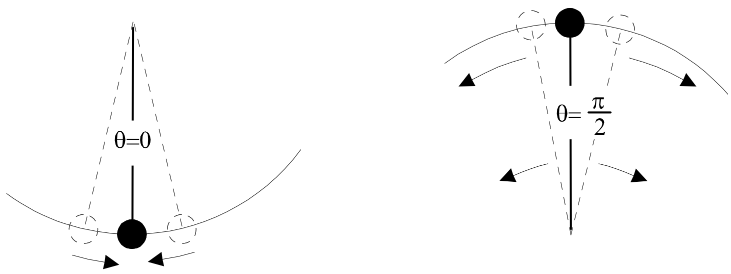
The Pendulum Equation

Let's conclude by considering a pendulum of length l that forms an angle θ at time t with a vertical line.



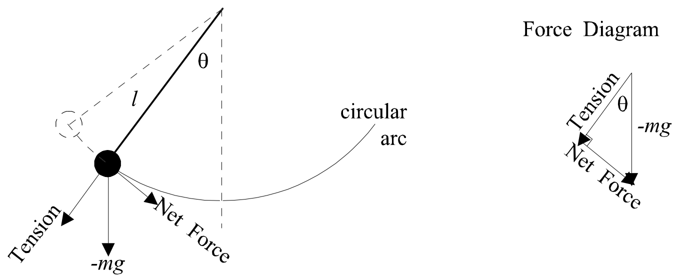
5-6: A pendulum of length l

The pendulum has an equilibrium at $\theta = 0$, which is stable because small variations lead to small oscillations about $\theta = 0$. In contrast, the equilibrium $\theta = \pi$ at the top of the pendulum's arc is *unstable* since a small deviation from $\theta = \pi$ will cause the pendulum to swing toward the bottom of its arc.



5-7: $\theta = 0$ is stable, $\theta = \frac{\pi}{2}$ is unstable

Let's look at this phenomenon using a phase portrait of the pendulum's angle θ at time t. To begin with, suppose a mass m is located at the end of a pendulum of length l subject to a constant gravitational acceleration $g > 0$.



5-8: Force of gravity in two components

The force of gravity, $-mg$, causes both a tension in the pendulum and motion tangential to the circular arc. The tension is balanced by the pendulum arm. Thus, the net force at angle θ that causes the motion of the pendulum is given by

$$Net\ Force = -mg\sin(\theta)$$

Since the length of an arc on the circle is $l\theta$, the velocity of the pendulum is $v = l\theta'$ and the acceleration is $a = l\theta''$. Newton's law, $F = ma$, thus implies the equation of motion is

$$ml\theta'' = -mg\sin(\theta)$$

which in turn simplifies to the *pendulum equation*

$$\frac{d^2\theta}{dt^2} = \frac{-g}{l}\sin(\theta) \qquad (7.60)$$

Let's find the equilibria and the phase portrait of the pendulum equation. To begin with, we let $y = \frac{d\theta}{dt}$, so that the resulting system is

$$\frac{d\theta}{dt} = y, \qquad \frac{dy}{dt} = \frac{-g}{l}\sin(\theta)$$

Since $\sin(\theta) = 0$ when $\theta = 0, \pm\pi, \pm 2\pi, \ldots$, the equilibria of the pendulum are of the form $(n\pi, 0)$ where n is an integer.

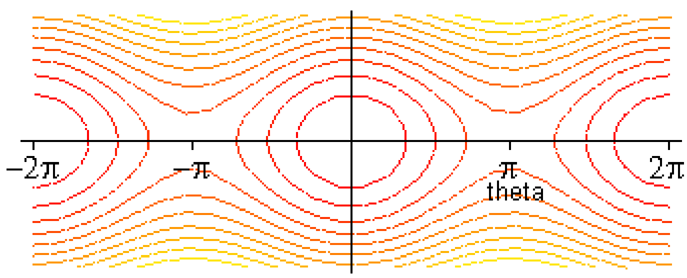
The phase portrait is the set of integral curves of

$$\frac{dy}{d\theta} = \frac{-g\sin(\theta)}{ly}$$

Separation leads to $ydy = \frac{-g}{l}\sin(\theta)$, so that integration leads to

$$\begin{aligned} \int ydy &= \frac{-g}{l}\int \sin(\theta)\, d\theta \\ \frac{y^2}{2} &= \frac{g}{l}\cos(\theta) + H \end{aligned}$$

where H is a constant. Thus, the curves $\frac{y^2}{2} = \frac{g}{l}\cos(\theta) + H$ form the phase portrait of the pendulum



5-9: Phase portrait for the pendulum

Clearly, $\theta = 0$ is a stable equilibrium, while $\theta = \pi$ appears to be unstable, just as we suspected.

Exercises

Find the equilibria and phase portrait of the following systems:

1. $\frac{dx}{dt} = y, \ \frac{dy}{dt} = -x$	2. $\frac{dx}{dt} = y, \ \frac{dy}{dt} = x - 2$
3. $\frac{dx}{dt} = x - 1, \ \frac{dy}{dt} = y + 1$	4. $\frac{dx}{dt} = y^2, \ \frac{dy}{dt} = x + 2$
5. $\frac{dx}{dt} = y^2, \ \frac{dy}{dt} = x - 1$	6. $\frac{dx}{dt} = y, \ \frac{dy}{dt} = x + 3$
7. $\frac{dx}{dt} = y, \ \frac{dy}{dt} = \sin(x)$	8. $\frac{dx}{dt} = 1, \ \frac{dy}{dt} = xy$
9. $\frac{dx}{dt} = \sin(y), \frac{dy}{dt} = \tan(x)$	10. $\frac{dx}{dt} = y, \ \frac{dy}{dt} = \ln(x+1)$
11. $\frac{dx}{dt} = \sqrt{y}, \ \frac{dy}{dt} = y\cos(x)$	12. $\frac{dx}{dt} = y, \ \frac{dy}{dt} = x$

Find the phase portrait of the following second order differential equations:

13. $x'' + x = 0$	14. $x'' = 1$
15. $x'' = 4x$	16. $x'' = x^2$
17. $x'' = \dfrac{2x(x')^2}{x^2+1}$	18. $x'' = \dfrac{2x(x')^2}{x^2-1}$
19. $x'' + (x')^2 x + x = 0$	20. $x'' - (x')^2 x + x = 0$

Use the method of quadratures to find solutions to the following second order equations:

21. $x'' + x = 0$	22. $x'' = 1$
23. $x'' = 4x$	24. $x'' = x^2$
25. $x'' = \dfrac{2x(x')^2}{x^2+1}$	26. $x'' = \dfrac{2x(x')^2}{x^2-1}$

27. A projectile near the earth's surface has a constant acceleration of 32 feet per second per second. Thus, if $r(t)$ is the height of a projectile at time t, then

$$\frac{d^2r}{dt^2} = -32$$

What is the phase portrait of this equation? Use the method of quadratures to find the actual solution.

28. If an object moves along a line through the center of the earth, then its distance $r(t)$ from the center of the earth at time t satisfies

$$\frac{d^2r}{dt^2} = \frac{-k}{r^2}$$

What is the phase portrait of this equation?

29. Suppose x is a population that is growing exponentially, and y is a population whose rate of growth is proportional to both x and y (the mass action law). Then

$$\frac{dx}{dt} = kx \qquad \textit{and} \qquad \frac{dy}{dt} = pxy$$

where k and p are constants. What is the phase portrait of this system? What are the equilibria?

30. Suppose that the population in exercise 29 is growing logistically. Then

$$\frac{dx}{dt} = kx\,(1-x) \qquad \textit{and} \qquad \frac{dy}{dt} = pxy$$

where k and p are constants. What is the phase portrait of this system? What are the equilibria?

31. If x is the population of human hosts of a possible viral infection and y is the viral population itself, then the model

$$\frac{dx}{dt} = [a - g\,(y)]\,x, \qquad \frac{dy}{dt} = bxy - cy$$

is a crude model of the viral infection of its human hosts.

(a) What is the phase portrait and equilibria when $g\,(y) = 0$ for all y?

(b) What is the phase portrait and equilibrium when $g\,(y) = y^2$ for all y?

32. Protein-mRNA interactions can be described by the system of equations

$$\frac{dM}{dt} = \frac{1}{1+E} - a, \qquad \frac{dE}{dt} = M - b$$

where a and b are constants. What is the phase portrait of this system? What are the equilibria?

33. Let x denote the number of excited photons and y denote the number of stimulated atoms in a laser. Then a good model of the laser is given by

$$\frac{dx}{dt} = axy - bx, \qquad \frac{dy}{dt} = -axy - cy + p$$

where a is the gain coefficient, b is the rate of absorption of photons due to mirror imperfections, c is the rate of spontaneous emission, and p is the rate of production of stimulated atoms (i.e., a pump). What are the equilibria of the model? What is the phase portrait when $p = 0$ (i.e., no atoms currently being stimulated)?

34. In exercise 33, suppose instead that $a = 0$. Interpret the model in this case. Also, find the phase portrait and the equilibria of the model.

35. At the end of the section, we showed that the phase portrait of the pendulum of length l is formed by the curves

$$\frac{y^2}{2} = \frac{g}{l}\cos(\theta) + H$$

where H is constant. Suppose that $y = 0$ when $\theta = \theta_0$ for a constant initial angle θ_0. Show that

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}\left(\cos(\theta) - \cos(\theta_0)\right)}$$

and thus that the solution is of the form

$$\int \frac{d\theta}{\sqrt{\cos(\theta) - \cos(\theta_0)}} = \left(\sqrt{\frac{2g}{l}}\right) t + C$$

36. Try it out! (Group Learning Project) A pendulum of length L can be used to measure the acceleration due to gravity at the earth's surface. Here is a proposal for a 3 member group.

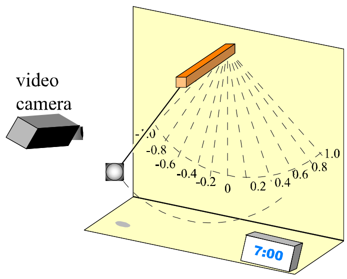
(a) **Member 1:** Present the pendulum equation and show that

$$\frac{1}{2}\left(\theta'\right)^2 + \frac{g}{l}\cos(\theta) = H$$

Then explain why $Y = gX + H$ if

$$Y = \frac{1}{2}\left(\theta'\right)^2, \qquad X = \frac{-\cos(\theta)}{l}$$

(b) **Member 2:** Use a CBL system or videotape a pendulum swinging in front of a panel which has been marked off in small angles. Also include a clock showing seconds:



Measure θ and θ' at different times t, where t is in seconds (note: you may need to approximate θ' by ratios of changes in θ over small changes in time.)

(c) **Member 3:** Use the data collected by member 3 to produce a new data set with

$$Y = \frac{1}{2}\left(\theta'\right)^2, \qquad X = \frac{-\cos(\theta)}{L}$$

Then fit the new data set to (**??**). The slope of the fit is the acceleration due to gravity, which will be in meters per sec^2 if L is measured in meters or in feet per sec^2 if L is measured in feet.

37. Given a system

$$\frac{dx}{dt} = f(x, y), \qquad \frac{dy}{dt} = g(x, y)$$

the *x-nullclines* are the curves where $\frac{dx}{dt} = 0$ and the *y-nullclines* are the curves where $\frac{dy}{dt} = 0$. What are the nullclines for the laser model

$$\frac{dx}{dt} = axy - bx, \qquad \frac{dy}{dt} = -axy - cy + p$$

38. What are the nullclines for the viral infection model

$$\frac{dx}{dt} = [a - g(y)]\,x, \qquad \frac{dy}{dt} = bxy - cy$$

when $g(y) = y^2$?

7.6 Harmonic Oscillation

The Harmonic Oscillator

In this section, we want to explore a second order autonomous differential equation that appears frequently in applications. To do so, let us first suppose that $y(t)$ is the harmonic oscillation

$$y(t) = a\cos(\omega t) + b\sin(\omega t)$$

where a, b, and ω are constants. Differentiating results in

$$\begin{aligned} y'(t) &= -\omega a\sin(\omega t) + \omega b\cos(\omega t) \\ y''(t) &= -\omega^2 a\cos(\omega t) - \omega^2 b\sin(\omega t) \end{aligned}$$

so that factoring yields $y'' = -\omega^2(a\cos(\omega t) + b\sin(\omega t))$, or equivalently, $y'' = -\omega^2 y$. Thus, $y(t)$ is a solution to the *harmonic oscillator* differential equation

$$y'' + \omega^2 y = 0 \tag{7.61}$$

Furthermore, $y(0) = a$ and $y'(0) = b\omega$, which is the same as $b = y'(0)/\omega$. In fact, the computations above can be extended to the following theorem.

Theorem 6.1: If p, q, and ω are constant, then the *only solution* to the harmonic oscillator problem

$$y'' + \omega^2 y = 0, \qquad y(0) = p, \qquad y'(0) = q$$

is given by the harmonic oscillation

$$y = p\cos(\omega t) + \frac{q}{\omega}\sin(\omega t) \tag{7.62}$$

Often harmonic oscillations are displacements about an equilibrium position. Thus, the dependent variable is often interpreted as a distance either above or below that equilibrium.

EXAMPLE 1 Find the period, amplitude, and extrema of the solution to

$$y'' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 2 \tag{7.63}$$

where y is a displacement, in centimeters, from equilibrium. Then sketch the graph of the oscillation.

Solution: Since $p = 1$, $\omega = 2$, and $q = 2$, theorem 6.1 (specifically, equation (7.62)) implies that the solution to (7.63) is

$$y(t) = 1\cos(2t) + \frac{2}{2}\sin(2t) = \cos(2t) + \sin(2t)$$

The amplitude is $A_m = \sqrt{2}$, the period is $T = \pi$, and the derivative is

$$y'(t) = -2\sin(2t) + 2\cos(2t)$$

Since $y'(t) = 0$ implies that $2\sin(2t) = 2\cos(2t)$, which becomes

$$\tan(2t) = 1$$

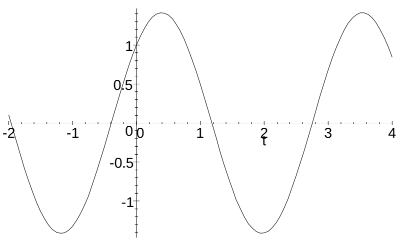
Table 6.1 implies that

$$2t = \frac{\pi}{4} + n\pi$$

where n is an integer. Thus, the extrema are located at

$$t = \frac{\pi}{8} + \frac{n\pi}{2}$$

which leads to the graph shown below:



6-1

The importance of theorem 6.1 cannot be overstated, because the harmonic oscillator differential equation occurs frequently in applications.

EXAMPLE 2 Carbon Monoxide (CO) is a diatomic molecule with a reduced mass of $m_r = 1.14 \times 10^{-26}$ kg and a molecular bond force constant of $k = 1858$ Newton-meters (the bond force constant is a measure of the "stiffness" of the molecular bond). It can be shown that if $y(t)$ is the displacement from equilibrium of a vibrating CO molecule, then

$$m_r y'' + ky = 0 \tag{7.64}$$

If $y(0) = 1.3$ angstroms (1 angstrom $= 10^{-10}$ meters) and $y'(0) = 0$, then what is the solution to (7.64)? What is its period and frequency?

Solution: We can write (7.64) as

$$y'' + \frac{k}{m_r} y = 0$$

which implies that $\omega = \sqrt{k/m_r} = \sqrt{1858/(1.14 \times 10^{-26})} = 4.037 \times 10^{14}$ radians per second. Since $p = y(0) = 1.3 \times 10^{-10}$ meters and $q = 0$, the solution is

$$y(t) = 1.3 \times 10^{-10} \cos\left(4.037 \times 10^{14}\, t\right)$$

The period of the oscillation is

$$T = \frac{2\pi}{4.037 \times 10^{14}} = 1.5564 \times 10^{-14} \text{ sec}$$

and the frequency of the oscillation is

$$f = \frac{1}{1.5564 \times 10^{-14}} = 64.25 \times 10^{12} \; hertz$$

which is 64.25 Tetraherz. To put this in perspective, the *infrared spectrum* ranges from approximately 10^{11} to 10^{13} hertz, so the vibrational frequency for carbon monoxide is in the infrared spectrum.

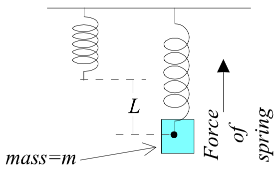
Check your Reading *Where does the first minimum after the origin occur in example 2?*

Hooke's Law

Hooke's law says that if a mass m stretches a linear spring a distance L beyond its natural length, then the restoring force of the spring is

$$F_{restore} = -kL$$

where k is a constant.

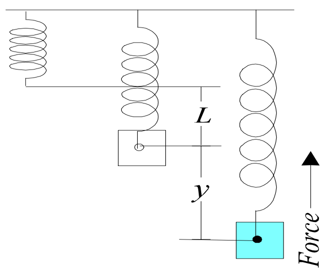


6-2: Mass m stretches a spring a distance L

At *equilibrium*—that is, when the mass is not moving—the mass's weight balances the restoring force. Since mg is the weight of the mass (where g is the acceleration due to gravity), at equilibrium we have

$$kL = mg \tag{7.65}$$

Suppose now that at time t the spring is stretched a distance y beyond equilibrium:



6-3: A Stretched Spring

The restoring force is $F_{restore} = -k\,[L + y]$, so that Newton's second law yields

$$\begin{array}{ccccc} \text{Mass} \times \text{accel} & = & F_{restore} & + & \text{Weight} \\ my'' & = & -k\,[L+y] & + & mg \\ my'' & = & -kL - ky & + & mg \end{array}$$

Since $kL = mg$, this reduces to $my'' = -ky$ or equivalently,

$$my'' + ky = 0 \tag{7.66}$$

If (7.66) is divided by m, the result is of the form (7.61) with $\omega^2 = k/m$.

EXAMPLE 3 A mass of 0.2 kg pulls a spring to an equilibrium position 10 cm beyond its natural length. The string is then pulled an additional 1 cm (= 0.01 m) and released. Use theorem 6.1 to find the solution which describes the resulting oscillation.

Solution: To find k, we use (7.65) with $L = 0.1$ m, $g = 9.8$ m/sec^2, and $m = 0.2$ kg.

$$k \cdot 0.1 = 0.2 \cdot 9.8 \qquad \Longrightarrow \qquad k = \frac{0.2 \cdot 9.8}{0.1} = 19.6$$

If y is the distance from equilibrium at time t, then (7.66) becomes

$$0.2y'' + 19.6y = 0$$

Since $\frac{19.6}{0.2} = 98.0$, this reduces to the harmonic oscillator equation

$$y'' + 98y = 0$$

Because the mass is not moving immediately before it is released, the initial velocity is 0 and consequently, the equation of motion is

$$y'' + 98y = 0, \qquad y(0) = 0.01 \text{ m}, \quad y'(0) = 0 \tag{7.67}$$

According to Theorem 6.1 with $\omega^2 = 98$, $p = 0.01$, and $q = 0$, the solution is

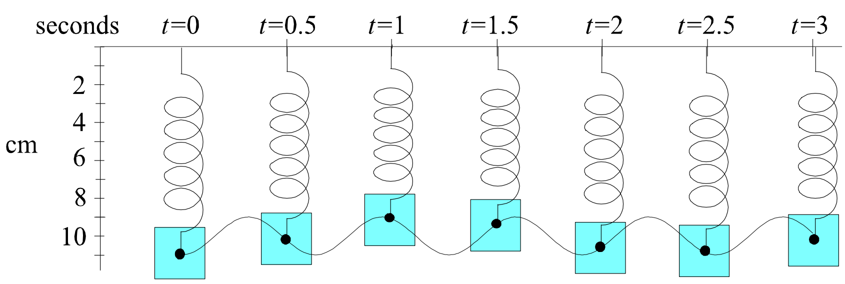
$$y(t) = 0.01 \cos\left(\sqrt{98}t\right) + \frac{0}{\sqrt{98}} \cdot \sin\left(\sqrt{98}t\right)$$

Thus, the solution is $y(t) = 0.01 \cos\left(\sqrt{98}t\right)$.

Actually, the solution for example 5 should be written in the form

$$y(t) = 0.01 \cos\left(\sqrt{98}t\right) + 0.1$$

That is, the mass oscillates from between 9 and 11 cm about the equilibrium of 10 cm. Moreover, since $\omega = \sqrt{98}$ and $T = 2\pi/\sqrt{98}$, the oscillation is at a frequency of $f = 1/T = \sqrt{98}/2\pi = 1.576$ hz, which is a little more than $1\frac{1}{2}$ up-down cycles per second.



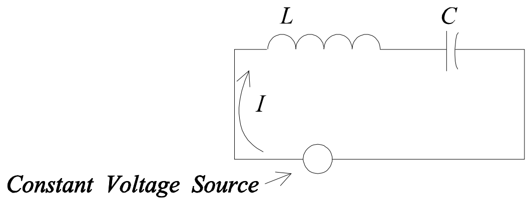
6-4: Spring mass exhibits a harmonic oscillation

Check your Reading *What is the initial length of the spring-mass in example 5?*

Harmonic Oscillators in other Settings

There are a number of other settings where a harmonic oscillator differential equation occurs. For example, electrical circuits often produce harmonic oscillations.

To illustrate, let's consider the current flowing through a circuit with a lumped capacitance C, a lumped inductance L, and a constant voltage source E.



6-5: An electric circuit that produces harmonically oscillating current.

Kirkoff's laws in electrical circuit theory imply that

$$V_{ind} + V_{cap} = V_{source}$$

where V_{cap}, V_{ind}, and V_{source} denote the voltages across the capacitor, the inductor, and the voltage source, respectively. However, it can be shown that the current across a capacitor is proportional to the time rate of change of the voltage V_{cap} and that the voltage V_{ind} across an inductor is proportional to the time rate of change of the current:

$$V_{ind} = L\frac{dI}{dt} \qquad and \qquad I = C\frac{dV_{cap}}{dt}$$

As a result, Kirkoff's law becomes

$$LI' + V_{cap} = E$$

where E is the constant voltage source.

Differentiating again leads to

$$LI'' + \frac{dV_{cap}}{dt} = 0$$

since E is constant. Finally, since $V'_{cap} = I/C$, we obtain

$$LI'' + \frac{1}{C}I = 0 \tag{7.68}$$

This is a harmonic oscillator equation with $\omega^2 = (LC)^{-1}$.

EXAMPLE 4 Find the current $I(t)$ in amps for a circuit in which $L = 0.4$ henries, $C = 0.7 \times 10^{-6}$ farads, $I(0) = 2$ amps, and $I'(0) = 0$. What is the period, amplitude, and frequency of the oscillation?

Solution: The equation (7.68) for $L = 0.4$ and $C = 0.7 \times 10^{-6}$ is

$$0.4I'' + \frac{1}{0.7 \times 10^{-6}}I = 0$$

Dividing throughout by 0.4 thus leads to

$$I'' + (3,571,428)\,I = 0$$

Thus, $\omega^2 = 3,571,428$, so that $\omega = 1890$ and

$$I(t) = a\cos(1890t) + b\sin(1890t)$$

However, $I(0) = 2$ implies that $a = 2$ and $I'(0) = 0$ implies that $b = 0$. Thus,

$$I(t) = 2\cos(1890t) \quad amps$$

The amplitude is $A_m = 2$, the period is $T = 2\pi/\omega = 2\pi/1890$, and the frequency is

$$f = \frac{1}{T} = \frac{1890}{2\pi} = 300.803\ hz$$

That is, the current oscillates about 300 times each second.

Check your Reading *Can we hear an oscillation of 300 hertz?*

Phase Portrait for Harmonic Oscillators

Let's write the harmonic oscillator in the form

$$x'' + \omega^2 x = 0$$

and let's let $y = x'$. Since $x'' = -\omega^2 x$, the phase portrait for the harmonic oscillator is

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -\omega^2 x$$

Thus, the origin is the equilibrium and the phase portrait is

$$\frac{dy}{dx} = \frac{-\omega^2 x}{y}$$

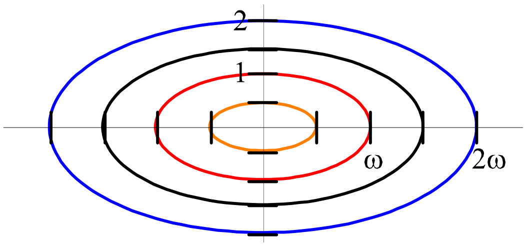
Separation of variables leads to

$$\begin{aligned} ydy &= -\omega^2 xdx \\ \int ydy &= -\omega^2 \int xdx \\ \frac{y^2}{2} &= -\omega^2 \frac{x^2}{2} + H \end{aligned} \tag{7.69}$$

where H is a constant known as the *energy* of the oscillation. As a result, we obtain

$$\frac{x^2}{2/\omega^2} + \frac{y^2}{2} = H$$

which is a family of ellipses in which the horizontal axis is ω times longer than the vertical axis.



6-6: Phase portrait for the Harmonic Oscillator

EXAMPLE 5 Use the phase portrait to obtain the general solution in theorem 6.1.

Solution: Since $y = x'$, we can write (7.69) in the form

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 &= 2H - \omega^2 x^2 \\ \frac{dx}{dt} &= \omega\sqrt{\frac{2H}{\omega^2} - x^2} \end{aligned}$$

The resulting equation is separable, so we obtain

$$\begin{aligned} \frac{dx}{\sqrt{\frac{2H}{\omega^2} - x^2}} &= \omega dt \\ \int \frac{dx}{\sqrt{\frac{2H}{\omega^2} - x^2}} &= \int \omega dt \\ \sin^{-1}\left(\frac{x}{2H/\omega^2}\right) &= \omega t + C \\ \frac{x}{2H/\omega^2} &= \sin(\omega t + C) \\ x(t) &= \frac{\omega^2}{2H}\sin(\omega t + C) \end{aligned}$$

Let's now expand using the sine of the sum identity:

$$x(t) = \frac{\omega^2}{2H}\sin(C)\cos(\omega t) + \frac{\omega^2}{2H}\cos(C)\sin(\omega t)$$

Thus, if we let $a = \omega^2 \sin(C)/(2H)$ and let $b = \omega^2\cos(C)/(2H)$, then the result is the harmonic oscillation

$$x(t) = a\cos(\omega t) + b\sin(\omega t)$$

Exercises

Use theorem 6.1 to determine the solutions to each of the following harmonic oscillator initial value problems. Then find the period, frequency, and amplitude of the oscillation. Finally, locate the extrema and sketch the graph of the oscillation.

1. $y'' + y = 0$, $y(0) = 1$, $y'(0) = 0$
2. $y'' + y = 0$, $y(0) = 0$, $y'(0) = 1$
3. $y'' + 4y = 0$, $y(0) = 2$, $y'(0) = 4$
4. $y'' + 4y = 0$, $y(0) = 3$, $y'(0) = 6$
5. $y'' + 3y = 0$, $y(0) = 1$, $y'(0) = 3$
6. $3y'' + y = 0$, $y(0) = 2$, $y'(0) = -2$

Find the harmonic oscillator equation and the resulting solution given a spring-mass system with mass m and Hooke's constant k. Then determine the period, frequency, and amplitude of the oscillation. Finally, graph the oscillation with a *graphing calculator*

7.	$m = 2$ kg, $k = 8\ \frac{kg}{\sec^2}$ $y(0) = 0.5$ m, $y'(0) = 0$	8.	$m = 2$ kg, $k = 8\ \frac{kg}{\sec^2}$ $y(0) = 1$ m, $y'(0) = 0$
9.	$m = 2$ kg, $k = 4\ \frac{kg}{\sec^2}$, $y(0) = 0.03$ m, $y'(0) = 0$	10.	$m = 0.3$ kg, $k = 3\ \frac{kg}{\sec^2}$, $y(0) = 0.02$ m, $y'(0) = 0$
11.	$m = 4$ kg, $k = 4\pi^2 \frac{kg}{\sec^2}$, $y(0) = 0$, $y'(0) = 0.02 \frac{\text{m}}{\sec}$	12.	$m = 0.2$ kg, $k = 30\ \frac{kg}{\sec^2}$, $y(0) = 0$, $y'(0) = 0$

In exercises 13 - 16, use (7.68) to determine the current in the circuit.

13.	$L = 1$ henry, $C = 0.25 \times 10^{-6}$ farads $I(0) = 2$ amps, $I'(0) = 0$	14.	$L = 1$ henry, $C = 10^{-5}$ farads $I(0) = 5$ amps, $I'(0) = 0$
15.	$L = 4$ henries, $C = 10^{-6}$ farads $I(0) = 10$ amps, $I'(0) = 0$	16.	$L = 2$ henries, $C = 10^{-6}$ farads $I(0) = 0$, $I'(0) = 2$ amps/sec

17. Show that the period of a spring-mass system is given by

$$T = 2\pi\sqrt{\frac{k}{m}} \tag{7.70}$$

18. **Try it out!** Hang a known mass m onto the end of a spring, and then wait until it stops moving. The change in length is L, and k can be now be determined. Use (7.70) in exercise 17 to calculate T. Then measure the time required for 20 oscillations of the spring and divide by 20 to estimate T directly. Does the calculated value of T match the estimated value of T?

19. **Try it out!** The acceleration due to gravity can be measured using a spring-mass system. Show that

$$k = \frac{mg}{L}$$

so that the period of the spring is

$$T = 2\pi\sqrt{\frac{L}{g}} \tag{7.71}$$

Then measure L and T for a given spring (see exercise 18), substitute into (7.71) and solve for g.

20. For a given pendulum, measure the period of the spring as in exercise 18. How accurately does (7.71) produce the length L of the pendulum?

21. A spring-mass oscillates up and down in a harmonic oscillation described by

$$y(t) = 3\cos(15t)$$

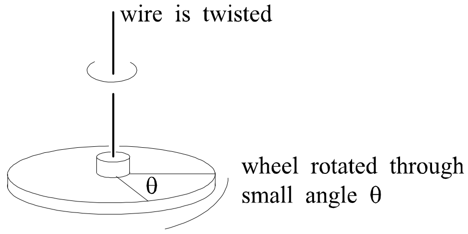
What is the period of the oscillation? How fast is the spring-mass moving initially? What harmonic oscillator differential equation does it satisfy? What initial conditions does it have?

22. A spring-mass oscillates up and down in a harmonic oscillation described by

$$y(t) = 80\cos(880\pi t) + 80\sin(880\pi t)$$

What is the period of the oscillation? What is the amplitude? How fast is the spring-mass moving initially? What harmonic oscillator differential equation does it satisfy? What initial conditions does it have? What does its graph look like?

23. A massive wheel is attached to a thin metal wire, and the wheel is turned through an angle θ, which twists the wire through an angle θ as well.



6-7: Massive Wheel Attached to a Thin Metal Wire

If the angle θ is sufficiently small, then the twist exerts a torque τ that is proportional to θ. That is

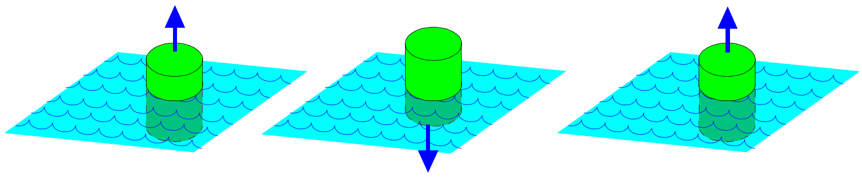
$$\tau = -\kappa\theta$$

where κ is the torsion constant. If I denotes the moment of inertia of the massive wheel, then the torque will be related to the angular acceleration by

$$\tau = I\frac{d^2\theta}{dt^2}$$

Solve the resulting harmonic oscillator equation. What is the period of the oscillation? What is the frequency?

24. A right cylindrical cork with weight w and radius R bobs up and down on the surface of the water:



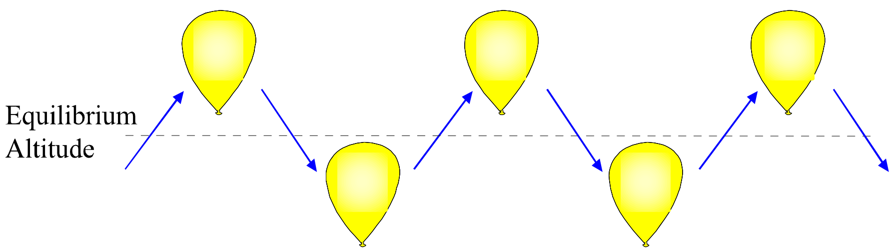
6-8: Bobbing Cork

Archimedes principle says that if y is the displacement from equilibrium at time t, then

$$\begin{aligned} \text{weight} \times \text{accel} &= \text{bouyant Force} \times \text{gravitational accel} \\ wy'' &= -4\pi R^2 \rho_w g\, y \end{aligned}$$

where $\rho_w = 62.5$ lbs per cubic foot is the density of the water and $g = 32$ feet per sec^2 is the acceleration due to gravity. What is the period and frequency of a bobbing cork that weigh 1.5 pounds and has a radius of 3 inches?

25. In calm air with negligible drag, a helium balloon will rise to an *equilibrium altitude,* which is an altitude where the density of the air outside of the balloon matches the density of the air inside the balloon. It will then oscillate above and below that equilibrium altitude.



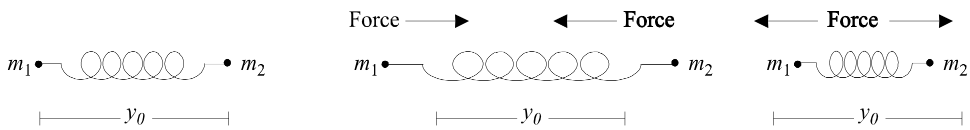
6-9: Bouncing Balloon

For a certain balloon, the displacement from equilibrium at time t satisfies

$$z'' = \frac{-9.8}{10000} z$$

What is $z(t)$ explicitly if $z(0) = 2$ m and $z'(0) = 0$? What is the period, amplitude, and frequency of the oscillation?

26. Write to Learn: * In a diatomic molecule like carbon monoxide, the equilibrium distance y_0 between the nuclei of the two atoms in the molecule is the distance at which the molecule does not vibrate. However, if the distance y between the nuclei is not y_0, then the binding force causes the molecule to vibrate.



6-10: An internuclear force that is "spring-like."

This internuclear force is "spring-like" in nature. Indeed, if $y(t)$ is the separation at time t, then $y(t)$ satisfies

$$m_r y'' + k(y - y_0) = 0 \tag{7.72}$$

where k is the bond force constant and

$$m_r = \frac{m_1 m_2}{m_1 + m_2}$$

is the reduced mass of the molecule. Explain why if $y(0) = p$ and $y'(0) = q$, then the solution to (7.72) is

$$y(t) = (p - y_0)\cos(\omega t) + \frac{q}{\omega}\sin(\omega t) + y_0$$

where $\omega = \sqrt{k/m_r}$. How is the frequency of the oscillation related to the reduced mass and the bond force constant?

7.7 Slope Fields and Equilibria

Slope Fields of First Order Differential Equations

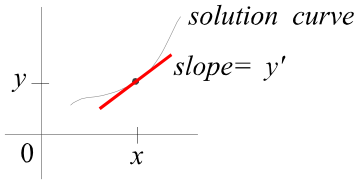
There are many differential equations that cannot be solved by separation or any other means. Such differential equations must instead be studied *qualitatively,* which means that we seek to discover properties of the solutions without actually determining the solutions themselves.

A non-autonomous differential equation is an equation of the form

$$y' = f(x, y) \tag{7.73}$$

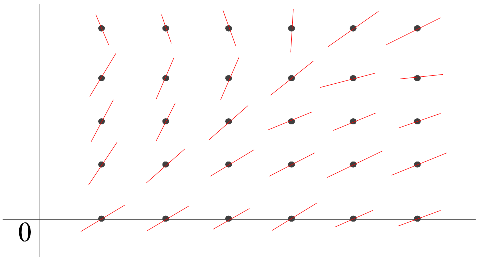
where $f(x, y)$ is some expression in x and y. A *slope field* for a non-autonomous equation is based on the fact that a short line segment through (x, y) with slope

y' approximates the solution in the vicinity of (x, y).



7-1: Line segment is a small section of a tangent line

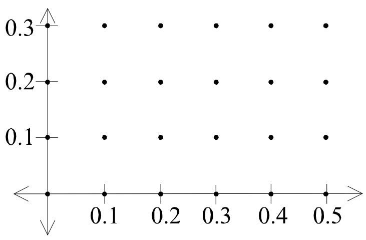
Let's suppose that we choose a grid of points (x, y) in some region of the xy-plane. If a short line segment with slope y' is placed at each point in the grid, then the result, which is known as a *slope field* for the differential equation (7.73), can be used to visualize the solutions to (7.73).



7-2: Many line segments form a slope field

In addition, the line segments in a slope field are often drawn with arrowheads on the right to indicate the direction in which x is increasing.

EXAMPLE 1 Construct a slope field for $y' = 2x - y$ on the grid shown below:



7-3: Grid for a slope field

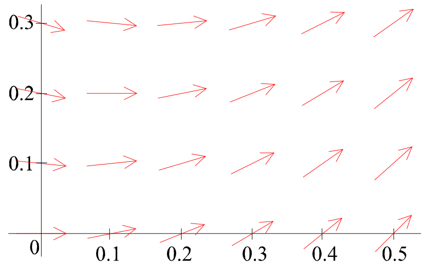
Solution: The slope function is $f(x, y) = 2x - y$. When $x = 0$, then the slope is

$$Slope = 2 \cdot 0 - y = -y$$

Thus, at $(0, 0)$ the slope is 0, at $(0, 0.1)$ the slope is -0.1, and so on. Now let's consider $x = 0.1$:

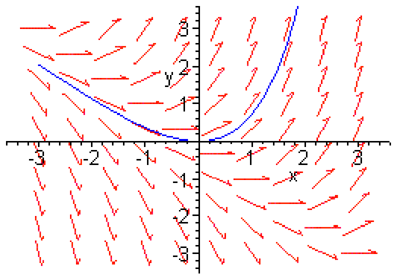
$$\begin{aligned}
at\ (0.1, 0) &: x = 0.1, y = 0.0 : Slope = 2 \cdot 0.1 - 0 = 0.2 \\
at\ (0.1, 0.1) &: x = 0.1, y = 0.1 : Slope = 2 \cdot 0.1 - 0.1 = 0.1 \\
at\ (0.1, 0.2) &: x = 0.1, y = 0.2 : Slope = 2 \cdot 0.1 - 0.2 = 0 \\
at\ (0.1, 0.3) &: x = 0.1, y = 0.3 : Slope = 2 \cdot 0.1 - 0.3 = -0.1
\end{aligned}$$

Similarly, at $(0.2, 0)$, the slope is 0.4, at $(0.2, 0.1)$, the slope is 0.3 and so on. Below is the slope field for $y = 2x - y$ produced by the computer algebra system *Maple*.



7-4: Slope field for $y' = 2x - y$

Slope fields are often produced by a calculator or computer as a means of visualizing the solution to a differential equation. For example, the slope field of $y' = x+y$ shown in figure 4-9 was produced with the computer algebra system *Maple*.



7-5: Solution to $y' = x + y$, $y(0) = 0$.

The blue curve is a solution to $y' = x + y$ with initial condition $y(0) = 0$.

Check your Reading *Will your graphing calculator produce slope fields?*

Equilibria of Autonomous Equations

Slope fields are particularly useful for autonomous equations, which are differential equations of the form

$$y' = f(y) \tag{7.74}$$

In particular, if C is a number for which $f(C) = 0$, then $y = C$ is a constant function which is also a solution to $y' = f(y)$. We call $y = C$ an *equilibrium solution* of (7.74). Slope fields of autonomous equations yield information about the *equilibria* of a differential equation.

EXAMPLE 2 Find the equilibria of the logistic equation

$$y' = 0.1y\left(1 - \frac{y}{1000}\right)$$

Solution: The equation is of the form $y' = f(y)$ where

$$f(y) = 0.1y\left(1 - \frac{y}{1000}\right)$$

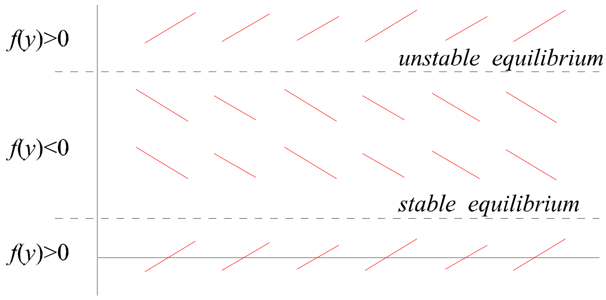
and $f(y) = 0$ when $y = 0$ or when $1 - \frac{y}{1000} = 0$, which is when $y = 1000$. Thus, the equilibria are $y = 0$ and $y = 1000$.

Moreover, an equilibrium $y = C$ of $y' = f(y)$ is said to be *stable* if

$$\lim_{t \to \infty} y(t) = C \tag{7.75}$$

for any solution $y(t)$ which is sufficiently close to C at some time t. An equilibrium which is not stable is said to be *unstable.*

Slope fields can be used to determine if an equilibrium is stable or unstable. However, the slope field is often constructed by hand in such instances. To do so, we first draw the equilibria as horizontal lines. We then test $f(y)$ between the equilibria. If $f(y) < 0$ on some interval (a, b), then short line segments with negative slopes are drawn between $y = a$ and $y = b$. If $f(y) > 0$ on (c, d), then short line segments with positive slopes are drawn between $y = c$ and $y = d$.



7-6: Equilibrium Analysis

If solutions that start near an equilibrium approach that equilibrium, then the equilibrium is *stable.* Otherwise, the equilibrium is unstable.

EXAMPLE 3 Use a slope field to determine if the equilibria of

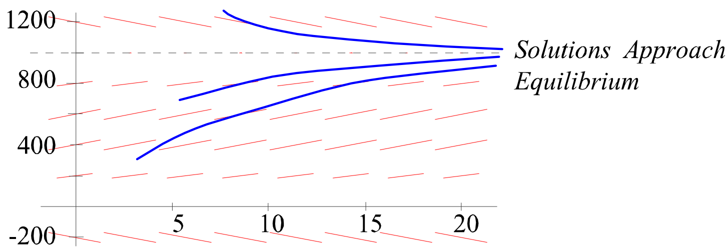
$$y' = 0.1y\left(1 - \frac{y}{1000}\right) \tag{7.76}$$

are stable or unstable.

Solution: In example 2, we saw that the equilibria are $y = 0$ and $y = 1000$. Notice now that

$$\begin{array}{rlcl} f(-1) = & -0.1\left(1 - \frac{-1}{1000}\right) = -0.1001 < 0 & \Longrightarrow & f(y) < 0 \ for \ y < 0 \\ f(1) = & 0.1\left(1 - \frac{1}{1000}\right) = 0.0999 > 0 & \Longrightarrow & f(y) > 0 \ for \ y \ in \ (0, 1000) \\ f(1001) = & 0.1(1001)\left(1 - \frac{1001}{1000}\right) = -0.001 < 0 & \Longrightarrow & f(y) < 0 \ for \ y > 1000 \end{array}$$

Thus, we draw line segments with negative slope below $y = 0$, line segments with positive slope between $y = 0$ and $y = 1000$, and line segments with negative slope for $y > 1000$.



7-7: Solutions that begin sufficiently close to stable equilibrium approach that equilibrium

Solutions above $y = 1000$ are decreasing, and consequently, they must approach 1000. Solutions between $y = 0$ and $y = 1000$ are increasing, and consequently, they also must approach 1000. Thus, $y = 1000$ is a stable equilibrium, while $y = 0$ is an unstable equilibrium.

Check your Reading *In example 3, what does a solution below* 0 *do as* x *approaches* ∞?

Applications of Equilibria

Slope fields can be used to determine the approximate behavior of a system when it is near equilibrium.

EXAMPLE 4 Suppose that an intravenously-infused solution delivers a certain drug at a constant rate of 100cc per unit volume of solution per hour, and suppose that the body reduces the concentration by about 10% each hour. Find the equilibrium concentration and show that it is stable.

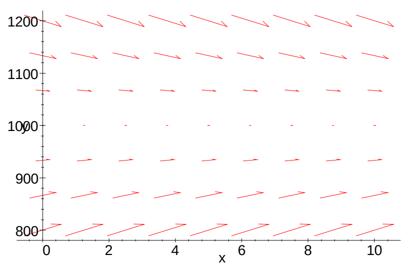
Solution: If $y(t)$ denotes the concentration of the drug at time t, then

rate of change of concentration		rate of removal		rate of infusion
y'	$=$	$-0.1y$	$+$	100

Thus, the equilibrium is the solution to

$$0 = -0.1y + 100$$

which is $y = 1000$ cc per unit volume. A slope field for the model is given below:



7-8: Slope field for drug concentration

Clearly, the equilibrium is stable.

Let's consider another class of examples. If $y(t)$ is the population of a certain renewable resource at time t that is being *harvested* at a rate $H(y,t)$ at a population of y and at time t, then $y(t)$ can often be considered to be the solution to the modified logistic equation

$$y' = ky\left(1 - \frac{y}{N}\right) - H(y,t) \tag{7.77}$$

where N is the carrying capacity of the resource in the event of no harvesting and k is the intrinsic growth rate. Slope fields and equilibria are important in this application because (7.77) can be solved in closed form only for certain choices of the harvesting function $H(t)$.

EXAMPLE 5 A forest with a carrying capacity of 1000 trees per acre and an intrinsic growth rate of 0.1 % increase per year is being harvested at a rate of 20 trees per acre per year. Produce and interpret the slope field for this model. If there are initially 500 trees per acre, about how many trees per acre will there be after 2 years using $h = \frac{1}{12}$ (i.e., h about the length of one month)?

Solution: The model (7.77) with $k = 0.1$, $N = 1000$, and $H = 20$ yields

$$y' = 0.1y\left(1 - \frac{y}{1000}\right) - 20 \tag{7.78}$$

The equilibria of (7.78) are the solutions to

$$0 = 0.1y(1 - 0.001y) - 20$$

since $1/1000 = 0.001$. Expanding yields

$$-0.0001y^2 + 0.1y - 20 = 0$$

Solutions to

$$ay^2 + by + c = 0$$

are given by

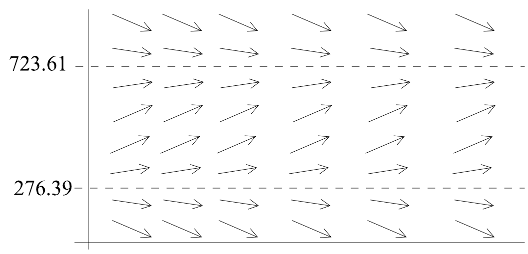
$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To solve the equation, we use the quadratic formula:

$$y = \frac{-0.1 \pm \sqrt{(0.1)^2 - 4(-0.0001)(-20)}}{2 \cdot (-0.0001)} = 276.39,\ 723.61$$

Thus, the equilibria are $y = 276.39$ and $y = 723.61$.

A slope field for (7.78) is shown below:



7-9: Slope field for tree harvesting

Clearly, $y = 723.61$ trees per acre is a stable equilibrium, while the equilibrium $y = 276.39$ is unstable. Thus, as long as there are more than 277 trees per acre in the forest, then the forest population will grow to an equilibrium of 732.61 trees per acre. But if the number of trees per acre drops below 276.39, then harvesting will cause the forest to die out.

Check your Reading *What happens in example 5 if there are initially more than 723.61 trees per acre?*

The Lotka–Volterra Equations

Suppose that x is the population at time t of a species of *prey* and suppose that y is the population at time t of a species of *predators*. If we assume an ample food supply, then the prey species grows exponentially if there are no predators, which means that

$$\frac{dx}{dt} = Ax,$$

where $A > 0$ is the *intrinsic growth rate constant* for the prey population. If we further assume that the prey are the predators sole food source, then the predator population will decay exponentially in the absence of prey, which means that

$$\frac{dy}{dt} = -By$$

where $B > 0$ is the *intrinsic extinction rate constant* for the predator population. Finally, let's assume that change due to predation of both species is proportional to the population of both species. It follows that the predator-prey model is thus given by

$$\frac{dx}{dt} = Ax - Cxy, \qquad \frac{dy}{dt} = -By + Dxy \tag{7.79}$$

where $C > 0$ and $D > 0$ are the interaction constants for the two species. The predator-prey equations in (7.79) are also known as the *Lotka-Volterra equations* after the two mathematicians that originally introduced them.

The equilibria are the points at which $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ simultaneously. Thus, the equilibria are solutions to

$$\begin{aligned} Ax - Cxy &= 0 \quad \textit{and} \quad -By + Dxy = 0 \\ x(A - Cy) &= 0 \quad \textit{and} \quad y(-B + Dx) = 0 \end{aligned}$$

which can be shown to be $(0, 0)$ and $\left(\frac{B}{D}, \frac{A}{C}\right)$. The *phase portrait* is formed by the solutions to the differential equation

$$\frac{dy}{dx} = \frac{-By + Dxy}{Ax - Cxy} \tag{7.80}$$

Although (7.80) is separable, the closed form solution is not all that helpful (see exercise 43). Thus, we instead supply numerical values for the parameters and use slope fields to study the phase portrait.

For example, between 1845 and 1935, the Hudson Bay Company of Canada kept records of the number of lynx and hare pelts they sold. If x denotes the hare population at time t and y denotes the lynx population at time t in years, then their data suggests a Lotka-Volterra model of the form

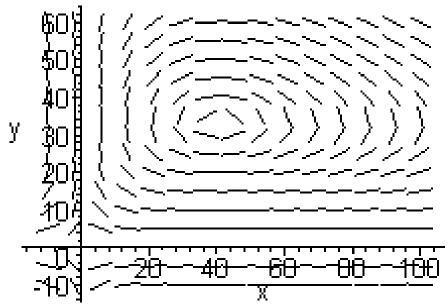
$$\frac{dx}{dt} = x - 0.03xy, \qquad \frac{dy}{dt} = -0.4y + 0.01xy$$

The equilibria are consequently $(0, 0)$ and $\left(\frac{0.4}{0.01}, \frac{1}{0.03}\right) = (40, 33.3)$, which says that a population of 40 hares and 33.3 lynx's results in a constant number of hares and lynx's for all time.

The slope field for the phase portrait is the slope field for the equation

$$\frac{dy}{dx} = \frac{-0.4y + 0.01xy}{x - 0.03xy}$$

which is shown in figure 6-10:



7-10: Slope field for Lotka-Volterra Phase Portrait

The slope field shows that phase curves are closed cycles about the equilibrium. Thus, for initial populations near the equilibrium populations, the lynx-hare populations oscillate.

It must be pointed out, however, that while the equilibrium $(40, 33.3)$ matches the data from the Hudson bay company, the parameters $A = 1$, $B = 0.4$, $C = 0.03$, and $D = 0.01$ are not realistic. For example, $A = 1$ implies that in the absence of lynx, the hare population would more than double once every 8 months. Thus, while the Lotka-Volterra model is very instructive, it is not quite sophisticated enough to model predator-prey interactions in the real world (see exercises 44 and 45).

Exercises:

Each of the following equations is autonomous. Identify any equilibria of the differential equation and then use a slope field to determine if they are stable. Assume $H > 0$, $N > 0$ and $k > 0$ are constants.

1.	$y' = y\,(1-y)$	2.	$y' = y\,(2-y)$
3.	$y' = 2y\,(1-y)$	4.	$y' = 2y\,(10-y)$
5.	$y' = 0.05y\left(1-\dfrac{y}{500}\right)$	6.	$y' = 0.5y\left(1-\dfrac{y}{100}\right)$
7.	$y' = 3y-y^2$	8.	$y' = y^2-3y$
9.	$y' = y^2-3y+2$	10.	$y' = 2y^2+6y-9$
11.	$y' = \dfrac{y-1}{y^2+1}$	12.	$y' = e^y-1$
13.	$y' = \sin(y)\,,$	14.	$y' = \sin(y)$
15.	$y' = ky$	16.	$y' = ky$
17.	$y' = y\left(1-\dfrac{y}{N}\right)$	18.	$y' = y\left(1-\dfrac{y}{N}\right)+H$

Construct a slope field for each of the following equtions.

19.	$y' = x-y$	20.	$y' = xy$
21.	$y' = e^{-x}-y$	22.	$y' = \sin(x)+y$
23.	$y' = x^2$	24.	$y' = \cos(x)$
25.	$y' = 0,$	26.	$y' = 1$
27.	$y' = \dfrac{x-2y}{-x+y}$	28.	$y' = \dfrac{x^2-y^2}{2xy}$

29. **Harvesting:** Suppose that the number of trees in a certain forest increases at a rate of 5% per year, suppose that the carrying capacity of the forest is 1,000 trees per acre, and suppose that 200 trees per acre are harvested from the forest every year.

(a) Find the equilibria of the process.
(b) Use a slope field to determine which are stable and which are unstable.
(c) Interpret your findings in (b).

30. **Harvesting:** A certain fish population doubled the first year it was introduced into a lake where that species of fish had disappeared. It is estimated that the maximum fish population for the lake is 10,000 fish, and it is estimated that fishermen will take about 4,000 fish per year out of the lake.

(a) Find the equilibria of the fish population.
(b) Use a slope field to determine which are stable and which are unstable.
(c) Interpret your findings in (b).

In exercises 31 through 40, you must set up a differential equation model, find its equilibria, and determine if the equilibria are stable or unstable.

31. A certain drug is infused intravenously at a rate of 500cc per hour. If the body removes 25% of the drug per hour, then how much of the drug will be in the body after a long period of time?

32. A certain drug is administered intravenously at a rate of 2cc per hour. Suppose that the body reduces the drug concentration by 15% per hour. How much of the drug will be in the body after 24 hours?

33. Suppose that the velocity v in feet per second of a raindrop with a diameter of 0.002 inches satisfies

$$v' = 32 - 118.44v$$

What is the terminal velocity of the raindrop?[12]

34. As a certain object falls through the air, the atmosphere slows its progress in proportion to its velocity. If the object is slowed at a rate of 10% per second, what is its terminal velocity?

35. Acme College accepts 600 new students each year, and every year, 30% of the students are lost to graduation, poor grades, transfer, etcetera. If the college continues in this way for a long period of time, then how large—in numbers of students—will the college eventually become?

36. Each month, Crooked Credit Corporation increases your credit card balance by 2%, and each month you pay the minimum monthly payment of $100. What will your balance eventually be? Will you ever pay off the credit card?[13]

[12] Adapted from "Falling Raindrops" by Walter J. Meyer in Applications of Calculus, Philip Straffin, Editor, MAA Notes number 29, The Mathematical Association of America, 1993.

[13] Credit card companies don't actually do this, but if you use your credit card every month, any credit card usage can degenerate to the point where you are essentially only paying off the interest on what you owe.

37. A dam is built across a stream flowing at a rate of 10,000 gallons of water each day. If 10% of the water is lost each day to evaporation, seepage through the soil, overflow of the dam, etcetera, then how large, in gallons, will the pond behind the dam eventually become?

38. A fifteen year mortgage at an annual interest rate of 7.2% (i.e., a monthly rate of $\frac{0.072}{12} = 0.006$) requires a monthly payment of \$1561. If $y(t)$ is the balance at time t in months and if $y(0) = \$171,000$, then what will the approximate balance be in one year? In five years? (Hint: use the fact that $y' \approx$ Interest - Payment).

39. Suppose that for the mortgage in exercise 38 that the borrower instead pays \$1,800 per month. What will the approximate balance be in one year? In five years?

40. A thirty year mortgage at an annual interest rate of 7.2% (i.e., a monthly rate of 0.6%) requires a monthly payment of \$1,166.52. If $y(t)$ is the balance at time t in months and if $y(0) = \$171,000$, then what will the approximate balance be in one year? In five years?

41. There is a great deal of evidence to suggest that once a population drops below a certain critical value m, the population will become extinct. We can incorporate the extinction factor into population growth by modifying the logistic equation into

$$y' = ky\left(1 - \frac{y}{N}\right)\left(1 - \frac{y}{m}\right) \tag{7.81}$$

where $0 < m < N$.

(a) What are the equilibria of (7.81)?

(b) Explain why $y = m$ is always unstable and all other equilibria are stable.

(c) Sketch a typical slope field of (7.81).

(d) Interpret the results in (c).

(e) * Can you solve (7.81) using separation of variables?

42. A special kind of unstable equilibrium is the *saddle equilibrium*, in which solutions approach the equilibrium from one side but not from the other. Construct slope field diagrams for the following and then identify their saddle equilibria.

(a)	$y' = y^2$	(b)	$y' = y^2 - 2y + 1$
(c)	$y' = y^3 - 4y^2$	(d)	$y' = y^4 - 4y^2$

43. Find the phase portrait of the Lotka-Volterra equations by solving

$$\frac{dy}{dx} = \frac{-By + Dxy}{Ax - Cxy}$$

where A, B, C, and D are constants. Can you infer that there is a closed curve solution from the closed form expression for the phase portrait?

44. What would the equilibria and phase-portrait slope field for the Lotka-Volterra equations be if $A = 0.2$, $B = 0.4$, $C = 0.03$, and $D = 0.01$?

45. Vito Volterra modified the Lotka-Volterra model in an attempt to explain the results of a 1924 statistical study of fish-shark populations in the Adriatic Sea that showed that the shark population increased dramatically during World War I and then decreased immediately thereafter. In particular, if A, B, C, and D are as in the Lotka-Volterra model and if E, F are fishing rate constants, then

$$\frac{dx}{dt} = Ax - Cxy - Ex, \qquad \frac{dy}{dt} = -By + Dxy - Fy$$

where x and y are the population of fish and sharks, respectively, at time t. What are the equilibria for this model if $E < A$? Explain why the slope field for this model is qualitatively the same as the slope field of the Lotka-Volterra model?

46. In the Lotka-Volterra model, we assumed exponential growth of the prey population. Suppose that in the *absence* of predators, the prey population grows logistically according to the law

$$\frac{dx}{dt} = Ax\left(1 - \frac{x}{N}\right)$$

where N is the carrying capacity for the prey. What is the resulting model? What are the equilibria? What does the slope field look like?

Self Test

A variety of questions are asked in a variety of ways in the problems below. Answer as many of the questions below as possible before looking at the answers in the back of the book.

1. Answer each statement as true or false. If the statement is false, then state why or give a counterexample.

   (a) The differential equation $\frac{dy}{dt} = 2y + t$ is a separable equation.

   (b) The differential equations $xyy' = 1$ is a separable equation.

   (c) The solution to $y' = 3y$, $y(0) = 2$ is $y = 3e^{2x}$.

   (d) A savings account balance grows exponentially.

   (e) Carbon-14 decays exponentially to Carbon 12.

   (f) In the partial fraction expansion of

   $$\frac{2x-7}{(x-5)(x+2)}$$

   we need only consider the form

   $$\frac{A}{x-5} + \frac{B}{x+2}.$$

   (g) In the partial fraction expansion of

   $$\frac{2x-7}{(x^2+5)(x+2)}$$

   we need only consider the form

   $$\frac{A}{x^2+5} + \frac{B}{x+2}.$$

   (h) $y' = 2y^2 - 5y$ is a logistic model.

   (i) $y' = y(5-y)$ is a logistic equation.

   (j) A separable equation of the form

   $$\frac{dy}{dx} = k\frac{g(x)}{f(y)}$$

   has solutions of the form $F(y) = kG(x) + C$ that can be fit to data to determine k and C.

   (k) Every differential equation of the form $y' = f(x, y)$ has an equilibrium solution.

   (l) A harmonic oscillation is a solution to the differential equation $y' + \omega^2 y = 0$.

   (m) The short line segments in a slope field diagram are practically the same as a short section of an actual solution curve.

2. If a savings account balance doubles every 10 years, then what is the approximate annual interest rate assuming continuous compounding?

   (a) 5% (b) 6% (c) 7% (d) 8% (e) 9%

3. Which of the following is a solution of

$$y' = 4x\sqrt{y}$$

that is **not** produced by separation of variables.

(a) $y = \left(x^2+2\right)^2$ (b) $y = x^3$ (c) $y = x^4$ (d) $y = 0$ (e) $y = 4x$

4. Which of the following is the same as $\displaystyle\int \frac{2x+1}{x^2-7x+12}dx$:

(a) $\displaystyle\int \frac{1}{x-4}dx - \int \frac{1}{x-3}dx$ (b) $\displaystyle\int \frac{3}{x-4}dx + \int \frac{5}{x-3}dx$

(c) $\displaystyle\int \frac{5}{x-4}dx + \int \frac{11}{x-3}dx$ (d) $\displaystyle\int \frac{9}{x-4}dx - \int \frac{7}{x-3}dx$

5. Solve the separable differential equation $\dfrac{dy}{dx} = x\left(1+y^2\right)$

(a) $\tan^{-1}(y) = \dfrac{x^2}{2} + C$ (b) $\ln\left(1+y^2\right) = \dfrac{x^2}{2} + C$

(c) $\tan(y) = \dfrac{x^2}{2} + C$ (d) $\dfrac{3}{3y+y^3} = \dfrac{x^2}{2} + C$

6. Evaluate the antiderivative

$$\int \frac{x-3}{x^2-x-2}dx$$

7. Find solutions to the differential equation

$$y' = \frac{x}{y(x+1)}$$

8. Find the solution to the differential equation $y' = (2y+1)x$, $y(0) = 1$

9. Fit the given data to the solutions of the differential equation to estimate k.

$$\frac{dy}{dx} = k\frac{x}{y},$$

x	0	1	2	5
y	2.0	2.2	2.9	5.2

10. In this exercise we consider two strains of Ecoli bacteria.

(a) Strain one consists of a bacteria with initial amount P_1 at time $t = 0$. This strain doubles its population size every 15 minutes (0.25 hours). What is the model for $y_1(t)$, which is the population of strain one at time t.(Recall section 3.3)

(b) The population size of the second strain of bacteria is estimated by measuring the area covered by the colony. The following measurements are taken

t (hours)	0	0.25	0.5	1.0
Area (sq. cm.)	1	1.7	2.8	8.3

Determine the model for $y_2(t)$, the population of strain two at time t.

(c) What is the doubling time for strain two?

(d) Which strain of bacteria grows faster?

11. Sketch the slope field of each of the equation $y' = -0.38y + 4$
12. What second order differential equation has solutions of the form

$$y(t) = a\cos(\omega t) + b\sin(\omega t) + M$$

where a, b, ω, and M are constants?

13. Solve the logistic equation $y' = 0.2y(3 - y)$ and then compute the limit of $y(t)$ as t approaches ∞. Then sketch the slope field of the equation and identify stable and unstable equilibria. What is the relationship between the limit of the closed form solution and the equilibria.
14. A falling object is subject to air resistance proportional to its velocity. Its speed is decreasing at 25% of its velocity per second. Use $g = 32$ ft./sec.^{2}.

    (a) What is the model for the event?

    (b) What is the terminal velocity for the object?

15. The Mississippi-Missouri river system has a mean flow rate of 18,000 cubic meters/sec. Suppose a dam were to be built across the Mississippi and suppose that it is estimated that 12% of the water in the reservoir behind the dam will be lost per day due to evaporation and outflow release. Let $y(t)$ denote the volume of water contained behind the dam at time t.

    (a) What is the model for $y(t)$? (Hint: first make sure rates are in the same units)

    (b) Solve the differential equation.

    (c) Approximately what volume of water will the reservoir have to hold?

16. A large cylindrical tank $6'$ in diameter filled initially with $1'$ of water has a 1 square inch round hole in the bottom of the tank. When the hole is unplugged the following data is recorded, where $y(t)$ is the height of the water in the tank at time t

t (min)	0	0.5	1.0	1.5
y (feet)	1	0.94	0.88	0.82

    (a) Find the model for $y(t)$. (Hint: Use Toricelli's law)

    (b) How may minutes does it take for the tank to completely drain?

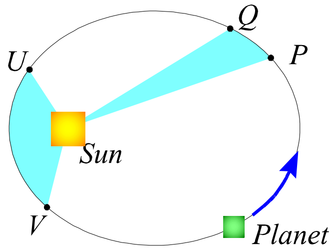
    (c) How many minutes does it take for the tank to empty half its contents?

    (d) At what rate is the water draining from the tank when half it has emptied half its contents?

The Next Step... Why Calculus?

Calculus (in particular, the fundamental theorem of calculus) was discovered independently by Sir Isaac Newton and Gottfried Liebniz in the late seventeenth century. But what motivated Newton to develop his version of the Calculus? Indeed, what is it about calculus that sets it apart from all fields of mathematics that came before? The answer to that question requires that we revisit the first mathematical model based on calculus—namely, the motion of the planets.

In the early 1600's, Johannes Kepler concluded a half-century of turmoil in astronomy by showing that a planet orbits the sun in an elliptical orbit with the sun at one focus and that the planet travels faster in its orbit when it is closer to the sun.

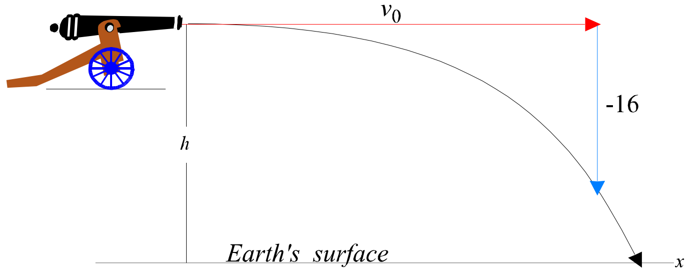


NS-1: Planet travels from P to Q in same amount of time it travels from U to V.

However, planets speeding up and slowing down as they traveled in egg-shaped orbits made astronomy even more mysterious than before. After years of arguing whether the earth or the sun was at the center of the known universe, the answer supplied by Kepler's laws was "Neither!" In fact, the motion of the planets now seemed stranger than ever with almost know explanation as to why the planets would behave in such ways.

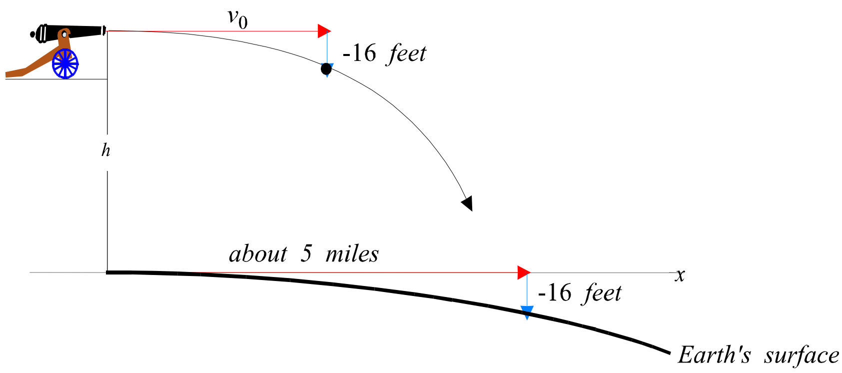
Kepler himself was not able to explain the cause of such motion, and for a half-century afterwards, scientists tried without success to explain how such orbits could occur. In fact, the mystery was not unraveled until 1665, when a plague in England closed Cambridge University and sent a young mathematician named Isaac Newton home to his uncle's farm.

To explain Keplerian motion, Newton built on the work of one of Kepler's contemporaries, the mathematician and scientist Galileo Galilei. At the end of the sixteenth century, Galileo showed that objects in free fall near the earth's surface have parabolic trajectories. For example, if a cannonball is fired horizontally with a muzzle velocity of v_0 from a height h above the earth, then after one second the cannon ball will have traveled a distance v_0 horizontally and will have fallen 16 feet vertically.



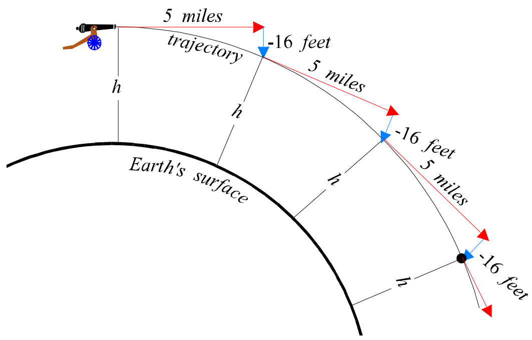
NS-2: Galilean projectile motion

Newton built upon Galileo's work by first realizing that the cannonball is falling toward a *curved* earth. In fact, at a distance of a little under five miles from the cannon, the earth's surface drops 16 feet below the x-axis.



NS-3: Galilean projectile motion above a curved earth

Thus, if v_0 is about 5 miles per second, then the cannonball's height h after one second is about the same as its initial height, which implies that motion in the "next" second can also be approximated by a Galilean trajectory with $v_0 \approx 5$, and then the next, and the next, and so on.



NS-4: A Patchwork of Galilean projectile motions

That is, Newton surmised that Keplerian orbits are "practically the same" as a patchwork of Galilean trajectories, each of which is valid only for a short distance. A *Galilean motion* is along a parabolic arc of the form

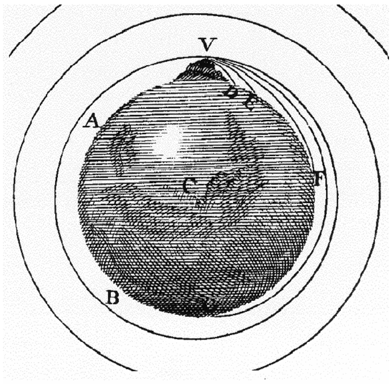
$$y = R_e + h - \frac{gx^2}{2v_0^2} \tag{7.82}$$

where $g = 32 \frac{ft}{\sec^2}$ is the acceleration due to gravity at the earth's surface and R_e is the radius of the earth, whereas a *Keplerian orbit* about the earth is an ellipse of the form

$$x^2 + y^2 = (p - \varepsilon y)^2 \tag{7.83}$$

where ε is the *eccentricity* of the orbit and p is the *parameter* of the orbit. Thus, Newton invented calculus so that he could model a Keplerian orbit locally by a Galilean orbit, and as a result, he was able to deduce his law of Universal gravitation and show that it explained all the "strangeness" inherent in Kepler's

3 laws. [14]



NS-5: Newton's original diagram from the *Principia*

Write to Learn If the cannonball has an initial velocity of $v_0 = 5$ miles per second and an initial height of 3965 miles from the center of the earth, then its Galilean parabolic trajectory is

$$y = 3965 - \frac{gx^2}{50}$$

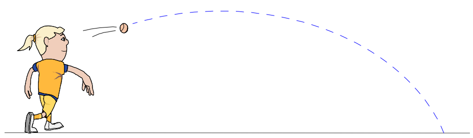
where $g = 32\ \frac{ft}{\sec^2} = \frac{32}{5280}\ \frac{miles}{\sec^2}$, and its Keplerian motion is

$$x^2 + y^2 = \left(4125 - \frac{32y}{793}\right)^2$$

Use differentiation on the Galilean trajectory and implicit differentiation on the Keplerian motion to show that $y(0)$, $y'(0)$, and $y''(0)$ are the same in both cases. Write a short essay explaining how this relates to the discussion above.

Write to Learn Locally, we assume the earth's surface is flat. Thus, if we throw a baseball, it will slow until it reaches a maximum altitude and then accelerate as it falls toward the earth.

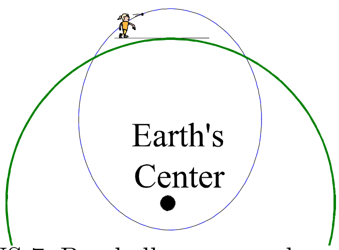
Local:



NS-6: Baseball on a flat earth

However, if all of the earth's mass were located at its center so that the surface of the earth did not get in the way, then the baseball would orbit the earth's center in a highly elliptical orbit.

Global:



NS-7: Baseball on a round earth

[14] It must also be pointed out that Calculus was discovered at about the same time by Gottfried Liebniz of Germany, to whom we owe much of the notation in calculus we now use.

Indeed, if v_0 is the horizontal velocity at the top of its arc and h is the maximum height of the ball, then the eccentricity of the ball's orbit is

$$\varepsilon = \frac{v_0^2}{g\left(R_e + h\right)} - 1$$

and its parameter is given by $p = (1+\varepsilon)(R_e + h)$. Write a short essay in which you determine the Keplerian path of a ball with a horizontal speed of 88 feet per second when it reaches its maximum height of 10 feet, and in that essay, draw an analogy between the relative speed of the baseball and the speeding up and slowing down of a planet as it orbits the sun.

Write to Learn Use the fact that Galilean trajectories (7.82) and Keplerian trajectories (7.83) as functions of x have the same y-coordinate, the same derivative and the same second derivative at $x = 0$ to derive the fact that

$$\varepsilon = \frac{v_0^2}{g\left(R_e + h\right)} - 1$$

and that $p = (1+\varepsilon)(R_e + h)$. Write a short essay explaining your computations and your results.

Write to Learn * Although we cannot consider the full scope of Newton's law of Universal Gravitation at this point, we can use the methods in this chapter to explore *vertical* motion of an object near the earth. In particular, if r is the height of an object at time t, then

$$m\frac{d^2 r}{dt^2} = \frac{-GMm}{r^2}$$

where m is the mass of the object, $M = 5.98 \times 10^{24} kg$ is the mass of the earth, and

$$G = 6.67 \times 10^{-11}\ \frac{Nm^2}{kg^2}$$

is the universal gravitational constant. Show that the phase curves are of the form

$$H = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{GM}{r}$$

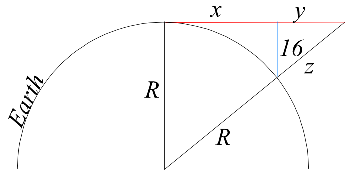
where H is a constant. What do the actual solutions look like when $H < 0$? When $H = 0$? When $H > 0$? Explain the signficance of all 3 cases.

Write to Learn Go to the library and/or surf the internet to learn more about the work of Galileo, Kepler and Newton. Then write a paper describing how Newton was led to generalize Galileo's projectile motion in order to explain Kepler's laws.

Group Learning In this group learning exercise, we "empirically" determine the value of v_0 necessary for the cannonball to achieve a circular orbit (assuming that the earth is a perfect sphere). Complete the following computations, and then present your findings to the class.

(a) **Compute the radius of the earth:** On a certain day, the sun rises in Knoxville at 7:14 a.m., but it rises in Nashville about 11 minutes later. If we assume that Nashville is 190 miles due west of Knoxville, then the surface of the earth is moving at a speed of $\frac{190}{11} = 17.273\ \frac{miles}{\min}$. Multiply by $24 \cdot 60 = 1440$ minutes in a day. The result is the circumference of the earth. Divide by 2π to obtain the radius of the earth.

(b) **Compute the curvature of the earth.** Let R denote the radius of the earth. Convert it into feet. (1 mile = 5280 feet). Use the Pythagorean theorem and the diagram below to determine the distance x in feet which must be traversed horizontally before the earth's surface is 16 feet *below* the horizontal line.



NS-8: x is the distance traveled tangentially until a 16 foot drop

(c) **Compute the distance an object falls in one second.** According to Galileo, the height $r(t)$ at time t of the cannonball is

$$r(t) = h - \frac{1}{2}gt^2$$

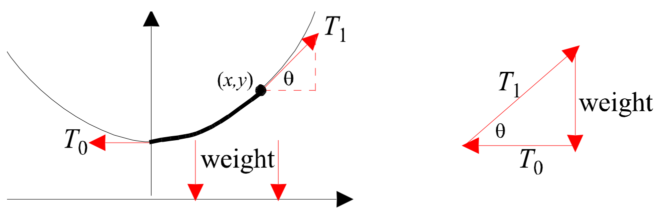
How far does the object fall in the first second if $g = 32$ feet per second per second?

(d) **Combine your computations into a presentation which explains why circular motion about the earth requires the initial velocity of about $v_0 = x$ feet per second, where x is as in (b)**.

Advanced Contexts:

Another important model—one that has been mentioned several times in the text—is that a cable hangs in the shape of a *hyperbolic cosine.* Let's conclude by using differential equations to show that this is true.

If the section of a hanging cable between its lowest point and a point (x, y) is not moving, then the force of its weight induces tensions T_0 and T_1 at either end of the section.



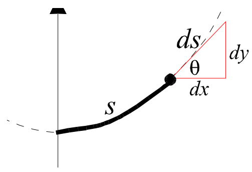
NS-9: Hanging Cable with weight diagram

Moreover, the tensions balance the weight of the cable, leading to the right triangle shown above. Thus,

$$\frac{weight}{T_0} = \tan(\theta)$$

since weight is a measure of the force of gravity. Notice, however, that

$$\frac{dy}{dx} = \tan(\theta)$$



Thus, we have

$$\frac{weight}{T_0} = \frac{dy}{dx}$$

If we now let s denote the distance from the lowest point to (x, y) and if let w denote the weight per unit length of the cable, then the weight of the section of the cable between the lowest point and the point (x, y) is

$$weight = ws$$

when w is constant. Letting y' denote dy/dx thus yields

$$T_0 y' = ws$$

so that application of the derivative yields

$$T_0 y'' = w\frac{ds}{dx} \tag{7.84}$$

However, in the triangle above, it is clear that

$$\frac{ds}{dx} = \sec(\theta) = \sqrt{1 + \tan^2(\theta)} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

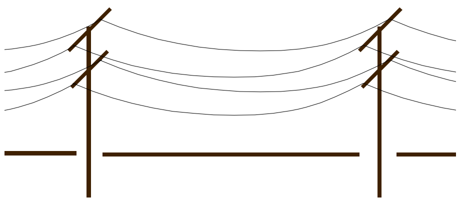
Replacing ds/dx in (7.84) by $\sqrt{1 + (y')^2}$ thus yields

$$y'' = \frac{w}{T_0}\sqrt{1 + (y')^2}$$

Usually, we let $a = w/T_0$, so that our differential equation becomes

$$y'' = a\sqrt{1 + (y')^2} \tag{7.85}$$

Let's solve (7.85) and thus show that a hanging cable, which is also known as a *catenary,* hangs in the shape of the hyperbolic cosine.



NS-10: Power lines hang in the shape of a catenary

1. Let $p = y'$. Then $p' = y''$ and (7.85) becomes

$$p' = a\sqrt{1 + p^2} \tag{7.86}$$

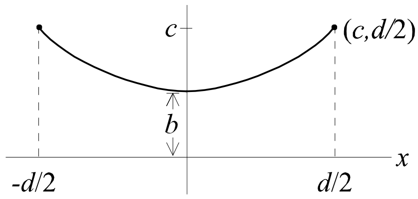
   which is a separable differential equation. Solve the equation for $p(x)$.

2. Let $p = y'$ and then integrate to find y.

3. Show that if the curve implied by the cable passes through the points $(c, d/2)$, $(0, b)$, and $(-c, d/2)$, then the equation of the curve is

$$y = b + \frac{1}{a}(\cosh(ax) - 1)$$

where a is the solution to the equation $\cosh\left(\frac{1}{2}ad\right) = 1 + a\,(c - b)$.



NS-11: Catenary with equation $y = b + \frac{1}{a}\left[\cosh(ax) - 1\right]$

8. SEQUENCES AND SERIES

Nearly all modern computers are descendants of the ENIAC (**E**lectronic **N**umerical **I**ntegrator **a**nd **C**omputer), a product of World War II scientific and engineering research which was unveiled on Valentine's Day in 1946. In particular, the ENIAC was a marriage of the electrical engineering of John W. Mauchly and J. Presper Eckert and the initial concepts of computer programming outlined by the mathematician John von Neumann. Many others also contributed, and to this day, there is some controversy as to the various roles played by the mathematicians and engineers who developed the ENIAC.

However, there is no controversy about the fact that the modern computer was developed as a tool for modern mathematics. The physicists, engineers and mathematicians who worked on ENIAC understood that they were creating a tool that would revolutionize computation, and indeed, computers have truly become the "logical engine" of mathematics envisioned by the ENIAC's developers.

In this chapter, we introduce some of the ideas in calculus which have made the computer a powerful tool in science and technology. In particular, we examine infinitely long lists of numbers, which are called *sequences,* and we explore infinitely long sums of numbers, which are called *series.* Moreover, we use sequences and series to explore an idea at the core of modern science—the decomposition of a function into a infinite series of sines and cosines.

8.1 Sequences

General Term of a Sequence

When the inputs of a function are restricted to the positive integers, then that function is called a *sequence.* Often we use the letter n to represent the input since n is generally used to denote a positive integer. For example, if n denotes a positive integer, then the function $f(n) = n^2$ is a sequence.

A sequence is a function whose inputs are restricted to the positive integers.

However, we generally do not use function notation to describe sequences, but instead we use the notation a_n to denote the *general term* of the sequence. For example, in place of $f(n) = n^2$, we instead write

$$a_n = n^2$$

and then we use $a_n = n^2$ to generate a_n for each n:

$$\begin{array}{cccccc} n=1 & n=2 & n=3 & \ldots, & \text{general term} & \ldots \\ 1, & 4, & 9, & \ldots, & n^2, & \ldots \end{array}$$

The notation "..." is known as an *ellipsis* and is interpreted to mean "and the pattern continues." Placing an ellipsis at the end of the list of number above indicates that the sequence is infinitely long.

In closed form, a sequence as a whole with general term a_n is denoted by $\{a_n\}_{n=1}^{\infty}$. However, in expanded form the sequence $\{a_n\}_{n=1}^{\infty}$ is an infinite list

$$a_1, a_2, \ldots, a_n, \ldots$$

Morever, the number of terms listed in the beginning is not important. That is, $\{a_n\}_{n=1}^{\infty}$ can also be written

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

and if the first few terms reveal a pattern, then the general term can be omitted altogether

EXAMPLE 1 Write the sequence $\{2^{-n}\}_{n=1}^{\infty}$ in expanded form.

Solution: Since the general term is $a_n = 2^{-n}$, letting $n = 1$ leads to $\frac{1}{2}$, letting $n = 2$ leads to $\frac{1}{4}$, and so on. Thus, the expanded form of $\{2^{-n}\}_{n=1}^{\infty}$ is

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots$$

EXAMPLE 2 Write the sequence $\left\{\frac{1}{n^2}\right\}_{n=1}^{\infty}$ in expanded form.

Solution: Since $a_n = 1/n^2$, the expanded form of the sequence is

$$1, \frac{1}{4}, \frac{1}{9}, \ldots, \frac{1}{n^2}, \ldots \tag{8.1}$$

Moreover, the expanded form can also be written

$$1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}, \frac{1}{49}, \ldots, \frac{1}{n^2}, \ldots$$

and since the first six or seven terms reveals a pattern, the general term can be omitted altogether:

$$1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}, \frac{1}{49}, \ldots$$

Check your Reading *What are the first four terms of the sequence* $\{3^{-n}\}_{n=0}^{\infty}$?

Graphs and Limits of Sequences

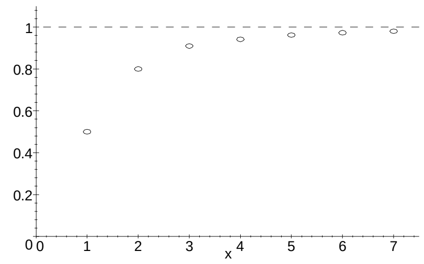
Since a sequence a_n is a function, its graph is the set of input-output pairs of the form (n, a_n), where n is a positive integer. For example, the sequence $\left\{\frac{n^2}{n^2+1}\right\}_{n=1}^{\infty}$ is given by

$$\frac{1}{2}, \frac{4}{5}, \frac{9}{10}, \frac{16}{17}, \frac{25}{26}, \frac{36}{37}, \frac{49}{50}, \ldots$$

and the corresponding input-output pairs are

$$\left(1, \frac{1}{2}\right), \left(2, \frac{4}{5}\right), \left(3, \frac{9}{10}\right), \left(4, \frac{16}{17}\right), \left(5, \frac{25}{26}\right), \left(6, \frac{36}{37}\right), \left(7, \frac{49}{50}\right), \ldots$$

If these pairs are plotted as points in the xy-plane, the result is the graph in figure 1-1.



1-1: Graph of a Sequence

Moreover, since a_n is a function, it can have a horizontal asymptote. Indeed, in the example above, the numbers in the sequence approach the horizontal asymptote $y = 1$. As a result, we say that the sequence $\left\{ \frac{n^2}{n^2+1} \right\}_{n=1}^{\infty}$ has a limit of 1 and we write

$$\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$$

This leads us to the following definition:

Definition 1.1: A sequence $\{a_n\}_{n=1}^{\infty}$ has a limit L, which we write as $\lim_{n\to\infty} a_n = L$, if given any $\varepsilon > 0$, there is a positive integer N such that if $n \geq N$, then $\quad |a_n - L| < \varepsilon$

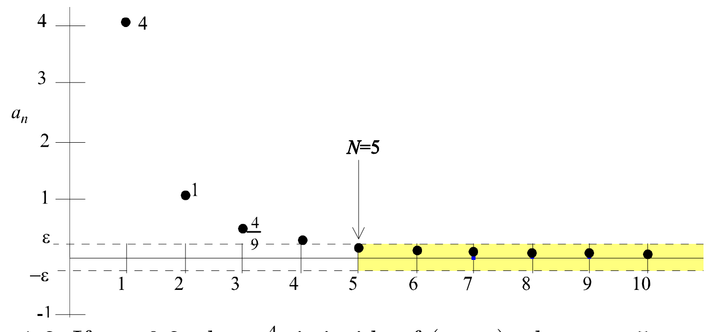
That is, N can be chosen so that all the terms a_n for $n \geq N$ are inside the target interval $(L - \varepsilon, L + \varepsilon)$.

EXAMPLE 3 Show that the sequence

$$4, \frac{4}{4}, \frac{4}{9}, \ldots, \frac{4}{n^2}, \ldots$$

has a limit of 0.

Solution: Given any $\varepsilon > 0$, we can find an N large enough to force $\frac{4}{n^2}$ to be inside of $(-\varepsilon, \varepsilon)$ for all $n \geq N$. For example, $\frac{4}{n^2}$ is within $\varepsilon = 0.2$ of 0 for all $n \geq 5$, as is shown in the figure below:



1-2: If $\varepsilon = 0.2$, then $\frac{4}{n^2}$ is inside of $(-\varepsilon, \varepsilon)$ when $n \geq 5$

Thus, the limit of the sequence must be 0.

Limits of sequences have properties similar to limits of function in general.

Theorem: If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergent sequences and k is a constant, then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n \qquad \lim_{n\to\infty}(ka_n) = k\lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty}(a_n - +b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n \qquad \lim_{n\to\infty} k = k$$

$$\lim_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) \qquad \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad if \ \lim_{n\to\infty} b_n \neq 0$$

In addition, if $\lim_{n\to\infty} a_n = L$ and f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

Also, there is a Squeeze theorem for sequences which says that

$$\text{if } a_n \leq c_n \leq b_n \text{ and if } \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L,$$
$$\text{then } \lim_{n\to\infty} c_n = L$$

EXAMPLE 4 Evaluate the following limit, if it exists:

$$\lim_{n\to\infty}\frac{4+4\cos(n)}{n^2}$$

Solution: To begin with, we can write this as the sum of two limits

$$\lim_{n\to\infty}\frac{4+4\cos(n)}{n^2} = \lim_{n\to\infty}\frac{4}{n^2} + \lim_{n\to\infty}\frac{4\cos(n)}{n^2}$$

In example 3, we showed that the first limit is 0. Moreover, since $-1 \leq \cos(n) \leq 1$, we have

$$\frac{-4}{n^2} \leq \frac{4\cos(n)}{n^2} \leq \frac{4}{n^2}$$

Example 3 implies $\lim_{n\to\infty}\frac{-4}{n^2} = \lim_{n\to\infty}\frac{4}{n^2} = 0$. Thus, the squeeze theorem implies that

$$\lim_{n\to\infty}\frac{4\cos(n)}{n^2} = 0$$

Check your Reading *So what is the value of the limit*

$$\lim_{n\to\infty}\frac{4+4\cos(n)}{n^2}$$

Limits with L'hopital's Rule

If we can find a function $f(x)$ such that $a_n = f(n)$ for all n, then

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) \tag{8.2}$$

when both limits exist. Moreover, (8.2) allows us to use L'hopital's rule to evaluate limits of sequences.

EXAMPLE 5 Find the limit of the sequence

$$\frac{1}{2}, \frac{4}{5}, \frac{9}{10}, \frac{16}{17}, \frac{25}{26}, \ldots, \frac{n^2}{n^2+1}, \ldots \tag{8.3}$$

Solution: The general term of the sequence is $a_n = \frac{n^2}{n^2+1}$, so that by (8.2), we have

$$\lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{x\to\infty} \frac{x^2}{x^2+1}$$

The latter limit is of the form $\frac{\infty}{\infty}$, so that we get

$$\lim_{x\to\infty} \frac{x^2}{x^2+1} \stackrel{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{2x}{2x} = 1$$

As shown earlier, the limit of the sequence (8.3) is 1.

EXAMPLE 6 Find the limit of the sequence

$$e^{-1}, 4e^{-2}, 9e^{-3}, 16e^{-4}, 25e^{-5}, \ldots$$

Solution: The general term is $a_n = n^2e^{-n}$, so that the limit of the sequence is

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2e^{-n} = \lim_{n\to\infty} x^2e^{-x}$$

Since the limit is of the form $\infty \cdot 0$, we transform it and use L'hopital's rule:

$$\lim_{n\to\infty} x^2e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} \stackrel{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{2x}{e^x} \stackrel{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$$

Thus, the limit of the sequence with general term $a_n = n^2e^{-n}$ is 0.

Check your Reading *Find the limit of the sequence*

$$\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \ldots, \frac{n}{2n+1}, \ldots$$

The Monotone Convergence Theorem

When the limit of a sequence cannot be evaluated using L'hopital's rule, we often resort to a theoretical but very useful result. In particular, if there is a number B such that

$$a_n \leq B$$

for all positive integers n, then B is said to be an *upper bound* on the sequence $\{a_n\}_{n=1}^{\infty}$. For example, the sequence

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots \tag{8.4}$$

has an upper bound of 1 since for all positive n we have

$$\frac{n}{n+1} < 1$$

The *least upper bound* of a sequence is the smallest possible upper bound for the sequence. For example, the sequence (8.4) has upper bounds of 1, 2, π and so on. However, the number 1 is the smallest possible upper bound on (8.4), so it is the least upper bound. Finally, if

$$a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n \leq a_{n+1} \leq \ldots$$

then the sequence $\{a_n\}_{n=1}^{\infty}$ is said to be *increasing.* Combining these concepts leads to the following result:

Monotone Convergence Theorem: If an increasing sequence has an upper bound, then the limit of the sequence is its least upper bound.

Likewise, if $a_1 \geq a_2 \geq \ldots \geq a_n \geq a_{n+1} \geq \ldots$, then $\{a_n\}_{n=1}^{\infty}$ is said to be *decreasing.* Moreover, if a decreasing sequence has a *lower bound,* then the limit of the sequence is its *greatest lower bound.*

To determine if a sequence is increasing or decreasing, we examine the difference $a_n - a_{n+1}$. If $a_n - a_{n+1} \leq 0$ for all n, then $a_n \leq a_{n+1}$ and the sequence is *increasing.* However, if $a_n - a_{n+1} \geq 0$, then the sequence is *decreasing.*

EXAMPLE 7 Given that $|x| < 1$, determine if the sequence

$$|x|, |x|^2, |x|^3, \ldots, |x|^n, \ldots \tag{8.5}$$

is increasing or decreasing, and then apply the monotone convergence theorem.

Solution: The general term is $a_n = |x|^n$, so that $a_{n+1} = |x|^{n+1}$. Thus, the difference $a_n - a_{n+1}$ becomes

$$a_n - a_{n+1} = |x|^n - |x|^{n+1} = |x|^n (1 - |x|) \geq 0$$

As a result, $a_n \geq a_{n+1}$, which implies that (8.5) is decreasing. Since the greatest lower bound of (8.5) is 0, we have

$$\lim_{n\to\infty} |x|^n = 0 \; if \;\; |x| < 1$$

$n!$ is the product of the integers from 1 up to and including n.

Sequences may also involve the *factorial function, $n!$*, which is defined by

$$n! = n \cdot (n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

For example, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ and $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. The most important property of the factorial function is that

$$(n+1)! = (n+1) \cdot n! \tag{8.6}$$

For example, $4! = 4 \cdot 3!$ and $5! = 5 \cdot 4!$.

EXAMPLE 8 Determine if the sequence

$$\frac{2}{1!}, \frac{3}{2!}, \frac{4}{3!}, \frac{5}{4!}, \ldots, \frac{n+1}{n!}, \ldots \tag{8.7}$$

is increasing or decreasing, and then apply the Monotone convergence theorem.

Solution: The general term is $a_n = \frac{n+1}{n!}$, so that $a_{n+1} = \frac{n+2}{(n+1)!}$. This leads to

$$a_n - a_{n+1} = \frac{n+1}{n!} - \frac{n+2}{(n+1)!} = \frac{n+1}{n+1} \cdot \frac{n+1}{n!} - \frac{n+2}{(n+1)!}$$

The property (8.6) thus yields a common denominator and allows us to simplify:

$$a_n - a_{n+1} = \frac{(n+1)^2 - (n+2)}{(n+1)!} = \frac{n^2+n-1}{(n+1)!}$$

Since $n^2+n-1 > 0$ for $n \geq 1$, we have $a_n \geq a_{n+1}$. Thus, the sequence (8.7) is decreasing.

Moreover, every term in the sequence (8.7) is positive, so it has a lower bound of 0. In fact, it can be shown that 0 is actually its greatest lower bound, so that the monotone convergence theorem implies that

$$\lim_{n\to\infty} \frac{n+1}{n!} = 0$$

Exercises:

Find the general term of the sequence

1. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$
2. $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots$
3. $\frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{7}{16}, \ldots$
4. $\frac{-1}{2}, \frac{2}{6}, \frac{-3}{24}, \frac{5}{120}, \ldots$
5. $\frac{-9}{2}, \frac{3}{4}, \frac{-1}{8}, \frac{1}{48}, \ldots$
6. $\frac{1}{1}, \frac{4}{2\cdot 1}, \frac{9}{3\cdot 2\cdot 1}, \frac{16}{4\cdot 3\cdot 2\cdot 1}, \ldots$

Graph the first four terms of each sequence and then evaluate the limit

7. $\lim\limits_{n\to\infty} \frac{1}{n+2}$
8. $\lim\limits_{n\to\infty} \frac{n}{n+2}$
9. $\lim\limits_{n\to\infty} \frac{n^2-1}{n^2+1}$
10. $\lim\limits_{n\to\infty} \frac{n^2}{n^2+2}$
11. $\lim\limits_{n\to\infty} \frac{3n^2+2n+1}{(2n+1)^2}$
12. $\lim\limits_{n\to\infty} \frac{(n-1)^2+1}{(2n+1)^2}$
13. $\lim\limits_{n\to\infty} \frac{n^2}{n+2}$
14. $\lim\limits_{n\to\infty} \frac{n^3}{(n+2)^2}$
15. $\lim\limits_{n\to\infty} \left(\frac{n}{n+2} - \frac{n^2+7}{(2n+1)^2}\right)$
16. $\lim\limits_{n\to\infty} \left(\frac{n}{n+2} + \frac{n}{\sqrt{n^2+1}}\right)$
17. $\lim\limits_{n\to\infty} \tanh(n)$
18. $\lim\limits_{n\to\infty} \ln(n)$
19. $\lim\limits_{n\to\infty} e^{-n}$
20. $\lim\limits_{n\to\infty} \operatorname{sech}(n)$

Determine if the sequence is increasing or decreasing by (a) graphing the first few terms of the sequence and then (b) by determining the sign of $a_n - a_{n+1}$. Does the sequence have a limit? Explain.

21. $a_n = \dfrac{1}{n}$

22. $a_n = \dfrac{1}{n^2}$

23. $a_n = \dfrac{2n+1}{n+3}$

24. $a_n = \dfrac{n-2}{4n+3}$

25. $a_n = \dfrac{4^n}{3^{n+1}}$

26. $a_n = \dfrac{4^{n+1}}{5^n}$

27. $a_n = \dfrac{4}{n!}$

28. $a_n = \dfrac{n+2}{n!}$

29. $a_n = \dfrac{n-1}{n!}$

30. $a_n = \dfrac{n^2+2n}{n!}$

31. Explain why the following is true:

$$\lim_{n\to\infty} x^n = \begin{cases} 0 & |x| < 1 \\ \infty & |x| > 1 \end{cases}$$

32. Evaluate the limit

$$\lim_{n\to\infty} \frac{n}{n+1}$$

and then use the result to explain why the sequence

$$\frac{-1}{2}, \frac{2}{3}, \frac{-3}{4}, \frac{4}{5}, \frac{-5}{6}, \ldots, (-1)^n \frac{n}{n+1}, \ldots$$

does not have a limit.

33. Suppose $a_n \geq 0$ for all n and suppose that

$$\lim_{n\to\infty} a_n = 0$$

Does the sequence $\{(-1)^n a_n\}_{n=1}^{\infty}$ converge? Explain.

34. Suppose $a_n \geq 0$ for all n and suppose that

$$\lim_{n\to\infty} a_n = L$$

where $L \neq 0$. Does the sequence $\{(-1)^n a_n\}_{n=1}^{\infty}$ converge? Explain.

35. Use the definition of the limit of a sequence to prove that

$$\text{if } \lim_{n\to\infty} a_n = K \quad \text{and} \quad \lim_{n\to\infty} b_n = L, \quad \text{then} \quad \lim_{n\to\infty} (a_n + b_n) = K + L$$

36. Use the definition of the limit of a sequence to prove that

$$\text{if } \lim_{n\to\infty} a_n = L \quad \text{and } r \text{ is a number,} \quad \text{then} \quad \lim_{n\to\infty} (ra_n) = rL$$

37. **Write to Learn:** What does the sequence $\{a_n\}_{n=1}^{\infty}$ look like if $a_1 = \frac{1}{10}$ and for $n \geq 2$ we have

$$a_{n+1} = a_n + \frac{n}{10^n}$$

Write a short essay in which you not only answer this question, but in which you also use the Monotone convergence theorem to prove that the sequence converges.

38. **Write to Learn:**Write a short essay in which you explain why a repeating decimal must be the limit of a sequence $\{a_n\}_{n=1}^{\infty}$ of the form

$$a_{n+1} = a_n + \frac{r}{10^{mn}}, \qquad a_1 = \frac{r}{10^m}$$

for fixed positive integers r and m with $r \leq 10^m$. Then use the Monotone convergence theorem to prove that all such sequences converge.

39. Let L be the least upper bound of a bounded increasing sequence $\{a_n\}_{n=1}^{\infty}$.

(a) For a given $\varepsilon > 0$, explain why there must be at least one positive integer N such that

$$a_N > L - \varepsilon$$

(b) Explain why it must follow that

$$a_n > L - \varepsilon \tag{8.8}$$

for all $n \geq N$.

(c) Show that (8.8) reduces to

$$|a_n - L| < \varepsilon$$

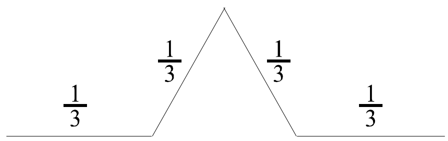
What is the value of the limit $\lim_{n\to\infty} a_n$?

40. Let G be the greatest lower bound of a bounded decreasing sequence $\{a_n\}_{n=1}^{\infty}$. Show that

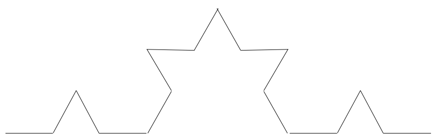
$$\lim_{n\to\infty} a_n = G$$

(Hint: let $b_n = -a_n$ and see previous exercise).

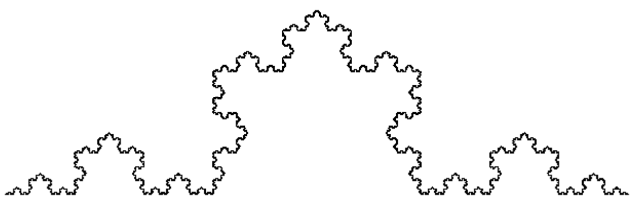
41. The Koch curve is started by dividing a line segment of length 1 into three identical parts, removing the middle third and replacing it with two segments that also have a length of $\frac{1}{3}$.



The middle third of each of these line segments is subsequently removed and replaced with two segments of the same length. The resulting curve has a length of 16/9.



If this process is continued ad-infinitum, the result is the *Koch curve*.



How long is the Koch curve?

8.2 Linear Recursion

Linear Recursion

There are many applications of sequences in which the coefficients of a sequence $\{y_n\}_{n=0}^{\infty}$ satisfy an equation of the form

$$y_{n+1} = f(y_n)$$

Such equations are called *recursions* and are important in many applications, as we will see in the next section.

EXAMPLE 1 For $y_0 = 2$, generate the next 4 terms (i.e., y_1, y_2, y_3, and y_4) of the sequence of iterates of the recursion

$$y_{n+1} = 2y_n^2 - 5$$

Solution: The recursion implies that

$$y_1 = 2y_0^2 - 5 = 8 - 5 = 3 \qquad and \qquad y_2 = 2y_1^2 - 5 = 15$$

Likewise, $y_3 = 2y_2^2 - 5 = 2(15)^2 - 5 = 445$ and

$$y_4 = 2y_3^2 - 5 = 2(445)^2 - 5 = 396,045$$

In this section, we concentrate on *linear recursions,* which are recursions of the form

$$y_{n+1} = ry_n + a \tag{8.9}$$

where r and a are constants. A linear recursion may also be called a *linear dynamical system* or a *linear difference equation.*

EXAMPLE 2 If $y_0 = 2$, what are the next four iterates of the linear recursion

$$y_{n+1} = \frac{1}{2}y_n + \frac{1}{2}$$

Solution: Since $y_0 = 2$, the recursion leads to

$$\begin{aligned}
y_1 &= \frac{1}{2}y_0 + \frac{1}{2} = \frac{2}{2} + \frac{1}{2} = \frac{3}{2} \\
y_2 &= \frac{1}{2}y_1 + \frac{1}{2} = \frac{3/2}{2} + \frac{1}{2} = \frac{3}{4} + \frac{1}{2} = \frac{5}{4} \\
y_3 &= \frac{1}{2}y_2 + \frac{1}{2} = \frac{5/4}{2} + \frac{1}{2} = \frac{5}{8} + \frac{1}{2} = \frac{9}{8} \\
y_4 &= \frac{1}{2}y_3 + \frac{1}{2} = \frac{9/8}{2} + \frac{1}{2} = \frac{9}{16} + \frac{1}{2} = \frac{17}{16}
\end{aligned}$$

Thus, the sequence of iterates of $y_{n+1} = \frac{1}{2}y_n + \frac{1}{2}$ with $y_0 = 2$ is

$$2, \frac{3}{2}, \frac{5}{4}, \frac{9}{8}, \frac{17}{16}, \ldots$$

Linear recursions often appear in contexts in which y_n is the *state* of the system after n time periods, and the equation (8.9) is a formula which *linearly* relates the *next* state of the system to the current state of the system.

EXAMPLE 3 Americanville has an initial population of 10,000 people in 1990. If the birth rate is 5% annually and the death rate is 1% annually, then find the linear recursion (8.9) that relates the current year's population to next years population. What will the population be in 2010?

Solution: If y_n denotes the population in the n^{th} year since 1990, then the number of births in the n^{th} year is $0.05y_n$ and the number of deaths in the n^{th} year is $0.01y_n$. Thus,

$$\begin{array}{ccccccc} \text{Next Year} & = & \text{Current} & + & \text{Births} & - & \text{Deaths} \\ y_{n+1} & = & y_n & + & 0.05y_n & - & 0.01y_n \end{array}$$

Thus, $y_{n+1} = 1.04y_n$. Since $y_0 = 10,000$, we have $y_1 = 1.04\,(10,000) = 10,400$ and

$$y_2 = 1.04y_1 = 1.04\,(10400) = 10,816$$

Similarly, $y_3 = 1.04y_2 = 11,249$ and $y_4 = 1.04y_3 = 11,699$. Since 2010 is the $n = 20^{th}$ year since 1990, we continue in the same fashion until we obtain y_{20}, which is

$$y_{20} = 1.04y_{19} = 21,911$$

Check your Reading *Why is $y_0 = 10,000$ in example 2?*

Geometric Progression

The general term of a sequence of iterates of a linear recursion can be expressed in closed form by the *formula for geometric progression.* Before introducing this formula, however, we must first establish an identity involving $1+r+\ldots+r^{n-2}+r^{n-1}$. In particular, if we expand the product

$$(1-r)\left(1+r+r^2+\ldots+r^{n-1}\right)$$

then the result is

$$\begin{array}{rcccccc} (1-r)\left(1+r+r^2+\ldots+r^{n-1}\right) = 1 & +r & +r^2 & +\ldots & +r^{n-1} & \\ & -r & -r^2 & -\ldots & -r^{n-1} & -r^n \end{array}$$

That is, we have

$$(1-r)\left(1+r+r^2+\ldots+r^{n-1}\right) = 1-r^n$$

which implies that

$$1+r+r^2+\ldots+r^{n-1} = \frac{1-r^n}{1-r} \tag{8.10}$$

Let's now apply this to linear recursions. If the initial value of (8.9) is y_0, then $y_1 = ry_0 + a$ and $y_2 = ry_1 + a$. Substituting y_1 into the equation for y_2 then yields

$$y_2 = r\,(\,ry_0 + a\,) + a = r^2 y_0 + ar + a$$

Likewise, $y_3 = ry_2 + a = r\left(r^2 y_0 + ar + a\right) + a$. This simplifies as well, and indeed,

$$y_3 = r^3 y_0 + ar^2 + ar + a \qquad \text{and} \qquad y_4 = r^4 y_0 + ar^3 + ar^2 + ar + a$$

As a result, the general term of a linear recursion is of the form

$$y_n = r^n y_0 \ + \ ar^{n-1} + \ldots + ar^2 + ar + a$$

Moreover, the general term can be factored into

$$y_n = r^n y_0 + a\left(1 + r + r^2 + \ldots + r^{n-1}\right)$$

As a result, the identity (8.10) implies the following:

Geometric Progression: The general term of the sequence of iterates of

$$y_{n+1} = ry_n + a$$

for an initial value y_0 is given by

$$y_n = r^n\,y_0 + a\left(\frac{1 - r^n}{1 - r}\right) \tag{8.11}$$

Sequences generated by (8.11) are also known as *geometric progressions.*

EXAMPLE 4 Find the general term of the recursion

$$y_{n+1} = \frac{1}{2} y_n + \frac{1}{2}, \quad y_0 = 2$$

Then use the general term to generate y_1, y_2, y_3, and y_{10}.

Solution: The recursion is of the form $y_{n+1} = ry_n + a$ where $a = r = \frac{1}{2}$. Thus, (8.11) yields

$$y_n = \frac{y_0}{2^n} + \frac{1}{2}\left(\frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}\right)$$

Simplifying the denominator then yields

$$y_n = \frac{2}{2^n} + \frac{1}{2}\left(\frac{1 - \frac{1}{2^n}}{\frac{1}{2}}\right) = \frac{1}{2^{n-1}} + 1 - \frac{1}{2^n}$$

Since $\frac{1}{2^{n-1}} = \frac{2}{2^n}$, we further simplify to

$$y_n = \frac{2}{2^n} + 1 - \frac{1}{2^n} = 1 + \frac{1}{2^n}$$

When $n = 0$, we obtain $y_0 = 1 + \frac{1}{2^0} = 1 + 1 = 2$. Likewise,

$$\begin{aligned} y_1 &= 1 + \frac{1}{2^1} = 1 + \frac{1}{2} = \frac{3}{2} \\ y_2 &= 1 + \frac{1}{2^2} = 1 + \frac{1}{4} = \frac{5}{4} \\ y_3 &= 1 + \frac{1}{2^3} = 1 + \frac{1}{8} = \frac{9}{8} \end{aligned}$$

which matches what we saw in example 2. Moreover, when $n = 10$, we obtain

$$y_{10} = 1 + \frac{1}{2^{10}} = 1 + \frac{1}{1024} = \frac{1025}{1024}$$

which in decimal form is $y_{10} = 1.0009765625$.

Check your Reading *What would you hypothesize is the limit of the sequence in example 3?*

Limits and Applications

If $|r| < 1$, then $r = e^{-\beta}$ for some number $\beta > 0$, so that

$$\lim_{n\to\infty} r^n = \lim_{n\to\infty} e^{-\beta n} = 0$$

Thus, if $|r| < 1$, then

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} r^n\, y_0 + a \lim_{n\to\infty} \left(\frac{1 - r^n}{1 - r} \right) = 0 + a\frac{1 - 0}{1 - r}$$

This yields the following:

Theorem 2.2: If $|r| < 1$ and $y_{n+1} = ry_n + a$, then

$$\lim_{n\to\infty} y_n = \frac{a}{1 - r}$$

If $|r| \geq 1$, then the limit of $\{y_n\}_{n=0}^{\infty}$ does not exist.

For instance, example 4 yields

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \left(1 + \frac{1}{2^n} \right) = 1$$

which is the same as the result in theorem 2.2 with $a = 0.5$ and $r = 0.5$.

EXAMPLE 5 Find the general term and compute the limit of

$$y_{n+1} = 0.7y_n + 3, \quad y_0 = 1 \tag{8.12}$$

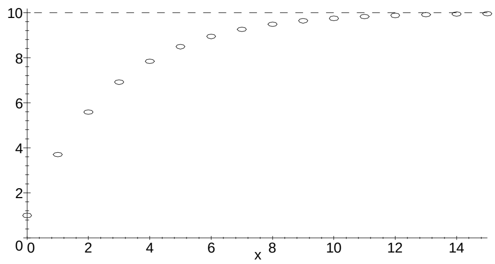
Solution: Since $r = 0.1$ and $a = 9$, the formula (8.11) implies that the general term of the sequence is

$$y_n = (0.7)^n\, y_0 + 3\left(\frac{1 - (0.7)^n}{1 - (0.7)} \right) = (0.7)^n + 10\left(1 - (0.7)^n\right) \tag{8.13}$$

since $y_0 = 0$. As a result, the limit of the recursion (8.12) is

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \left[(0.7)^n + 10\left(1 - (0.7)^n\right)\right] = 0 + 10\left(1 - 0\right) = 10 \quad (8.14)$$

The sequence and the limiting value of 10 are shown below:



2-1: Sequence has a limit of 10

Geometric progressions and their limits are important in a wide number of applications, as we shall see in the remainder of this section.

EXAMPLE 6 Each month, the Gotham Gazette loses 5% of its subscribers. However, a sustained advertising campaign produces 1000 new subscribers each month. If both trends continue indefinitely, about how many subscribers will the Gotham Gazette eventually have?

Solution: Let y_n denote the number of subscribers in the n^{th} month. Then

Next Month		95% of This Month		New Subscribers
y_{n+1}	$=$	$0.95y_n$	$+$	1000

However, the general term for $y_{n+1} = 0.95y_n + 1000$ is

$$y_n = (0.95)^n\ y_0 + 1000\left(\frac{1 - (0.95)^n}{1 - 0.95}\right)$$

The limit as n approaches ∞ of y_n is

$$\begin{aligned}\lim_{n\to\infty} y_n &= \lim_{n\to\infty}\left((0.95)^n\ y_0 + 1000\left(\frac{1 - (0.95)^n}{0.05}\right)\right)\\ &= 0\cdot y_0 + 1000\left(\frac{1-0}{0.05}\right)\\ &= 20,000\end{aligned}$$

Thus, the Gotham Gazette can expect to have about 20,000 subscribers if both the trend of losing 5% and also the trend of gaining 1000 new subscribers continues indefinitely.

Check your Reading *If the Gotham Gazette has 20,000 readers currently, about how many will they have a year later?*

Applications to Finance

Linear recursions occur frequently in financial applications. In particular, if y_n denotes the balance during the n^{th} period of an investment and if i is the interest rate per period, then $i \cdot y_n$ is the amount of interest paid in the n^{th} period and

$$\begin{array}{ccccccc} \text{next period} & = & \text{previous period} & + & \text{interest} & + & \text{payment} \\ y_{n+1} & = & y_n & + & i \cdot y_n & + & a \end{array}$$

which simplifies to $y_{n+1} = (1+i)\, y_n + a$. Thus, $r = 1 + i$ and a is the amount of each payment.

EXAMPLE 7 Find the balance after 1 year of a loan of $100,000 for 25 years at an interest rate of 9% annually if the monthly payments are $839.20.

Solution: Since the annual interest rate is 9% and the periods are months, the interest rate per period is

$$i = \frac{9\%}{12} = 0.75\% = 0.0075$$

If y_n denotes the loan balance after n months, then

$$\begin{array}{ccccccc} \text{next month} & = & \text{previous month} & + & \text{interest} & - & \text{payment} \\ y_{n+1} & = & y_n & + & 0.0075\, y_n & - & 839.20 \end{array}$$

which simplifies to the linear recursion

$$y_{n+1} = 1.0075 y_n - 839.20$$

Since the initial balance is $100,000, (8.11) yields a general term of

$$y_n = (1.0075)^n \cdot 100,000 - 839.20 \left(\frac{1 - (1.0075)^n}{1 - 1.0075} \right)$$

Thus, after one year, the loan balance will be

$$y_{12} = (1.0075)^{12} \cdot 100,000 - 839.20 \left(\frac{1 - (1.0075)^{12}}{1 - 1.0075} \right) = 98,884.32$$

We can also use linear recursions to determine the payment for a given loan. In particular, in order to pay off the loan in N equal payments, we simply leave a as a parameter, set $y_N = 0$, and solve for a.

EXAMPLE 8 Find the monthly payment required for an auto loan of $20,000 over 5 years at 12% APR compounded monthly.

Solution: Since $i = \frac{0.12}{12} = 0.01$, the loan balance y_n after n months satisfies

$$\begin{array}{ccccccc} \text{next month} & = & \text{previous month} & + & \text{interest} & + & \text{payment} \\ y_{n+1} & = & y_n & + & 0.1\, y_n & + & a \end{array}$$

where the monthly payment, a, is unknown. The result is the linear recursion

$$y_{n+1} = 1.01y_n + a$$

Since the initial balance is $y_0 = \$20,000$, (8.11) yields a general term of

$$y_n = (1.01)^n \cdot 20,000 + a\frac{1-(1.01)^n}{1-1.01}$$

Since 5 years is $n = 60$ months, paying off the loan means $y_{60} = 0$. Thus,

$$0 = (1.01)^{60} \cdot 20000 + a\frac{1-(1.01)^{60}}{1-1.01}$$

Solving for a yields $a = -444.89$. Thus, the monthly payment for the auto loan is

$$\$444.89 \quad \text{per} \quad \text{month}$$

which thus tells us that a \$20,000 automobile purchased with a 5 year loan at 12% actually costs $60 \times \$444.89 = \$26,693.40$.

Exercises:

For the given y_0, generate the next 4 terms (i.e., y_1, y_2, y_3, and y_4) of the sequence of iterates of each recursion.

1. $y_{n+1} = 2y_n(1-y_n)$, $y_0 = 0.1$
2. $y_{n+1} = y_n(1-y_n)$, $y_0 = 0.5$
3. $y_{n+1} = 3.2y_n(1-y_n)$, $y_0 = 0.5$
4. $y_{n+1} = 3.5y_n(1-y_n)$, $y_0 = 0.5$
5. $y_{n+1} = 3.83y_n(1-y_n)$, $y_0 = 0.5$
6. $y_{n+1} = 4y_n(1-y_n)$, $y_0 = 0.5$
7. $y_{n+1} = y_n^3 + 0.75y_n$, $y_0 = 0.2$
8. $y_{n+1} = y_n^3 - 1.44y_n$, $y_0 = 0.2$
9. $y_{n+1} = y_n - \ln|y_n|$, $y_0 = 0.99$
10. $y_{n+1} = y_n + \ln|y_n|$, $y_0 = 0.99$
11. $y_{n+1} = \sin\left(\frac{\pi}{3}y_n\right)$, $y_0 = 1$
12. $y_{n+1} = \sin\left(\frac{\pi}{2}y_n\right)$, $y_0 = 0.5$

Find the general term of the following linear recursions given that $y_0 = 3$ in each. Use the general term to produce the next four terms of the sequence. Then use the general term to determine if the sequence has a limit.

13. $y_{n+1} = 2y_n$
14. $y_{n+1} = 4y_n$
15. $y_{n+1} = \frac{1}{2}y_n$
16. $y_{n+1} = \frac{1}{3}y_n$
17. $y_{n+1} = 0.2y_n + 7$
18. $y_{n+1} = 0.2y_n - 5$
19. $y_{n+1} = -y_n + 1$
20. $y_{n+1} = -0.1y_n + 11$
21. $y_{n+1} = -\frac{1}{5}y_n - 1$
22. $y_{n+1} = -2y_n - 1$
23. $y_{n+1} = \frac{5}{4}y_n + \frac{1}{8}$
24. $y_{n+1} = \frac{4}{5}y_n + \frac{7}{10}$

25. Suppose that we borrow \$100,000 for 25 years at an interest rate of 9% annually, but instead of making the required monthly payments of \$839.20, we instead pay \$1000 each month. What will the loan balance be at the end of the first year? At the end of the second year?

26. A fifteen year mortgage at an annual interest rate of 7.25% requires a monthly payment of \$1561. If $y(t)$ is the balance at time t in months and if $y(0) = \$171,000$, then what will the balance be in one year? In five years?

27. Suppose that for the mortgage in exercise 26 that the borrower instead pays \$1,800 per month. What will the approximate balance be in one year? In five years? How much less is this than the corresponding balances in exercise 27?

28. A thirty year mortgage at an annual interest rate of 7.25% requires a monthly payment of $\$1,166.52$. If $y(t)$ is the balance at time t in months and if $y(0) = \$171,000$, then what will the balance be in one year? In five years?

29. What will the monthly payments be for a 30 year loan of \$200,000 home loan given a 7.5% APR? Compute the product of the amount of each payment and the number of payments to determine the ultimate cost of the home.

30. What will the monthly payments be for a 20 year loan of \$200,000 home loan given a 7.5% APR? Compute the product of the amount of each payment and the number of payments to determine the ultimate cost of the home.

31. Suppose we know that we can afford to pay \$500.00 each month for an automobile. If we borrow the money for the car at 12% interest over four years, how expensive a car can we afford (Hint: solve for y_0 given that $y_{48} = 0.$)

32. Which has lower payments, a loan of \$200,000 for 30 years with monthly payments given a 7.5% APR or a loan of \$200,000 for 15 years with 6% APR.

In exercises 33 through 38, you must construct a linear recursion and then find its limit.

33. Acme College accepts 600 new students each year, and every year, 30% of the students are lost to graduation, poor grades, transfer, etcetera. If the college continues in this way for a long period of time, then how large—in numbers of students—will the college eventually become? (i.e., set up the linear recursion and find its limit).

34. Each month, Crooked Credit Corporation increases your credit card balance by 2%, and each month you pay the minimum monthly payment of \$100. What will the balance eventually be? Will you ever pay off the credit card?

35. A certain drug is infused intravenously at a rate of 500cc per hour. If the body removes 25% of the drug per hour, then after a long period of time, how much of the drug will be in the body at any given time?

36. A certain drug is administered intravenously at a rate of 2cc per hour. Suppose that the body reduces the drug concentration by 15% per hour. How much of the drug will be in the body after 24 hours? What is the equilibrium solution?

37. If c_n denotes the concentration of Argon in the n^{th} expelled breath and q denotes the ratio of atmospheric air to lung capacity exchanged in each breath, then

$$c_{n+1} = (1 - q)\, c_n + q\gamma$$

where γ is the concentration of Argon in the atmosphere ($\gamma = 0.93\% = 0.0093$). If $q = 18\%$, then what is the equilibrium concentration of Argon expelled in each breath?

38. A certain species of moth inhabiting a certain island breeds annually and then dies. If 80% of the offspring survive to reproduce, and if approximately 700 moths immigrate to the island from surrounding islands each year, what will the moth population of the island eventually be?

39. In this exercise, we derive the linear recursion formula (8.11) in a different manner.

(a) Notice that we can write the linear recursion formula as

$$y_{n+1} = ry_n + a\frac{1-r}{1-r} \tag{8.15}$$

Use this to generate y_1 in terms of y_2. How is it related to (8.11) when $n = 1$?

(b) Substitute the expression for y_1 into the expression for

$$y_2 = ry_1 + a\frac{1-r}{1-r}$$

Simplify to the form implied by (8.11) when $n = 2$.

(c) Substitute (8.11) for y_n in (8.15) and simplify. What is the result?

40. **Write to Learn:** Write a short essay which explains why if m and b are positive integers such that $b < 10^m$, then the linear recursion

$$y_{n+1} = \frac{1}{10^m}y_n + \frac{b}{10^m}$$

generates a repeating decimal. For example, if $y_0 = 0$, $m = 1$ and $b = 7$, then

$$y_{n+1} = \frac{1}{10}y_n + \frac{7}{10}, \quad y_0 = 0$$

generates the sequence $y_0 = 0$, $y_1 = 0.7$, $y_2 = 0.77$, $y_3 = 0.777$, and so on. How would we determine the rational number represented by this repeating decimal?

8.3 Discrete Dynamical Systems

Fixed Points and Web Diagrams

Linear recursions are a special type of recursion, in that a *recursion* is a sequence $\{x_n\}_{n=1}^{\infty}$ generated by a formula of the form

$$x_{n+1} = f(x_n) \tag{8.16}$$

Recursions in which x_0 varies across a wide range of values are called *discrete dynamical systems*. In this section, we explore recursions and discrete dynamical systems graphically, numerically, and analytically.

To begin with, a number p such that $f(p) = p$ is called a *fixed point* of $f(x)$. A fixed point in a dynamical system is an analogue of an *equilibrium* in the slope field of a differential equation.

EXAMPLE 1 What are the fixed point(s) of the recursion

$$x_{n+1} = 2.5x_n - x_n^2$$

Solution: The fixed points are solutions to $f(x) = x$, where $f(x) = 2.5x - x^2$. Thus,

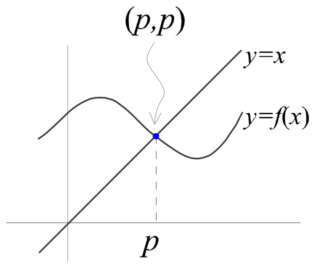
$$\begin{aligned} 2.5x - x^2 &= x \\ 1.5x - x^2 &= 0 \\ x(1.5 - x) &= 0 \end{aligned}$$

so that the fixed points are 0 and 1.5. To check our work, let us notice that if $x_0 = 1.5$, then

$$\begin{aligned} x_1 &= 2.5x_0 - x_0^2 = 2.5 \cdot 1.5 - 1.5^2 = 3.75 - 2.25 = 1.5 \\ x_2 &= 2.5x_1 - x_1^2 = 2.5 \cdot 1.5 - 1.5^2 = 3.75 - 2.25 = 1.5 \end{aligned}$$

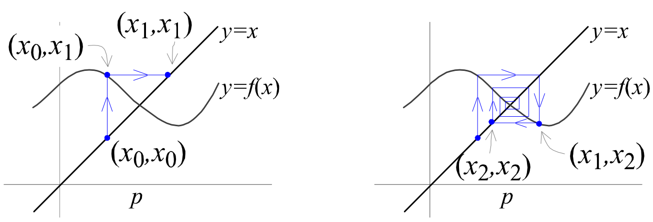
and so on. Similarly, if $x_0 = 0$, then $x_1 = x_2 = \ldots = 0$.

We can explore the behavior of a discrete dynamical system in the vicinity of a fixed point by using a *web diagram*. A web diagram is constructed by drawing both the line $y = x$ and the curve $y = f(x)$. If $f(x)$ has a fixed point at p, then the line $y = x$ intersects $y = f(x)$ at $x = p$.



3-1: $f(p) = p$

Let us now choose a point (x_0, x_0) on the line $y = x$ (see below right). Moving vertically to the curve $y = f(x)$ means moving to the point (x_0, x_1), where $x_1 = f(x_0)$, and then moving horizontally to $y = x$ takes us to the point (x_1, x_1).



3-2: Web Diagrams

Moving vertically again to $y = f(x)$ yields (x_1, x_2) where

$$x_2 = f(x_1)$$

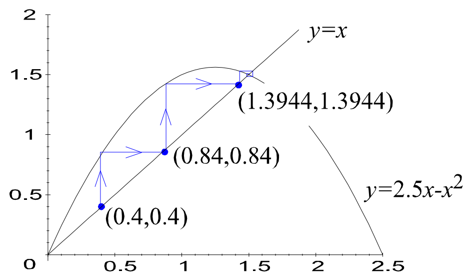
and then moving horizontally to $y = x$ takes us to (x_2, x_2). Continuing ad infinitum thus gives us a graphical representation of (8.16). Indeed, if the web diagram "closes " in on the fixed point (p, p) (as it does above right), then

$$\lim_{n\to\infty} x_n = p$$

EXAMPLE 2 For $x_0 = 0.4$, construct a web diagram for the iterates of the discrete dynamical system

$$x_{n+1} = 2.5x_n - x_n^2$$

Solution: To construct a web diagram, we graph the function $y = 2.5x - x^2$ along with the line $y = x$. We then begin at $(x_0, x_0) = (0.4, 0.4)$, move vertically to $y = 2.5x - x^2$, and then move horizontally to $(0.84, 0.84)$ on $y = x$. We then move vertically to $y = 2.5x - x^2$, move horizontally to $(1.3944, 1.3944)$ on $y = x$, and so on.



3-3: Fixed Point is attracting

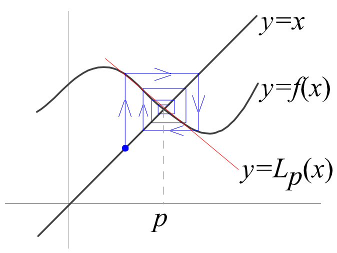
Since it appears that the sequence is "closing in" on the fixed point $(1.5, 1.5)$, we can conclude that

$$\lim_{n\to\infty} x_n = 1.5$$

Check your Reading *What is x_2 if $x_0 = 0.4$ and $x_{n+1} = 2.5x_n - x_n^2$?*

The Fixed Point Theorem

Suppose that $f(x)$ is differentiable at a fixed point p, and let $L_p(x)$ denote the linearization of $f(x)$ at p. A web diagram illustrates that iteration of $f(x)$ is practically the same as iteration of $L_p(x)$ as long as x_n is sufficiently close to p.



3-4: Iteration with a linear Approximation of f

That is, if x_n is sufficiently close to p, then $x_{n+1} = f(x_n)$ is practically the same as

$$x_{n+1} = L_p(x_n) \tag{8.17}$$

Since $L_p(x) = f(p) + f'(p)(x-p)$ and since $f(p) = p$, the recursion (8.17) is a of the form

$$x_{n+1} = p + f'(p)(x_n - p)$$

Now suppose that $x_0 = p + h$ where h is very close to 0. Then $x_1 = f(x_0)$ is practically the same as

$$x_1 = p + f'(p)(p + h - p) = p + f'(p)h$$

Moreover, $x_2 = f(x_1)$ is practically the same as

$$\begin{aligned} x_2 &= p + f'(p)(x_1 - p) \\ &= p + f'(p)(p + f'(p)h - p) \\ &= p + [f'(p)]^2 h \end{aligned}$$

Likewise, it is easy to show that $x_n = p + [f'(p)]^n h$ for all $n \geq 0$.

If $|f'(p)| < 1$, then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}(p) + h \lim_{n\to\infty}[f'(p)]^n = p + h \cdot 0$$

We summarize this result in the falling theorem.

Fixed Point Theorem: If $f(p) = p$ and if $|f'(p)| < 1$, then the sequence $x_{n+1} = f(x_n)$ satisfies

$$\lim_{n\to\infty} x_n = p$$

for x_0 chosen sufficiently close to p.

If $|f'(p)| < 1$, then we say that p is an *attracting fixed point* of f, and if $|f'(p)| > 1$, then we say that p is a *repelling fixed point* of f.

EXAMPLE 3 Find and classify the fixed points of

$$x_{n+1} = x_n^3 - 0.96x_n \tag{8.18}$$

as either attracting or repelling. Illustrate with a web diagram.

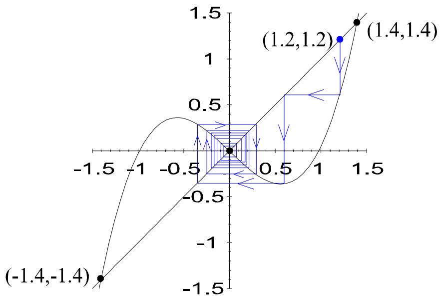
Solution: Since $f(x) = x^3 - 0.96x$ the fixed points are solutions to $f(x) = x$.

$$\begin{aligned} x^3 - 0.96x &= x \\ x^3 - 1.96x &= 0 \\ x\left(x^2 - 1.96\right) &= 0 \\ x(x-1.4)(x+1.4) &= 0 \end{aligned}$$

Thus, the fixed points are $x = -1.4, 0, 1.4$. However, $f'(x) = 3x^2 - 0.96$, so that

$$\begin{aligned} |f'(-1.4)| &= \left|3(-1.4)^2 - 0.96\right| = 4.92 > 1, \qquad p = -1.4 \textit{ is repelling} \\ |f'(0)| &= \left|3(0)^2 - 0.96\right| = 0.96 < 1, \qquad p = 0 \textit{ is attracting} \\ |f'(1.4)| &= \left|3(1.4)^2 - 0.96\right| = 4.92 > 1, \qquad p = 1.4 \textit{ is repelling} \end{aligned}$$

This is illustrated in the web diagram below for $x_0 = 1.2$.



3-5: An attracting Fixed Point

Check your Reading *What is* x_1 *if* $x_0 = 1.4$ *and* $x_{n+1} = x_n^3 - 0.96x_n$?

Discrete Dynamical Systems in Applications

Population growth is often modeled by equations of the form

$$x_{n+1} = f(x_n)$$

where x_n is the number of individuals in a given population after n time steps. For example, *logistic growth* is often modeled by a recursion of the form

$$x_{n+1} = x_n + bx_n\left(1 - \frac{x_n}{N}\right) \tag{8.19}$$

where N is called the *carrying capacity* of the system.

EXAMPLE 4 Find and classify the fixed points of (8.19) when $b = 2$ and $N = 4$.

Solution: Since $x_{n+1} = 2x_n\left(1 - \frac{x_n}{4}\right)$ implies that $f(x) = 2x\left(1 - \frac{x}{4}\right)$, the fixed points satisfy

$$2x\left(1 - \frac{x}{4}\right) = x$$

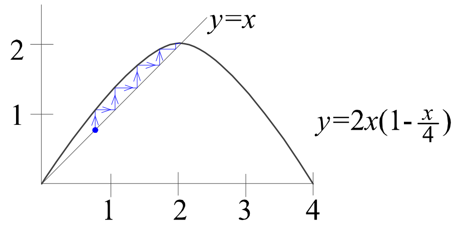
Clearly, $x = 0$ is a fixed point. If $x \neq 0$, then

$$2 - \frac{x}{2} = 1 \qquad \Longrightarrow \qquad x = 2$$

However, $f(x) = 2x - \frac{2x^2}{4}$ implies that $f'(x) = 2 - x$. Thus,

$$\begin{aligned} |f'(0)| &= |2 - 0| = 2 > 1, \qquad p = 0 \textit{ is repelling} \\ |f'(0)| &= |2 - 2| = 0 < 1, \qquad p = 2 \textit{ is attracting} \end{aligned}$$

Example 4 says that if a population is doubling every time period, then it will rapidly approach its carrying capacity.



3-6: Logistic Growth

There are also many other discrete dynamical systems that are used for modeling populations, as will be explored in the exercises.

Discrete dynamical systems are also used in a number of other applications. For example, *second order linear* difference equations, which are equations of the form

$$y_{n+1} = ay_n + by_{n-1}$$

can be rewritten as *first order discrete dynamical systems* by letting

$$F_n = \frac{y_n}{y_{n-1}} \tag{8.20}$$

Let's look at a famous example of this type of discrete dynamical system.

EXAMPLE 5 In 1202 A.D., Fibonacci showed that if a pair of rabbits every pair of rabbits produces a pair of rabbits after one month and another pair of rabbits after 2 months, then the number y_n of pairs of rabbits after n generations satisfies

$$y_{n+1} = y_n + y_{n-1} \tag{8.21}$$

The sequence generated by the second order linear recursion (8.21) when $y_0 = y_1 = 1$ is called the *Fibonacci sequence.* Write the first four terms of the Fibonacci sequence and then use (8.20) to rewrite it as a first order discrete dynamical system. Then find and classify the fixed points.

Solution: To begin with, if $y_0 = y_1 = 1$, then

$$\begin{aligned} y_2 &= y_1 + y_0 = 1 + 1 = 2 \\ y_3 &= y_2 + y_1 = 2 + 1 = 3 \\ y_4 &= y_3 + y_2 = 3 + 2 = 5 \end{aligned}$$

and so on. In general, the *Fibonacci numbers* are the numbers in the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

in which each term is the sum of the two previous terms.

Next, application of (8.20) implies that F_{n+1} satisfies

$$F_{n+1} = \frac{y_{n+1}}{y_n} = \frac{y_n + y_{n-1}}{y_n} = 1 + \frac{y_{n-1}}{y_n}$$

However, $F_n = \frac{y_n}{y_{n-1}}$ then implies that

$$F_{n+1} = 1 + \frac{1}{F_n} \tag{8.22}$$

Thus, the *ratios* of successive Fibonacci numbers satisfies the recurrence relation (8.22). Since $F_{n+1} = f(F_n)$ where $f(x) = 1 + \frac{1}{x}$, the fixed point of (8.22) satisfies

$$1 + \frac{1}{x} = x \qquad \Longrightarrow \qquad x + 1 = x^2$$

Thus, $x^2 - x - 1 = 0$, which via the quadratic formula has a solution of

$$x = \frac{1 + \sqrt{5}}{2}$$

since ratios of Fibonacci numbers are positive. Moreover, $f'(x) = -x^{-2}$, which implies that

$$\left| f'\left(\frac{1+\sqrt{5}}{2}\right)\right| = \left(\frac{1+\sqrt{5}}{2}\right)^{-2} = 0.382 < 1$$

The fixed point theorem consequently leads us to conclude that

$$\lim_{n\to\infty} F_n = \frac{1+\sqrt{5}}{2} \tag{8.23}$$

i.e., ratios of successive Fibonacci numbers approach a limit of $\frac{1+\sqrt{5}}{2}$.

The number $\Phi = \frac{1+\sqrt{5}}{2}$ is often called the *golden ratio* or the golden mean. Like π, the golden ratio Φ occurs frequently in applications. In fact, it was intentionally incorporated by the ancient Greeks into much of their art and architecture.

Moreover, (8.23) also has an alternative interpretation. The recursion (8.22) implies that

$$F_n = 1 + \frac{1}{F_{n-1}} = 1 + \frac{1}{1 + \frac{1}{F_{n-2}}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{F_{n-3}}}}$$

and so on. If we continue indefinitely, then we obtain

$$1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\ddots}}} = \lim_{n\to\infty} F_n = \frac{1+\sqrt{5}}{2}$$

The expression on the left is called a *continued fraction*, and the fixed point theorem thus implies that it has a value of $\Phi = \frac{1+\sqrt{5}}{2}$.

Check your Reading *What comes after 34 in the Fibonacci sequence?*

Periodic Points and Bifurcation

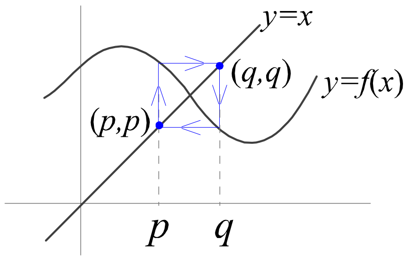
In addition to fixed points, discrete dynamical systems can also have *cycles*, where a cycle of length N is a set of points

$$x_0, x_1, \ldots, x_{N-1}$$

such that $x_i \neq x_j$ if $i \neq j$, $x_{j+1} = f(x_j)$, and $f(x_{N-1}) = x_0$. For example, points p and q form a 2-cycle if $p \neq q$ and $f(p) = q$ while $f(q) = p$, which is to say that under an recursion $x_{n+1} = f(x_n)$, we have

$$p \to q \to p \to q \to \ldots$$

In a web diagram, a 2 cycle corresponds to a rectangle with vertices (p,p), (p,q), (q,p), and (p,p).



3-7: A 2-cycle

The 2-cycle is *attracting* if x_0 sufficiently close to p implies $x_{2n} \to p$ (or if x_0 sufficiently close to q implies $x_{2n} \to q$).

To find period 2 points, we notice that $q = f(p)$ and $p = f(q)$ implies that $p = f(f(p))$. Thus, period 2 points are those solutions to

$$f(f(x)) = x$$

which are not fixed points of $f(x)$. Since $\frac{d}{dx} f(f(x)) = f'(f(x)) f'(x)$, we have

$$\frac{d}{dx} f(f(x))\Big|_{x=p} = f'(f(p)) f'(p) = f'(q) f'(p)$$

so that by the fixed point theorem, a 2-cycle is *attracting* if

$$|f'(q) f'(p)| < 1$$

and is *repelling* if $|f'(q) f'(p)| > 1$.

EXAMPLE 6 Find and classify the period 2 points of

$$x_{n+1} = \frac{1}{2} - 2x_n^2$$

as either attracting or repelling. Illustrate with a web diagram.

Solution: The recursion is of the form $x_{n+1} = f(x_n)$ where $f(x) = \frac{1}{2} - 2x^2$. The equation $f(f(x)) = x$ reduces to

$$\frac{1}{2} - 2[f(x)]^2 = x \qquad \Longrightarrow \qquad \frac{1}{2} - 2\left(\frac{1}{2} - 2x^2\right)^2 = x$$

which simplifies to

$$4x^2 - 8x^4 = x \tag{8.24}$$

Factoring leads to

$$x(2x-1)\left(4x^2 + 2x - 1\right) = 0$$

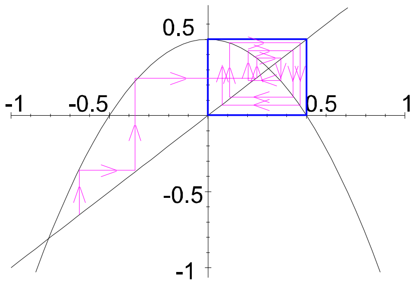
which implies that $x = 0$, $x = \frac{1}{2}$, $-\frac{1}{4} + \frac{1}{4}\sqrt{5}$, or $-\frac{1}{4} - \frac{1}{4}\sqrt{5}$. However, $f(0) = \frac{1}{2}$ and also

$$f\left(\frac{1}{2}\right) = \frac{1}{2} - 2\left(\frac{1}{4}\right) = 0$$

Thus, $0 \to \frac{1}{2} \to \ldots$ is a 2-cycle. Moreover, $f'(x) = -4x$, which implies that

$$f'(0)\, f'\left(\frac{1}{2}\right) = 0 \cdot (-2) = 0$$

and since $0 < 1$, the 2-cycle is attracting.



3-8: An attracting 2-cycle

Similarly, the period 3 points are solutions to

$$f(f(f(x))) = x$$

which are not fixed points or period 2 points, and the period 4 points are solutions to

$$f(f(f(f(x)))) = x$$

which are not fixed, period 2, or period 3 points.

Exercises:

Determine and classify the fixed points of each of the following discrete dynamical systems. Use a web diagram to illustrate your conclusions.

1.	$x_{n+1} = \frac{1}{2}x_n + 1$	2.	$x_{n+1} = 3x_n + 2$
3.	$x_{n+1} = 2x_n(1 - x_n)$	4.	$x_{n+1} = x_n(1 - x_n)$
5.	$x_{n+1} = 3.2x_n(1 - x_n)$	6.	$x_{n+1} = 3.5x_n(1 - x_n)$
7.	$x_{n+1} = 3.83x_n(1 - x_n)$	8.	$x_{n+1} = 4x_n(1 - x_n)$
9.	$x_{n+1} = 2 - \frac{1}{2}x_n^2$	10.	
11.	$x_{n+1} = x_n^2 - 3x_n + 2$	12.	$x_{n+1} = \frac{3}{4}x_n^2 - \frac{5}{2}x_n + 2$
13.	$x_{n+1} = 3\left\|x_n - \frac{1}{2}\right\| + \frac{3}{2}$	14.	
15.	$x_{n+1} = x_n^3 + 0.75x_n$	16.	$x_{n+1} = x_n^3 - 1.44x_n$
17.	$x_{n+1} = 2x_n^2 - x_n^3$	18.	$x_{n+1} = x_n^4 - 3x_n$
19.	$x_{n+1} = x_n - \ln\|x_n\|$	20.	$x_{n+1} = x_n + \ln\|x_n\|$
21.	$x_{n+1} = \sin\left(\frac{\pi}{3}x_n\right)$	22.	$x_{n+1} = \sin\left(\frac{\pi}{2}x_n\right)$

Determine and classify the 2 cycles of each of the following discrete dynamical systems. Use a web diagram to illustrate your conclusions.

23.	$x_{n+1} = 2 - \frac{1}{2}x_n^2$	24.	$x_{n+1} = x_n^2 - 1$
25.	$x_{n+1} = x_n^2 - 3x_n + 2$	26.	$x_{n+1} = \frac{37}{12}x_n(1 - x_n)$
27.	$x_{n+1} = 3\left\|x_n - \frac{1}{2}\right\| + \frac{3}{2}$	28.	$x_{n+1} = 3.1x_n(1 - x_n)$

29. If y_n is the population of a certain collection of sandhill cranes, then

$$y_{n+1} = y_n + 0.0987 y_n \left(1 - \frac{y_n}{194.6}\right)$$

Determine and classify the fixed points of the sandhill crane population. Explain the significance of the result.

30. The population P_n (in millions) of Sweden from 1800 to 1920 can be modeled by

$$y_{n+1} = y_n - 1.535 + 0.023\,(y_n - 1.535)\left(1 - \frac{y_n - 1.535}{6.336}\right)$$

where $y_0 = 2.302$. Determine and classify the fixed points of the population of Sweden. Explain the significance of the result.

31. In *Ricker's population model,* the population x_n after n time steps satisfies the recursion

$$x_{n+1} = x_n e^{b(1 - x_n/N)} \tag{8.25}$$

where b and N are positive constants. Determine and classify the fixed points of (8.25). What is the significant of N?

32. In still another population model, the population x_n after n time steps satisfies the recursion

$$x_{n+1} = \frac{k x_n}{b + x_n}$$

where b and k are positive constants. Determine and classify the fixed points of (8.25). Then explain why we must require $k > b$ in order for the population model to make sense.

33. If y_n represents the percentage of red blood cells in circulation on day n that were produced by the marrow on day n, then y_n satisfies the recursion

$$y_{n+1} = \frac{pr}{1 - r + y_n} \tag{8.26}$$

where p is the ratio of the number produced to the number lost and r is the percentage of red blood cells removed by the spleen each day. Determine and classify the fixed points of (8.26). What is the significant about the model when $p = 1$?

34. If y_n denotes the number of terminal segments of a certain algae plant after n growth stages, then

$$y_{n+1} = (1 + q)\, y_n + r y_{n-1}$$

where q is the frequency with which a terminal segment produces a pair of daughter terminal segments and r is the frequency with which next-to-terminal segments produce a single daughter segment. Use the ratio

$$F_n = \frac{y_n}{y_{n-1}} \tag{8.27}$$

to transform (8.27) into a first order recursion. What is the fixed point of F_n? Is it attracting or repelling? What is the significance of the fixed point.

35. If $f(p) = p$ and if $|f'(p)| = 1$, then p is sometimes called a *saddle point* of f. Show that the fixed point of the recursion

$$x_{n+1} = 2 - x_n$$

is a saddle point. Use a web diagram to explore the behavior of the iterates in the vicinity of this fixed point.

36. For what value of b does the recursion

$$x_{n+1} = bx_n(1 - x_n)$$

have a saddle point (see exercise 35 for description of saddle points)?

37. Show that if $x_0 = 5$ and if

$$x_{n+1} = 1 + \frac{2}{x_n}$$

then the sequence $\{x_n\}_{n=0}^{\infty}$ has a limit of p, where p is the *continued fraction*

$$p = 1 + \cfrac{2}{1 + \cfrac{2}{1 + \cfrac{2}{\ddots}}}$$

What is the value of p?

38. Show that a recursion of the form

$$x_{n+1} = \frac{ax_n + b}{cx_n + d}, \quad c \neq 0$$

results in a continued fraction. (Hint: rewrite the recursion in the form

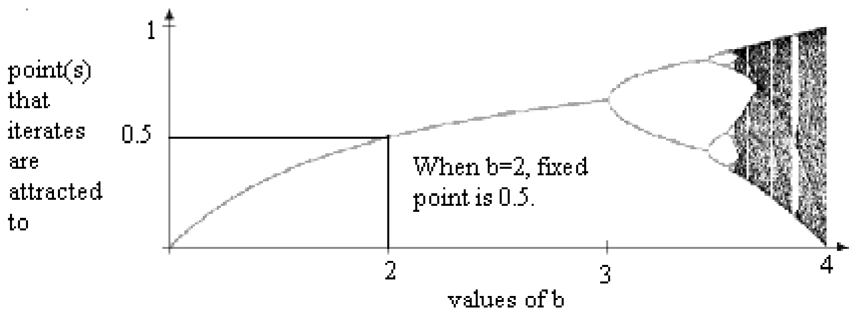
$$x_{n+1} = \frac{a}{c} + \frac{K}{cx_n + d}$$

where $K = b - \frac{a}{c}d$). What relationship between a, b, c, and d is necessary for the continued fraction to converge?

39. In the recursion given by

$$x_{n+1} = bx_n(1 - x_n) \tag{8.28}$$

the parameter b is called the *bifurcation parameter.* A *bifurcation diagram* is a picture of the set of attracting points at a given value of b.



Write a short essay in which you use web diagrams to explain how the bifurcation diagram is related to the recursion (8.28).

40. Write to Learn: A discrete dynamical system $x_{n+1} = f(x_n)$ is said to be *chaotic* if it has the following 4 properties.[1]

(a) **Sensitive Dependence on Initial Conditions.** Points close together cannot remain close together. Specifically, there exists a number $c > 1$ such that

$$|f(a) - f(b)| \geq c\,|a - b|$$

whenever a and b are sufficiently close.

(b) **Periodic Points are Repelling:** A periodic point p is a number such that if $x_0 = p$, then $x_n = p$ for some $n > 0$. A periodic point is repelling if $|f'(x_0) f'(x_1) \cdot \ldots \cdot f'(x_{n-1})| > 1$ when $x_0 = x_n = p$ but $x_j \neq p$ for any $j = 1, \ldots, n-1$.

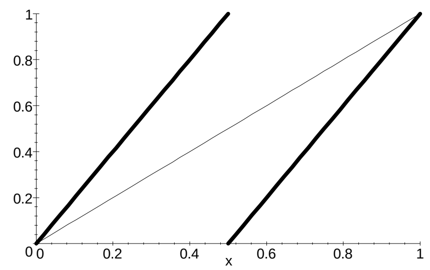
(c) **Bounded Sequences of Iterates:** There is an interval $[a, b]$ containing all the periodic points such that if x_0 is in $[a, b]$, then x_n is in $[a, b]$ for all $n > 0$.

(d) ***Existence of Wandering Orbits:** There is a sequence of iterates $\{x_n\}_{n=0}^{\infty}$ in an interval $[a, b]$ containing all the periodic points such that if p is a periodic point and $\varepsilon > 0$, then there exists infinitely many iterates x_n such that $|x_n - p| < \varepsilon$.

Write a short essay illustrating each of these phenomena for $x_{n+1} = f(x_n)$ when

$$f(x) = \begin{cases} 2x & if \quad x \leq 0.5 \\ 2x - 1 & if \quad x > 0.5 \end{cases}$$

which is graphed below along with the line $y = x$.



(Hints: These will help with 1-4 above)

(a) Explain why $|a - b| < 0.5$ means that $c = 2$

(b) What is $f'(x)$ for all $x \neq 0.5$?

(c) Explain why x_0 in $[0, 1]$ implies that x_n is in $[0, 1]$ for all n.

(d) *A decimal in base 2 is of the form $0.d_1d_2d_3d_4\ldots_2$ where $d_j = 0$ or 1 and

$$0.d_1d_2d_3d_4\ldots_2 = \frac{d_1}{2} + \frac{d_2}{2^2} + \frac{d_3}{2^3} + \frac{d_4}{2^4} + \ldots$$

For example, the base 2 decimal 0.01101_2 is the number

$$0.01101_2 = \frac{0}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{0}{2^4} + \frac{1}{2^5} = \frac{13}{32}$$

Explain why the periodic points of f are the repeating decimals in base 2, and then consider the sequence generated by

$$x_0 = 0.101001000100001\ldots_2$$

[1] There is no universally accepted definition of chaos, but these 4 ideas capture the essence of what most mathematicians would call chaos.

8.4 Euler's and Newton's Methods

Euler's Method

Sequences are often used in numerical estimation. In this section, we will explore two such uses of sequences—estimating solutions to differential equations and estimating integrals.

Let's begin with a simple method for estimating the solution to initial value problems of the form

$$y' = f(x, y), \quad y(a) = y_a \tag{8.29}$$

where y_a is a number and where $f(x, y)$ is some expression in x and y. First, let us construct a sequence of input values of the form

$$x_0 = a, \quad x_2 = a + h, \quad x_3 = a + 2h, \quad \ldots, \quad x_n = a + nh, \ldots$$

for a given value of h. Notice that $x_1 = x_0 + h$, $x_2 = x_1 + h$, and in general, $x_{n+1} = x_n + h$.

If h is close to 0, then (8.29) implies that

$$\frac{y(x_n + h) - y(x_n)}{h} \approx f(x_n, y(x_n))$$

However, $y(x_{n+1}) = y(x_n + h)$, so that

$$y(x_{n+1}) \approx y(x_n) + hf(x_n, y(x_n))$$

If we now let $y_0 = y(a)$ and

$$y_{n+1} = y_n + hf(x_n, y_n) \tag{8.30}$$

then y_n is an approximation of $y(x_{n+1})$ when h is sufficiently small. We say that $\{y_n\}_{n=0}^{\infty}$ is the sequence of *Euler estimates* of the solution to (8.29).

EXAMPLE 1 Find the first 6 terms of the sequence of Euler estimates with $h = 0.1$ to the solution of

$$y' = 2x - y, \quad y(0) = 0.1 \tag{8.31}$$

Solution: Since $f(x, y) = 2x - y$, Euler's method is of the form

$$y_{n+1} = y_n + 0.1(2x_n - y_n) = 0.2x_n + 0.9y_n$$

Since $h = 0.1$, the x-coordinates for (8.31) are

$$x_0 = 0, \ x_1 = x_0 + 0.1 = 0.1, \ x_2 = x_1 + 0.1 = 0.2, \ \ldots$$

To determine the sequence of Euler estimates y_n, we begin with the initial value of $y_0 = 0.1$. It then follows that

$$y_1 = 0.2x_0 + 0.9y_0 = 0.2(0) + 0.9(0.1) = 0.09$$

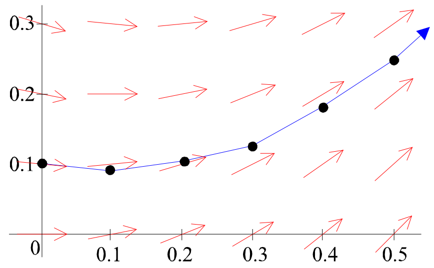
Similarly, estimates for y_2, y_3, and so on are as follows:

$$\begin{aligned} y_2 &= 0.2x_1 + 0.9y_1 = 0.2(0.1) + 0.9(0.09) = 0.101 \\ y_3 &= 0.2x_2 + 0.9y_2 = 0.2(0.2) + 0.9(0.101) = 0.1309 \\ y_4 &= 0.2x_3 + 0.9y_3 = 0.2(0.3) + 0.9(0.1309) = 0.17781 \\ y_5 &= 0.2x_4 + 0.9y_4 = 0.2(0.4) + 0.9(0.17781) = 0.240029 \end{aligned}$$

Thus, the sequence of Euler estimates is of the form

$$0.1, 0.09, 0.101, 0.1309, 0.17781, 0.240029, \ldots$$

Plotting the (x_n, y_n) pairs and connecting them with line segments thus leads to an estimate of the solution to (8.31).



4-1: Euler Approximation of a Solution

Estimates produced with Euler's method tend to become less accurate as n increases. For example, Euler estimates of $y' = y$, $y(0) = 1$ do not grow as fast as the actual solution of $y = e^x$, as shown in example 2.

EXAMPLE 2 Find the sequence of Euler estimates with $h = 0.2$ of the solution to $y' = y$, $y(0) = 1$.

Solution: Since $f(x, y) = y$, Euler's method is

$$y_{n+1} = y_n + hy_n = (1 + h)\, y_n = (1.2)\, y_n$$

If $y_0 = 1$, then $y_1 = 1.2y_0 = 1.2$ and $y_2 = 1.2y_1 = (1.2)^2$. It follows that the n^{th} Euler estimate is of the form $y_n = (1.2)^n$, so that in closed form the sequence of Euler estimates is $\{(1.2)^n\}_{n=0}^{\infty}$, or equivalently,

$$1, 1.2, (1.2)^2, (1.2)^3, \ldots, (1.2)^n, \ldots$$

Since $x_0 = 0$, $x_1 = 0.2$, $x_3 = 0.4$, and so on, we can compare the Euler estimates to the actual values of $y = e^x$ at the numbers x_n.

n	x_n	Euler $= y_n$	Actual $= e^{x_n}$
0	0	1	1
1	0.2	1.2	1.22
2	0.4	1.44	1.492
3	0.6	1.728	1.8221
4	0.8	2.0736	2.22554
5	1.0	2.48832	2.718281

Clearly, the Euler estimates do not grow as fast as e^x does.

Check your Reading *How close is y_{10} to $e^2 = 7.389...$?*

Euler's Method Near Stable Equilibria

If $y = C$ is an equilibrium of an autonomous differential equation $y' = f(y)$, then $f(C) = 0$. Thus, if $y_n = C$, then

$$y_{n+1} = y_n + hf(y_n) = y_n + hf(C) = y_n$$

which means that if $y_0 = C$, then $y_n = C$ for all n. That is, if $y = C$ is an equilibrium, then Euler's method is exact if $y_0 = C$.

Moreover, if C is a stable equilibrium of $y' = f(y)$, then choosing an initial value y_0 sufficiently close to an equilibrium $y = C$ implies that

$$\lim_{n\to\infty} y_n = C$$

and in fact, Euler estimates tend to *improve* as they approach a stable equilibrium.

EXAMPLE 3 Find the sequence of Euler estimates for $h = 0.1$ of the solution to

$$y' = 2y(5-y) \qquad y(1) = 4 \tag{8.32}$$

What is the limit of the sequence of Euler estimates?

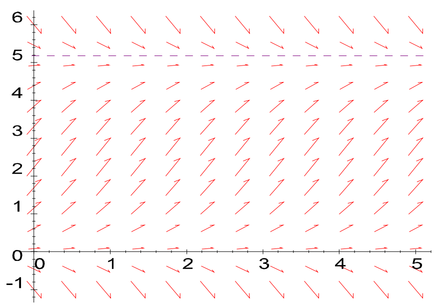
Solution: Since $f(y) = 2y(5-y)$, Euler's method is of the form

$$y_{n+1} = y_n + 2y_n(5-y_n)\,0.1$$

Since $y_0 = y(1) = 4$, we have

$$\begin{aligned} y_1 &= y_0 + 2y_0(5-y_0)\cdot 0.1 = 4 + 2\cdot 4(5-4)\cdot 0.1 = 4.8 \\ y_2 &= y_1 + 2y_1(5-y_1)\cdot 0.1 = 4.8 + 2\cdot 4.8(5-4.8)\cdot 0.1 = 4.992 \\ y_3 &= y_2 + 2y_2(5-y_2)\cdot 0.1 = 4.992 + 2\cdot 4.992(5-4.992)\cdot 0.1 = 4.99998 \end{aligned}$$

and so on. A slope field for (8.32) is shown below:



4-2: Euler's method near a stable equilibrium

If $y(a)$ is sufficiently close to 5 for any a, then $y(t)$ approaches 5 as t approaches ∞. Thus, $y = 5$ is a stable equilibrium of (8.32), and correspondingly, the limit of the sequence of Euler estimates is

$$\lim_{n\to\infty} y_n = 5$$

Since much of science and engineering is the study of behavior near stable equilibria, Euler's method is valuable in spite of its lack of precision in general. Indeed, we see it used in a wide variety of applications.

EXAMPLE 4 Suppose that an intravenously-infused solution delivers a certain drug at a constant rate of 100cc per unit volume of blood per hour. If the initial concentration of the drug is 800cc per unit volume, what will be the approximate concentration an hour later? (use $h = 6$ minutes $= 0.1$ hours).

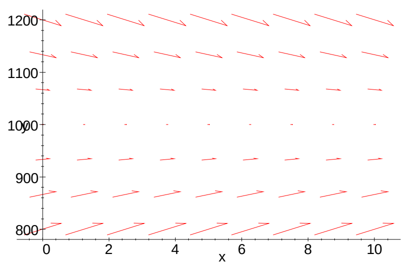
Solution: If $y(t)$ denotes the concentration of the drug at time t, then

rate of change of concentration		rate of removal		rate of infusion
y'	$=$	$-0.1y$	$+$	100

Thus, the equilibrium is the solution to

$$0 = -0.1y + 100$$

which is $y = 1000$ cc per unit volume. A slope field for the model is given below:



4-3: Euler's method approximates infusion

Clearly, the equilibrium is stable. Euler's method is of the form

$$y_{n+1} = y_n + 0.1(-0.1y_n + 100) = 0.99y_n + 10$$

Since $y_0 = 800$, we have

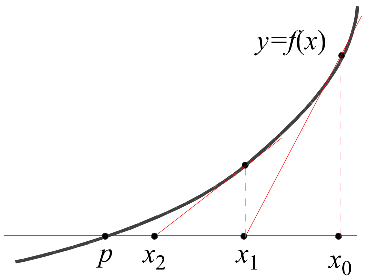
$$\begin{aligned} y_1 &= 0.99y_0 + 10 = 0.99 \cdot 800 + 10 = 802 \\ y_2 &= 0.99y_1 + 10 = 0.99 \cdot 802 + 10 = 803.98 \\ y_3 &= 0.99y_2 + 10 = 0.99 \cdot 804 + 10 = 805.94 \\ &\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ y_{10} &= 0.99y_9 + 10 = 0.99 \cdot 817.23 + 10 = 819.12 \end{aligned}$$

Thus, there will be a concentration of about 819.12 cc per unit volume in about one hour.

Check your Reading *How close is y_{10} to $e^2 = 7.389..$*

Newton's Method

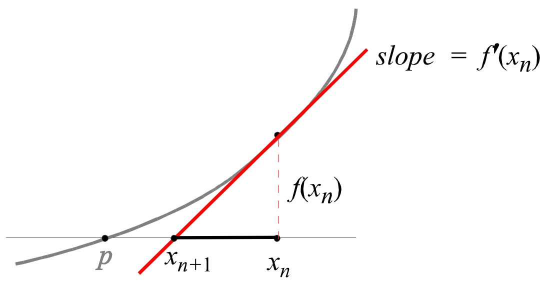
If $f(p) = 0$ and $f'(p) \neq 0$, then sequences can be used to approximate the value of p. In particular, if x_0 is an initial estimate of p, then let us let x_1 denote the zero of the *tangent line* to $y = f(x)$ at x_0.



4-4: $x_0, x_1, x_2, \ldots$ are increasingly better estimates of p

The tangent line to $y = f(x)$ at x_1 can then be used to predict a better estimate x_2, and so on.

In general, if x_n denotes the n^{th} approximation of p, then x_{n+1} is the root of the tangent line to $y = f(x)$ at an input x_n:



4-5: x_{n+1} is determined by x_n

As a result, a run of $x_n - x_{n+1}$ leads to a rise of $f(x_n)$, which means that

$$\frac{f(x_n)}{x_n - x_{n+1}} = slope\ of\ tangent\ line = f'(x_n)$$

We now solve for x_{n+1}:

$$x_n - x_{n+1} = \frac{f(x_n)}{f'(x_n)} \quad \Longrightarrow \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The result is the method described below:

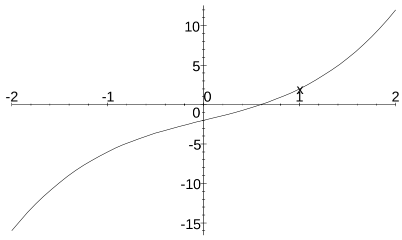
Newton's Method: Suppose that $f(p) = 0$ and that x_0 is sufficiently close to p. If we let

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{8.33}$$

then the sequence $\{x_n\}_{n=0}^{\infty}$ converges to p.

Moreover, when Newton's method works, it tends to work very well.

EXAMPLE 5 Use Newton's method to estimate the zero of $f(x) = x^3 + 3x - 2$:



4-6: A zero in the interval $[0, 1]$

Solution: Since $f'(x) = 3x^2 + 3$, Newton's method yields

$$x_{n+1} = x_n - \frac{x_n^3 + 3x_n - 2}{3x_n^2 + 3}$$

Let us suppose that we choose the initial estimate to be $x_0 = 1$. Then to 10 decimal places the approximations generated by Newton's method are

$$\begin{aligned}
x_1 &= x_0 - \frac{x_0^3 + 3x_0 - 2}{3x_0^2 + 3} = 1 - \frac{1 + 3 - 2}{3 + 3} = 0.667 \\
x_2 &= x_1 - \frac{x_1^3 + 3x_1 - 2}{3x_1^2 + 3} = 0.667 - \frac{(0.667)^3 + 3(0.667) - 2}{3(0.667)^2 + 3} = 0.59829 \\
x_3 &= x_2 - \frac{x_2^3 + 3x_2 - 2}{3x_2^2 + 3} = 0.5983 - \frac{(0.5983)^3 + 3(0.5983) - 2}{3(0.5983)^2 + 3} = 0.596074 \\
x_4 &= x_3 - \frac{x_3^3 + 3x_3 - 2}{3x_3^2 + 3} = 0.596074 - \frac{(0.596074)^3 + 3(0.596074) - 2}{3(0.596074)^2 + 3} = 0.59607164 \\
x_5 &= x_4 - \frac{x_4^3 + 3x_4 - 2}{3x_4^2 + 3} = 0.59607164 - \frac{(0.59607164)^3 + 3(0.59607164) - 2}{3(0.59607164)^2 + 3} = 0.560716379
\end{aligned}$$

Moreover, x_6, x_7, and x_8 are the same as x_5 up to 10 decimal places.

Check your Reading *To 10 decimal places, what is the solution to $x^3 + 3x - 2 = 0$?*

Convergence of Newton's Method

As a discrete dynamical system, Newton's method for a function $f(x)$ is of the form

$$x_{n+1} = N(x_n) \qquad \text{where} \qquad N(x) = x - \frac{f(x)}{f'(x)}$$

Thus, if $f(p) = 0$ and $f'(p) \neq 0$, then

$$N(p) = p - \frac{f(p)}{f'(p)} = p - \frac{0}{f'(p)} = p$$

That is, a zero of f is a fixed point of N.

Moreover, the quotient rule yields

$$N'(x) = 1 - \frac{[f'(x)]^2 - f(x) f''(x)}{[f(x)]^2}$$

so that if $f(p) = 0$ and $f'(p) \neq 0$, then

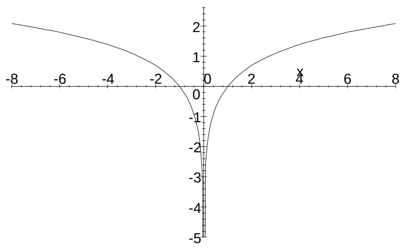
$$N'(p) = 1 - \frac{[f'(p)]^2 - 0}{[f'(p)]^2} = 1 - 1 = 0$$

Since $|N'(p)| = 0 < 1$, a zero of f is an attracting fixed point of N. Thus, if x_0 is chosen sufficiently close to p and if $f'(p) \neq 0$, then the sequence $\{x_n\}_{n=0}^{\infty}$ of iterates of Newton's method satisfies

$$\lim_{n\to\infty} x_n = p$$

However, Newton's method does not always converge. To begin with, if $f'(p) = 0$, then Newton's method cannot be used at all. In addition, if x_0 is not sufficiently close to p to begin with, then Newton's method may not converge.

EXAMPLE 6 Apply Newton's method to $f(x) = \ln|x|$ for $x_0 = e^2$.



4-7: Newton's method fails for $x_0 = e^2 \approx 7.39$

Solution: Since $f'(x) = \frac{1}{x}$, Newton's method becomes

$$x_{n+1} = x_n - \frac{\ln|x_n|}{1/x_n}$$

which simplifies to

$$x_{n+1} = x_n - x_n \ln|x_n|$$

If we begin with $x_0 = e^2$, then

$$\begin{aligned} x_1 &= x_0 - x_0 \ln|x_0| = e^2 - e^2 \ln\left|e^2\right| = e^2 - 2e^2 = -e^2 \\ x_2 &= x_1 - x_1 \ln|x_1| = -e^2 + e^2 \ln\left|-e^2\right| = -e^2 + 2e^2 = e^2 \end{aligned}$$

Likewise, $x_3 = -e^2$ and $x_4 = e^2$, so that the sequence generated by Newton's method is

$$e^2, -e^2, e^2, -e^2, \ldots, (-1)^n e^2, \ldots$$

and consequently, Newton's method does not converge to either of the zeroes of $\ln|x|$.

Exercises:

Use Euler's method for the given value of h to find the first four terms of the sequence of Euler estimates. Alternatively, your instructor may ask you to produce and graph Euler approximations with your calculator.

1.	$y' = x - y,$	$y(0) = 1.1$, $h = 0.1$	2.	$y' = xy,$	$y(0) = 1.1$, $h = 0.01$
3.	$y' = e^{-x} - y,$	$y(0) = 0.9$, $h = 0.1$	4.	$y' = \sin(x) + y,$	$y(0) = 1$, $h = 0.1$
5.	$y' = x^2,$	$y(-1) = 1.2$, $h = 0.1$	6.	$y' = \cos(x)$	$y(0) = 1$, $h = 0.1$
7.	$y' = 0,$	$y(1) = 2$, $h = 0.1$	8.	$y' = 1,$	$y(1) = 0$, $h = 0.1$
9.	$y' = e^{-x} y(1 - y),$	$y(0) = 1$, $h = 0.1$	10.	$y' = y(1 - y) + \cos(x),$	$y(0) = 0$, $h = 0.1$

Find an stable equilibrium of each of the following autonomous equations. Then choose y_0 close to the equilibrium and apply Euler's method. Does the sequence of Euler estimates approach the stable equilibrium?

11.	$y' = y(1 - y)$	2.	$y' = y(2 - y)$
13.	$y' = 2y(1 - y)$	4.	$y' = 2y(10 - y)$
15.	$y' = 0.05y\left(1 - \frac{y}{500}\right)$	6.	$y' = 0.5y\left(1 - \frac{y}{100}\right)$
17.	$y' = y^2 - 3y + 2$	18.	$y' = 2y^2 + 6y - 9$

Each of the following functions has a single zero in the interval $[-2, 4]$. Use Newton's method to approximate that zero to four decimal places of accuracy.

19.	$f(x) = x^2 - 3$	18.	$f(x) = 2x^2 + 15x - 50$
21.	$f(x) = x^5 + x - 1$	20.	$f(x) = 2x^3 + 15x - 50$
23.	$f(x) = 4x - 8$	24.	$f(x) = 11x + 3$
25.	$f(x) = \cos(x/2)$	26.	$f(x) = x - e^{-x}$
27.	$f(x) = \ln x$	28.	$f(x) = \tanh(x)$
29.	$f(x) = \dfrac{\sin(x)}{x}$	30.	$f(x) = e^x - e^{-2x}$

31. Find the recursion generated by applying Newton's method to $f(x) = (x - a)^n$. What is the general term of that recursion? What does it converge to?

32. Show that Newton's method applied to $f(x) = \frac{1}{x} - a$ results in

$$x_{n+1} = 2x_n - ax_n^2$$

How is this related to estimation of the reciprocal of a?

33. It is easy to see that $f(x) = x^{1/3}$ has a zero at 0. Apply Newton's method to $f(x) = x^{1/3}$ with $x_0 = 1$. Does the sequence have a limit of 0? What happens?

34. Apply Newton's method to $f(x) = \ln(|x|)$ with an initial estimate of $x_0 = e^3$. What happens?

35. If $f(x)$ has more than one zero, then choosing x_0 close to a given zero may still cause Newton's method to converge to a more distant zero of $f(x)$.

(a) Construct the Newton's method recursion for $f(x) = 4x^4 - 4x^2$

(b) Choose different initial approximations in $[-1, 1]$. Which zero do they converge to? Do any of them ever converge to the middle root of zero?

36. If a function $f(x)$ has no real zeroes, then the sequence of iterates of Newton's method may be *chaotic.* For example, consider Newton's method for the function

$$f(x) = x^4 + 1$$

Does the sequence of iterates converge? How do the Newton's method iterates behave?

37. A forest with a carrying capacity of 1000 trees per acre and an intrinsic growth rate of 0.1 % increase per year is being harvested at a rate of 20 trees per acre per year. If there are initially 500 trees per acre, about how many trees per acre will there be after 2 years? Use Euler's method with $h = \frac{1}{12}$ (i.e., h about the length of one month)? (Hint: see page (569))

38. A dam is built across a stream flowing at a rate of 10,000 gallons of water each day. If 10% of the water is lost each day to evaporation, seepage through the soil, overflow of the dam, etcetera, then how large, in gallons, will the pond behind the dam eventually become? About how big will it be after one month? (use $h = 1$ day for 30 days)

39. **Write to Learn:** Suppose that $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), and suppose that $f'(x) \neq 0$ for all x in (a, b). Write a short essay in which you apply Newton's method to

$$h(x) = f(b) - f(a) - f'(x)(b - a)$$

What is significant about the zero of $h(x)$? What is the

40. Show that Euler's method applied to $y' = f(x)$ over a regular partition of an interval $[a, b]$ results in the left endpoint approximation of $\int_a^b f(x)\,dx$.

8.5 Infinite Series

Sequences of Partial Sums

A series is the sum of an infinite set of numbers. Hence, the plus signs.

One of the most important applications of sequences is in the study of infinite series, where an *infinite series* is an infinitely long sum of a set of numbers $a_1, a_2, a_3, \ldots$. That is, an infinite series is an expression of the form

$$a_1 + a_2 + \ldots + a_n + \ldots \tag{8.34}$$

To make sense of (8.34), we define the n^{th} partial sum of the series to be

$$s_n = a_1 + a_2 + \ldots + a_n$$

If the sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ has a limit—that is, if there is a number L such that

$$\lim_{n\to\infty} s_n = L$$

then we say that the series *converges.* Moreover, we say that L is the *sum of the series,* and we write

$$a_1 + a_2 + \ldots + a_n + \ldots = L$$

EXAMPLE 1 Compute the sum of the series

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} + \frac{1}{6\cdot 7} + \ldots \tag{8.35}$$

Solution: The first partial sum is $s_1 = \frac{1}{2}$ and the second partial sum is

$$s_2 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{2}{3}$$

Moreover, the third and fourth partial sums are

$$s_3 = \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4} = \frac{3}{4} \qquad \text{and} \qquad s_4 = \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5} = \frac{4}{5}$$

It is easy to see that the general term of the sequence of partial sums is

$$s_n = \frac{n}{n+1}$$

Thus, the limit of the partial sums s_n yields the sum of the series (8.35).

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} + \ldots = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

Recall that Σ denotes the summation operation, so that we may write

$$a_1 + a_2 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n$$

Because Σ is the Greek letter "sigma," this is called the *sigma* notation of the series. Moreover, n is the *index* and a_n is the general term of the series. For example,

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \ldots + \frac{1}{n\,(n+1)} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n\,(n+1)}$$

Sigma notation allows us to state the primary properties of convergent series in a compact and precise form.

Theorem 4.1 If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge and k is constant, then

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \qquad \sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty}(ka_n) = k\sum_{n=1}^{\infty}a_n$$

EXAMPLE 2 Given that $\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1$, what is the sum of the series

$$\sum_{n=1}^{\infty}\frac{3}{n\,(n+1)}$$

Solution: The theorem above allows us to write

$$\sum_{n=1}^{\infty}\frac{3}{n\,(n+1)} = 3\sum_{n=1}^{\infty}\frac{1}{n\,(n+1)} = 3\cdot 1 = 3$$

A series diverges if its sequence of partial sums fails to have a limit.

Equally important, if the sequence of partial sums of a series does not have a limit, then we say that the series *diverges*. That is, if the sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ does not have a limit, then

$$a_1 + a_2 + \ldots + a_n + \ldots \quad \textit{diverges}$$

EXAMPLE 3 Determine if the following series converges or diverges:

$$-2 + 4 - 4 + 4 - 4 + 4 - 4 + \ldots$$

Solution: The first few partial sums are

$s_1 = -2, \qquad s_2 = -2+4 = 2, \qquad s_3 = -2+4-4 = -2, \qquad s_4 = -2+4-4+4 = 2$

from which we see that the sequence of partial sums is

$$-2,\, 2, -2,\, 2, -2,\, 2, \ldots,\ (-1)^n\, 2,\ \ldots$$

However, the limit of the sequence of partial sums does not exist. That is,

$$\lim_{n\to\infty}\left[(-1)^n\, 2\right] \quad \textit{does not exist}$$

Consequently, the series

$$-2 + 4 - 4 + 4 - 4 + 4 - 4 + \ldots \qquad \textit{diverges}$$

Check your Reading *Explain why* $\lim_{n\to\infty}\left[(-1)^n\, 2\right]$ *does not exist.*

Geometric Series

One of the more important series in calculus is the *geometric series*

$$1 + x + x^2 + \ldots + x^n + \ldots \tag{8.36}$$

where x is called the *common ratio* of the series. The partial sums of the series are of the form

$$s_n = 1 + x + x^2 + \ldots + x^n$$

However, the identity (8.10) in the previous section implies that

$$s_n = 1 + x + x^2 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

If $|x| < 1$, then $\lim_{n\to\infty} x^{n+1} = 0$ (see page **??**), so that

$$1 + x + \ldots + x^n + \ldots = \lim_{n\to\infty} (1 + x + \ldots + x^n) = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1 - x}$$

If $|x| \geq 1$, then $\lim_{n\to\infty} x^{n+1}$ diverges.

Theorem 5.1: If $|x| \geq 1$, the geometric series (8.36) diverges. However, if $|x| < 1$, then

$$1 + x + x^2 + \ldots + x^n + \ldots = \frac{1}{1 - x} \tag{8.37}$$

For example, a repeating decimal is actually a geometric series, and in such cases, theorem 5.1 tells us how to convert the repeating decimal into its rational number representation.

EXAMPLE 4 Find the rational number representation for the repeating decimal[2]

$$0.33333\ldots$$

Solution: We begin by noticing that the repeating decimal can be written as

$$0.3333\ldots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \ldots$$

Factoring out the term $\frac{3}{10}$ yields

$$0.3333\ldots = \frac{3}{10}\left(1 + \frac{1}{10} + \frac{1}{100} + \ldots + \frac{1}{10^n} + \ldots\right)$$

The series inside the parentheses is of the form (8.37) with $x = \frac{1}{10}$. Thus, theorem 3.1 implies that

$$0.3333\ldots = \frac{3}{10}\left(\frac{1}{1 - \frac{1}{10}}\right) = \frac{3}{10} \cdot \frac{10}{9} = \frac{3}{9} = \frac{1}{3}$$

EXAMPLE 5 Find the rational number representation for the repeating decimal

$$0.212121\ldots$$

[2] Notice that a non-repeating decimal can also be written as an infinite series, but that series will not be a geometric series. For example, the number π is given by

$$\pi = 3.14159\ldots = 3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{100} + \frac{5}{1000} + \frac{9}{10000} + \ldots$$

which is not a geometric series.

Solution: We begin by noticing that the repeating decimal can be written as

$$0.212121\ldots = \frac{21}{100} + \frac{21}{10000} + \frac{21}{100000} + \ldots$$

Factoring out the term $a = \frac{21}{100}$ yields

$$0.212121\ldots = \frac{21}{100}\left(1 + \frac{1}{100} + \frac{1}{10000} + \ldots + \left(\frac{1}{100}\right)^n + \ldots\right)$$

The series inside the parentheses is of the form (8.37) with $x = \frac{1}{100}$. Thus, theorem 3.1 implies that

$$0.212121\ldots = \frac{21}{100}\left(\frac{1}{1-\frac{1}{100}}\right) = \frac{21}{100}\left(\frac{1}{\frac{99}{100}}\right) = \frac{21}{100}\left(\frac{100}{99}\right) = \frac{21}{99} = \frac{7}{33}$$

Check your Reading

Verify example 5 by computing $\frac{7}{33}$ on a calculator.

More Geometric Series

Geometric series do not occur solely in the study of repeating decimals. Indeed, theorem 3.1 has a wide range of applications.

EXAMPLE 6 Find the sum of the geometric series

$$\frac{2}{3} - \frac{2}{6} + \frac{2}{12} - \frac{2}{24} + \ldots$$

Solution: To begin with, we can factor out $a = \frac{2}{3}$ to obtain

$$\frac{2}{3}\left(1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots\right)$$

Since $4 = 2^2$, $8 = 2^3$ and so on, the series inside the parentheses is a geometric series:

$$\frac{2}{3}\left(1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots + \frac{(-1)^n}{2^n} + \ldots\right)$$

The common ratio $x = \frac{-1}{2}$ satisfies $|x| < 1$, so that by (8.37) we obtain

$$\frac{2}{3}\left(1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots + \frac{(-1)^n}{2^n} + \ldots\right) = \frac{2}{3}\left(\frac{1}{1-\frac{-1}{2}}\right) = \frac{4}{9}$$

Consequently, we have

$$\frac{2}{3} - \frac{2}{6} + \frac{2}{12} - \frac{2}{24} + \ldots = \frac{4}{9}$$

Finally, let us consider an example where we apply theorem 3.1 to a geometric series of functions.

EXAMPLE 7 Apply theorem 3.1 to the series

$$1+e^{-t}+e^{-2t}+\ldots+e^{-nt}+\ldots$$

Solution: To begin with, the series is a geometric series with common ratio $x=e^{-t}$ since it can be written

$$1+e^{-t}+\left(e^{-t}\right)^2+\ldots+\left(e^{-t}\right)^n+\ldots$$

As a result, when $|e^{-t}|<1$, then

$$1+e^{-t}+\left(e^{-t}\right)^2+\ldots+\left(e^{-t}\right)^n+\ldots=\frac{1}{1-e^{-t}}$$

Check your Reading *For what values of t is $|e^{-t}|<1$?*

The Divergence Test

Finally, let us derive a general test for determining if a series diverges. In general, notice that if

$$s_n=a_1+a_2+\ldots+a_{n-1}+a_n$$

then the $n-1$ partial sum is

$$s_{n-1}=a_1+a_2+\ldots+a_{n-1}$$

It is then easy to see that $s_n-s_{n-1}=a_n$, so that if the series converges—i.e., if the limit of the partial sums exists and is equal to some number L—then

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}(s_n-s_{n-1})=\lim_{n\to\infty}s_n-\lim_{n\to\infty}s_{n-1}=L-L=0$$

That is, if the infinite series

$$a_1+a_2+\ldots+a_n+\ldots$$

converges, then it must follow that

$$\lim_{n\to\infty}a_n=0$$

If we turn this around, then we get the following result:

Theorem 3.2: If $\lim\limits_{n\to\infty}a_n\neq 0$ or if $\lim\limits_{n\to\infty}a_n$ does not exist, then $a_1+a_2+\ldots+a_n+\ldots$ diverges

We call theorem 3.2 the *divergence test* because it is a method for determining if a series *diverges.*

EXAMPLE 8 Apply the divergence test to

$$1+2+4+8+\ldots+2^n+\ldots$$

Solution: Since the general term is 2^n, theorem 3.2 leads us to

$$\lim_{n\to\infty} 2^n = \infty$$

Since the limit is not 0, the series

$$1+2+4+8+\ldots+2^n+\ldots \textit{ diverges}$$

EXAMPLE 9 Apply theorem 3.2 to the series

$$\frac{2}{1}+\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\frac{6}{5}+\frac{7}{6}+\frac{8}{7}+\ldots$$

Solution: We identify that $a_1 = 2$, $a_2 = \frac{3}{2}$, $a_3 = \frac{4}{3}$ and in general $a_n = \frac{n+1}{n}$. Since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$$

the divergence theorem implies that the series diverges.

Exercises:

Given that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ *and* $\sum_{n=1}^{\infty} \frac{1}{10^n} = \frac{1}{9}$*, find the sum of the following series:*

1. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n^2}+\frac{1}{10^n}\right)$
2. $\displaystyle\sum_{n=1}^{\infty}\frac{5}{n^2}$
3. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n^2}-\frac{3}{10^n}\right)$
4. $\displaystyle\sum_{n=1}^{\infty}\frac{3}{10^n}$

Construct the sequence of partial sums, and then determine the general term of the sequence of partial sums. Does the series converge?

5. $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}+\ldots$
6. $1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\ldots$
7. $\frac{3}{4}+\frac{3}{16}+\frac{3}{64}+\frac{3}{256}+\ldots$
8. $\frac{3}{2}-\frac{3}{4}+\frac{3}{8}-\frac{3}{16}+\ldots$
9. $1+1+1+1+\ldots$
10. $4-4+4-4+4-4+\ldots+4\,(-1)^{n+1}+\ldots$
11. $1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\frac{1}{4}+\ldots$
12. $1+2-2+3-3+4-4+\ldots$
13. $2-\frac{3}{4}-\frac{4}{8}-\frac{5}{16}-\frac{6}{32}-\frac{7}{64}-\ldots$
14. $1-\frac{2}{3}+\frac{2}{9}-\frac{2}{27}+\frac{2}{81}-\frac{2}{243}+\ldots$
15. $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\frac{5}{6!}+\ldots$
16. $1+\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\frac{2}{81}+\frac{2}{243}+\ldots$

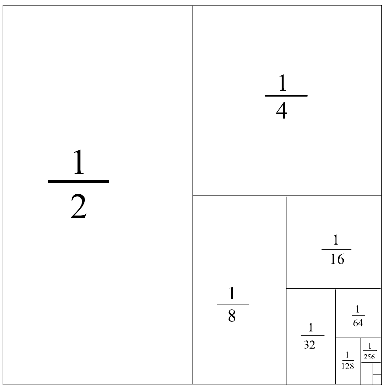
Determine if the series converges or diverges. If it converges, determine what it *converges to. If it diverges, explain why it diverges.*

17. $0.1111\ldots$
18. $0.4444\ldots$
19. $0.454545\ldots$
20. $0.232323\ldots$
21. $\frac{2}{3}-\frac{2}{15}+\frac{2}{75}-\frac{2}{375}+\ldots$
22. $\frac{1}{2}+\frac{1}{6}+\frac{1}{18}+\frac{1}{54}+\ldots$
23. $\frac{1}{10}+\frac{4}{30}+\frac{16}{90}+\frac{64}{270}+\ldots$
24. $\frac{3}{2}+\frac{9}{4}+\frac{27}{16}+\frac{81}{64}+\ldots$
25. $-8+4-2+1+\ldots$
26. $\frac{\pi^2}{e^2}+\frac{\pi^3}{e^3}+\frac{\pi^4}{e^4}+\frac{\pi^5}{e^5}+\ldots$
27. $\frac{1}{2}-\frac{3}{4}+\frac{7}{8}-\frac{15}{16}+\ldots$
28. $1+\frac{2}{3}+\frac{3}{5}+\frac{4}{7}+\ldots$
29. $1+e^{-1}+e^{-2}+e^{-3}+\ldots$
30. $\sin\left(\frac{\pi}{2}\right)+\sin\left(\frac{2\pi}{2}\right)+\sin\left(\frac{3\pi}{2}\right)+\sin\left(\frac{4\pi}{2}\right)+\ldots$

Use the geometric series formula to find the sum of each series and to determine the values of t for which it converges.

31. $1+e^t+e^{2t}+\ldots+e^{nt}+\ldots$
32. $1+\cos(t)+\cos^2(t)+\ldots+\cos^n(t)+\ldots$
33. $1+t^2+t^4+\ldots+t^{2n}+\ldots$
34. $1+\tan(t)+\tan^2(t)+\ldots+\tan^n(t)+\ldots$

35. Interpret the following diagram with a geometric series.



36. Notice that the repeating decimal $0.99999\ldots$ can be interpreted as either

$$\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\frac{9}{10000}+\ldots \tag{8.38}$$

or as the limit of the sequence of finite decimals

$$0.9, 0.99, 0.999, 0.9999, \ldots \tag{8.39}$$

(a) Explain the relationship between (8.38) and (8.39).

(b) Show that the general term of (8.39) is

$$s_n = 1-\frac{1}{10^{n+1}}$$

(c) Explain what is meant by the equation

$$0.9999\ldots = 1$$

37. A *binary decimal expansion* is defined to be

$$d_0.d_1d_2d_3\ldots d_n\ldots_{base\,2} = d_0 + \frac{d_1}{2} + \frac{d_2}{2^2} + \frac{d_3}{2^3} + \ldots + \frac{d_n}{2^n} + \ldots$$

Use the geometric series formula to reduce $0.10101010\ldots_{base\,2}$ to a fraction.

38. A *hexadecimal expansion* is defined to be

$$d_0.d_1d_2d_3\ldots d_n\ldots_{base\,16} = d_0 + \frac{d_1}{16} + \frac{d_2}{16^2} + \frac{d_3}{16^3} + \ldots + \frac{d_n}{16^n} + \ldots$$

Use the geometric series formula to reduce $0.9999\ldots_{base\,16}$ to a fraction.

39. **Write to Learn:** In this exercise, we derive theorem 3.1 in a different manner.

(a) Let $s_n = 1 + r + r^2 + \ldots + r^n$. Show that s_n satisfies the recursion

$$s_{n+1} = rs_n + 1$$

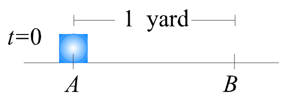
(b) Explain why $s_0 = 1$ and then explain why

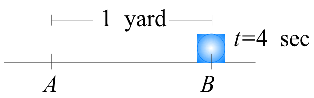
$$s_n = r^n + \frac{1 - r^n}{1 - r}$$

(c) What is $\lim_{n\to\infty} s_n$ when $|r| < 1$? What is $\lim_{n\to\infty} s_n$ when $|r| \geq 1$?

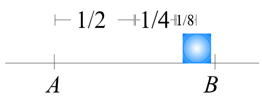
(d) In a short essay, explain how this is related to theorem 5.1.

40. **Write to Learn:** A ball rolls with a constant speed of 1 yard every 4 seconds. Thus, at time $t = 0$ it is at position A, and at time $t = 4$ it is at position B:





However, we could argue that the ball rolls halfway from A to B in half the time, then it rolls half again in half again the time, and so on.



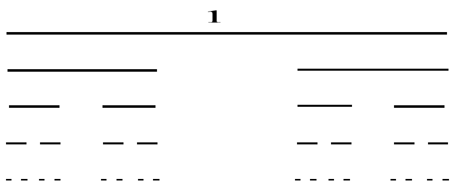
In a short essay, use geometric series to explain how we can still conclude the ball is eventually at position B.

41. **Write to Learn:** Some may argue that exercise 40 implies that the ball never quite reaches position B. Write a short essay in which you argue that even though the ball rolls halfway to B, then half again to B, and so on, its position after 2 seconds requires that its position must be B after one second.

42. The *Cantor set* is the set of points that remains if the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ is removed from $[0, 1]$, the intervals $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$ is removed from the result, the four intervals

$$\left(\frac{1}{27}, \frac{2}{27}\right), \left(\frac{7}{27}, \frac{8}{27}\right), \left(\frac{19}{27}, \frac{20}{27}\right), \left(\frac{25}{27}, \frac{26}{27}\right)$$

are removed from the subsequent result, and so on ad infinitum.

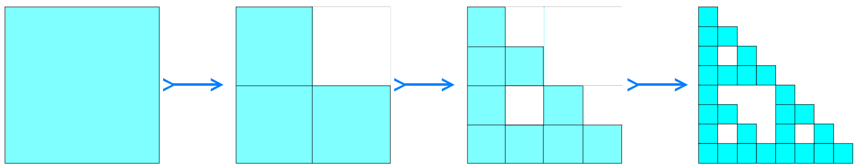


(a) Name three different numbers which are in the Cantor set.

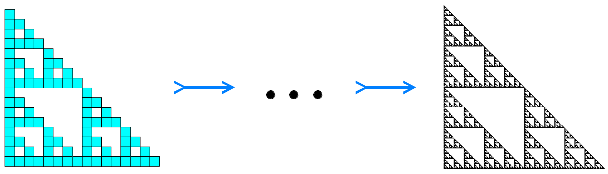
(b) What is the total of the lengths of the open intervals that are removed?

(c) Based on the result in (b), what would you say is the "length" of the cantor set itself (i.e., the "length" of the set that remains after all the open intervals are removed).

43. The *Sierpinski gasket* is the geometric figure formed by first partitioning a square into four identical squares and then the square in the upper right corner is removed. Each remaining square is similarly partitioned, and then the square in the upper right corner of each is removed.



The process is continued ad-infinitum. The figure which remains once all "upper right corner" squares have been removed is called the sierpinski gasket.



Assuming the initial square has sides of length 1, what is the total area of the squares that are removed from the initial square? As a result, what do you conclude is the "area" of the seirpinski gasket?

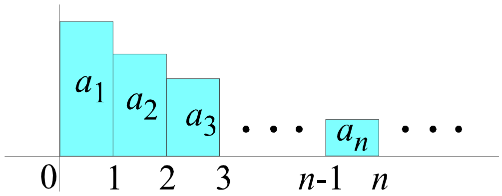
8.6 The Integral Test

The Integral Test

In this section, we develop a convergence test for *positive term series*, which are series of the form

$$a_1 + a_2 + \ldots + a_n + \ldots \tag{8.40}$$

in which $a_n \geq 0$ for all n. In particular, a positive term series can be interpreted to be the total area of a collection of rectangles with width 1 and height a_n.



6-1: Series as area of a collection of rectangles

The partial sum $s_n = \sum_{k=1}^{n} a_k$ thus represents the area of the first n rectangles.

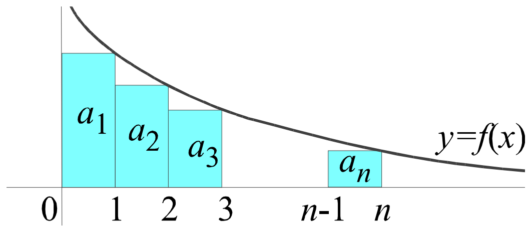
In particular, $s_1 = a_1$ is the area of the 1st rectangle, $s_2 = a_1 + a_2$ is the area of the 1st 2 rectangles, and so on. However, $s_1 \leq s_2$ because s_2 represents a greater area, and similarly, $s_2 \leq s_3$. Indeed, the higher the index n, the greater the area enclosed, which means that the sequence of partial sums forms an *increasing sequence*:

The partial sums of a positive term series forms an increasing sequence.

$$s_1 \leq s_2 \leq s_3 \leq \ldots \leq s_n \leq \ldots$$

As a result, the sequence of partial sums will have a limit if it has an upper bound.

Suppose now that there is a positive, continuous, decreasing function $f(x)$ for which $f(k) = a_k$. As is shown in the figure below, the first n rectangles are necessarily below the graph of $f(x)$:



(8.41)

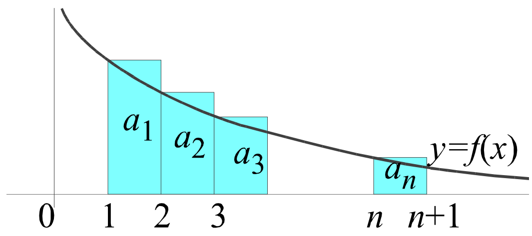
6-2: If $a_n \leq f(n)$ and f decreasing

Thus, we have the following upper bound on the n^{th} partial sum

$$a_1 + a_2 + \ldots + a_n \leq a_1 + \int_1^n f(x)\, dx$$

Similarly, translating the rectangles one unit to the right implies that

$$\int_1^n f(x)\, dx \leq a_1 + a_2 + \ldots + a_n$$



so that we have the following:

$$\int_1^n f(x)\, dx \leq \sum_{k=1}^{n} a_n \leq a_1 + \int_1^n f(x)\, dx \tag{8.42}$$

That is, we can use an integral to produce an upper and lower bound on the n^{th} partial sum.

EXAMPLE 1 Produce upper and lower bounds on the sequence of partial sums of

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

Solution: If we let $f(x) = \frac{1}{x^2}$, then $f(x)$ is decreasing, continuous, positive, and $f(k) = \frac{1}{k^2}$. Moreover,

$$\int_1^n f(x)\,dx = \int_1^n \frac{1}{x^2}dx = \left.\frac{-1}{x}\right|_1^n$$

so that we have $\int_1^n f(x)\,dx = 1 - \frac{1}{n}$. Since $a_1 = \frac{1}{1}$, this implies that

$$\begin{aligned} \int_1^n f(x)\,dx &\leq \sum_{k=1}^n \frac{1}{k^2} \leq a_1 + \int_1^n f(x)\,dx \\ 1 - \frac{1}{n} &\leq \sum_{k=1}^n \frac{1}{k^2} \leq 2 - \frac{1}{n} \end{aligned}$$

For example, if $n = 10$, then we obtain

$$0.9 \leq \sum_{k=1}^{10} \frac{1}{k^2} \leq 1.9$$

Check your Reading *Do you think the sum is closer to upper bound or to the lower bound?*

The Integral Test

If $\int_1^\infty f(x)\,dx$ exists, then (8.42) yields

$$\sum_{k=1}^n a_n \leq a_1 + \int_1^n f(x)\,dx \leq a_1 + \int_1^\infty f(x)\,dx \tag{8.43}$$

which implies that the sequence $\{\sum_{k=1}^n a_n\}_{n=1}^\infty$ is bounded. Moreover, $\{\sum_{k=1}^n a_n\}_{n=1}^\infty$ is also increasing, so that the monotone convergence theorem implies that $\{\sum_{k=1}^n a_n\}_{n=1}^\infty$ converges. However, if $\int_1^\infty f(x)\,dx$ diverges, then (8.42) implies that

$$\infty = \lim_{n\to\infty} \int_1^n f(x)\,dx \leq \lim_{n\to\infty} \sum_{k=1}^n a_k$$

That is, if $\int_1^\infty f(x)\,dx$ diverges, then so also does the series. More generally, we have the following.

The Integral Test: If $f(x)$ is positive, continuous and decreasing on $[N,\infty)$ for some integer N, and if $f(n) = a_n$ for all $n \geq N$, then

$$\begin{aligned} &\sum_{n=1}^\infty a_n \quad \text{converges if} \quad \int_N^\infty f(x)\,dx \quad \text{converges} \\ &\sum_{n=1}^\infty a_n \quad \text{diverges if} \quad \int_N^\infty f(x)\,dx \quad \text{diverges} \end{aligned}$$

However, let us not forget that the integral test is derived from (8.42). Indeed, many mathematicians believe that the proof of the integral test—i.e., the inequality (8.42)—is more important than the conclusion of the integral test.

EXAMPLE 2 Apply the integral test to the series

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots+\frac{1}{n^2}+\ldots=\sum_{n=1}^{\infty}\frac{1}{n^2} \tag{8.44}$$

Solution: We first observe that $f(x)=\frac{1}{x^2}$ is decreasing, positive and continuous on $[1,\infty)$ and also that $f(n)=\frac{1}{n^2}$. As a result, we evaluate the integral of $f(x)=x^{-2}$ over $[1,\infty)$:

$$\int_1^{\infty}\frac{1}{x^2}dx=\lim_{R\to\infty}\int_1^R x^{-2}dx=\lim_{R\to\infty}\left.\frac{x^{-1}}{-1}\right|_1^R=\lim_{R\to\infty}\left(\frac{-1}{R}+\frac{1}{1}\right)=1$$

Since the integral converges, the series (8.44) also converges.

EXAMPLE 3 Use the integral test to determine if the following series converges or diverges:

$$\frac{1}{2}+\frac{2}{5}+\frac{3}{10}+\ldots+\frac{n}{n^2+1}+\ldots=\sum_{n=1}^{\infty}\frac{n}{n^2+1} \tag{8.45}$$

We notice that $f(x)=\frac{x}{x^2+1}$ is decreasing, positive and continuous on $[1,\infty)$ and that $f(n)=\frac{n}{n^2+1}$. As a result, we evaluate

$$\int_1^{\infty}\frac{x}{x^2+1}dx=\lim_{R\to\infty}\int_1^R\frac{x}{x^2+1}dx$$

$\ln(x)$ goes to ∞ as x approaches ∞

We use the substitution $u=x^2+1$, $\frac{1}{2}du=xdx$ with limits of integration $u(1)=1^2+1=2$ and $u(R)=R^2+1$, which gives us

$$\begin{aligned}\int_1^{\infty}\frac{x}{x^2+1}dx &= \lim_{R\to\infty}\int_2^{R^2+1}\frac{du}{u}\\ &= \lim_{R\to\infty}\ln|u|\Big|_2^{R^2+1}\\ &= \lim_{R\to\infty}\left[\ln\left(R^2+1\right)-\ln(2)\right]\\ &= \infty\end{aligned}$$

Thus, the series (8.45) diverges.

EXAMPLE 4 Apply the integral test to

$$1+\frac{1}{e}+\frac{1}{e^2}+\ldots+\frac{1}{e^n}+\ldots=\sum_{n=0}^{\infty}e^{-n} \tag{8.46}$$

Solution: We notice that $f(x)=e^{-x}$ is decreasing, positive and continuous on $[0,\infty)$ and that $f(n)=e^{-n}$. As a result, we evaluate the integral of $f(x)=e^{-x}$ over $[0,\infty)$:

$$\int_0^{\infty}e^{-x}dx=\lim_{R\to\infty}-e^{-x}\Big|_0^R=\lim_{R\to\infty}\left(-e^{-R}+e^0\right)=1 \tag{8.47}$$

Thus, the series (8.46) converges.

Check your Reading *Use (8.41) to explain why $\int_1^\infty f(x)\,dx$ need not be the same as $\sum_{n=1}^{\infty} a_n$.*

p-Series

Finally, let us apply the integral test to an important class of series. For a constant p, a series of the form

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^p} \tag{8.48}$$

A p-series converges if $p > 1$

is called a *p-series.* If $p > 1$, then we notice that $f(x) = \frac{1}{x^p}$ is decreasing on $[1, \infty)$, so that we have

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{R\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^R = \frac{1}{-p+1} \lim_{R\to\infty} \left(\frac{1}{R^{p-1}} - 1\right) = \frac{1}{p-1}$$

That is, a p-series converges when $p > 1$. Moreover, for all $0 < p < 1$, the integral

$$\int_1^\infty \frac{1}{x^p} = \lim_{R\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^R = \frac{1}{-p+1} \lim_{R\to\infty} \left(R^{1-p} - 1\right) = \infty$$

A p-series diverges if $p \leq 1$

and when $p = 1$, the series also diverges (see below). When $p \leq 0$, the divergence test implies that the p-series (8.48) diverges. That is, we can conclude that a p-series diverges when $p \leq 1$.

EXAMPLE 5 Does the following series converge or diverge:

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots + \frac{1}{\sqrt{n}} + \ldots$$

Solution: We can rewrite the series as

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots + \frac{1}{\sqrt{n}} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

which is a p-series with $p = \frac{1}{2}$. Since $\frac{1}{2} < 1$, the series diverges.

Check your Reading *Does the p-series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converge or diverge?*

The Harmonic Series

When $p = 1$, the series (8.48) is called the *harmonic series.* That is, harmonic series is given by

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n}$$

We notice that $f(x) = \frac{1}{x}$ is decreasing on $[1, \infty)$ so that we evaluate the following integral:

$$\int_1^\infty \frac{1}{x}dx = \lim_{R\to\infty} \int_1^R x^{-1}dx = \lim_{R\to\infty} \ln|x||_1^R = \lim_{R\to\infty} (\ln|R| - \ln|1|) = \infty$$

Thus, the harmonic series diverges to ∞, which we write as

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots=\infty \tag{8.49}$$

Moreover, the harmonic series illustrates that the partial sums of a series can converge or diverge very slowly. As a result, tests for convergence are to be preferred over numerical computations with partial sums.

Indeed, in spite of the fact that the harmonic series sums to ∞, the 2000^{th} partial sum of (8.49) is

$$s_{2000}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots+\frac{1}{2000}=8.178368$$

In the table below, we have listed a sampling of the partial sums of the harmonic series with indices up to 8005.

n	s_n	n	s_n	n	s_n
2000	8.178368	4000	8.87139	8000	9.564475
2001	8.178868	4001	8.87164	8001	9.564600
2002	8.179367	4002	8.87189	8002	9.564725
2003	8.179867	4003	8.87214	8003	9.564850
2004	8.180366	4004	8.87239	8004	9.564975
2005	8.180864	4005	8.87264	8005	9.565100

Any of the three columns would seem to indicate a convergent series. Indeed, $s_{4000}=8.87139$ and $s_{4005}=8.87264$ differ by only 0.00125. However, the harmonic series does not converge, and even though the partial sums grow very slowly, they do become infinitely large as n approaches ∞.

Exercises:

Use (8.42) to find upper and lower bounds on the partial sums of the given series. What are the bounds when $n=10$*?*

1. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$
2. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3}}$
3. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$
4. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+1}$
5. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$
6. $\displaystyle\sum_{n=1}^{\infty}n^{-\pi}$

If the positive term series is not a p-series, use the integral test to determine if the series converges or diverges. If the series is a p-series, use the value of p to *determine if the series converges or diverges.*

7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+1}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

11. $\displaystyle\sum_{n=1}^{\infty} n^{-\pi}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+1}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$

14. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^2-1}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^3}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+2n+1}$

17. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+1}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^4+1}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^4+n}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

22. $\displaystyle\sum_{n=1}^{\infty} \operatorname{sech}^2(n)$

23. $\displaystyle\sum_{n=1}^{\infty} \tan^{-1}(n)$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{e^{2n}+1}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}$

27. $\displaystyle\sum_{n=1}^{\infty} e^{-n}\sin\left(e^{-n}\right)$

28. $\displaystyle\sum_{n=1}^{\infty} n\sin\left(\tfrac{1}{n}\right)$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2-0.25}}$

31. Computer Algebra System: Use a computer algebra system to aid in the evaluation of the improper integral implied by the integral test:

(a) $\displaystyle\frac{1}{1\cdot 2}+\frac{1}{2\cdot 5}+\frac{1}{3\cdot 10}+\frac{1}{4\cdot 17}+\ldots+\frac{1}{n\left(n^2+1\right)}+\ldots$

(b) $\displaystyle\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\ldots+\frac{1}{(2n-1)\,2n}+\ldots$

(c) $\displaystyle\frac{1}{e+1}+\frac{1}{e^2+1}+\frac{1}{e^3+1}+\ldots+\frac{1}{e^n+1}+\ldots$

(d) $\displaystyle\frac{1}{\cosh(1)}+\frac{1}{\cosh(2)}+\frac{1}{\cosh(3)}+\ldots+\frac{1}{\cosh(n)}+\ldots$

32. A *logarithmic p-series* is a series of the form

$$\sum_{n=2}^{\infty} \frac{1}{n\left[\ln(n)\right]^p}$$

For what values of p does a logarithmic p-series converge? For which does it diverge?

33. Consider the series

$$\frac{1}{e^{1/5}}+\frac{2}{e^{2/5}}+\frac{3}{e^{3/5}}+\ldots+\frac{n}{e^{n/5}}+\ldots=\sum_{n=1}^{\infty} ne^{-n/5}$$

(a) Graph the function $f(x) = xe^{-x/5}$ and select an interval on which it is decreasing.

(b) Evaluate $\int_N^{\infty} xe^{-x}dx$ where N is an integer chosen from the interval determined in (a).

(c) Does the series converge or diverge?

34. In this exercise, we explore the series

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\ldots+\frac{1}{n(n+1)}+\ldots$$

(a) Compute s_1, s_2, and so on until you discover a pattern for the partial sums. Does the series converge?

(b) Based on your answer in (a), does the following integral converge or diverge?

$$\int_1^{\infty}\frac{1}{x(x+1)}dx$$

(c) Evaluate the integral

$$\int_1^{\infty}\frac{1}{x(x+1)}dx$$

Does the integral converge or diverge?

35. In this exercise, we explore the divergence of the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots$$

(a) Show that

$$\ln(n)\le 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\le 1+\ln(n)$$

(b) Use the inequality in (a) to estimate the size of the n^{th} partial sum of the harmonic series when $n = 1000$, when $n = 1,000,000$ and when $n = 1,000,000,000$. About how long until the n^{th} partial sum is greater than 100? (Hint: solve $\ln(n) = 100$).

36. **Euler's Constant:** The slow divergence of the harmonic series motivated Euler to study the sequence whose general term is

$$s_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\ln(n)$$

(a) Use the identity in part (b) of exercise 35 to show that $s_n \ge 0$ for all n.

(b) Use the fact that $\int_n^{n+1}\frac{1}{x}dx = \ln(n+1)-\ln(n)$ to show that

$$\frac{1}{n+1} < \ln(n+1)-\ln(n)$$

(c) Use the result in (b) to show that the sequence of s_n's is a decreasing sequence.

(d) A decreasing sequence which is bounded below must converge. Thus, there is a number γ called *Euler's Constant* such that

$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\ln(n)\right)=\gamma$$

Estimate γ by computing s_n for various values of n.

37. An Asymptotic Inequality: Explain why if $\int_1^\infty f(x)\,dx$ diverges, then (8.42) implies that

$$1 \leq \frac{a_1 + \ldots + a_n}{\int_1^n f(x)\,dx} \leq \frac{a_1}{\int_1^n f(x)\,dx} + 1$$

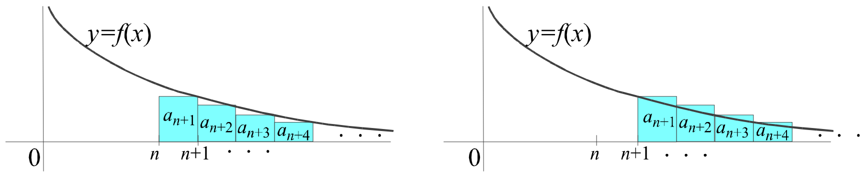
Then use the squeeze theorem to show that

$$\lim_{n\to\infty} \left(\frac{a_1 + \ldots + a_n}{\int_1^n f(x)\,dx} \right) = 1$$

38. Write to Learn: A useful result in estimating the sum of a series is that if $f(x)$ is positive, continuous, and decreasing with $f(n) = a_n$, then

$$\int_{n+1}^\infty f(x)\,dx \leq L - \sum_{k=1}^n a_k \leq \int_n^\infty f(x)\,dx$$

where $L = \sum_{k=1}^\infty a_k$. Explain this result using the pictures below:



6-3: Implies inequality in exercise 38

39. The inequality in 37 shows that the series $1 + 2 + 3 + 4 + \ldots$ satisfies

$$\lim_{n\to\infty} \left(\frac{1 + 2 + \ldots + n}{\int_1^n xdx} \right) = 1$$

Use this to show develop an estimate of $1+2+3+4+\ldots+n$ for large values of n.

40. The inequality in 37 shows that the series $\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \ldots$ satisfies

$$\lim_{n\to\infty} \left(\frac{\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n}}{\int_1^n \sqrt{x}dx} \right) = 1$$

Use this to show develop an estimate of $\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n}$ for large values of n.

8.7 The Comparison and Other Tests

The Comparison Test

If $\sum_{n=1}^\infty a_n$ is a positive term series, then its partial sums $s_n = a_1 + \ldots + a_n$ form an increasing sequence. Thus, the series will converge if the sequence of partial sums has an upper bound. To find such a bound, let us suppose that $a_n \leq b_n$ for all $n \geq 1$ and let's suppose that $\sum_{n=1}^{\infty} b_n$ converges. It follows that

$$a_1 + \ldots + a_n \leq b_1 + \ldots + b_n \leq \sum_{n=1}^{\infty} b_n$$

for all n. That is, $\sum_{n=1}^{\infty} b_n$ is a bound on the partial sums of $\sum_{n=1}^{\infty} a_n$, thus implying that $\sum_{n=1}^{\infty} a_n$ converges.

Conversely, let us suppose that $\sum_{n=1}^{\infty} a_n$ diverges. Then the inequality

$$a_1 + \ldots + a_n \leq b_1 + \ldots + b_n$$

implies that the partial sums of $\sum_{n=1}^{\infty} b_n$ do not have an upper bound. Hence, $\sum_{n=1}^{\infty} b_n$ also diverges. As a result, we have the following theorem.

Comparison Test: If there exists an integer N such that $a_n \leq b_n$for all $n \geq N$, then $\sum_{n=1}^{\infty} a_n$ converges if $\sum_{n=1}^{\infty} b_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges if $\sum_{n=1}^{\infty} a_n$ diverges.

A positive term series is often compared to the series obtained by ignoring all but the highest order terms in the numerator and in the denominator. As a result, a given series is often compared to a p-series. (recall that a p-series converges if $p > 1$ but diverges if $p \leq 1$).

Compare a given series to a series with only the highest order terms from the numerator and the denominator.

EXAMPLE 1 Apply the comparison test to

$$\sum_{n=1}^{\infty} \frac{n-1}{n^3 + 2n + 1}$$

Solution: We compare to the series involving only the dominant terms in the numerator and denominator. That is, we compare the given series to $\sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$. To do so, we must construct an inequality between the general terms of the two series. We begin with an obvious inequality involving the denominator:

$$n^3 + 2n + 1 \geq n^3$$

The reciprocal then leads to

$$\frac{1}{n^3 + 2n + 1} \leq \frac{1}{n^3}$$

Since $n - 1 \leq n$, we multiply on the left by $n - 1$ and on the right by n:

$$\frac{n-1}{n^3 + 2n + 1} \leq \frac{n}{n^3}$$

As a result, we have

$$\sum_{n=1}^{\infty} \frac{n-1}{n^3 + 2n + 1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

The larger series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series (i.e., $p = 2$), which implies that the smaller series $\sum_{n=1}^{\infty} \frac{n-1}{n^3+2n+1}$ also converges.

EXAMPLE 2 Determine if the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{2n+1}{2n^2-1}$$

Solution: In order to use the comparison test, we first note that we want to compare to the series

$$\sum_{n=1}^{\infty} \frac{2n}{2n^2} = \sum_{n=1}^{\infty} \frac{1}{n}$$

which is a divergent p-series ($p = 1$). To do so, we begin with the obvious:

$$2n^2 \geq 2n^2 - 1$$

Reconstruct the general term of the given series on the left of the inequality.

Taking the reciprocals on both sides then yields

$$\frac{1}{2n^2} \leq \frac{1}{2n^2-1} \tag{8.50}$$

Since $2n \leq 2n+1$, multiplying (8.50) on the left by $2n$ and on the right by $2n+1$ does not change the inequality:

$$\frac{2n}{2n^2} \leq \frac{2n+1}{2n^2-1}$$

Applying the summation operator to both sides yields

$$\sum_{n=1}^{\infty} \frac{1}{n} \leq \sum_{n=1}^{\infty} \frac{2n+1}{2n^2-1}$$

Since the smaller series diverges, the larger series $\sum_{n=1}^{\infty} \frac{2n+1}{2n^2-1}$ also diverges.

There are series for which the inequalities do not work out so nicely. However, these series can often be *reindexed* to facilitate the comparisons, as is explored in exercises 35-38, after which the comparison test works nicely.

Check your Reading *What series will be compared with the series*

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n}$$

Absolute Convergence

Suppose we have a series $\sum_{n=1}^{\infty} a_n$ where the a_n's may be either positive or negative. It is easy to see that

$$-|a_n| \leq a_n \leq |a_n|$$

Adding $|a_n|$ throughout yields

$$0 \leq a_n + |a_n| \leq 2|a_n|$$

Applying the summation operator yields

$$\sum_{n=1}^{\infty}(a_n + |a_n|) \leq 2\sum_{n=1}^{\infty}|a_n|$$

so that if the larger series $2\sum_{n=1}^{\infty}|a_n|$ converges, then so also does the smaller series, $\sum_{n=1}^{\infty}(a_n + |a_n|)$. Moreover, since

$$a_n = a_n + |a_n| - |a_n|$$

applying the summation operator yields

$$\sum_{n=1}^{\infty}a_n = \sum_{n=1}^{\infty}(a_n + |a_n| - |a_n|) = \sum_{n=1}^{\infty}(a_n + |a_n|) - \sum_{n=1}^{\infty}|a_n|$$

Thus, if $\sum_{n=1}^{\infty}|a_n|$ converges, then so also does $\sum_{n=1}^{\infty}a_n$. This leads us to the following definition

Definition 7.2: If $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty}a_n$ is said to *converge absolutely.*

Moreover, we can use the comparison test to determine whether or not a series converges absolutely.

EXAMPLE 3 Test the following series for absolute convergence:

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots + \frac{(-1)^{n+1}}{n^2} + \ldots = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} \qquad (8.51)$$

Solution: The series itself is not a positive term series. However, taking the absolute value of each term in the series yields

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

which is a convergent p-series (i.e., $p = 2$). Hence the original series (8.51) converges *absolutely.*

EXAMPLE 4 Test the following series for absolute convergence:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n - 2^n}$$

Solution: Since $2^n > n$ for all positive integers n, the series of absolute values is of the form

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n - 2^n}\right| = \sum_{n=1}^{\infty}\frac{1}{2^n - n}$$

We begin the comparison by finding an expression *smaller* than $2^n - n$. To do so, we must subtract a term *larger* than n. Since also $2^{n-1} > n$, we subtract 2^{n-1} to obtain

$$2^n - n > 2^n - 2^{n-1}$$

Since $2^n - 2^{n-1} = 2 \cdot 2^{n-1} - 2^{n-1}$ reduces to 2^{n-1}, we have

$$\sum_{n=1}^{\infty} \frac{1}{2^n - n} < \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$$

The last series is a geometric series with common ratio $\frac{1}{2}$, which implies that it converges. As a result, the original series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n - 2^n} \text{ converges absolutely}$$

Check your Reading *Compute 2^{2-1} and compare it with 2. Compute 2^{3-1} and compare it with 3. Compute 2^{4-1} and compare it with 4. Can we conclude that 2^{n-1} is always bigger than n?*

Alternating Series

An *alternating series* is a series in which the signs of the terms alternate. That is, if $a_n > 0$, then

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^{n+1} a_n + \ldots$$

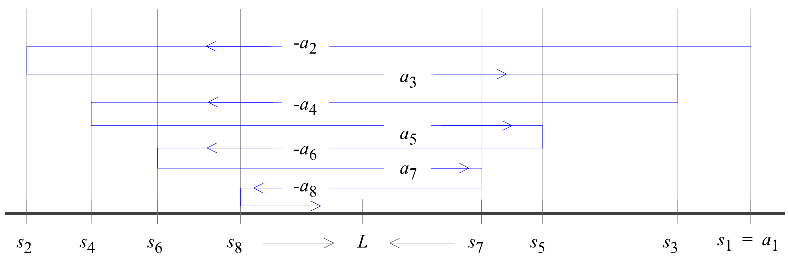
is an *alternating series.* If we also require that the sequence $\{a_n\}_{n=1}^{\infty}$ be decreasing with a limit of 0, then the partial sums have an interesting pattern. To begin with, $s_2 = a_1 - a_2$ is less than $s_1 = a_1$. Since $a_2 \geq a_3$, adding a_3 to s_2 will not quite get us back to s_1. That is,

$$s_3 = a_1 - a_2 + a_3 \leq a_1 - a_2 + a_2 = a_1 = s_1$$

Likewise, subtracting a_4 from s_3 will not quite make it back down to s_2. That is, $a_3 \geq a_4$ implies that

$$s_4 = a_1 - a_2 + a_3 - a_4 \geq a_1 - a_2 + a_3 - a_3 = a_1 - a_2 = s_2$$

Thus, the odd partial sums form a decreasing sequence bounded below by the increasing sequence of even partial sums:



Clearly then, the limits of both sequences exist, and since $s_{n+1} - s_n = (-1)^{n+1} a_n$, the difference between the two limits approaches zero, thus implying that the limit of even partial sums and the limit of odd partial sums both have the same limit L.

Notice also that the n^{th} partial sum can be no farther from L than the next term in the series. Thus, we have the following:

Alternating Series Test: If $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is an alternating series in which

$$\text{i)}\quad a_n \geq a_{n+1}\ for\ \text{all } n \qquad \text{and} \quad \text{ii)}\quad \lim_{n\to\infty} a_n = 0$$

then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges to a number L that satisfies

$$|s_n - L| < a_{n+1}$$

Moreover, if $a_n \geq 0$, then $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is also an alternating series to which the alternating series test can be applied.

EXAMPLE 5 The *alternating harmonic series* is the alternating series given by

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{(-1)^n}{n} + \ldots = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$

Test for convergence or divergence. If it converges, compute s_{10} and estimate its accuracty.

Solution: It is straightforward to show that

$$\frac{1}{n} \geq \frac{1}{n+1} \quad for\ all\ n \qquad and \qquad \lim_{n\to\infty}\frac{1}{n} = 0$$

Thus, the *alternating harmonic series converges.* Moreover,

$$s_{10} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{8} - \frac{1}{9} + \frac{1}{10}$$

which is approximately $s_{10} = 0.73056$. Moreover, s_{10} must differ from the sum of the series by no more than a_{11}, so that we have

$$|0.73056 - L| < \frac{1}{11} = 0.0909$$

That is, the sum of the series L must satisfy

$$0.63966 < L < 0.82146$$

Check your Reading *Does the series* $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\cos(nx)}{n^2}$ *converge absolutely for all* x*?*

Conditional Convergence

When a series has both positive and negative terms, the comparison test can be used only to determine whether or not the series converges absolutely. However, it is possible that the given series converges without converging absolutely, as is implied by the following definition:

Definition 7.3: If the series $\sum_{n=1}^{\infty} a_n$ converges but the series $\sum_{n=1}^{\infty} |a_n|$ diverges, then the series $\sum_{n=1}^{\infty} a_n$ is said to *converge conditionally.*

For example, in example 5 we demonstrated the convergence of the *alternating harmonic series,*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

However, the series of absolute values of the alternating series is of the form

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

That is, the series of absolute values produces the *harmonic series*, which was shown to diverge in section 8.4. As a result, we can conclude that the alternating harmonic series does **not** converge absolutely.

That is, the alternating series converges, even though it does not converge absolutely. That is, the alternating harmonic series *converges conditionally.*

Series which only converge conditionally are seldom used in applications because they often converge very slowly and also because a rearrangement of the terms might not converge at all. For example, the alternating harmonic series can be rearranged by grouping the odd and even terms separately:

$$\left(1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} + \ldots\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{2n} + \ldots\right) \qquad (8.52)$$

Each of the two series in (8.52) then compares to a p-series with $p = 1$, which implies that both series diverge. That is, the alternating harmonic series can be rearranged into the sum of two *divergent* series.

Indeed, it can be shown that a conditionally convergent series can be rearranged so that it converges to any given number. As a result, we will work with absolutely convergent series whenever possible in this textbook.

Exercises:

Which of the following series converge, and which diverge?

1. $\sum_{n=1}^{\infty} \frac{n-2}{n^3}$
2. $\sum_{n=1}^{\infty} \frac{n^2-4}{n^4+1}$
3. $\sum_{n=1}^{\infty} \frac{n}{2n-1}$
4. $\sum_{n=1}^{\infty} \frac{2n-5}{n^4+3n+2}$
5. $\sum_{n=1}^{\infty} \frac{8n^2+3n+7}{2n^3-1}$
6. $\sum_{n=1}^{\infty} \frac{6n^3-7n-1}{9n^5+13n^3+2}$
7. $\sum_{n=1}^{\infty} \frac{(n-1)^2}{n^4+2n+1}$
8. $\sum_{n=2}^{\infty} \left(\frac{n-1}{n^2-1}\right)^2$
9. $\sum_{n=2}^{\infty} \frac{n^{1/2}}{n^{3/2} - n^{1/2}}$
10. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n-1}}$
11. $\sum_{n=1}^{\infty} \frac{1}{2^n+3}$
12. $\sum_{n=1}^{\infty} \frac{2^n}{5^n+n}$
13. $\sum_{n=2}^{\infty} \frac{1+\cos(n)}{n^2}$
14. $\sum_{n=2}^{\infty} \frac{1}{2e^n-1}$
15. $\sum_{n=1}^{\infty} \frac{1}{(n+1)!}$ (Hint: $(n+1)! \geq n^2+n$)

Determine if the given series converges absolutely. If it does not, then write "does not converge absolutely."

16. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

17. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$

18. $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n-1}$

19. $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^4+2n}$

20. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+1}$

21. $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$

22. $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$

23. $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^5+2n}$

24. $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n-\cos(n)}$

Use the Alternating series test to determine if the following converge. If they converge, compute the 10^{th} partial sum. How close is the 10^{th} partial sum to the actual sum of the series?

25. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

26. $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{\sqrt{n}+1}$

27. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$

28. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{e^n}$

29. $\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right)$

30. $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right)$

31. In this exercise, we examine the conditionally convergent series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + \frac{(-1)^{n+1}}{2n-1} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \tag{8.53}$$

(a) Explain why (8.53) does not converge absolutely but does converge conditionally.

(b) Rearrange (8.53) into the difference of two divergent series.

32. In this exercise, we examine the conditionally convergent series

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \ldots + \frac{(-1)^{n+1}}{2n} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n} \tag{8.54}$$

(a) Show that (8.54) does not converge absolutely but does converge conditionally.

(b) Rearrange (8.54) into the difference of two divergent series.

33. In this exercise, we examine the conditionally series

$$1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \ldots \tag{8.55}$$

(a) Show that (8.55) does not converge absolutely.

(b) Rearrange (8.55) into the sum of two conditionally convergent series.

34. In this exercise, we examine the series

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \ldots \tag{8.56}$$

(a) Does (8.56) converge or diverge?
(b) Rearrange (8.56) into the difference of two divergent series.

35. Reindexing: In this exercise, we test for the convergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{4n^2 - 1} \tag{8.57}$$

(a) Explain why the comparison test cannot be directly applied to the series as it is written:
(b) Write out the first four terms of (8.57) and the first four terms of the series

$$\sum_{k=1}^{\infty} \frac{1}{4\left(k+1\right)^2 - 1}$$

Are they the same? Explain.
(c) Show that the series in (b) can be written as

$$\sum_{k=1}^{\infty} \frac{1}{4k^2 + 8k + 3}$$

and then apply the comparison test. Does (8.57) converge?

36. Reindexing: Show that

$$\sum_{n=2}^{\infty} \frac{1}{(2n-1)\,2n} = \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \ldots = \sum_{n=1}^{\infty} \frac{1}{(2n+1)\,(2n+2)}$$

and then apply the comparison test to determine if the series converges or diverges.

37. Reindexing: Show that the comparison test cannot be applied directly to

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

Reindex the series by letting $n = k + 1$ and test for convergence.

38. Reindexing: Show that the comparison test cannot be applied directly to

$$\sum_{n=2}^{\infty} \frac{1}{2n + 1}$$

Reindex the series by letting $n = k - 1$ and test for convergence.

39. The geometric series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots \tag{8.58}$$

can be rearranged into the sum of two geometric series:

$$\left(\frac{1}{2} + \frac{1}{8} + \frac{1}{32} \ldots + \frac{1}{2}\left(\frac{1}{4^n}\right) + \ldots\right) + \left(\frac{1}{4} + \frac{1}{16} + \ldots + \frac{1}{4^n} + \ldots\right) \tag{8.59}$$

Show that (8.58) and (8.59) converge to the same number.

40. In this exercise, we look at rearrangements of the absolutely convergent series

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots+\left(\frac{-1}{2}\right)^n+\ldots=\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n \tag{8.60}$$

(a) Use the geometric series test to show that (8.60) converges.

(b) Construct the series of absolute values of terms in (8.60) and show that it also converges.

(c) Use the geometric series test to determine the sum of the series

$$1+\frac{1}{4}+\frac{1}{16}\ldots+\left(\frac{1}{4}\right)^n+\ldots$$

(d) Use the geometric series test to determine the sum of the series

$$\frac{-1}{2}-\frac{1}{8}-\frac{1}{32}-\ldots$$

(hint: factor out $-1/2$).

(e) Rearrange the series in (8.60) so that it is the sum of the two series in (c) and (d). Combine the sums of the series in (c) and (d) and compare to the sum in (a).

41. **Write to Learn:** In a short essay, show that the partial sums of the series

$$\frac{1}{2}-\frac{1}{2}+\frac{1}{3}-\frac{1}{3}+\frac{1}{4}-\frac{1}{4}+\ldots \tag{8.61}$$

are of the form

$$S_n=\begin{cases}\frac{2}{n-1} & if \quad n \text{ is odd}\\ 0 & if \quad n \text{ is even}\end{cases}$$

and then explain why $\lim_{n\to\infty} S_n = 0$. Does the series (8.61) converge absolutely? What can we say about the series?

42. **Write to Learn:** In exercise 41, we showed that the following series converges conditionally

$$\frac{1}{2}-\frac{1}{2}+\frac{1}{3}-\frac{1}{3}+\frac{1}{4}-\frac{1}{4}+\ldots \tag{8.62}$$

In a short essay, explain why the series

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{2}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}-\frac{1}{3}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}-\frac{1}{4}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}-\ldots$$

is a rearrangement of (8.62). Use the following partial sums to suggest that the rearrangement converges, but does not converge to 0.

$$\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}-\tfrac{1}{2}+\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}=1.09285714$$
$$\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}-\tfrac{1}{2}+\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}-\tfrac{1}{3}+\tfrac{1}{8}+\tfrac{1}{9}+\tfrac{1}{10}=1.09563492$$
$$\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}-\tfrac{1}{2}+\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}-\tfrac{1}{3}+\tfrac{1}{8}+\tfrac{1}{9}+\tfrac{1}{10}-\tfrac{1}{4}+\tfrac{1}{11}+\tfrac{1}{12}+\tfrac{1}{13}=1.0968004218$$

Self Test

A variety of questions are asked in a variety of ways in the problems below. Answer as many of the questions below as possible before looking at the answers in the back of the book.

1. Answer each statement as true or false. If the statement is false, then state why or give a counterexample.

   (a) An increasing sequence must converge.

   (b) A convergent sequence must be increasing.

   (c) The decimal $1.2345678910111213\ldots$ is the limit of the sequence

   $$\{1.2\dot{3}\ldots n\}_{n=2}^{\infty}$$

   (d) A geometric series satisfies a linear recursion

   (e) If $x_{n+1} = f(x_n)$ and $|f'(p)| < 1$, then x_0 close to p implies that

   $$\lim_{n\to\infty} x_n = p$$

   (f) Every function has at least one fixed point.

   (g) If $f(p) = 0$, then Newton's method converges to p.

   (h) The series $\sum_{n=1}^{\infty} a_n$ converges only if the sequence $\{a_n\}_{n=1}^{\infty}$ converges.

   (i) The partial sums of a positive term series form an increasing sequence.

   (j) If $a_n \geq 0$ and if $f(x) \geq 0$ satisfies

   $$\sum_{k=1}^{n} a_n \leq \int_1^n f(x)\,dx$$

   for all positive integers n, then the convergence of $\int_1^n f(x)\,dx$ implies the convergence of the series $\sum_{n=1}^{\infty} a_n$

   (k) The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges only if $p > 1$

   (l) The comparison test can be used to test for the convergence of a conditionally convergent series.

2. Suppose today I give you a dollar and then each day after I give you half of what I gave you the day before. How much money will I have eventually given to you?

   (a) \$2 (b) \$256 (c) \$2048 (d) ∞

3. The series $\displaystyle\sum_{n=27}^{\infty} \frac{2^n}{n^2}$

   (a) converges (b) diverges (c) alternates (d) test is inconclusive

4. The series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

   (a) converges (b) diverges (c) alternates (d) test is inconclusive

5. The series $\sum_{n=1}^{\infty} \frac{3^n + 4}{5^n}$

(a) converges (b) diverges (c) alternates (d) test is inconclusive

6. Determine the limit, if it exists, of the sequence

$$\frac{1}{1}, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \ldots$$

7. Suppose you have an account from which you withdraw \$100 each month. If the interest on the account is 0.5% per month, then what minimum initial balance is necessary in order for the account to always have more than \$100 in it?

8. Discrete Populations can even be modeled by equations of the form

$$y_{n+1} = k\, y_n \log_{10}\left(\frac{P}{y_n}\right)$$

where y_n is the population after n generations, and where k and P are positive constants. Find and classify the fixed points of this population model.

9. Use Newton's method to estimate the zero of

$$f(x) = x + \ln(x)$$

10. Apply Euler's method to the differential equation

$$y' = -y^2, \ y(0) = 1$$

Do the estimates become better or worse as n increases? Explain.

11. Test for convergence using the comparison test:

$$\sum_{n=0}^{\infty} \frac{n-1}{n^3 + 3n + 1}$$

12. Use the integral test to test the convergence of the series

$$\sum_{n=1}^{\infty} ne^{-n}$$

13. Use a diagram to show that

$$\int_1^{n+1} \frac{dx}{2x-1} dx \leq \sum_{k=1}^{n} \frac{1}{2k-1} \leq 1 + \int_1^n \frac{dx}{2x-1}$$

and then use this to explain why

$$S_n = \sum_{k=1}^{n} \frac{1}{2k-1} - \frac{1}{2}\ln(2n+1)$$

is bounded below by 0 (i.e., $S_n \geq 0$ for all n). Then show that $S_n \leq S_{n+1}$ for all n and that $S_n \leq 1$ for all n. Then explain why the sequence $\{S_n\}_{n=1}^{\infty}$ converges.

14. **Write to Learn:** Let a_0 and b_0 be positive numbers and define

$$a_{n+1} = \frac{a_n + b_n}{2} \quad \textit{and} \quad b_{n+1} = \sqrt{a_n b_n}$$

Show that if $f_n = \frac{a_n}{b_n}$, then

$$f_{n+1} = \frac{1}{2}\left(\sqrt{f_n} + \frac{1}{\sqrt{f_n}}\right) \tag{8.63}$$

Find the fixed point(s) of (8.63) and show that they are stable. Then in a short essay, explain why (8.63) implies that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

when both limits exist.[3]

15. *Test for convergence:

$$\sum_{n=1}^{\infty} \frac{n + 2^n}{n^2 2^n}$$

16. *Test for convergence:

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$$

The Next Step... The Riemann Zeta Function

Geometric series occur naturally in the study of geometric progression and linear recursion. But do p-series actually occur in applications? Indeed, they do, and our next step is to show that p-series are important in a wide variety of applications.

The answer is that the number and the variety of applications is truly amazing! For example, the *Riemann Zeta function* for positive integer values $p > 1$ can be defined

$$\zeta(p) = \frac{1}{(p-1)!}\int_0^\infty \frac{t^{p-1}}{e^t - 1}dt \tag{8.64}$$

where $(p-1)! = p(p-1)\cdot\ldots\cdot 3\cdot 2\cdot 1$ is the product of the first p positive integers. The Riemann Zeta function can also be defined for non-integer values of p, as we will see below.

Integrals of the form (8.64) occur frequently in both quantum mechanics and statistical mechanics. For example, Planck used the concept of a quantum of energy to show that the total intensity radiated by a blackbody at thermal equilibrium is

$$I_{total} = \frac{2\pi h}{c^2}\int_0^\infty \frac{\nu^3}{e^{h\nu/kT} - 1}d\nu \tag{8.65}$$

where c is the speed of light in a vacuum, h is Planck's constant, k is Boltzmann's constant, and T is the temperature of the black body. In addition, the Riemann zeta function is foundational in the study of prime numbers.

However, let's notice that the integral (8.64) is equivalent to

$$\zeta(p) = \frac{1}{(p-1)!}\int_0^\infty \frac{t^{p-1}e^{-t}}{(e^t - 1)e^{-t}}dt = \frac{1}{(p-1)!}\int_0^\infty \frac{1}{1 - e^{-t}}\, t^{p-1}e^{-t}dt$$

[3] The common limit of a_n and b_n is called the *arithmetic-geometric* mean of a_0 and b_0. It is important in many applications, such as the one in the *advanced contexts* at the end of chapter 6.

Since $e^{-t} < 1$ for all $t > 0$, we can use the geometric series to write

$$\frac{1}{1-e^{-t}} = \sum_{m=0}^{\infty} \left(e^{-t}\right)^m = \sum_{n=0}^{\infty} e^{-mt}$$

Consequently, we have

$$\zeta(p) = \frac{1}{(p-1)!} \int_0^{\infty} \sum_{m=0}^{\infty} e^{-mt}\, t^{p-1} e^{-t} dt = \frac{1}{(p-1)!} \int_0^{\infty} \sum_{m=0}^{\infty} t^{p-1} e^{-(m+1)t} dt$$

Now let us assume that the integral and the sum can be switched (a rather large assumption, but one that can nonetheless be shown to be valid after a great deal of justification). Then

$$\zeta(p) = \sum_{m=0}^{\infty} \frac{1}{(p-1)!} \int_0^{\infty} t^{p-1} e^{-(m+1)t} dt$$

Let's now re-index with $n = m+1$, so that instead of $m = 0$ to ∞, we have $n = 1$ to ∞:

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{(p-1)!} \int_0^{\infty} t^{p-1} e^{-nt} dt \tag{8.66}$$

Although this may seem a bit complicated at this point, let evaluate it for $p = 3$:

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{2!} \int_0^{\infty} t^{3-1} e^{-nt} dt = \sum_{n=1}^{\infty} \frac{1}{2 \cdot 1} \int_0^{\infty} t^2 e^{-nt} dt$$

The improper integral must be evaluated using tabular integration:

$$\begin{array}{ccc} u & & dv \\ \hline t^2 & & e^{-nt} \\ & + \searrow & \\ 2t & & \frac{-1}{n} e^{-nt} \\ & - \searrow & \\ 2 & & \frac{1}{n^2} e^{-nt} \\ & + \searrow & \\ 0 & & \frac{-1}{n^3} e^{-nt} \end{array}$$

The result is that

$$\int_0^{\infty} t^2 e^{-nt} dt = \lim_{R\to\infty} \left[\frac{-t^2}{n} e^{-nt} - \frac{2t}{n^2} e^{-nt} - \frac{2}{n^3} e^{-nt} \right]_0^R = \frac{2}{n^3}$$

As a result, we have

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{2} \int_0^{\infty} t^2 e^{-nt} dt = \sum_{n=1}^{\infty} \frac{1}{2} \frac{2}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

That is, $\zeta(3)$ is a p-series with $p = 3$.

Likewise, it can be shown that

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p} \tag{8.67}$$

for all $p > 1$. Consequently, every application involving the Riemann Zeta function are actually application of p-series. Indeed, the importance of the p-series in applications cannot be overstated.

Write to Learn Write a short essay in which you use $p = 2$ in the formula (8.66) to show that $\zeta(2)$ is a p-series. Include in the essay the computation of a partial sum of

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{n^2} + \ldots$$

as an approximation of $\zeta(2)$.

Write to Learn Write a short essay in which you use $u = \frac{hv}{kT}$, $du = \frac{h}{kT}d\nu$, $u(0) = 0$, and $u(\infty) = \infty$ to rewrite (8.65) as

$$I_{total} = \frac{2\pi k^4 T^4}{c^2 h^3} \int_0^\infty \frac{u^3}{e^u - 1} du$$

Then use the discussion above to expression I_{total} in terms of a p-series.

Write to Learn In general, the Riemann Zeta function is defined

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^\infty \frac{t^{x-1}}{e^t - 1} dt, \quad x > 1$$

where the *gamma function* is defined $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$. Use the methods above to show that we can rewrite $\zeta(x)$ as

$$\zeta(x) = \frac{1}{\Gamma(x)} \sum_{n=1}^{\infty} \int_0^\infty t^{x-1} e^{-nt}\, dt$$

Then let $u = nt$, $du = ndt$, $u(0) = 0$ and $u(\infty) = \infty$ to show that $\zeta(x)$ reduces to a p-series.

Write to Learn Go do the library and/or search the internet to find out as much as you can about the Riemann Zeta function. Write a formal report detailing your findings.

Group Learning One of the most important identities of the Riemann zeta function is that if n is an integer, then

$$\zeta(1 - n) = \frac{2}{(2\pi)^n} \cos\left(\frac{n\pi}{2}\right) (n-1)!\, \zeta(n)$$

Each member of the group should choose one of $\zeta(-1)$, $\zeta(-2)$, $\zeta(-3)$, and $\zeta(-4)$ and should use the identity above to express the respective values in terms of convergent p-series.

Advanced Contexts:

The p-series representation (8.67) of $\zeta(x)$ only works for $x > 1$, but $\zeta(x)$ itself actually exists for a much larger range of x. To see this, let us notice that if we alternate signs in a positive term series, then in its sum with the original series the odd terms cancel.

$$\left(1 + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \ldots\right) + \left(-1 + \frac{1}{2^x} - \frac{1}{3^x} + \frac{1}{4^x} - \ldots\right) = \frac{2}{2^x} + \frac{2}{4^x} + \ldots$$

Since $\zeta(x) = 1 + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \ldots$, we have

$$\zeta(x) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^x} = \frac{2}{2^x} + \frac{2}{4^x} + \ldots + \frac{2}{(2n)^x} + \ldots$$

However, the series on the right is the sum over positive even integers, which are of the form $2n$, so that we can write

$$\zeta(x)+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^x}=\sum_{n=1}^{\infty}\frac{2}{(2n)^x}$$

which in turn leads to

$$\zeta(x)+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^x}=\sum_{n=1}^{\infty}\frac{2}{2^x n^x}=\frac{2}{2^x}\sum_{n=1}^{\infty}\frac{1}{n^x}$$

Since $\zeta(x)=\sum_{n=1}^{\infty}\frac{1}{n^x}$, we thus have

$$\zeta(x)+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^x}=2^{1-x}\zeta(x)$$

which we solve for $\zeta(x)$ to obtain

$$\zeta(x)=\frac{1}{1-2^{1-x}}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^x}\tag{8.68}$$

It can be shown that the series in (8.68) converges for all $x>0$, so that (8.68) is defined for all $x>0$ such that $x\neq 1$.

Moreover, unlike the p-series representation, the representation (8.68) of the zeta function raises the possibility that $\zeta(x)$ may have zeroes for some values of $x>0$. Indeed, one of the most famous unsolved problems in mathematics is the *Riemann Hypothesis,* which conjectures that if $\zeta(z)=0$ and z is not a negative number, then z is an imaginary number of the form

$$z=\frac{1}{2}+i\sigma$$

where σ can be either positive or negative and i is the imaginary unit, $i^2=-1$. Because of the importance of $\zeta(x)$ in physics, engineering, and mathematics, solving the Riemann Hypothesis would likely have a profound effect on those fields. As an added incentive, the Riemann Hypothesis is a Millennium problem of the Clay Mathematics Institute which offers a prize of 1 million dollars to the first person or persons to solve the Riemann Hypothesis.

1. Use the alternating series test to show that $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^x}$ converges for all $x>0$.
2. ** Another representation of the zeta function is given by

$$\zeta(x)=\frac{1}{1-2^{1-x}}\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!}\frac{(-1)^k}{(k+1)^x}$$

   Can you show that it reduces to (8.68)? (Hint: $\frac{n!}{k!(n-k)!}$ is called a *binomial coefficient* because it is the k^{th} entry in the n^{th} row of Pascal's triangle. You may also want to explore *Euler's series transformation*).
3. ** The value of $\zeta(-1)$ is important in *renormalization theory.* Thus, we need a representation for $\zeta(x)$ that converges for negative numbers as well. Can you show that the series

$$\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!}\frac{(-1)^k}{(k+1)^x}$$

converges for all x? What is the resulting expression for $\zeta(-1)$ (see previous exercise)? (Hint: you will need to compare to a geometric series, and you may need to explore the library and the internet to find additional properties of the binomial coefficients).

9. TAYLOR'S SERIES

A computer is only capable of performing elementary arithmetic. All higher mathematics must be implemented with software.

Because they are based on binary logic, calculators and computers can only add, subtract, multiply and divide—and they do not divide all that well! That is, a computer does not naturally estimate $\ln(2)$ with the decimal approximation 0.6931472, but rather it must be programmed to produce such approximations using only arithmetic and binary logic.

However, a polynomial is by definition a function that involves only addition, subtraction and multiplication. That is, polynomials can be computed exactly with a computer and are thus ideal for use as approximations to more complicated functions.

In many applications, the polynomials used in approximations are Taylor polynomials, where the n^{th} *Taylor polynomial of* a function $f(x)$ centered at a number p is defined to be

$$T_n(x) = f(p) + f'(p)(x-p) + \frac{f''(p)}{2!}(x-p)^2 + \ldots + \frac{f^{(n)}(p)}{n!}(x-p)^n \quad (9.1)$$

where $f^{(n)}(p)$ is the n^{th} derivative of $f(x)$ at $x = p$ and where the factorial function $n!$ is the product of the positive integers from 1 up to n. Indeed, the fundamental theorem of calculus allows us to extend Taylor polynomials into one of the most important mathematical tools ever introduced into science—the Taylor's series expansion of a function.

Taylor's series are unrivaled in their versatility, as they appear in settings ranging from numerical approximation in computer programming to theoretical descriptions of subatomic particles in quantum mechanics. We will see some of these applications in the pages to come, along with many of the other ideas and applications that have made Taylor's series foundational to much of the science and mathematics of the past three hundred years.

9.1 Taylor Polynomials

Generalizing the Tangent Line Concept

In this section, we generalize the tangent line concept to higher degree polynomials. Specifically, the n^{th} *Taylor polynomial of* a function $f(x)$ centered at a number p is defined to be

$$T_n(x) = f(p) + f'(p)(x-p) + \frac{f''(p)}{2!}(x-p)^2 + \ldots + \frac{f^{(n)}(p)}{n!}(x-p)^n \quad (9.2)$$

where $f^{(n)}(p)$ is the n^{th} derivative of $f(x)$ at $x = p$ and where $n!$, pronounced "n factorial," is the product of the positive integers from 1 up to n.

To find the n^{th} Taylor polynomial of a function $f(x)$ for given n and p, we construct a table in which the first column contains the numbers 0 up to n and the second column contains the derivatives of $f(x)$ of order 0 up to order n, with

$f^{(0)}(x)$ being the function itself. A third column is constructed from the second column by substituting the given p into each of the derivatives of $f(x)$, and then the third column is used to construct the Taylor polynomial using (9.2). Moreover, *we do not simplify our computations*, because in later sections we will be trying to identify patterns in the computations, and simplification destroys patterns.

Simplifying computations destroys patterns.

EXAMPLE 1 Find the third degree Taylor polynomial of $f(x) = x^4$ centered at $p = 2$.

Solution: We begin with a table whose first row is the "zero-th" derivative of the function—i.e., the function itself. We then complete the column by computing the first, second, and third derivatives of $f(x)$, which gives us

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	x^4	
1	$4x^3$	
2	$4 \cdot 3x^2$	
3	$4 \cdot 3 \cdot 2x$	

We complete the table by evaluating each of the entries in the second column at $p = 2$, which yields

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	x^4	2^4
1	$4x^3$	$4 \cdot 2^3$
2	$4 \cdot 3x^2$	$4 \cdot 3 \cdot 2^2$
3	$4 \cdot 3 \cdot 2x$	$4 \cdot 3 \cdot 2 \cdot 2$

From (9.2), we see that the third degree Taylor polynomial of $f(x)$ centered at $p = 2$ is

$$T_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$$

Since $2! = 2 \cdot 1$ and $3! = 3 \cdot 2 \cdot 1$, filling in from the table yields

$$T_3(x) = 2^4 + 4 \cdot 2^3 (x-2) + \frac{4 \cdot 3 \cdot 2^2}{2 \cdot 1}(x-2)^2 + \frac{4 \cdot 3 \cdot 2 \cdot 2}{3 \cdot 2 \cdot 1}(x-2)^3$$

Cancellation and simplification then yields

$$T_3(x) = 16 + 32(x-2) + 24(x-2)^2 + 8(x-2)^3$$

EXAMPLE 2 Find $T_2(x)$ centered at 1 of $f(x) = \ln(x)$.

Solution: We begin with the table of derivative information:

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln(x)$	$\ln(1)$
1	x^{-1}	1
2	$-x^{-2}$	-1

and since $\ln(1) = 0$, filling in from the table yields

$$\begin{aligned} T_2(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 \\ &= 0 + 1(x-1) - \frac{1}{2}(x-1)^2 \\ &= (x-1) - \frac{1}{2}(x-1)^2 \end{aligned}$$

It is important to note that the n^{th} degree Taylor polynomial of an n^{th} degree polynomial is equal to the polynomial itself, regardless of where the approximation is centered.

EXAMPLE 3 Find $T_2(x)$ centered at $p = 3$ when $f(x) = x^2$.

Solution: The table of derivatives is

n	$f^{(n)}(x)$	$f^{(n)}(3)$
0	x^2	9
1	$2x$	6
2	2	2

As a result, the Taylor polynomial $T_2(x)$ is

$$T_2(x) = 9 + 6(x-3) + \frac{2}{2!}(x-3)^2$$

If we expand the Taylor polynomial in example 3, we obtain

$$\begin{aligned} T_2(x) &= 9 + 6x - 18 + x^2 - 6x + 9 \\ &= x^2 + 6x - 6x + 9 + 9 - 18 \\ &= x^2 \end{aligned}$$

Check your Reading *What is $T_5(x)$ of $f(x) = x^5$ centered at $p = 0$?*

Linearization and Quadratic Approximation

The first two Taylor polynomials are given special names and notations. The first Taylor polynomial is denoted by $L_p(x)$ and is of the form

$$T_1(x) = L_p(x) = f(p) + f'(p)(x-p) \tag{9.3}$$

The function $L_p(x)$ is called the *linearization* of $f(x)$ at p since its graph is the tangent line to $f(x)$ at $x = p$.

picture

EXAMPLE 4 Find $L_1(x)$ of $f(x) = x^2$. Then sketch its graph and the graph of $f(x)$ over $[-2, 2]$

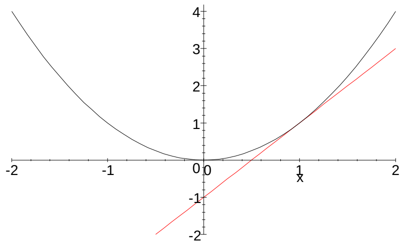
Solution: The table of derivatives is

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	x^2	1
1	$2x$	2

As a result, the Taylor polynomial is

$$L_1(x) = 1 + 2(x - 1)$$

which simplifies to $L_1(x) = 2x - 1$, so that $y = 2x - 1$ is tangent to $y = x^2$ when $p = 1$.



The second Taylor polynomial at p is often denoted $Q_p(x)$. It is defined

$$T_2(x) = Q_p(x) = f(p) + f'(p)(x - p) + \frac{f''(p)}{2!}(x - p)^2 \tag{9.4}$$

and is known as the *quadratic approximation to* $f(x)$ when $x = p$, and its graph is a *parabola* which is tangent to $f(x)$ when $x = p$.

EXAMPLE 5 Find the linearization and quadratic approximation—i.e., the first and second Taylor polynomials—at $x = 0$ of

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

Solution: To do so, we construct a table of derivatives up to order 2:

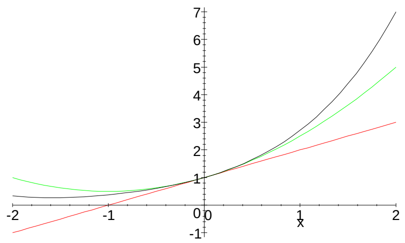
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$	1
1	$1 + \frac{2x}{2} + \frac{3x^2}{6} + \frac{4x^3}{24}$	1
2	$\frac{2}{2} + \frac{6x}{6} + \frac{12x^2}{24}$	1

Substituting into (9.3) yields

$$T_1(x) = L_0(x) = 1 + 1(x - 0) = 1 + x$$

and substituting into (9.4) yields

$$T_2(x) = Q_0(x) = 1 + 1(x-0) + \frac{1}{2}(x-0)^2 = 1 + x + \frac{x^2}{2}$$



1-1: $f(x)$ is in black, T_1 is red, and T_2 is green

Check your Reading *Is $T_1(x)$ the linearization of $T_2(x)$ at $p = 0$?*

Application to Differential Equations

Taylor polynomials can be used to approximate solutions to differential equations. To do so, we simply complete our table beginning in the second column with the derivative y' and beginning in the first column with the given initial value:

EXAMPLE 6 Find the quadratic approximation $T_2(x)$ of the solution to

$$y' = xy, \quad y(0) = 1 \tag{9.5}$$

Solution: Since we will need $y''(0)$, we begin by computing y'' using implicit differentiation:

$$\frac{d}{dx}y' = \frac{d}{dx}(xy)$$

Application of the product rule results in

$$\begin{aligned} y'' &= \left(\frac{d}{dx}x\right)y + x\left(\frac{d}{dx}y\right) \\ y'' &= y + xy' \end{aligned} \tag{9.6}$$

Since $y(0) = 1$, we can now write the table of derivatives

n	$y^{(n)}(x)$	$y^{(n)}(0)$
0		$y(0) = 1$
1	$x\,y$	
2	$y + xy'$	

Since $y' = xy$ and $y(0) = 1$, we have $y'(0) = 0 \cdot y(0) = 0$. Likewise, we obtain $y''(0)$ using $y(0) = 1$, $y'(0) = 0$ and the fact that $y'' = y + xy'$.

n	$y^{(n)}(x)$	$y^{(n)}(0)$
0		$y(0) = 1$
1	$x\,y$	$0 \cdot y(0) = 0$
2	$y + xy'$	$y(0) + 0 \cdot y'(0) = 1 + 0 = 1$

The completed table then yields the quadratic approximation of the solution of (9.5), which is

$$T_2(x) = 1 + 0(x-0) + \frac{1}{2}(x-0)^2 = 1 + \frac{x^2}{2}$$

EXAMPLE 7 Find the quadratic approximation of the solution to

$$y' = x^5 + 2y^2, \qquad y(1) = 3 \tag{9.7}$$

Solution: To do so, we first compute y'' using implicit differentiation:

$$y'' = \frac{d}{dx}\left(x^5 + 2y^2\right) = 5x^4 + 4yy'$$

$y'(0)$ in the second row of the third column is used in the third row of the third column.

Thus, our table of derivatives is

n	$y^{(n)}(x)$	$y^{(n)}(1)$
0		$y(1) = 3$
1	$x^5 + 2y^2$	$1^5 + 2 \cdot 3^2 = 19$
2	$5x^4 + 4yy'$	$5 \cdot 1 + 4 \cdot 3 \cdot 19 = 233$

and our quadratic approximation is

$$T_2(x) = 1 + 19(x-1) + \frac{233}{2}(x-1)^2$$

Check your Reading *Explain why* $\frac{d}{dx}\left(x^5 + 2y^2\right) = 5x^4 + 4yy'$

Second Order Differential Equations

Taylor approximations are especially important for second order differential equations. To do so, $y(0)$ and $y'(0)$ must be given, so that for a second order differential equation, we need only find $y''(0)$.

EXAMPLE 8 Find the quadratic approximation of the solution to

$$y'' = (x-2)y' - 2y, \qquad y(0) = 3, \qquad y'(0) = -4$$

Solution: At $x = 0$, the differential equation says that

$$y''(0) = (0-2)y'(0) - 2y(0)$$

Substituting $y(0) = 3$ and $y'(0) = -4$ leads to

$$y''(0) = -2 \cdot 3 - 2 \cdot (-4) = 2$$

Thus, the second Taylor polynomial is

$$\begin{aligned} T_2(x) &= y(0) + y'(0)x + \frac{y''(0)}{2}x^2 \\ &= 3 - 4x + \frac{2}{2}x^2 \end{aligned}$$

and as a result, the quadratic approximation of the solution is

$$T_2(x) = x^2 - 4x + 3$$

In fact, in many applications—in numerical analysis and physics, for example—it is not unusual for the solutions to second order differential equations to be polynomials. In such cases, the Taylor polynomials are not only approximations, but often are themselves the solutions.

For example, the Taylor polynomial $y = x^2 - 4x + 3$ we constructed in example 8 is actually a solution to the original differential equation

$$y'' = (x-2)\,y' - 2y$$

To see this, let's notice that $y' = 2x - 4$ and $y'' = 2$, so that substitution leads to

$$2 = (x-2)(2x-4) - 2\left(x^2 - 4x + 3\right)$$

We now FOIL (i.e., use the distributive law) to obtain

$$\begin{aligned} 2 &= \left(2x^2 - 4x - 4x + 8\right) - \left(2x^2 - 8x + 6\right) \\ 2 &= 2x^2 - 8x + 8 - 2x^2 + 8x - 6 \\ 2 &= 2 \end{aligned}$$

That is, $y = x^2 - 4x + 3$ reduces $y'' = (x-2)\,y' - 2y$ to an identity (i.e., it makes the differential equation true), so that it is indeed a solution.

Exercises:

Find the given Taylor polynomial centered at the given value p. Then graph the Taylor polynomial and the function on an interval containing p.

1. $T_2(x)$ for $f(x) = x^2 + 3x + 1, \quad p = 0$
2. $T_2(x)$ for $f(x) = x^3 + 3x^2 - 5x + 1, \quad p = 1$
3. $T_3(x)$ for $f(x) = \sin(x), \quad p = 0$
4. $T_3(x)$ for $f(x) = \tan(x), \quad p = 0$
5. $T_4(x)$ for $f(x) = \ln(x), \quad p = 1$
6. $T_4(x)$ for $f(x) = \ln(x-1), \quad p = 2$
7. $T_3(x)$ for $f(x) = e^x, \quad p = 0$
8. $T_3(x)$ for $f(x) = \cos(x), \quad p = \frac{\pi}{4}$
9. $T_4(x)$ for $f(x) = \sin\left(x^2\right), \quad p = 0$
10. $T_3(x)$ for $f(x) = e^x, \quad p = 1$

Find the linearization and Quadratic approximation of the given functions at the given input. Then graph the function, its linearization, and its Quadratic approximation.

11. $f(x) = x^3,\ p = 1$
12. $f(x) = x^5,\ p = 1$
13. $f(x) = e^x,\ p = 0$
14. $f(x) = \ln(x),\ p = 1$
15. $f(x) = \sqrt{2 - x^2},\ p = 1$
16. $f(x) = \sqrt{2 - x^2},\ p = 0$
17. $f(x) = \cosh(x),\ p = 0$
18. $f(x) = \sinh(x),\ p = 0$

Find the quadratic approximation to the solution of each of the following initial value problems.

19. $y' = 6x^2,\ y(0) = 4$
20. $y' = \sin\left(x^2\right),\ y\left(\sqrt{\pi}\right) = 2$
21. $y' = 3y, \quad y(0) = 2$
22. $y' = y^2, \quad y(0) = -5$
23. $y' = x^2 + y, \quad y(2) = 3$
24. $y' = xy,\ y(3) = 3$
25. $y' = y + e^x, \quad y(\ln(2)) = 1$
26. $y' = y^2 + e^x, \quad y(0) = 2$
27. $y' = x^2 y + xy^2,\ y(0) = 1$
28. $y' = \sqrt{1 - y^2},\ y(0) = 1$
29. $y'' = 2y + y', \quad y(0) = 3, \quad y'(0) = 2$
30. $y'' + y = 0, \quad y(0) = 1, \quad y'(0) = 0$

31. The following explores the relationship between the upper half of the circle of radius R

$$f(x) = \sqrt{R^2 - x^2}$$

and its quadratic approximation at 0 for various values of R.

(a) Show that the quadratic approximation of $f(x)$ at 0 is

$$T_2(x) = R - \frac{x^2}{2R}$$

(b) Graph $f(x)$ and $T_2(x)$ when $R = 0.5$ on $[-0.25, 0.25]$

(c) Graph $f(x)$ and $T_2(x)$ when $R = 2.25$ on $[-1.125, 1.125]$

(d) Graph $f(x)$ and $T_2(x)$ when $R = 10$ on $[-5, 5]$

32. **Write to Learn:** In astronomical telescopes, parabolic mirrors are used to collect and focus incoming light on the eyepiece. However, in constructing low cost astronomical telescopes, a circle is used to approximate a parabola since spherical mirrors cost less to manufacture than parabolic mirrors. A rule of thumb is that spherical mirrors are only used on telescopes with mirrors less than 4.5 inches in diameter. Write a short essay which describes why spherical mirrors can be used on small telescopes but should not be used on larger telescopes. (hint: see previous exercise).

33. A special case of *Legendre's differential equation* is given by

$$(1 - x^2)y'' - 2xy' + 6y = 0, \qquad y(0) = \frac{-1}{2}, \qquad y'(0) = 0 \tag{9.8}$$

(a) Show that $T_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$

(b) Show that $y = \frac{1}{2}\left(3x^2 - 1\right)$ is a solution to (9.8).

34. The *Hermite polynomials* are solutions to the differential equation

$$y'' - 2xy' + 2ny = 0, \qquad y(0) = 1, \qquad y'(0) = 0 \tag{9.9}$$

(a) Show that $T_2(x) = 1 - nx^2$. What is $T_2(x)$ when $n = 2$?

(b) Show that $y = 1 - 2x^2$ is a solution to (9.9) when $n = 2$.

35. The *Chebyshev polynomials* are solutions of the differential equation

$$\left(1 - x^2\right) y'' - xy' + n^2 y = 0, \qquad y(0) = 1, \qquad y'(0) = 0 \tag{9.10}$$

for $n = 0, 1, 2, \ldots$.

(a) Show that $T_2(x) = 1 - 2x^2$ when $n = 2$.

(b) Show that $y = 1 - 2x^2$ is a solution to (9.10) when $n = 2$.

(c) Find $T_3(x)$ when $n = 3$, and then show by substitution into (9.10) that it is actually a solution to (9.10) when $n = 3$.

(d) The Chebyshev polynomials can also be defined

$$T_n(x) = \cos\left(n \cos^{-1}(x)\right) \tag{9.11}$$

where $\cos^{-1}(x)$ is the inverse cosine. Show that (9.11) simplifies to $T_2(x) = 1 - 2x^2$ when $n = 2$ and to the answer in (c) when $n = 3$.

36. The *Laguerre polynomials* $L_n^\alpha(x)$ are solutions to the equation

$$xy'' + (\alpha + 1 - x) y' + ny = 0, \qquad y(0) = \frac{1}{2}(1 + \alpha)(2 + \alpha), \qquad y'(0) = -2 - \alpha$$

What is $L_2^\alpha(x)$?

37. In this exercise we examine the initial value problem

$$y' = \frac{2y}{x+1}, \quad y(0) = 1 \tag{9.12}$$

(a) Show that the quadratic approximation of the solution is $T_2(x) = 1 + 2x + x^2$.

(b) Show directly that $y = 1 + 2x + x^2$ is the solution to (9.12).

(c) Solve (9.12) using separation of variables. Is it the same as the functions in (a) and (b)?

38. Find $T_5(x)$ centered at $p = 0$ for $f(x) = (x+1)^5$. How is this related to Pascal's triangle expansion of $(x+1)^5$?

39. L'hopital's rule: Assume that $g''(p) \neq 0$ and use the quadratic approximations of $f(x)$ and $g(x)$ to show that when $f(p) = g(p) = f'(p) = g'(p) = 0$, then

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \frac{f''(p)}{g''(p)}$$

40. L'hopital's rule: Assume that $g'''(p) \neq 0$ and use cubic approximations of $f(x)$ and $g(x)$ to show that when $f(p) = g(p) = f'(p) = g'(p) = f''(p) = g''(p) = 0$, then

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \frac{f'''(p)}{g'''(p)}$$

41. The 0^{th} order Bessel function is defined

$$J_0(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin\theta)\, d\theta$$

(a) What is $J_0(0)$?

(b) To compute the derivative of $J_0(x)$, we differentiate under the integral sign. That is,

$$J_0'(x) = \frac{d}{dx} \frac{1}{\pi} \int_0^{\pi} \cos(x \sin\theta)\, d\theta = \int_0^{\pi} \left[\frac{d}{dx} \cos(x \sin\theta) \right] d\theta$$

What is $J_0'(0)$?

(c) Similarly, find $J_0''(0)$. What is the second Taylor polynomial $T_2(x)$ of $J_0(x)$ centered at 0.

42. The 1^{st} order Bessel function is defined

$$J_1(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin\theta - \theta)\, d\theta$$

(a) What is $J_1(0)$?

(b) To compute the derivative of $J_1(x)$, we differentiate under the integral sign. That is,

$$J_1'(x) = \frac{d}{dx} \frac{1}{\pi} \int_0^{\pi} \cos(x \sin\theta - \theta)\, d\theta = \int_0^{\pi} \left[\frac{d}{dx} \cos(x \sin\theta - \theta) \right] d\theta$$

What is $J_1'(0)$?

(c) Similarly, find $J_1''(0)$. What is the second Taylor polynomial $T_2(x)$ of $J_1(x)$ centered at 0.

43. In this exercise, we explore the fact that if $T_n(x)$ is the n^{th} Taylor polynomial of $f(x)$, then the portion of $[T_n(x)]^2$ that forms an n^{th} degree polynomial is the n^{th} Taylor polynomial of $[f(x)]^2$.

(a) Show that the 2^{nd} Taylor polynomial of $f(x) = \cos(x)$ centered at 0 is

$$T_2(x) = 1 - \frac{x^2}{2} \tag{9.13}$$

(b) Find $T_2(x)$ centered at 0 of the function

$$g(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

(c) Show that $g(x) = \cos^2(x)$, and then square the polynomial in (9.13). How is it related to the result in (b)?

44. In this exercise, we explore the fact that if $T_n(x)$ is the n^{th} Taylor polynomial of $f(x)$, then the portion of $[T_n(x)]^2$ that forms an n^{th} degree polynomial is the n^{th} Taylor polynomial of $[f(x)]^2$.

(a) Show that the 3^{rd} Taylor polynomial of $f(x) = \sin(x)$ centered at 0 is

$$T_2(x) = x - \frac{x^3}{3!} \tag{9.14}$$

(b) Find $T_3(x)$ centered at 0 of the function

$$g(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

(c) Show that $g(x) = \sin^2(x)$, and then square the polynomial in (9.13). How is it related to the result in (b)?

9.2 Families of Taylor Polynomials

General Formulas for Taylor Polynomials

The collection of all possible Taylor polynomials of a function $f(x)$ is called the *family of Taylor polynomials* of the function. To determine the family of Taylor polynomials of a function, we first identify a *pattern* relating the derivatives to the index n, and once that pattern is established, the definition

$$T_n(x) = f(p) + f'(p)(x-p) + \frac{f''(p)}{2!}(x-p)^2 + \ldots + \frac{f^{(n)}(p)}{n!}(x-p)^n$$

is used to develop a formula for the family of Taylor polynomials. That is, the pattern is a function mapping the index n to $f^{(n)}(p)$, and the family of Taylor polynomials is a function mapping n to the n^{th} Taylor polynomial, $T_n(x)$.

EXAMPLE 1 Construct the family of Taylor polynomial approximations to $f(x) = e^x$ centered at $x = 0$.

Solution: We begin with a table of derivatives of $f(x) = e^x$:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^x	$e^0 = 1$
1	e^x	$e^0 = 1$
2	e^x	$e^0 = 1$
3	e^x	$e^0 = 1$
$\vdots$	$\vdots$	$\vdots$
n	e^x	$e^0 = 1$

Clearly, $f^{(n)}(0) = 1$ for all n, so that the Taylor polynomial is

$$\begin{aligned} T_n(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \ldots + \frac{f^{(n)}(0)}{n!}(x-0)^n \\ &= 1 + 1\cdot x + \frac{1}{2!}x^2 + \ldots + \frac{1}{n!}x^n \end{aligned}$$

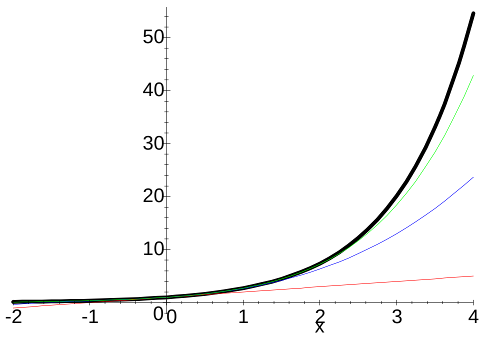
Thus, the family of Taylor polynomials of e^x centered at 0 is

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} \tag{9.15}$$

Formulas like (9.15) allows us to generate Taylor polynomials of a given degree n. For example, (9.15) implies that the first, third and fifth Taylor polynomials of e^x centered at 0 are

$$\begin{aligned} T_1(x) &= 1 + x \\ T_3(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \\ T_5(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \end{aligned}$$

Moreover, as n becomes larger, the polynomials $T_n(x)$ become better and better approximations of e^x, as is shown in the graph below in which e^x is in heavy black and $T_1(x), T_3(x), T_5(x)$ are in red, blue and green, respectively:



2-1: Taylor Approximations of e^x centered at $p = 0$

Check your Reading *Use (9.15) to generate $T_7(x)$.*

The Geometric Series

The geometric series we considered in section 7-3 has its own family of Taylor polynomial approximations.

EXAMPLE 2 Construct the family of Taylor polynomials centered at 0 of

$$f(x) = \frac{1}{1-x}$$

Solution: We begin by constructing a table of the first five or six derivatives and then substituting 0 *without any simplification!*

When looking for a pattern, do not simplify intermediate calculations.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1-x)^{-1}$	1
1	$(1-x)^{-2}$	1
2	$2(1-x)^{-3}$	2
3	$3\cdot 2(1-x)^{-4}$	$3\cdot 2$
4	$4\cdot 3\cdot 2(1-x)^{-5}$	$4\cdot 3\cdot 2$
5	$5\cdot 4\cdot 3\cdot 2(1-x)^{-6}$	$5\cdot 4\cdot 3\cdot 2$

Our goal is to relate the indices in the first column to the derivatives in the third column. Since $0! = 1$, $1! = 1$, $2! = 2\cdot 1$ and so on, we have

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1-x)^{-1}$	$1 = 0!$
1	$(1-x)^{-2}$	$1 = 1!$
2	$2(1-x)^{-3}$	$2 = 2!$
3	$3\cdot 2(1-x)^{-4}$	$3\cdot 2 = 3!$
4	$4\cdot 3\cdot 2(1-x)^{-5}$	$4\cdot 3\cdot 2 = 4!$
5	$5\cdot 4\cdot 3\cdot 2(1-x)^{-6}$	$5\cdot 4\cdot 3\cdot 2 = 5!$

so that the general form of $f^{(n)}(0)$ is

$$f^{(n)}(0) = n!$$

As a result, the Taylor polynomials centered at 0 are given by

$$\begin{aligned} T_n(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \ldots + \frac{f^{(n)}(0)}{n!}(x-0)^n \\ &= 1 + x + \frac{2!}{2!}x^2 + \ldots + \frac{n!}{n!}x^n \\ &= 1 + x + x^2 + \ldots + x^n \end{aligned}$$

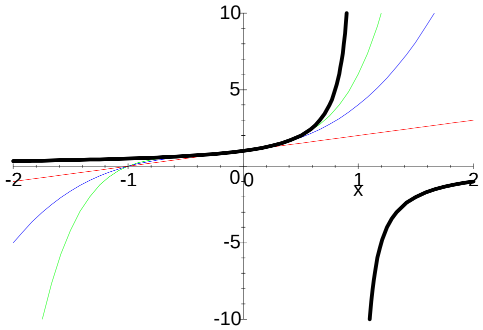
That is, the formula for the n^{th} Taylor polynomial of $f(x) = 1/(1-x)$ is

$$T_n(x) = 1 + x + x^2 + \ldots + x^n \qquad (9.16)$$

Families of Taylor polynomials are good approximations of a function $f(x)$ only on a finite interval containing the center $x = p$. For example, in the graph below, the function $f(x) = \frac{1}{1-x}$ in heavy black is plotted against

$$\begin{aligned} T_1(x) &= 1 + x \\ T_3(x) &= 1 + x + x^2 + x^3 \\ T_5(x) &= 1 + x + x^2 + x^3 + x^4 + x^5 \end{aligned}$$

in red, blue and green, respectively:



2-2: Taylor approximations of $(1-x)^{-1}$ centered at $p=0$

Clearly, the three polynomials $T_1(x)$, $T_3(x)$ and $T_5(x)$ are **not** good approximations of $f(x)$ when x is in the interval $[1,2]$. Indeed, $T_n(x)$ is an increasingly better approximation of $f(x)=\frac{1}{1-x}$ only over the interval $(-1,1)$.

Check your Reading *How are the Taylor polynomials of $f(x)=\frac{1}{1-x}$ centered at $p=0$ related to the geometric series?*

Resist the Urge to Simplify

Because the simplification of individual computations can destroy the pattern we seek, we must resist the urge to simplify when attempting to relate the index n to the n^{th} derivative. Instead, we should seek ways of simultaneously rewriting entire columns of coefficients.

EXAMPLE 3 Construct the family of Taylor polynomials to $f(x)=\ln(x)$ centered at $x=1$. We begin with a table of derivatives:

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln(x)$	0
1	x^{-1}	1
2	$-1x^{-2}$	-1
3	$(-1)(-2)x^{-3}$	$(-1)^2(1\cdot 2)$
4	$(-1)(-2)(-3)x^{-4}$	$(-1)^3(1\cdot 2\cdot 3)$
5	$(-1)(-2)(-3)(-4)x^{-5}$	$(-1)^4(1\cdot 2\cdot 3\cdot 4)$

In the third column of the table, we simultaneously collected the "negative signs" into powers of -1. However, we stopped short of simplifying $(-1)^2$ to 1, $(-1)^3$ to -1, $(-1)^4$ to 1, and so on.

In each row, -1 in the third column is raised to one less than the index. Similarly, the products in the third row end at the integer which is one less than the index, so that the general form of $f^{(n)}(1)$ is

$$f^{(n)}(1)=(-1)^{n-1}(1\cdot 2\cdot 3\cdot\ldots\cdot(n-1))=(-1)^{n-1}(n-1)!$$

As a result, the family of Taylor polynomials of $\ln(x)$ centered at $x=1$ is

$$\begin{aligned} T_n(x) &= f(1)+f'(1)(x-1)+\frac{f''(1)}{2!}(x-1)^2+\ldots+\frac{f^{(n)}(1)}{n!}(x-1)^n \\ &= 0+1(x-1)+\frac{-1}{2!}(x-1)^2+\ldots+\frac{(-1)^{n-1}(n-1)!}{n!}(x-1)^n \end{aligned}$$

Since $n! = n \cdot (n-1)!$, this further simplifies to

$$T_n(x) = (x-1) - \frac{1}{2}(x-1)^2 + \ldots + \frac{(-1)^{n-1}}{n}(x-1)^n$$

EXAMPLE 4 Construct the family of Taylor polynomials of the sine function centered at $x = 0$.

Solution: We begin with a table of derivatives:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin(x)$	0
1	$\cos(x)$	1
2	$-\sin(x)$	0
3	$-\cos(x)$	-1
4	$\sin(x)$	0
5	$\cos(x)$	1

To identify the pattern in the third column, we must describe the even and odd terms separately, which we do by using indices of the form "$2n$" to describe even terms and indices of the form "$2n+1$" to describe odd terms. Since the even coefficients in the third column are zero, we have

Even terms are indexed by "$2n$" and odd terms are indexed by "$2x+1$"

$$f^{(2n)}(0) = 0$$

for all n. However, $f^{(1)}(0) = 1$, $f^{(3)}(0) = -1$, $f^{(5)}(0) = 1$, and so on. That is, the odd coefficients alternate in sign, which is written as

$$f^{(2n+1)}(0) = (-1)^n$$

Thus, the Taylor polynomials centered at 0 of $\sin(x)$ are of the form

$$T_{2n+1}(x) = f'(1)(x-0) + \frac{f'''(0)}{3!}(x-0)^3 + \ldots + \frac{f^{(2n+1)}(0)}{(2n+1)!}(x-0)^{2n+1}$$

Notice that we only include odd indices since all even-indexed terms are 0. Substituting $f^{(2n+1)}(0) = (-1)^n$ then gives us

$$T_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Likewise, it can be shown that the Taylor polynomials centered at 0 of $\cos(x)$ are

$$T_{2n}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{(-1)^n x^{2n}}{(2n!)} \tag{9.17}$$

That is, the cosine function Taylor polynomials involve only even indexed terms.

Check your Reading *Generate $T_8(x)$ for $\cos(x)$ centered at 0 using (9.17).*

Taylor Polynomials Near Vertical Asymptotes

Because polynomials do not have vertical asymptotes, polynomial approximations are poor approximations in the vicinity of a vertical asymptote. Correspondingly, if the n^{th} derivative of $f(x)$ has a vertical asymptote, then $T_n(x)$ and all higher order Taylor polynomials are poor approximations of $f(x)$ on any closed interval containing that asymptote.

EXAMPLE 5 Find the second and third Taylor polynomials centered at $p = 1$ of $f(x) = x^{4/3}$.

Solution: To do so, we first construct a table:

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^{4/3}$	1
1	$\frac{4}{3}x^{1/3}$	$\frac{4}{3}$
2	$\frac{4}{9}x^{-2/3}$	$\frac{4}{9}$
3	$\frac{-8}{27}x^{-5/3}$	$\frac{-8}{27}$

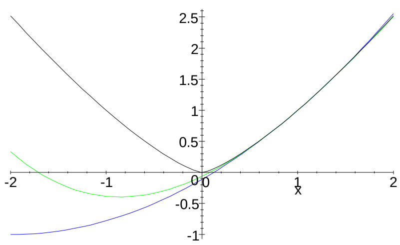
As a result, the second Taylor polynomial centered at 1 is

$$\begin{aligned} T_2(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 \\ &= 1 + \frac{4}{3}(x-1) + \frac{4}{9}\frac{1}{2}(x-1)^2 \end{aligned}$$

and the third Taylor polynomial centered at 1 is

$$\begin{aligned} T_3(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \\ &= 1 + \frac{4}{3}(x-1) + \frac{4}{9}\frac{1}{2}(x-1)^2 - \frac{8}{27}\frac{1}{6}(x-1)^3 \end{aligned}$$

The graphs of $f(x) = x^{4/3}$, $T_2(x)$ and $T_3(x)$ are shown below in black, blue and green, respectively.



2-3: Taylor polynomials are not always good approximations

When $x > 0$, it is clear that $T_2(x)$ and $T_3(x)$ are good approximations of $f(x)$. However, $T_2(x)$ and $T_3(x)$ are **not** good approximations of $f(x)$ when $x \leq 0$. This is because $f''(x)$, $f'''(x)$, and all subsequent derivatives of $f(x)$ have a vertical asymptote at 0, even though $f(x)$ itself does not have a vertical asymptote.

Exercises:

Find the general form of the family of Taylor polynomials centered at p. The results from exercises 1-14 will be used in section 8-6.

1.	$f(x) = e^{2x},\ p = 0$	2.	$f(x) = e^{x},\quad p = 1$
3.	$f(x) = \ln(1-x),\ p = 0$	4.	$f(x) = \ln(2-x),\quad p = 1$
5.	$f(x) = x^{-1},\quad p = 1$	6.	$f(x) = x^{-1},\quad p = -1$
7.	$f(x) = x^{-2},\quad p = 1$	8.	$f(x) = x^{-2},\quad p = -1$
9.	$f(x) = \cosh(x),\quad p = 0$	10.	$f(x) = \sinh(x),\quad p = 0$
11.	$f(x) = \sin(x),\quad p = \pi$	12.	$f(x) = \cos(x),\quad p = 2\pi$
13.	$f(x) = \sin(x),\quad p = \frac{\pi}{4}$	14.	$f(x) = \cos(x),\quad p = \frac{\pi}{4}$
15.	$f(x) = \tan^{-1}(x),\quad p = 0$	16.	$f(x) = \ln\left(x^2+1\right),\quad p = 0$
17.	$f(x) = \frac{1}{2}\ln\left\|\frac{x-1}{x+1}\right\|,\quad p = 0$	18.	$f(x) = e^{-x^2},\quad p = 0$

For each function given below, do the following:

- *Find $T_2(x)$, $T_3(x)$ and $T_4(x)$. You may be able to use results from above.*
- *Graph $f(x)$, $T_2(x)$, $T_3(x)$ and $T_4(x)$ over $[-3,3]$.*
- *Identify any intervals on which the family of Taylor polynomials does not appear to provide good approximations of $f(x)$.*

19.	$f(x) = x^{-1},\ p = 1$	20.	$f(x) = x^{-1},\quad p = -1$
21.	$f(x) = \ln(1-x),\ p = 0$	22.	$f(x) = \ln(2-x),\quad p = 1$
23.	$f(x) = x^{2/3},\quad p = 1$	24.	$f(x) = x^{3/5},\quad p = -1$
25.	$f(x) = \tan^{-1}(x),\quad p = 0$		

26. Explain why the Taylor polynomials centered at $p = 1$ are poor approximations of $f(x) = x^{1/3}$ for $x < 0$.

27. Let $f(x) = |x|$.

(a) Explain why $T_1(x) = f(x)$ for all $x \geq 0$ when $T_1(x)$ is centered at any p such that $p > 0$.

(b) Explain why $T_n(x) = T_1(x)$ for all $n \geq 1$ when $T_n(x)$ is centered at any p such that $p > 0$.

(c) Explain why $T_n(x)$ is not a good approximation of $f(x)$ when $x < 0$.

28. Computer Algebra System:[1] Use a Taylor polynomial to generate $T_7(x)$, $T_9(x)$ and $T_{11}(x)$ centered at 0. Then graph these along with the original function over the given interval:

(a)	$f(x) = \tan(x)$	(b)	$f(x) = \sin\left(e^{2x/3}\right)$	(c)	$f(x) = \tanh(x)$
	x in $[-3,3]$		x in $[-5,5]$		x in $[-5,5]$
	y in $[-15,15]$		y in $[-1,1]$		y in $[-1,1]$

29. The *Sine integral* is defined to be

$$\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t}\,dt$$

Its family of Taylor polynomials centered at $x = 0$ is

$$T_{2n+1}(x) = x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)\cdot(2n+1)!}$$

[1] such as the command "**taylor(**expression1,var,order[,point]**)** on the TI-89 and TI-92.

(a) What are $T_5(x)$, $T_7(x)$ and $T_9(x)$?
(b) Graph $T_5(x)$, $T_7(x)$ and $T_9(x)$ over $[-2,2]$. What would you expect $\mathrm{Si}(x)$ to look like over $[-2,2]$?
(c) **Computer Algebra System:**[2] Use a computer algebra system to generate $T_{21}(x), T_{25}(x)$ and $T_{29}(x)$ centered at 0 of $\mathrm{Si}(x)$, and then graph these three over $[-10,10]$. What would you expect $\mathrm{Si}(x)$ to look like over $[-10,10]$?

30. Let $J_0(x)$ denote the 0^{th} order Bessel function. The family of Taylor polynomials of $J_0(x)$ centered at 0 is given by

$$T_{2n}(x) = 1 - \frac{x^2}{4} + \frac{x^4}{4^2(2!)^2} - \frac{x^6}{4^3(3!)^2} + \ldots + \frac{(-1)^n x^{2n}}{4^n(n!)^2} \tag{9.18}$$

(a) Graph $T_4(x)$, $T_6(x)$ and $T_8(x)$ on $[-2,2]$. Do the graphs of any of the three cross the x-axis in this interval?
(b) **Computer Algebra System:**[3] Find $T_{24}(x)$ and plot it over $[0,10]$. How many times does its graph cross the x-axis on $[0,10]$?
(c) The family of Taylor polynomials of $J_0(x)$ can also be written as

$$T_{2n}(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots + \frac{(-1)^n x^{2n}}{2^2 \cdot 4^2 \cdot 6^2 \cdot \ldots \cdot (2n)^2}$$

Show that $T_4(x)$ and $T_6(x)$ with this formula are the same as those in (a).

31. Let $J_1(x)$ denote the 1^{st} order Bessel function. The family of Taylor polynomials of $J_1(x)$ centered at 0 is given by

$$T_{2n-1}(x) = \frac{2x}{4} - \frac{4x^3}{4^2(2!)^2} + \frac{6x^5}{4^3(3!)^2} - \ldots + \frac{(-1)^{n-1}(2n)x^{2n-1}}{4^n(n!)^2} \tag{9.19}$$

(a) Graph $T_3(x)$, $T_3(x)$ and $T_7(x)$ on $[-2,2]$. Do the graphs of any of the three cross the x-axis in this interval?
(b) **Computer Algebra System:**[4] Find $T_{23}(x)$ and plot it over $[0,10]$. How many times does its graph cross the x-axis on $[0,10]$?
(c) Use the fact that $J_1(x) = -J_0'(x)$ to derive (9.19) from (9.18).

32. In this exercise, we revisit the family of Taylor approximations of $f(x) = \frac{1}{1-x}$ centered at 0, which are

$$T_n(x) = 1 + x + x^2 + x^3 + x^4 + \ldots + x^n$$

(a) Evaluate $T_1(x)$, $T_3(x)$ and $T_5(x)$ at $x = -1$.
(b) Evaluate $T_2(x)$, $T_4(x)$ and $T_6(x)$ at $x = -1$.
(c) Compute $f(-1)$, and then explain why $T_n(-1)$ is not a good approximation of $f(-1)$ for any value of n.

33. Find the family of Taylor approximations of $f(x) = (x+1)^4$ centered at $x = 0$. How are $T_5(x)$, $T_6(x)$, ... related to $T_4(x)$.

34. Find the family of Taylor approximations of $f(x) = (x+1)^5$ centered at $x = 0$. How are $T_6(x)$, $T_7(x)$, ... related to $T_5(x)$.

[2] such as the command "**taylor(**expression1,var,order[,point]**)** on the TI-89 and TI-92.
[3] such as the command "**taylor(**expression1,var,order[,point]**)** on the TI-89 and TI-92.
[4] such as the command "**taylor(**expression1,var,order[,point]**)** on the TI-89 and TI-92.

35. Show that the family of Taylor approximations of $f(x) = e^x$ centered at p is

$$T_n(x) = e^p\left[1 + (x-p) + \frac{1}{2!}(x-p)^2 + \ldots + \frac{1}{n!}(x-p)^n\right]$$

36. Show that if a is a constant, then the family of Taylor approximations of $f(x) = e^{x+a}$ centered at $p = 0$ is

$$T_n(x) = e^a\left[1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}\right]$$

9.3 Taylor's Theorem

Estimating Error in Linearization Approximations

In this section, we use integration by parts and the fundamental theorem of calculus to develop a tool for determining how well a Taylor polynomial $T_n(x)$ centered at p approximates the function $f(x)$ that generated it. To do so, we begin with Taylor's theorem for $T_1(x)$ centered at p of a function $f(x)$ which is second differentiable over $[a, b]$.

In particular, for x in $[a, b]$ and p fixed, let's use integration by parts to evaluate

$$\int_p^x (x-t)\, f''(t)\, dt$$

Since t is the variable of integration, letting $u = x - t$ means that $du = -dt$. Moreover, letting $dv = f''(t)$ yields $v = f'(t)$, so that

$$\int_p^x (x-t)\, f''(t)\, dt = (x-t)\, f'(t)\Big|_p^x + \int_p^x f'(t)\, dt$$

However, $\int_p^x f'(t)\, dt = f(x) - f(p)$, so that we have

$$\int_p^x (x-t)\, f''(t)\, dt = (x-x)\, f'(x) - (x-p)\, f'(p) + f(x) - f(p)$$

Simplifying and rearranging terms thus yields

$$\int_p^x (x-t)\, f''(t)\, dt = f(x) \quad - \quad [\, f(p) + f'(p)(x-p)\,] \tag{9.20}$$

Since $T_1(x) = f(p) + f'(p)(x-p)$, this further simplifies to

$$f(x) - T_1(x) = \int_p^x (x-t)\, f''(t)\, dt \tag{9.21}$$

Applying absolute values and using a property of integrals then

$$|f(x) - T_1(x)| \le \int_p^x |(t-x)\, f''(t)|\, dt$$

Finally, since t, p, and x are all in $[a,b]$, we must have $|t-x| \leq |b-a|$, so that

$$|f(x) - T_1(x)| \leq (b-a)\int_a^b |f''(x)|\,dx \tag{9.22}$$

for all x in $[a,b]$. Moreover, the inequality (9.22) can be used to estimate the error in approximating $f(x)$ over $[a,b]$ by its linearization $T_1(x)$ centered at p.

EXAMPLE 1 Apply (9.22) to $f(x) = x^3$ over $[0.9, 1.1]$ if $p = 1$.

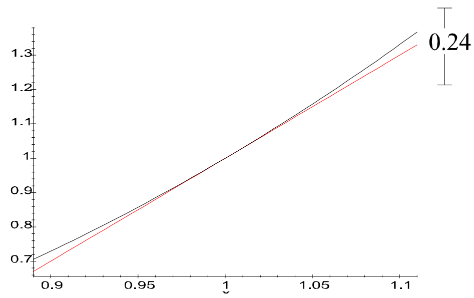
Solution: Since $f'(x) = 3x^2$ and $f'(1) = 3$, we have $T_1(x) = 3x - 2$. Moreover, (9.22) with $f''(x) = 6x$ yields

$$|f(x) - T_1(x)| \leq (1.1 - 0.9)\int_{0.9}^{1.1} |6x|\,dx$$

Substituting and evaluating the integral then yields

$$\left|x^3 - (3x-2)\right| \leq (0.2)\int_{0.9}^{1.1} 6x dx = 0.24$$

That is, the function x^3 differs from $3x - 2$ by less than 0.24 over the interval $[0.9, 1.1]$.



3-1: Error predicted to be less than 0.24

EXAMPLE 2 Estimate the error in approximating $f(x) = \sin(x)$ over $[-0.1, 0.1]$ by its linearization at $p = 0$.

Solution: The linearization is $T_1(x) = x$. Since $f''(x) = -\sin(x)$, Taylor's theorem (9.22) becomes

$$|f(x) - T_1(x)| \leq (0.1 - -0.1)\int_{-0.1}^{0.1} |-\sin(x)|\,dx$$

which after substituting $f(x) = \sin(x)$ and $T_1(x) = x$ yields

$$|\sin(x) - x| \leq (0.2)\int_{-0.1}^{0.1} |\sin(x)|\,dx$$

After evaluating the integral, we find that

$$|\sin(x) - x| \leq 0.002$$

Check your Reading *Graph both* $\sin(x)$ *and* x *over* $[-0.1, 0.1]$. *Can you tell the two apart?*

Taylor's Theorem

We can extend the identity in (9.21), which is

$$f(x) - T_1(x) = \int_p^x (x-t) f''(t)\, dt \tag{9.23}$$

to higher degree Taylor polynomials. To do so, let us notice that the family of Taylor polynomials $T_n(x)$ centered at p can be written in the form

$$T_n(x) = f(p) + f'(p)(x-p) + \ldots + \frac{f^{(n-1)}(p)}{(n-1)!}(x-p)^{n-1} + \frac{f^{(n)}(p)}{n!}(x-p)^n$$

That is, the "next" Taylor polynomial and the "previous" Taylor polynomial satisfy

$$T_n(x) = T_{n-1}(x) + \frac{f^{(n)}(p)}{n!}(x-p)^n \tag{9.24}$$

Thus, if we apply integration by parts to (9.23) with $u = f''(t)$, $du = f'''(t)$ and $dv = (x-t)\,dt$, $v = \frac{-1}{2}(x-t)^2$, then

$$f(x) - T_1(x) = \left.\frac{-f''(t)}{2}(x-t)^2\right|_p^x + \int_p^x \frac{(x-t)^2}{2} f'''(t)\, dt$$

Substituting the limits and simplifying then yields

$$f(x) - T_1(x) - \frac{f''(p)}{2!}(x-p)^2 = \int_p^x \frac{(x-t)^2}{2} f'''(t)\, dt$$

The formula (9.24) thus implies that

$$f(x) - T_2(x) = \int_p^x \frac{(x-t)^2}{2!} f'''(t)\, dt$$

Repeated integration by parts then leads to *Taylor's theorem,* which says that

$$f(x) - T_n(x) = \int_p^x \frac{(t-x)^n}{n!} f^{(n+1)}(t)\, dt \tag{9.25}$$

for all positive integers n. Properties of integrals can then be used to transform (9.25) into the inequality

$$|f(x) - T_n(x)| \le \int_p^x \frac{|t-x|^n}{n!} \left|f^{(n+1)}(t)\right| dt$$

and since $x, p,$ and t are in $[a, b]$, that we have

$$|f(x) - T_n(x)| \le \int_p^x \frac{|t-x|^n}{n!} \left|f^{(n+1)}(t)\right| dt \le \int_a^b \frac{|b-a|^n}{n!} \left|f^{(n+1)}(t)\right| dt$$

That is, we have the following:

Taylor's Inequality: If p is in $[a, b]$and if $f^{(n+1)}(x)$ is continuous on $[a, b]$, then

$$|f(x) - T_n(x)| \leq \frac{|b-a|^n}{n!} \int_a^b \left| f^{(n+1)}(x) \right| dx \tag{9.26}$$

where $T_n(x)$ is the n^{th} Taylor polynomial of $f(x)$ centered at p.

Moreover, (9.26) allows us to estimate the error in approximating $f(x)$ by $T_n(x)$ centered at some p in a given interval $[a, b]$.

EXAMPLE 3 Estimate the error in approximating $f(x) = x^{-1}$ over $[5, 6]$ by $T_5(x)$ centered at some point p in $[5, 6]$.

Solution: To do so, we notice that $n = 5$ and that $f^{(6)}(x) = 720x^{-7}$. As a result, Taylor's theorem (9.26) says that

$$\left| x^{-1} - T_5(x) \right| \leq \frac{|6-5|^5}{5!} \int_5^6 \left| 720x^{-7} \right| dx$$

Evaluating the integral then tells us that

$$\left| x^{-1} - T_5(x) \right| \leq 0.008333$$

Check your Reading *Why does $T_5(x)$ have to be centered at some p in $[5, 6]$?*

Remainder Estimation Formulas

Taylor's theorem is often used to establish *Remainder estimation formulas* for an entire family of Taylor polynomials over a given interval.

To illustrate, let us estimate the error in approximating $f(x) = e^x$ by $T_n(x)$ centered at $p = 0$ over the interval $[0, b]$, where b is constant. In the previous section, we showed that the n^{th} Taylor polynomial of e^x centered at 0 is

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$$

Moreover, if $f(x) = e^x$, then $f^{(n+1)}(x) = e^x$, so that (9.26) becomes

$$|e^x - T_n(x)| \leq \frac{|b-0|^n}{n!} \int_0^b e^x dx \leq \frac{b^n}{n!} \left(e^b - 1 \right)$$

Since $e^b - 1 \leq e^b$, this further simplifies to

$$\left| e^x - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} \right) \right| \leq \frac{b^n}{n!} e^b \tag{9.27}$$

EXAMPLE 4 Find a bound on the error in approximating e^x over $[0,2]$ by

$$T_{10}(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^{10}}{10!}$$

Solution: The inequality (9.27) implies that the error in the approximation over $[0,2]$ satisfies

$$\left| e^x - \left(1 + x + \frac{x^2}{2!} + \ldots + \frac{x^{10}}{10!}\right)\right| \leq \frac{2^{10}}{10!}e^2 = 0.00208$$

Alternatively, we can write this result as

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^{10}}{10!} \qquad \pm \quad 0.00208 \qquad (9.28)$$

for all x in $[0,2]$.

Because of the properties of the sine function, we need only develop an error formula for $f(x) = \sin(x)$ when x is in $\left[0, \frac{\pi}{4}\right]$. In the previous section, we showed that the Taylor polynomials of $\sin(x)$ centered at 0 are

$$T_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

As a result, Taylor's theorem over $\left[0, \frac{\pi}{4}\right]$ becomes

$$|\sin(x) - T_n(x)| \leq \frac{\left(\frac{\pi}{4} - 0\right)^{2n+1}}{(2n+1)!} \int_0^{\pi/4} \left| f^{(n)}(x)\right| dx$$

Since $f(x) = \sin(x)$, its n^{th} derivative is either a sine or a cosine, thus implying that $\left| f^{(n+1)}(x)\right| \leq 1$. As a result, we obtain

$$\left| \sin(x) - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)\right| \leq \frac{(\pi/4)^{2n+2}}{(2n+1)!} \qquad (9.29)$$

for all x in $[0, \pi/4]$. Likewise, it can be shown that if x is in $[0, \pi/4]$, then

$$\left| \cos(x) - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^n x^{2n}}{(2n)!}\right)\right| \leq \frac{(\pi/4)^{2n+1}}{(2n)!} \qquad (9.30)$$

EXAMPLE 5 Find a bound on the error in approximating $\sin(0.5)$ by a Taylor polynomial centered at 0 when $n = 4$.

Solution: When $n = 4$, (9.29) becomes

$$\left| \sin(x) - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}\right)\right| \leq \frac{\left(\frac{\pi}{4}\right)^{10}}{9!} \leq 0.00000025$$

Thus, since 0.5 is in $\left[0, \frac{\pi}{4}\right]$, this implies that

$$\left| \sin(0.5) - \left(0.5 - \frac{(0.5)^3}{3!} + \frac{(0.5)^5}{5!} - \frac{(0.5)^7}{7!} + \frac{(0.5)^9}{9!}\right)\right| \leq 0.00000025$$

which we can alternatively write as

$$\sin(0.5) = 0.5 - \frac{(0.5)^3}{3!} + \frac{(0.5)^5}{5!} - \frac{(0.5)^7}{7!} + \frac{(0.5)^9}{9!} \qquad \pm \quad 0.00000025$$

Check your Reading *Straightforward calculation shows that*

$$0.5-\frac{(0.5)^3}{3!}+\frac{(0.5)^5}{5!}-\frac{(0.5)^7}{7!}+\frac{(0.5)^9}{9!}=0.47942\,5539$$

Compare this to a calculator-produced approximation of $\sin(0.5)$ *(in radians!).*

MacLaurin Series Representations

In each of the three remainder estimation theorems above, the error term approaches 0 as n approaches ∞. For example, if x is in $[0,b]$, then (9.27) implies that

$$\lim_{n\to\infty}\left|e^x-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}\right)\right|\leq\lim_{n\to\infty}\left(\frac{b^n}{n!}e^b\right)$$

However, $b^n/n!$ approaches 0 as n approaches ∞. Thus, for any given value of x, we can conclude that

$$\lim_{n\to\infty}\left|e^x-\left(1+x+\frac{1}{2!}x^2+\ldots+\frac{1}{n!}x^n\right)\right|=0$$

The ellipsis at the end of an expression indicates a limit as n approaches infinity.

which from the definition of a series implies that

$$e^x=1+x+\frac{1}{2!}x^2+\ldots+\frac{1}{n!}x^n+\ldots \tag{9.31}$$

for all real numbers x.

Moreover, in the same fashion, we can show that the remainders for the sine and cosine functions also approach 0 for all x, so that as a result, we can also show that

$$\sin(x) = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+\ldots \tag{9.32}$$

$$\cos(x) = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots+\frac{(-1)^n x^{2n}}{(2n)!}+\ldots \tag{9.33}$$

for all real numbers x. Finally, we call the expansions (9.31), (9.32) and (9.33) the *Maclaurin's series expansions* of e^x, $\sin(x)$, and $\cos(x)$, respectively.

Exercises:

Estimate the error in approximating $f(x)$ *by* $L_p(x)$ *over the interval given. Graph* $f(x)$ *and* $L_p(x)$ *over the given interval to verify the error estimate. (Note:* $L_p(x)$ *is the same as* $T_1(x)$ *centered at* p*).*

1. $f(x)=x^2$ by $L_1(x)=2x-1$ over $[0.9,1.1]$
2. $f(x)=x^4$ by $L_1(x)=4x-3$ over $[0.9,1.1]$
3. $f(x)=\sqrt{x}$ by $L_1(x)=\frac{1}{2}x+\frac{1}{2}$ over $[0.99,1.01]$
4. $f(x)=\sqrt[3]{x}$ by $L_1(x)=\frac{1}{3}x+\frac{2}{3}$ over $[0.999,1.001]$
5. $f(x)=x^{-1}$ by $L_1(x)=2-x$ over $[0.9999,1.0001]$
6. $f(x)=x^{-2}$ by $L_1(x)=3-2x$ over $[0.9999,1.0001]$

7. $f(x) = e^x$ by $L_0(x) = 1 + x$ over $[-0.1, 0.1]$

8. $f(x) = \ln(x)$ by $L_1(x) = x - 1$ over $[0.9, 1.1]$

9. $f(x) = \tan(x)$ by $L_0(x) = x$ over $[-0.1, 0.1]$

10. $f(x) = \sinh(x)$ by $L_0(x) = x$ over $[-0.01, 0.01]$

11. $f(x) = \tanh(x)$ by $L_0(x) = x$ over $[-0.01, 0.01]$

12. $f(x) = \tan^{-1}(x)$ by $L_0(x) = x$ over $[-0.01, 0.01]$

Use (9.29), (9.30) and (9.27) to determine an error term for ???:

13. $e^1 = 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} + \frac{1^5}{5!} + \frac{1^6}{6!} + \frac{1^7}{7!} \quad \pm \quad ???$

14. $e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} + \frac{2^7}{7!} \quad \pm \quad ???$

15. $e^3 = 1 + 3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \frac{3^5}{5!} + \frac{3^6}{6!} + \frac{3^7}{7!} \quad \pm \quad ???$

16. $e^3 = 1 + 3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \ldots + \frac{3^{15}}{15!} \quad \pm \quad ???$

17. $\sin(0.1) = 0.1 - \frac{(0.1)^3}{3!} + \frac{(0.1)^5}{5!} - \frac{(0.1)^7}{7!} + \frac{(0.1)^9}{9!} \quad \pm \quad ???$

18. $\sin(0.5) = 0.5 - \frac{(0.5)^3}{3!} + \frac{(0.5)^5}{5!} - \ldots + \frac{(0.5)^{13}}{13!} \quad \pm \quad ???$

19. $\cos\left(\frac{\pi}{6}\right) = 1 - \frac{\left(\frac{\pi}{6}\right)^2}{2!} + \frac{\left(\frac{\pi}{6}\right)^4}{4!} - \frac{\left(\frac{\pi}{6}\right)^6}{6!} + \frac{\left(\frac{\pi}{6}\right)^8}{8!} \quad \pm \quad ???$

20. $\cos\left(\frac{\pi}{6}\right) = 1 - \frac{\left(\frac{\pi}{6}\right)^2}{2!} + \frac{\left(\frac{\pi}{6}\right)^4}{4!} - \ldots + \frac{\left(\frac{\pi}{6}\right)^{12}}{12!} \quad \pm \quad ???$

Use (9.26) to estimate the error bound for the given function over the given interval for the given value of n. Assume the Taylor polynomials are centered at a point in the given interval.

21. $\left|x^5 - T_3(x)\right| \leq ???$ over $[-1, 1]$

22. $\left|e^{2x} - T_4(x)\right| \leq ???$ over $[0, 1]$

23. $\left|\sqrt{x} - T_2(x)\right| \leq ???$ over $[1, 3]$

24. $\left|\sin(x) - T_5(x)\right| \leq ???$ over $\left[0, \frac{\pi}{6}\right]$

25. $\left|\tan^{-1}(x) - T_2(x)\right| \leq ???$ over $[0, 1]$

26. $\left|\sinh(x) - T_3(x)\right| \leq ???$ over $[-1, 1]$

27. In this exercise, we produce a remainder estimation theorem and a Maclaurin's series expansion of $f(x) = \cosh(x)$.

(a) Show that the family of Taylor polynomials centered at $p = 0$ is

$$T_{2n}(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!}$$

(b) The n^{th} derivative of $\cosh(x)$ will be either $\cosh(x)$ or $\sinh(x)$. Graph $2e^x$, $|\cosh(x)|$ and $|\sinh(x)|$ on $[-2, 2]$. Explain why $|\cosh(x)|$ and $|\sinh(x)|$ are less than $2e^x$.over any interval of the form $[0, x]$.

(c) Derive the remainder estimation theorem

$$\left|\cosh(x) - \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!}\right)\right| \leq \frac{2e^x}{(2n+1)!}|x|^{2n+1}$$

(d) Explain why

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!} + \ldots$$

for all real numbers x.

28. In this exercise, we produce a remainder estimation theorem and a Maclaurin's series expansion of $f(x) = \sinh(x)$.

(a) Show that the general form of the Taylor polynomials is

$$T_{2n+1}(x) = x + \frac{x^3}{2!} + \frac{x^3}{4!} + \ldots + \frac{x^{2n+1}}{(2n+1)!}$$

(b) The n^{th} derivative of $\cosh(x)$ will be either $\cosh(x)$ or $\sinh(x)$. Graph $2e^x$, $|\cosh(x)|$ and $|\sinh(x)|$ on $[-2, 2]$. Explain why $|\cosh(x)|$ and $|\sinh(x)|$ are less than $2e^x$ over any interval of the form $[0, x]$.

(c) Derive the remainder estimation theorem

$$\left|\sinh(x) - \left(x + \frac{x^3}{2!} + \frac{x^3}{4!} + \ldots + \frac{x^{2n+1}}{(2n+1)!}\right)\right| \leq \frac{2e^x}{(2n+2)!}|x|^{2n+2}$$

(d) Explain why

$$\sinh(x) = x + \frac{x^3}{2!} + \frac{x^3}{4!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + \ldots$$

for all real numbers x.

29. If a is constant, then the family of Taylor polynomials of $f(x) = e^{x+a}$ centered at $p = 0$ is

$$T_n(x) = e^a + e^a x + e^a \frac{x^2}{2!} + \ldots + e^a \frac{x^n}{n!}$$

Use Taylor's theorem and the fact that $f^{(n+1)}(x) = e^{x+a}$ to derive the Remainder estimation formula for $f(x) = e^{x+a}$.

30. Show that the remainder estimate in exercise 29 approaches 0 as n approaches ∞, and then explain why this allows us to write

$$e^{x+a} = e^a + e^a x + e^a \frac{x^2}{2!} + \ldots + e^a \frac{x^n}{n!} + \ldots$$

Finally, factor out e^a from the series and prove that $e^{x+a} = e^x e^a$.

31. In computer science, each arithmetic calculation is called a *floating point operation* or *flop.* How many flops are required to compute

$$T_{10}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}$$

if the numbers $2!, 3!, 4!, \ldots, 10!$ are stored in memory and are thus not computed?

32. A typical personal computer requires 10^{-8} seconds for each flop. How long does it take to compute $T_{10}(x)$ in exercise 31?

33. Each term in $T_{10}(x)$ after the first contains a factor of x^2. Thus, we can write

$$T_{10}(x) = 1 + x^2\left(\frac{-1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!} + \frac{x^6}{8!} - \frac{x^8}{10!}\right) \tag{9.34}$$

How many flops to compute (9.34) if x^2 is computed only one time and then substituted when necessary?

34. The more we factor, the faster $T_{10}(x)$ can be computed. Show that we can factor $T_{10}(x)$ into the optimal form

$$T_{10}(x) = 1 + x^2\left(\frac{-1}{2!} + x^2\left(\frac{1}{4!} + x^2\left(\frac{-1}{6!} + x^2\left(\frac{1}{8!} - \frac{x^2}{10!}\right)\right)\right)\right) \tag{9.35}$$

How many flops to compute (9.35) if x^2 is computed only one time and then substituted when necessary?

35. Apply integration by parts to the integral in

$$f(x) - T_2(x) = \int_p^x \frac{(x-t)^2}{2!} f'''(t)\,dt$$

and then use (9.24) to show that

$$f(x) - T_3(x) = \int_p^x \frac{(x-t)^3}{3!} f^{(4)}(t)\,dt$$

36. Apply integration by parts to the integral in

$$f(x) - T_n(x) = \int_p^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$$

and then use (9.24) to show that

$$f(x) - T_{n+1}(x) = \int_p^x \frac{(x-t)^{n+1}}{(n+1)!} f^{(n+2)}(t)\,dt$$

37. Remainder Estimation Theorem: Suppose that there exists a constant M such that $f^{(n+1)}(t) \le M$ for all t in $[p, x]$. Use Taylor's theorem to show that

$$|f(x) - T_n(x)| \le \frac{M}{(n+1)!}(x-p)^{n+1}$$

38. Suppose that there exists a c in $[p, x]$ such that

$$\int_p^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt = f^{(n+1)}(c)\int_p^x \frac{(x-t)^n}{n!}\,dt$$

Use this and Taylor's theorem to show that for fixed x, we have

$$f(x) = T_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-p)^{n+1}$$

39. Write to Learn: Write a short essay in which you use the remainder estimate in (9.29) to prove that

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots$$

40. Write to Learn: Write a short essay in which you use the remainder estimate in (9.30) to prove that

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots$$

9.4 Algebra and Counting

Four Important MacLaurin's Series

Memorize these. They will appear frequently from this point forward.

In the last section, we used Taylor's theorem to establish the MacLaurin's series of e^x, $\sin(x)$ and $\cos(x)$. These three combined with the geometric series are four of the more frequently occurring series in calculus:

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots, \qquad -\infty < x < \infty \\
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots, \qquad -\infty < x < \infty \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots, \qquad -\infty < x < \infty \\
\frac{1}{1-x} &= 1 + x + x^2 + \ldots + x^n + \ldots, \qquad -1 < x < 1
\end{aligned}$$

Indeed, many other MacLaurin series and many identities in mathematics are derived using these series.

EXAMPLE 1 Find the MacLaurin series expansion of $f(x) = e^{-x^2}$.

Solution: We begin with the series expansion of e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$$

and then we replace the input variable x by the expression $-x^2$, which results in

$$\begin{aligned}
e^{-x^2} &= 1 - x^2 + \frac{\left(-x^2\right)^2}{2!} + \ldots + \frac{\left(-x^2\right)^n}{n!} + \ldots \\
&= 1 - x^2 + \frac{x^4}{2!} + \ldots + \frac{(-1)^n x^{2n}}{n!} + \ldots
\end{aligned}$$

for all real numbers x.

EXAMPLE 2 Find the MacLaurin's series expansion of

$$\frac{x}{1+x^2}$$

Solution: To do so, we replace x by $-x$ in the geometric series to obtain

$$\frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \ldots + (-x)^n + \ldots$$

Thus, for x in $(-1,1)$, we have

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots + (-1)^n x^n + \ldots \tag{9.36}$$

We now replace x by x^2 to obtain

$$\begin{aligned} \frac{1}{1+x^2} &= 1 - x^2 + \left(x^2\right)^2 - \left(x^2\right)^3 + \left(x^2\right)^4 - \ldots + (-1)^n \left(x^2\right)^n + \ldots \\ &= 1 - x^2 + x^4 - x^6 + \ldots + (-1)^n x^{2n} + \ldots \end{aligned}$$

and finally, we multiply by x to obtain

$$\frac{x}{1+x^2} = x\left(1 - x^2 + x^4 - x^6 + \ldots + (-1)^n x^{2n} + \ldots\right)$$

which simplifies to

$$\frac{x}{1+x^2} = x - x^3 + x^5 - x^7 + \ldots + (-1)^n x^{2n+1} + \ldots \tag{9.37}$$

Check your Reading *Why is (9.37) true only when x is in $(-1,1)$?*

MacLaurin Series of Other Elementary Functions

We can use known Maclaurin's series expansions to find MacLaurin's series expansions of other elementary functions.

EXAMPLE 3 Find the MacLaurin's series expansion for $\cosh(x)$ using the fact that

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

Solution: If we replace x by $-x$ in the Maclaurin's series for e^x, then we obtain

$$\begin{aligned} e^{-x} &= 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \frac{(-x)^5}{5!} + \frac{(-x)^6}{6!} + \ldots \\ &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} + \ldots \end{aligned}$$

We then compute $e^x + e^{-x}$ by aligning similar powers of x and combining vertically:

$$\begin{array}{lllllllllllll} e^x & = & 1 & + & x & + & \frac{x^2}{2!} & + & \frac{x^3}{3!} & + & \frac{x^4}{4!} & + & \frac{x^5}{5!} & + & \ldots \\ e^{-x} & = & 1 & - & x & + & \frac{x^2}{2!} & - & \frac{x^3}{3!} & + & \frac{x^4}{4!} & - & \frac{x^5}{5!} & + & \ldots \\ \hline e^x + e^{-x} & = & 2 & & & + & 2 \cdot \frac{x^2}{2!} & & & + & 2 \cdot \frac{x^4}{4!} & & & + & \ldots \end{array}$$

Dividing the sum $e^x + e^{-x}$ by 2 thus yields

$$\frac{e^x + e^{-x}}{2} = \frac{1}{2}\left(2 + 2 \cdot \frac{x^2}{2!} + 2 \cdot \frac{x^4}{4!} + 2 \cdot \frac{x^6}{6!} + \ldots\right) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$$

As a result, the Maclaurin's series expansion of $\cosh(x)$ is

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!} + \ldots \qquad (9.38)$$

Moreover, the Maclaurin's series expansions for e^x, $\sin(x)$, and $\cos(x)$ allow us to extend calculus to any setting where addition, multiplication and division by an integer are defined.

EXAMPLE 4 Find the Maclaurin's series expansion for e^{ix}, where i is the imaginary unit.

Solution: To begin with, since $i^2 = -1$, we have $i^3 = i^2 \cdot i = -i$ and $i^4 = (i^2)^2 = (-1)^2 = 1$. Thus,

$$\begin{aligned} e^{ix} &= 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} \ldots \\ &= 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \frac{i^6x^6}{6!} + \frac{i^7x^7}{7!} \ldots \\ &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \ldots \end{aligned}$$

since also $i^5 = i^4 \cdot i = i$ and $i^6 = i^4 \cdot i^2 = -1$. Regrouping the series into its real and imaginary parts results in

$$e^{ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} - i\frac{x^7}{7!} + \ldots$$

Moreover, factoring out the i from the odd terms yields

$$e^{ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)$$

However, the real and imaginary parts are the Maclaurin's series expansions of $\cos(x)$ and $\sin(x)$, respectively. Thus, we have proven *Euler's formula,* which says that

$$e^{ix} = \cos(x) + i\sin(x) \qquad (9.39)$$

Check your Reading *Use (9.39) to show that $e^{i\pi} + 1 = 0$.*[5]

Products of MacLaurin Series

The keys to multiplying series are organization and the distributive law.

Products of series also lead to new MacLaurin's series expansions. In particular, the distributive law is used to compute products of series, and it is important to keep the computation organized by aligning similar terms.

[5] The identity $e^{i\pi} + 1 = 0$ is interesting because it involves all of the five elementary constants– $0, 1, \pi, e$, and i.

EXAMPLE 5 Find the MacLaurin's series expansion of

$$\frac{1+x+x^2}{1-x} \tag{9.40}$$

Solution: To do so, we write several terms of the geometric series and omit the general term:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \dots$$

Thus, (9.40) is the same as

$$\left(1+x+x^2\right)\left(1+x+x^2+x^3+x^4+x^5+\dots\right)$$

Applying the distributive law and organizing the computation with a table yields

$$\begin{array}{rcccccccc} 1\left(1+x+x^2+x^3+x^4+x^5+\dots\right) & = & 1 & +x & +x^2 & +x^3 & +x^4 & +x^5 & +\dots \\ +x\left(1+x+x^2+x^3+x^4+x^5+\dots\right) & = & & x & +x^2 & +x^3 & +x^4 & +x^5 & +\dots \\ +x^2\left(1+x+x^2+x^3+x^4+x^5+\dots\right) & = & & & x^2 & +x^3 & +x^4 & +x^5 & +\dots \end{array}$$

We then add vertically to obtain

$$\frac{1+x+x^2}{1-x} = 1 + 2x + 3x^2 + 3x^3 + 3x^4 + 3x^5 + \dots$$

EXAMPLE 6 Use Maclaurin Series to prove that $e^x e^{-x} = 1$.

Solution: Since

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad \text{and} \qquad e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

their product is

$$e^x e^{-x} = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right)$$

We then apply the distributive law and organize the computation into a table. However, since e^x and e^{-x} are expanded only to the third term, we only expand their product to the third term.

Expand a product to the same number of terms as the factors are expanded to.

$$\begin{array}{rcccccc} 1\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right) & = & 1 & -x & +\frac{x^2}{2!} & -\frac{x^3}{3!} & +\dots \\ +x\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right) & = & & x & -x^2 & +\frac{x^3}{2!} & +\dots \\ +\frac{x^2}{2!}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right) & = & & & \frac{x^2}{2!} & -\frac{x^3}{2!} & +\dots \\ +\frac{x^3}{3!}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right) & = & & & & \frac{x^3}{3!} & +\dots \end{array}$$

The sum of the second column after the equal's signs is $-x + x = 0$. The sum of the third column is

$$\frac{x^2}{2} - x^2 + \frac{x^2}{2} = 0$$

Likewise, the sum of the fourth column is 0, so that we have shown that

$$e^x e^{-x} = 1 + 0 + 0 + 0 + \dots$$

Check your Reading *Show that the sum of the fourth column after the equals sign is also 0.*

Application to Combinatorics

Products of series can also be used to *count* the number of objects in a given set. Indeed, the use of series as a tool for counting is one of the central themes of a field of mathematics known as *combinatorics.*

When the order of selection does not matter–i.e., when ab is the same as ba—each selection is called a *combination* of the letters.

To illustrate, let us notice that if asked to choose 1 letter from the group $\{a, b, c\}$, there are exactly 3 possible outcomes–i.e., a, b, and c. Moreover, if asked to choose 2 letters from $\{a, b, c\}$ when the order of the selection does not matter, we would discover 3 possible *combinations*—ab, ac, and bc. However, there is only 1 combination of all three letters, namely abc.

Surprisingly, the information above can also be obtained from the product

$$(1 + ax)(1 + bx)(1 + cx)$$

Indeed, application of the distributive law reveals that the product is

$$1 + (a + b + c)\, x + (ab + ac + bc)\, x^2 + abc\, x^3$$

The coefficient of x represents the 3 ways of choosing 1 letter from $\{a, b, c\}$, the coefficient of x^2 represents the 3 ways of choosing 2 letters from $\{a, b, c\}$, and the coefficient of x^3 is the only way of choosing all 3 letters from $\{a, b, c\}$.

If we are only interested in counting the number of ways of choosing n letters from a group of three, we let $a = b = c = 1$, which yields

$$(1 + x)^3 = 1 + 3x + 3x^2 + x^3$$

That is, the polynomial $1 + x$ represents the act of choosing or not choosing a single letter, so that $(1 + x)^3$ represents the number of ways of choosing or not choosing from 3 letters. As a result, the expansion

$$1 + 3x + 3x^2 + x^3$$

tells us that there is one way of choosing no letters, 3 ways of choosing only one letter, 3 ways of choosing two letters and one way of choosing all three letters.

To count when selections like $aabc$ or $bbbc$ are possible, we must use higher degree polynomials. That is, $1 + x + x^2$ allows up to two repetitions of a letter, $1 + x + x^2 + x^3$ allows up to three repetitions of a letter, and

$$1 + x + x^2 + \ldots + x^n + \ldots$$

allows an arbitrary number of repetitions of a letter. (i.e., there is no limit on the number of times the letter can be repeated). Conversely, the infinite series

$$x + x^2 + x^3 + \ldots + x^n + \ldots$$

implies the letter *must occur* at least once, the infinite series

$$x^2 + x^3 + \ldots + x^n + \ldots$$

implies the letter *must occur* at least twice, and so on.

EXAMPLE 7 How many ways are there of choosing 5 letters from the set $\{a, b, c\}$ when a and b can appear at most once, but c must appear at least twice.

Solution: The polynomial $1 + x$ represents the act of choosing a and it also represents the act of choosing b. However,

$$x^2 + x^3 + x^4 + x^5 + \dots$$

represents the act of choosing c since c must occur at least twice. As a result, the product

$$(1 + x)(1 + x)\left(x^2 + x^3 + x^4 + x^5 + \dots\right)$$

counts the number of ways of choosing a, b, and c. Since $(1 + x)(1 + x) = 1 + 2x + x^2$, we must compute

$$\left(1 + 2x + x^2\right)\left(x^2 + x^3 + x^4 + x^5 + \dots\right)$$

Applying the distributive law and organizing the computation with a table yields

$$\begin{array}{rclllll} 1\left(x^2 + x^3 + x^4 + x^5 + \dots\right) & = & x^2 & +\ x^3 & +\ x^4 & +\ x^5 & + \dots \\ +2x\left(x^2 + x^3 + x^4 + x^5 + \dots\right) & = & & +\ 2x^3 & +\ 2x^4 & +\ 2x^5 & + \dots \\ +x^2\left(x^2 + x^3 + x^4 + x^5 + \dots\right) & = & & & x^4 & +\ x^5 & + \dots \end{array}$$

Combining along each column yields

$$\left(1 + 2x + x^2\right)\left(x^2 + x^3 + x^4 + x^5 + \dots\right) = x^2 + 3x^3 + 4x^4 + 4x^5 + \dots$$

The coefficient of x^5 thus tells us that there are 4 ways of choosing 5 letters from $\{a, b, c\}$ if a and b can occur at most once and c must occur at least twice.

Exercises:

Obtain the Maclaurin's series of the given function from the MacLaurin's series of e^x, $\sin(x)$, $\cos(x)$ *and* $\frac{1}{1-x}$.

1. $f(x) = e^{-x}$
2. $f(x) = \sin\left(x^2\right)$
3. $f(x) = \dfrac{x}{1-x}$
4. $f(x) = \dfrac{x}{1+x^2}$
5. $f(x) = \cos\left(\sqrt{x}\right)$
6. $f(x) = \dfrac{\sin(x)}{x}$
7. $f(x) = \dfrac{1-\cos(x)}{x^2}$
8. $f(x) = \dfrac{e^x - 1}{x}$
9. $f(x) = \sinh(x)$
10. $f(x) = \sinh(2x)$
11. $f(x) = e^{-ix}$
12. $f(x) = \cos(ix)$
13. $f(x) = 2\cos^2(x)$
14. $f(x) = 2\sin^2(x)$
15. $f(x) = \dfrac{1+x}{1-x}$
16. $f(x) = \dfrac{2+x}{1-x}$
17. $f(x) = (x+1)\,e^x$
18. $f(x) = \dfrac{1+x^2}{1-x^2}$
19. $f(x) = \dfrac{1}{(1-x)^2}$
20. $f(x) = \dfrac{1}{(1-x)\left(1-x^2\right)}$

Use MacLaurin's series expansions to prove the following identities. You will need the expansions in (9.38) and in exercise 9.

21. $\cosh(ix) = \cos(x)$

22. $\sinh(ix) = i\sin(ix)$

23. $\cos(ix) = \cosh(x)$

24. $\sin(ix) = -i\sinh(x)$

25. $\dfrac{1}{1-x} + \dfrac{1}{1+x} = \dfrac{2}{1-x^2}$

26. $\dfrac{1}{1-x} - \dfrac{1}{1+x} = \dfrac{2x}{1-x^2}$

27. $e^{ix}e^{-ix} = 1$

28. $e^{x}e^{x} = e^{2x}$

29. $2\sin(x)\cos(x) = \sin(2x)$

30. $2\cos^2(x) = \cos(2x) + 1$

31. How many ways are there of choosing 5 letters from $\{a, b, c\}$ if a and b can occur at most once and c must occur at least once? Can you list all the different ways?

32. How many ways are there of choosing 5 letters from $\{a, b, c\}$ if a can occur at most once, b can occur at most twice, and c can occur an arbitrary number of times? Can you list all the different ways?

33. How many ways are there of choosing 100 letters from $\{a, b, c\}$ if a and b can occur at most once and c can occur any number of times?

34. Determine how many ways there are of choosing 7 letters from a three letter set $\{a, b, c\}$ if a can appear at most one time, but b and c can appear an arbitrary number of times. (Hint: the Maclaurin's series

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots + (n+1)x^n + \ldots$$

represents the number of ways two letters can appear an arbitrary number of times).

35. Computer Algebra System: Use a computer algebra system to expand the product which results from the following: A dean appoints a 9 member committee comprised of students, faculty, and staff. He decides that at least four of the committee members should be students, but no more than 2 of the committee members should be staff and no more than 4 of the committee members should be faculty. How many different student-faculty-staff compositions of the committee are possible.

36. Computer Algebra System: Use a computer algebra system to expand the product which results from the following: Acme

37. Use identity (9.39) to show that

$$e^{-ix} = \cos(x) - i\sin(x) \tag{9.41}$$

Then use (9.39) and (9.41) to show that

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \qquad \text{and} \qquad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

38. Use (9.39) to show that

$$e^{(a+ib)x} = e^{ax}\left[\cos(bx) + i\sin(bx)\right]$$

Then show that

$$\frac{d}{dx}e^{(a+ib)x} = (a+ib)\,e^{(a+ib)x}$$

How does this relate to the fact that $y = e^{kx}$ is a solution to $y' = ky$?

39. Matrices can be multiplied, added, and divided by an integer. In particular,

$$\left[\begin{array}{cc} a_1 & a_2 \\ b_1 & b_2 \end{array}\right]\left[\begin{array}{cc} c_1 & d_1 \\ c_2 & d_2 \end{array}\right] =$$

and the sum of two matrices is the sum of the corresponding coefficients. Use the Maclaurin's series expansion to compute e^{At} when

$$A = \left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right]$$

and when t is a variable.

40. Use the Maclaurin's series expansion to compute e^{At} when

$$A = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]$$

and when t is a variable.

When the order of selection is important—i.e., when ab is not considered the same as ba—then a selection of letters is called a permutation and counting is performed with polynomials of the form

$$a_0 + a_1 x + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + \ldots + a_n\frac{x^n}{n!} \tag{9.42}$$

where a_1 is the number of ways of making one choice, a_2 is the number of ways of making two choices, and so on. Exercises 41 to 43 introduce methods used to count permutations.

41. Write to Learn: In a short essay, explain why

$$\left(1 + x + \frac{x^2}{2!}\right)(1+x) = 1 + 2x + 3\frac{x^2}{2!} + 3\frac{x^3}{3!}$$

tells us that there are 3 ways of choosing 3 letters from a group of two letters when the order is important.

42. Write to Learn: In a short essay, determine the number of ways of choosing 5 letters from a set of three letters when two letters can appear at most once and the third letter can appear an arbitrary number of times, assuming that the order of selection is important. Then list those five ways.

43. Deoxyribonucleic acid, or DNA for short, is a chain in which each link is one of four possible chemicals: thymine, T; cytosine, C; adenine, A; and guanine, G. Assuming each can repeat an arbitrary number of times means expanding

$$\left(1 + x + \frac{x^2}{2!} + \ldots\right)\left(1 + x + \frac{x^2}{2!} + \ldots\right)\left(1 + x + \frac{x^2}{2!} + \ldots\right)\left(1 + x + \frac{x^2}{2!} + \ldots\right)$$

How many ways are there of constructing such a DNA chain with n links?

What happens when the "cos" button on a calculator is depressed? How does a computer compute e^x for a given value of x? Our next step is to demonstrate how Taylor polynomials can be converted into computer programs.

Computers represent real numbers in scientific notation, so that a number like 2,433,674 is known to the computer as

$$2.433674 \times 10^6$$

The decimal 2.433674 is the *mantissa* and is said to have 7 *significant digits.* The integer 6 is the *exponent* of 2,433,674.

The Institute for Electronics and Electrical Engineering (IEEE) 754 double-precision specification specifies that a real number is to be represented by a 64 bit string of ones and zeroes inside the computer. The first 52 bits is called the *mantissa* of the number, and since

$$2^{52} \approx 4.5 \times 10^{15},$$

double-precision numbers have fifteen significant digits. Thus, the goal of any algorithm is to produce an approximation in which the first fifteen digits of the mantissa are correct.

Unfortunately, the remainder estimation theorems deal with actual errors rather than number of correct digits. Moreover, Taylor polynomials of the elementary functions do not converge all that well unless x is close to 0. Consequently, almost all numerical algorithms involve some concept of scaling.

For example, let's develop a numerical algorithm for estimating e^x to 15 significant digits of accuracy when $x > 0$. To begin with, we choose a positive integer m such that

$$2^m \leq e^x < 2^{m+1}$$

To indicate how this is done, let us apply $\ln(x)$ throughout:

$$\ln(2^m) \leq x \leq \ln\left(2^{m+1}\right)$$

Since $\ln(2^m) = m\ln(2)$ and $\ln\left(2^{m+1}\right) = (m+1)\ln(2)$, we must choose x such that

$$m \leq \frac{x}{\ln(2)} \leq m+1$$

That is, m is the integer part of $x/\ln(2)$. We then let $t = x - m\ln(2)$, so that

$$x = m\ln(2) + t$$

As a result, t is in $[-0.7, 0.7]$ and

$$e^x = e^{m\ln(2)+t} = e^{m\ln(2)}e^t = 2^m e^t$$

Thus, to compute e^x for $x > 0$, we compute 2^m for a positive integer m, which is quite easy on a computer since it uses binary arithmetic. We then approximate e^t for t in $[-0.7, 0.7]$ using a Taylor polynomial.

Moreover, since e^t is in $[-0.7, 0.7]$, estimating e^t to fifteen significant digits means choosing n such that

$$\left| e^t - \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots + \frac{t^n}{n!}\right)\right| \leq 10^{-15}$$

However, the remainder estimation theorem says that

$$\left| e^t - \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots + \frac{t^n}{n!} \right) \right| \leq \frac{(0.7)^n}{n!} e^{0.7}$$

Experimenting with different values of n reveals that

$$\frac{(0.7)^{16}}{(16)!} e^{0.7} < 10^{-15}$$

As a result, we use the sixteenth Taylor polynomial

$$T_{16}(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots + \frac{t^n}{n!}$$

to approximate e^t over $[-0.7, 0.7]$.

In summary, our algorithm for estimating e^x for $x > 0$ is to

* Let m be the integer part of $x/\ln(2)$
* Let $t = x - m\ln(2)$
* Estimate $T_{16}(t)$
* Compute the product $2^m e^t$.

Write to Learn Write a program or pseudocode which computes e^x for $x > 0$. How would you handle the case of $x < 0$? Write a short paper describing your program.

Write to Learn Develop a method for estimating $\sin(x)$ to fifteen significant digits. Your first step in doing so will be to choose n and t such that

$$x = 2\pi n + t$$

since by periodicity, $\sin(x) = \sin(t)$. Your last step will be to find n such that

$$\left| \sin(x) - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right) \right| \leq \frac{(\pi/2)^{2n+2}}{(2n+1)!} \leq 10^{-15}$$

Write to Learn Search the library and the internet for implementations of numerical algorithms to estimate e^x and $\sin(x)$. Write a short paper reporting the results of your research.

Group Learning Search the library and the internet for implementations of numerical algorithms to estimate e^x, $\sin(x)$, $\cos(x)$, and $\ln(x)$. Give a group presentation in which each member of the group presents one of the algorithms.

Advanced Contexts:

Functions which can be approximated by Taylor polynomials often can also be approximated by rational functions as well. In particular, a rational function

$$R(x) = \frac{a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m}{1 + b_1 x + b_2 x^2 + \ldots + b_n x^n}$$

is called a *Pade' approximant* of a function $f(x)$ if

$$R(0)=f(0), \quad R'(0)=f'(0), \quad R''(0)=f''(0), \quad \ldots, \quad R^{(m+n)}(0)=f^{(m+n)}(0)$$

Pade' approximants often converge more quickly than Taylor's series, and thus, they are often used in numerical algorithms in place of Taylor polynomials.

Since the first n derivatives of $f(x)$ at $x=0$ are the same as the first n derivatives of $T_n(x)$ centered at 0, Taylor polynomials are often used to generate Pade' approximants. For example, a quadratic approximation

$$T_3(x)=c_0+c_1x+c_2x^2$$

has a corresponding Pade' approximant of

$$R(x)=\frac{1+a_1x+a_2x^2}{1+b_1x+b_2x^2}$$

where $a_0=c_0$ and where

$$c_1=R'(0), \qquad c_2=R''(0)$$

Likewise, a fourth order Taylor polynomial centered at $x=0$

$$T_4(x)=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4$$

has a Pade' approximant of

$$R(x)=\frac{a_0+a_1x+a_2x^2}{1+b_1x+b_2x^2}$$

where $a_0=c_0$ and where

$$c_1=R'(0), \quad c_2=R''(0), \quad c_3=R'''(0), \quad c_4=R''''(0)$$

1. The fourth Taylor polynomial of

$$f(x)=\frac{1}{1+x^2}$$

is $T_4(x)=1-x^2+x^4$, and a corresponding Pade' approximant is

$$R(x)=\frac{1-5x^2/12}{1+x^2/12}$$

Graph $f(x), T_4(x)$ and $R(x)$ over the interval $[-2,2]$. What do you notice?

2. * Find the Pade' approximant of the quadratic approximation of

$$f(x)=\frac{1}{1-x}$$

3. * The fourth Taylor polynomial of

$$f(x)=\frac{1}{1+x^2}$$

is $T_4(x)=1-x^2+x^4$. Determine the coefficients for the corresponding Pade' approximant

$$R(x)=\frac{a_0+a_1x+a_2x^2}{1+b_1x+b_2x^2}$$

9.5 Power Series

The Ratio Test for Power Series

A MacLaurin's series is a power series centered at $p = 0$.

MacLaurin's series are examples of what are known as power series, where a *power series centered at* p is a series of the form

$$a_0 + a_1 (x - p) + a_2 (x - p)^2 + \ldots + a_n (x - p)^n + \ldots$$

and where the numbers $a_0, a_1, \ldots, a_n, \ldots$ are called the *coefficients* of the series. Indeed, a Maclaurin's series is simply a power series centered at $p = 0$.

We usually define $(x - p)^0 = 1$ for all x and also define $0! = 1$. As a result, a power series can be written in sigma notation as

$$\sum_{n=0}^{\infty} a_n (x - p)^n$$

That is, we have two ways of expressing a power series:

$$\sum_{n=0}^{\infty} a_n (x - p)^n = a_0 + a_1 (x - p) + a_2 (x - p)^2 + \ldots + a_n (x - p)^n + \ldots$$

For example, the power series

$$1 + \frac{1}{2}(x - 1) + \frac{1}{4}(x - 1)^2 + \ldots + \frac{1}{2^n}(x - 1)^n + \ldots$$

can also be written in the form

$$\sum_{n=0}^{\infty} \frac{1}{2^n}(x - 1)^n$$

Likewise, a power series in sigma notation can also be written in expanded form:

$$\sum_{n=0}^{\infty} \frac{1}{n!}(x + 3)^n = \frac{1}{0!}(x + 3)^0 + \frac{1}{1!}(x + 3)^1 + \frac{1}{2!}(x + 3)^2 + \ldots$$

It is not uncommon for a power series to converge for some x's and diverge for others. Indeed, the purpose of this section is to introduce a tool for determining when a series converges and when it does not. In particular, we will see at the end of this section that when power series are compared to geometric series, the result is the following theorem:

Ratio Test: Given $\sum_{n=0}^{\infty} a_n (x - p)^n$, let ρ, pronounced "rho," be defined by

$$\rho = \lim_{n \to \infty} \left| \frac{a_{n+1} (x - p)^{n+1}}{a_n (x - p)^n} \right| \tag{9.43}$$

Then the power series $\sum_{n=0}^{\infty} a_n (x - p)^n$ converges absolutely when $\rho < 1$ and diverges when $\rho > 1$.

Hence, to determine the values of x for which a power series converges, we evaluate (9.43), set the result less than 1, and solve for x. The result is called the *open interval of convergence* of the power series, and the distance R from p to an endpoint of the interval of convergence is called the *radius of convergence.*

EXAMPLE 1 Find the open interval of convergence of the series

$$\sum_{n=1}^{\infty} n2^n \left(x-3\right)^n$$

Solution: Since $a_n = n2^n$, we have $a_{n+1} = (n+1)\,2^{n+1}$. Thus, (9.43) implies that

$$\rho = \lim_{n\to\infty} \left| \frac{(n+1)\,2^{n+1}\left(x-3\right)^{n+1}}{n2^n\left(x-3\right)^n} \right|$$

Since $2^{n+1}/2^n = 2$ and $\left(x-3\right)^{n+1}/\left(x-3\right)^n = (x-3)$, cancellation yields

$$\rho = \lim_{n\to\infty} \left| 2\,(x-3)\,\frac{n+1}{n} \right| = 2\,|x-3| \lim_{n\to\infty} \left| \frac{n+1}{n} \right|$$

Since $\lim\limits_{n\to\infty} \left|\frac{n+1}{n}\right| = 1$, we find that $\rho = 2\,|x-3|$. Setting $\rho < 1$ thus yields

$$2\,|x-3| < 1$$

Division by 2 and the definition of absolute values then yields

$$|x-3| < \frac{1}{2} \qquad \text{and} \qquad \frac{-1}{2} < x-3 < \frac{1}{2}$$

Finally, adding 3 throughout yields

$$\frac{-1}{2}+3 < x < \frac{1}{2}+3 \qquad \text{or} \qquad 2.5 < x < 3.5$$

Thus, the open interval of convergence is $(2.5, 3.5)$, and the radius of convergence, which is the distance from $p = 3$ to an endpoint of $(2.5, 3.5)$, is $R = 0.5$.

Check your Reading *Explain why* $\lim\limits_{n\to\infty} \left|\frac{n+1}{n}\right| = 1$.

An Alternative Representation for the Ratio Test

When the coefficients a_n are fractions, we often write (9.43) in the form

$$\rho = \lim_{n\to\infty} \left| a_{n+1}\left(x-p\right)^{n+1} \cdot \frac{1}{a_n\left(x-p\right)^n} \right| \tag{9.44}$$

That is, (9.44) is the limit of the product of the numerator and the *reciprocal* of the denominator. Moreover, the phrase "interval of convergence" refers to the open interval of convergence for the remainder of this section.

EXAMPLE 2 Find the interval and radius of convergence of

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2} \tag{9.45}$$

Solution: Since $\frac{n^2}{(x-1)^n}$ is the reciprocal of $\frac{(x-1)^n}{n^2}$, we interpret (9.44) in the form

$$\rho = \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{(n+1)^2} \frac{n^2}{(x-1)^n} \right|$$

Collecting common terms then yields

$$\rho = \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{(x-1)^n} \frac{n^2}{(n+1)^2} \right| = \lim_{n\to\infty} \left| \frac{(x-1)}{1} \left(\frac{n}{n+1} \right)^2 \right|$$

Factoring out $|x-1|$ and evaluating the limit using L'hopital's rule yields

$$\rho = |x-1| \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^2 = |x-1| \left(\lim_{n\to\infty} \frac{n}{n+1} \right)^2 = |x-1| \, (1)^2 = |x-1|$$

We now set $\rho < 1$, which gives us $|x-1| < 1$. Solving for x results in

$$\begin{aligned} -1 &< x-1 < 1 \\ 0 &< x < 2 \end{aligned}$$

Thus, $(0,2)$ is the interval of convergence of (9.45), and the radius of convergence is 1. Indeed, the ratio test implies that the series diverges if $x < 0$ or $x > 2$.

The ratio test is the tool of choice for finding intervals of convergence of power series because it works well with the factorials that are frequently encountered. Moreover, in working with factorials, we often use the property

$$(n+1)! = (n+1) \cdot n!$$

For example, $5! = 5 \cdot 4!$ since both represent the product $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

EXAMPLE 3 Determine the interval of convergence of

$$\sum_{n=0}^{\infty} \frac{n+1}{n!} x^n \tag{9.46}$$

Solution: We begin by computing ρ:

$$\rho = \lim_{n\to\infty} \left| \overbrace{\frac{(n+2)\,x^{n+1}}{(n+1)!}}^{n \text{ is increased by } 1} \quad \overbrace{\frac{n!}{(n+1)\,x^n}}^{reciprocal} \right|$$

Collecting similar terms yields

$$\rho = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \frac{n+2}{n+1} \frac{n!}{(n+1)!} \right|$$

Since $(n+1)! = (n+1)\ n!$, we can simplify to

$$\rho = \lim_{n\to\infty} \left| \frac{x}{1} \frac{n+2}{n+1} \frac{n!}{(n+1)\ n!} \right| = \lim_{n\to\infty} \left| \frac{x}{1} \frac{n+2}{n+1} \frac{1}{n+1} \right|$$

As a result, we have

$$\rho = |x| \left(\lim_{n\to\infty} \frac{n+2}{(n+1)^2} \right) = |x| \cdot 0 = 0$$

Thus, $\rho = 0$ for all x, and since $0 < 1$, the series (9.46) must converge for all x. That is, the interval of convergence is $(-\infty, \infty)$.

Check your Reading *What is the radius of convergence of (9.46)?*

More with Factorials

When the coefficients of series contain expressions like $(2n)!$, then the $n+1$ term will contain the factorial $(2n+2)!$, which has the property

$$(2n+2)! = (2n+2)(2n+1)(2n)! \tag{9.47}$$

EXAMPLE 4 Determine the interval of convergence of the series

$$1 - \frac{1}{2}x + \frac{4!}{2!2!}\left(\frac{x}{4}\right)^2 - \frac{6!}{3!3!}\left(\frac{x}{4}\right)^3 + \ldots + (-1)^n \frac{(2n)!}{(n!)^2}\left(\frac{x}{4}\right)^n + \ldots \tag{9.48}$$

Solution: First we compute ρ, which is given by

$$\rho = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(2n+2)!\, x^{n+1}}{[(n+1)!]^2\, 4^{n+1}} \quad \frac{(n!)^2\ 4^n}{(-1)^n (2n)!\, x^n} \right|$$

and which immediately simplifies to

$$\rho = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \frac{4^n}{4^{n+1}} \frac{(2n+2)!}{(2n)!} \left[\frac{n!}{(n+1)!} \right]^2 \right|$$

Using $(n+1)! = (n+1)\, n!$ and (9.47), we obtain

$$\begin{aligned} \rho &= \lim_{n\to\infty} \left| \frac{x}{4} \frac{(2n+2)(2n+1)(2n)!}{(2n)!} \left[\frac{n!}{(n+1)\, n!} \right]^2 \right| \\ &= \frac{|x|}{4} \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{1} \frac{1}{(n+1)^2} \end{aligned}$$

Since $2n+2 = 2(n+1)$, we then obtain

$$\rho = \frac{|x|}{4} \lim_{n\to\infty} \frac{2(n+1)(2n+1)}{(n+1)^2} = \frac{|x|}{4} \lim_{n\to\infty} \frac{2(2n+1)}{n+1} \tag{9.49}$$

and after evaluating the limit, we obtain $\rho = |x|$. Setting $\rho < 1$ yields $|x| < 1$ or

$$-1 < x < 1$$

Thus, the interval of convergence of (9.48) is $(-1, 1)$, and the radius of convergence is $R = 1$.

Check your Reading *Evaluate the limit in (9.49).*

Justification for the Ratio Test

Finally, let us show that the ratio test is essentially the comparison test in which a power series is compared to a geometric series. If $\rho < 1$, then it follows that there is a number $r < 1$ such that $\rho < r$. Thus, the definition of ρ implies that

$$\left|\frac{a_{n+1}\,(x-p)^{n+1}}{a_n\,(x-p)^n}\right| < r$$

for all $n \geq M$ for some M. Multiplication then yields

$$\left|a_{n+1}\,(x-p)^{n+1}\right| < r\left|a_n\,(x-p)^n\right| \tag{9.50}$$

That is, the "next" term is smaller than r times the "previous" term.

Moreover, the n^{th} term must be smaller than r times the "previous" term, which is the $n-1$ term, thus implying that

$$\left|a_{n+1}\,(x-p)^{n+1}\right| < r\left|a_n\,(x-p)^n\right| < r\cdot r\left|a_{n-1}\,(x-p)^{n-1}\right|$$

Assuming that $M = 0$, we can continue this pattern as follows:

$$\begin{aligned}\left|a_{n+1}\,(x-p)^{n+1}\right| \quad < r\left|a_n\,(x-p)^n\right| \quad &< \quad r^2\left|a_{n-1}\,(x-p)^{n-1}\right| \\ &< \quad r^3\left|a_{n-2}\,(x-p)^{n-2}\right| \\ &\quad\;\vdots \\ &< \quad r^{n+1}\left|a_{n-n}\,(x-p)^{n-n}\right|\end{aligned}$$

That is, $\left|a_{n+1}\,(x-p)^{n+1}\right| < r^{n+1}\left|a_0\right|$, which we reindex into the form

$$\left|a_n\,(x-p)^n\right| < r^n\left|a_0\right|$$

As a result, we have

$$\sum_{n=0}^{\infty}\left|a_n\,(x-p)^n\right| < \sum_{n=0}^{\infty} r^n\left|a_0\right| \tag{9.51}$$

and since the latter is a geometric series with common ratio $r < 1$, it converges and thus, so also does the series of absolute values of the power series. That is, if $\rho < 1$, then the power series $\sum_{n=0}^{\infty} a_n\,(x-p)^n$ converges absolutely.

Moreover, when $\rho > 1$, then we must have

$$\left|\frac{a_{n+1}\,(x-p)^{n+1}}{a_n\,(x-p)^n}\right| > 1$$

for all $n \geq M$. This, in turn, implies that

$$\left|a_{n+1}\,(x-p)^{n+1}\right| > \left|a_n\,(x-p)^n\right|$$

for all $n \geq M$. However, a series cannot converge unless its individual terms become arbitrarily close to 0, which implies that the series diverges when $\rho > 1$.

Thus, the ratio test implies absolute convergence when $\rho < 1$ and divergence when $\rho > 1$, with no information when $\rho = 1$. Moreover, when applied to a power series, the ratio test reveals that a power series always converges either at a single point, in a finite interval or for all real numbers.

Exercises:

Find the interval and radius of convergence of the following power series.

1. $\displaystyle\sum_{n=0}^{\infty} x^n$
2. $\displaystyle\sum_{n=0}^{\infty} nx^n$
3. $\displaystyle\sum_{n=0}^{\infty} n(x-1)^n$
4. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{n}$
5. $\displaystyle\sum_{n=0}^{\infty} \frac{(x+1)^n}{2^n}$
6. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n}$
7. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n (x-3)^n}{n!}$
8. $\displaystyle\sum_{n=0}^{\infty} \frac{(x+1)^n}{2^n n}$
9. $\displaystyle\sum_{n=0}^{\infty} \frac{n\,(x-2)^n}{10^n}$
10. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{n^2+1}$
11. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$
12. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!}$
13. $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!\, x^n}{(2n+1)!}$
14. $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)\,(n+1)!\, x^n}{(2n)!}$
15. $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} \frac{x^n}{2^n}$

Each of the following is a geometric series. Determine the interval of convergence of each using (a) the ratio test and then (b) the geometric series test.

16. $1 + x^2 + x^4 + \ldots + x^{2n} + \ldots$

17. $1 + 2(x+3) + 4(x+3)^2 + \ldots + 2^n (x+3)^n + \ldots$

18. $1 - \frac{1}{2}(x-1) + \frac{1}{4}(x-1)^2 - \ldots + \frac{(-1)^n}{2^n}(x-1)^n + \ldots$

19. $1 + \frac{2}{3}(x-3) + \frac{4}{9}(x-3)^2 + \ldots + \frac{2^n}{3^n}(x-3)^n + \ldots$

20. $3 + \frac{3}{5}(x-\pi) + \frac{3}{25}(x-\pi)^2 + \ldots + \frac{3}{5^n}(x-\pi)^n + \ldots$

21. Endpoints: Use the ratio test to show that the open interval of convergence of

$$\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$$

is $(2,4)$. Then answer the following:

(a) Does the series converge when $x = 4$? Explain.
(b) Does the series converge when $x = 2$? Explain.

22. Endpoints: Use the ratio test to show that the open interval of convergence of

$$\sum_{n=0}^{\infty} \frac{(x+4)^n}{n^2}$$

is $(-5,-3)$. Then answer the following:

(a) Does the series converge when $x = -5$? Explain.
(b) Does the series converge when $x = -3$? Explain.

23. Endpoints: Use the ratio test to show that the open interval of convergence of

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n}$$

is $(-1, 3)$. Then answer the following:

(a) Does the series converge when $x = 4$? Explain.
(b) Does the series converge when $x = 2$? Explain.

24. Endpoints: Use the ratio test to show that the open interval of convergence of

$$\sum_{n=0}^{\infty} \frac{(x-4)^n}{2^n n^2}$$

is $(2, 6)$. Then answer the following:

(a) Does the series converge when $x = 6$? Explain.
(b) Does the series converge when $x = 2$? Explain.

25. In this exercise, we explore the geometric series

$$1 + \frac{x-1}{2} + \left(\frac{x-1}{2}\right)^2 + \ldots + \left(\frac{x-1}{2}\right)^n + \ldots$$

(a) Use the ratio test to find the interval of convergence of the series.
(b) Use the geometric series test to determine what the series converges to when it converges.
(c) How far is it from the center of the series to the closest vertical asymptote? How is this related to the interval of convergence.
(d) Graph the function in (b) along with the functions

$$\begin{aligned} T_2(x) &= 1 + \frac{x-1}{2} + \left(\frac{x-1}{2}\right)^2 \\ T_3(x) &= 1 + \frac{x-1}{2} + \left(\frac{x-1}{2}\right)^2 + \left(\frac{x-1}{2}\right)^3 \\ T_4(x) &= 1 + \frac{x-1}{2} + \left(\frac{x-1}{2}\right)^2 + \left(\frac{x-1}{2}\right)^3 + \left(\frac{x-1}{2}\right)^4 \end{aligned}$$

26. In this exercise, we explore the geometric series

$$1 + 2x + 4x^2 + 8x^3 + \ldots + 2^n x^n + \ldots$$

(a) Use the ratio test to find the interval of convergence of the series.
(b) Use the geometric series test to determine what the series converges to when it converges.
(c) How far is it from the center of the series to the closest vertical asymptote? How is this related to the interval of convergence.
(d) Graph the function in (b) along with the functions

$$\begin{aligned} T_2(x) &= 1 + 2x + 4x^2 \\ T_3(x) &= 1 + 2x + 4x^2 + 8x^3 \\ T_4(x) &= 1 + 2x + 4x^2 + 8x^3 + 16x^4 \end{aligned}$$

27. Given the power series $\sum_{n=0}^{\infty} a_n (x-p)^n$, the radius of convergence is given by the limit

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

(a) Show that when $0 < R < \infty$, then

$$\sum_{n=0}^{\infty} a_n (x-p)^n$$

converges for all x such that $|x-p| < R$.

(b) Show that when $R = \infty$, then

$$\sum_{n=0}^{\infty} a_n (x-p)^n$$

converges for all x.

(c) Show that when $R = 0$, then

$$\sum_{n=0}^{\infty} a_n (x-p)^n$$

converges only when $x = p$.

28. Use the definition of the limit of a sequence as n approaches ∞ to prove that if $\rho < r$, then there exists M such that

$$\left| \frac{a_{n+1} (x-p)^{n+1}}{a_n (x-p)^n} \right| < r$$

for all $n \geq M$. (Hint: Let $\varepsilon = (r-\rho)/2$)

29. Show that $\sum_{n=0}^{\infty} a_n (x-p)^n$ and $\sum_{n=0}^{\infty} n a_n (x-p)^n$ have the same open interval of convergence.

30. Show that $\sum_{n=1}^{\infty} a_n (x-p)^n$ and

$$\sum_{n=1}^{\infty} \frac{a_n}{n} (x-p)^n$$

have the same open interval of convergence.

31. **Root Test:** Show that if

$$\sqrt{|a_n (x-p)^n|} \leq r, \qquad n = 0, 1, 2, \ldots$$

then $\sum_{n=0}^{\infty} a_n (x-p)^n$ satisfies (9.51). If $r < 1$, what does that tell us about the series?

32. Use the root test to show that $\sum_{n=0}^{\infty} a_n (x-p)^n$ and $\sum_{n=0}^{\infty} n a_n (x-p)^n$ have the same open interval of convergence.

Root Test: *The Root Test says that if*

$$\rho = \lim_{n\to\infty} \sqrt{|a_n (x-p)^n|}$$

then $\sum_{n=0}^{\infty} a_n (x-p)^n$ *converges if* $\rho < 1$ *and diverges if* $\rho > 1$. *Use the root test to test for the convergence of the following:*

33. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{3^n}$ 34. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n (x-1)^n}{4^n}$ 35. $\displaystyle\sum_{n=0}^{\infty} n^n x^n$

36. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n^n}$ 37. $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+1)^n}$ 38. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n^{n+1}}$

9.6 Taylor Series

Analytic Functions

If a function $f(x)$ can be written as a power series centered at p over some open interval of convergence about p, then $f(x)$ is said to be *analytic at* p. That is, $f(x)$ is analytic at p if there is an open interval (a,b) containing p such that

$$f(x) = a_0 + a_1 (x-p) + a_2 (x-p)^2 + a_3 (x-p)^3 + a_4 (x-p)^4 + \dots \tag{9.52}$$

for all x in (a,b).

Let's now determine the coefficients a_n. To begin with, notice that

$$f(p) = a_0 + a_1 (p-p) + a_2 (p-p)^2 + a_3 (p-p)^3 + a_4 (p-p)^4 + \dots$$

That is, $f(p) = a_0$. Differentiating (9.52) then yields

$$f'(x) = a_1 + 2a_2 (x-p) + 3a_3 (x-p)^2 + 4a_4 (x-p)^3 + 5a_5 (x-4)^4 + \dots$$

which implies that $f'(p) = a_1$. Moreover, if we continue differentiating, then we obtain

$$\begin{aligned} f''(x) &= 2a_2 + 3 \cdot 2a_3 (x-p) + 4 \cdot 3a_4 (x-p)^2 + 5 \cdot 4a_5 (x-4)^3 + \dots \\ f'''(x) &= 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4 (x-p) + 5 \cdot 4 \cdot 3a_5 (x-p)^2 + \dots \\ f^{(4)}(x) &= 4 \cdot 3 \cdot 2a_4 + 5 \cdot 4 \cdot 3 \cdot 2a_5 (x-p) + 6 \cdot 5 \cdot 4 \cdot 3a_6 (x-p)^2 + \dots \end{aligned}$$

which implies that $f''(p) = 2a_2$, $f'''(p) = 3!a_3$, and $f^{(4)}(p) = 4!a_4$. Thus, it follows that in general, $f^{(n)}(p) = n!a_n$, so that

$$a_n = \frac{f^{(n)}(p)}{n!}$$

Thus, we have the following:

Theorem 6.1 *:* If $f(x)$ is analytic at a point p, then

$$f(x) = f(p) + f'(p)(x-p) + \frac{f''(p)}{2!}(x-p)^2 + \dots + \frac{f^{(n)}(p)}{n!}(x-p)^n + \dots$$

for all x in some open interval (a,b) containing p.

We call the power series representation of $f(x)$ centered at p the *Taylor's series of $f(x)$ centered at p*. If we define $0! = 1$, then the Taylor's series of $f(x)$ centered at p can be written

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!}(x-p)^n$$

Moreover, when $p = 0$, the Taylor's series is a *MacLaurin's series.* For example, in section 8-3, we constructed the MacLaurin's series of e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Check your Reading *What is the Taylor's series centered at $p = 0$ of the function*

$$f(x) = \frac{1}{1-x}$$

Taylor's Series of Analytic Functions

Our goal in this section is to develop Taylor's series representations of analytic functions. This requires that we combine the methods for finding families of Taylor polynomials from section 8-2 with the determination of intervals of convergence from 8-5.

EXAMPLE 1 Find the Taylor's series expansion and interval of convergence of $f(x) = \ln(x)$ centered at $x = 1$.

Solution: We begin with a table of derivatives:

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln(x)$	0
1	x^{-1}	1
2	$-1x^{-2}$	$(-1)\ 1!$
3	$(-1)(-2)x^{-3}$	$(-1)^2\ 2!$
4	$(-1)(-2)(-3)x^{-4}$	$(-1)^3\ 3!$
5	$(-1)(-2)(-3)(-4)x^{-5}$	$(-1)^4\ 4!$

from which it follows that the general form of $f^{(n)}(1)$ is

$$f^{(n)}(1) = (-1)^{n-1}(1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)) = (-1)^{n-1}(n-1)!$$

As a result, the Taylor's series expansion of $\ln(x)$ centered at $x = 1$ is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!}(x-1)^n$$

Thus, the Taylor's series centered at $x = 1$ generated by $\ln(x)$ is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n \qquad (9.53)$$

Now we must determine the interval of convergence of (9.53). The ratio test implies that

$$\rho = \lim_{n\to\infty} \left| \frac{(-1)^n (x-1)^{n+1}}{n+1} \frac{n}{(-1)^{n-1} (x-1)^n} \right|$$

Regrouping the terms and simplifying yields

$$\begin{aligned} \rho &= \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{(x-1)^n} \frac{n}{n+1} \right| \\ &= |x-1| \lim_{n\to\infty} \frac{n}{n+1} \\ &= |x-1| \end{aligned}$$

Setting $\rho < 1$ thus yields

$$|x-1| < 1$$

$$\begin{aligned} -1 &< x-1<1 \\ 0 &< x<2 \end{aligned}$$

so that the interval of convergence is $(0,2)$. That is, we have shown that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n$$

converges for all x in $(0,2)$.

EXAMPLE 2 Find the Taylor's series expansion of $\ln(x)$ centered at $p = 2$.

Solution: We begin with the table of derivative information:

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$\ln x$	$\ln(2)$
1	x^{-1}	2^{-1}
2	$-x^{-2}$	-2^{-2}
3	$-1\cdot -2x^{-3}$	$-1\cdot -2\cdot 2^{-3}$
4	$-1\cdot -2\cdot -3x^{-4}$	$-1\cdot -2\cdot -3\cdot 2^{-4}$
$\vdots$	$\vdots$	$\vdots$
n	$(-1)^{n-1}(n-1)!\,x^{-n}$	$(-1)^{n-1}(n-1)!\,2^{-n}$

As a result, the pattern is that $f^{(n)}(2) = (-1)^{n-1}(n-1)!\,2^{-n}$ when $n \geq 1$. However, the pattern does not hold for the first term, which is $f(2) = \ln(2)$. Thus, we write the first term separately to obtain

$$\ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!\,2^{-n}}{n!} (x-2)^n$$

which simplifies to

$$\ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\,2^n} (x-2)^n$$

To find the interval of convergence, we first compute

$$\rho = \lim_{n\to\infty} \left| \frac{(-1)^n (x-2)^{n+1}}{(n+1)\ 2^{n+1}} \frac{n\ 2^n}{(-1)^{n-1} (x-2)^n} \right|$$

Rearranging and simplifying yields

$$\begin{aligned} \rho &= \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(x-2)^n} \frac{2^n}{2^{n+1}} \frac{n}{n+1} \right| \\ &= \frac{|x-2|}{2} \lim_{n\to\infty} \frac{n}{n+1} \\ &= \frac{1}{2} |x-2| \end{aligned}$$

Setting $\rho < 1$ and solving for x yields

$$\begin{aligned} \frac{1}{2} |x-2| &< 1 \\ |x-2| &< 2 \end{aligned}$$

$$\begin{aligned} -2 &< x-2 < 2 \\ 0 &< x < 4 \end{aligned}$$

so that we can conclude that

$$\ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\ 2^n} (x-2)^n$$

converges for all x in $(0,4)$.

Check your Reading *Why would we not expect a power series representation of* $\ln(x)$ *to converge when* $x = 0$ *regardless of where it is centered? That is, what is significant about* $\ln(x)$ *when* $x = 0$?

Alternating Pattern in the Taylor Coefficients

It is not unusual for the even and odd terms of a series to satisfy different patterns. Moreover, in such instances it is important to treat the even and odd terms separately by using $2n$ for the even indices and $2n+1$ for the odd indices.

EXAMPLE 3 Find the Taylor's series expansion of $f(x) = \cos(x)$ centered at $p = \pi$, along with its interval of convergence.

Solution: We begin with the table of derivative information

n	$f^{(n)}(x)$	$f^{(n)}(\pi)$
0	$\cos(x)$	$-1 = (-1)^1$
1	$-\sin(x)$	0
2	$-\cos(x)$	$1 = (-1)^2$
3	$\sin(x)$	0
4	$\cos(x)$	$-1 = (-1)^3$
5	$-\sin(x)$	0
6	$-\cos(x)$	$1 = (-1)^4$

Since the odd terms are all 0, so that we need only consider the even terms. However,

$$f(2\cdot 0)=(-1)^1, \quad f(2\cdot 1)=(-1)^2, \quad f(2\cdot 2)=(-1)^3, \quad f(2\cdot 3)=(-1)^4$$

Thus, our pattern is that $f^{(2n)}(\pi)=(-1)^{n+1}$ and that the odd coefficients are 0. As a result, the Taylor's series expansion of $\cos(x)$ centered at $p=\pi$ is

$$\sum_{n=0}^{\infty}\frac{f^{(2n)}(\pi)}{(2n)!}(x-\pi)^{2n}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n)!}(x-\pi)^{2n}$$

To find its interval of convergence, we evaluate the limit

$$\rho=\lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-\pi)^{2n+2}}{(2n+2)!}\frac{(2n)!}{(-1)^{n+1}(x-\pi)^{2n}}\right|$$

Collecting terms and simplifying yields

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{(x-\pi)^{2n+2}}{(x-\pi)^{2n}}\frac{(2n)!}{(2n+2)!}\right| \\ &= |x-\pi|^2\lim_{n\to\infty}\frac{(2n)!}{(2n+2)(2n+1)(2n!)} \\ &= |x-\pi|^2\lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)} \\ &= 0\end{aligned}$$

That is, the limit is equal to 0 regardless of the value of x, so that

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n)!}(x-\pi)^{2n} \qquad (9.54)$$

converges for all real numbers x.

Check your Reading *What is the radius of convergence of the series (9.54)?*

Infinitely Differentiable Does Not Imply Analytic

A function is said to be *infinitely differentiable* at a point p if it and all of its derivatives exist at p. In this last example, we show that a function can be infinitely differentiable at p without being analytic at p. In particular, the Taylor's series expansion centered at p may converge to a function other than the one that generated it.

EXAMPLE 4 Find the Taylor's series expansion centered at 0 of

$$f(x)=\begin{cases} e^{-1/x^2} & if \quad x\neq 0 \\ 0 & if \quad x=0\end{cases}$$

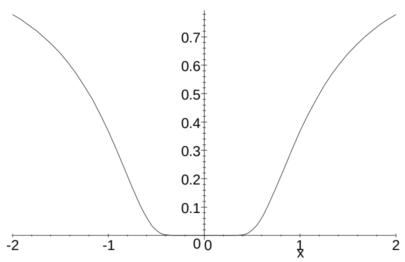
Solution: To begin with, notice that the first derivative at $x=0$ is given by

$$f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{e^{-1/h^2}}{h}$$

However, if we let $h = 1/R$, then as h approaches 0, R approaches ∞, so that we have

$$f'(0) = \lim_{R\to\infty} \frac{e^{-R^2}}{1/R} = \lim_{R\to\infty} Re^{-R^2} = 0$$

Similarly, it can be shown that $f^{(n)}(0) = 0$ for all positive integers n. That is, $f(x)$ is "infinitely flat" at the origin:



6-1: A function that is "infinitely flat" at the origin

And since $f^{(n)}(0) = 0$, the MacLaurin series of $f(x)$ is

$$0 + 0(x-0) + \frac{0}{2!}(x-0)^2 + \ldots + \frac{0}{n!}(x-0)^n + \ldots = 0$$

Consequently, $f(x)$ is **not** analytic on any open interval containing the origin, even though every derivative of $f(x)$ exists at the origin.

If a function can be infinitely differentiable at a point without being analytic at that point, how then do we determine when a function is actually analytic at a given value? For example, how do we know that the Taylor's series expansion of $f(x) = e^x$ centered at $p = 2$ actually converges to $f(x) = e^x$?

Fortunately, the last section in this chapter will give us a method for determining when a function is analytic. In particular, if a function $f(x)$ is the solution to a linear differential equation with polynomial coefficients, and if the initial values at a point p guarantee only one solution, then $f(x)$ is analytic at p. For example, $y = e^x$ is the only solution to $y' = y$, $y(2) = e^2$, so that e^x is indeed analytic at $p = 2$.

Exercises:

In section 8-2, you found the family of Taylor polynomials for each of the following. Now write down the Taylor's series expansion of each at the given value of p and find the interval of convergence of the series.

1. $f(x) = e^{2x}$, $p = 0$	2. $f(x) = e^x$, $p = 1$
3. $f(x) = \ln(1-x)$, $p = 0$	4. $f(x) = \ln(2-x)$, $p = 1$
5. $f(x) = x^{-1}$, $p = 1$	6. $f(x) = x^{-1}$, $p = -1$
7. $f(x) = x^{-2}$, $p = 1$	8. $f(x) = x^{-2}$, $p = -1$
9. $f(x) = \cosh(x)$, $p = 0$	10. $f(x) = \sinh(x)$, $p = 0$
11. $f(x) = \sin(x)$, $p = \pi$	12. $f(x) = \cos(x)$, $p = 2\pi$
13. $f(x) = \sin(x)$, $p = \frac{\pi}{4}$	14. $f(x) = \cos(x)$, $p = \frac{\pi}{4}$

Find the general form of the Taylor's series centered at the given value of p of each of the functions given below, and then determine the interval of convergence of each series.

15. $f(x) = x^{-1}$, $p = 2$
16. $f(x) = x^{-1}, \quad p = 3$
17. $f(x) = \ln(x)$, $p = 3$
18. $f(x) = \ln(x), \quad p = 4$
19. $f(x) = xe^{x} \quad p = 1$
20. $f(x) = e^{2x}$, $p = 1$
21. $f(x) = (1-x)^{-2}, \quad p = -1$
22. $f(x) = x^{-2}, \quad p = 3$
23. $f(x) = \cosh(x), \quad p = \ln(2)$
24. $f(x) = \sinh(x), \quad p = \ln(2)$

Use a computer algebra system to generate $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$, and so on. Use the result to determine the general term $f^{(n)}(0)$, use the general term to generate the Maclaurin's series of the function and then find its interval of convergence.

25. $f(x) = \ln|1+2x|$
26. $f(x) = \ln|1+3x|$
27. $f(x) = \tan^{-1}(x)$
28. $f(x) = \tan^{-1}(2x)$
29. $f(x) = \ln|1+x^2|$
30. $f(x) = \ln\left|\dfrac{1+x}{1-x}\right|$

31. If $x = q$ is the vertical asymptote of $f(x)$ closest to a point p in the domain of $f(x)$, the interval of convergence is $(-|q-p|, |q-p|)$. Use this idea to determine the interval of convergence of those functions in exercises 1-24 that have a vertical asymptote.

32. We can manipulate a geometric series to show that

$$\frac{1}{x^2+1} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Does $\frac{1}{x^2+1}$ have a vertical asymptote? Does its MacLaurin series converge for all x? Explain.

33. In this exercise, we use the fact (see (9.53) above) that

$$u(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (x-1)^n$$

is the Taylor's series centered at $p = 1$ generated by $v(x) = \ln(x)$.

(a) Show that $u'(x) = v'(x)$ for all x in $(0, 2)$. (Hint: apply geometric series formula to $u'(x)$).

(b) Explain why we can conclude that the alternating harmonic series converges to $\ln(2)$. That is, explain why

$$\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

for the standard arrangement of the alternating harmonic series.

34. In this exercise, we use Taylor's series to prove that

$$e^{a+b} = e^a e^b$$

(a) Show that the Taylor's series expansion of e^x centered at $x = b$ is

$$e^x = e^b \left[1 + (x-b) + \frac{(x-b)^2}{2!} + \frac{(x-b)^3}{3!} + \ldots + \frac{(x-b)^n}{n!} + \ldots \right]$$

(b) Let $x = a + b$ and use the result from (a) to show that $e^{a+b} = e^a e^b$.

(c) Notice that the result in (a) is true only if the initial value problem which implicitly defines the exponential function,

$$y' = y, \quad y(0) = 1$$

has only one solution. We showed that is does indeed have one solution on page **??** of the text.

35. Prove the following: If $f(x)$ is even and analytic at 0, then its MacLaurin's series is an infinite series of even powers of x. (Hint: first show that if $f(x)$ is odd, then $f'(0) = 0$).

36. Prove the following: If $f(x)$ is odd and analytic at 0, then its MacLaurin's series is an infinite series of odd powers of x.

37. **Newton's Method:** Suppose that $f(x)$ is analytic at p, that $f'(p) \neq 0$, and that $f(r) = 0$.

(a) Explain why

$$0 = f(p) + f'(p)(r-p) + \frac{f''(p)}{2!}(r-p)^2 + \ldots + \frac{f^{(n)}(p)}{n!}(r-p)^n + \ldots$$

(b) Show that

$$r = p - \frac{f(p)}{f'(p)} - \frac{f''(p)}{2!f'(p)}(r-p)^2 - \ldots - \frac{f^{(n)}(p)}{n!f'(p)}(r-p)^n + \ldots$$

(c) Explain why Newton's method yields a good approximation of r when $(r-p)$ is sufficiently small.

38. **Fixed Points:** Let $f(x)$ be analytic at p with $f'(p) \neq 1$.

(a) Explain why if r is a fixed point of $f(x)$, then

$$r = f(p) + f'(p)(r-p) + \frac{f''(p)}{2!}(r-p)^2 + \ldots + \frac{f^{(n)}(p)}{n!}(r-p)^n + \ldots$$

(b) Show that

$$r = \frac{f(p) - pf'(p)}{1 - f'(p)} + \frac{f''(p)}{1-f'(p)}\frac{(r-p)^2}{2!} + \ldots + \frac{f^{(n)}(p)}{1-f'(p)}\frac{(r-p)^n}{n!} + \ldots$$

(c) Explain why if $(r-p)$ is sufficiently small, then

$$\frac{f(p) - pf'(p)}{1 - f'(p)}$$

is a good approximation of the fixed point r.

39. **Write to Learn:** Explain in a short essay why if we define

$$f(x) = \begin{cases} e^{-1/x} & if \quad x > 0 \\ 0 & if \quad x \leq 0 \end{cases}$$

then the Taylor's series of $f(x)$ centered at 0 does not converge to $f(x)$ on any open interval containing 0.

40. Write to Learn: Explain in a short essay why if we define

$$f(x) = \begin{cases} x+1+e^{-1/x^2} & if \quad x > 0 \\ 0 & if \quad x \le 0 \end{cases}$$

then the MacLaurin's series of $f(x)$ does not converge to $f(x)$ on any open interval containing 0.

9.7 Calculus with Power Series

Term by Term Integration and Differentiation

New series can be obtained from derivatives and integrals of known series.

In this section, we study derivatives and integrals of power series, both for computational purposes and also to obtain new power series expansions from known power series representations. To do so, however, requires the following theorem:

Theorem 7.1: If there is a number $c > 0$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n (x-p)^n$$

for all x in the interval $(p-c, p+c)$, then the power series representation of $f(x)$ can be integrated *term by term,* which is to say that

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-p)^{n+1}$$

for all x in $(p-c, p+c)$. Likewise, the power series representation of $f(x)$ can be differentiated *term by term,* which is to say that

$$f'(x) = \sum_{n=0}^{\infty} a_n\, n\,(x-p)^{n-1}$$

for all x in $(p-c, p+c)$.

Although we cannot prove theorem 7.1, we will use it in this section to derive Taylor's series representations of certain analytic functions.[6]

EXAMPLE 1 Find the MacLaurin's series expansion of $\ln(1-x)$.

Solution: To do so, we first notice

$$\ln(1-x) = \int \frac{1}{1-x}dx$$

[6] Not all series can be differentiated term by term. Thus, assuming that theorem 7.1 is true is a rather large assumption and will need to be justified in a later course.

Integrating both sides of the geometric series thus yields

$$\int \frac{1}{1-x} dx = \int \left(1 + x + x^2 + \ldots + x^n + \ldots\right) dx$$

from which we obtain

$$\ln(1-x) = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{x^{n+1}}{n+1} + \ldots$$

To determine C, we let $x = 0$:

$$\ln(1) = C + 0 + 0 + \ldots + 0 + \ldots$$

Since $\ln(1) = 0$, we also have $C = 0$ and thus

$$\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{x^{n+1}}{n+1} + \ldots, \qquad -1 < x < 1$$

EXAMPLE 2 Find the Maclaurin's series of $(1-x)^{-2}$.

Solution: To do so, we first notice that

$$\frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2} \tag{9.55}$$

Thus, we apply the derivative operator to both sides of the geometric series:

$$\frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}\left[1 + x + x^2 + x^3 + \ldots + x^n + \ldots\right]$$

which results in

$$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + \ldots + nx^{n-1} + \ldots \tag{9.56}$$

Moreover, since the geometric series converges for x in $(-1, 1)$, the series (9.56) also converges in $(-1, 1)$.

Check your Reading *Show that (9.55) is true.*

Definite Integrals of Power Series

Since a power series can be antidifferentiated term by term, the fundamental theorem implies that definite integrals of power series can also be evaluated term by term.

EXAMPLE 3 Find the MacLaurin's series of $\tan^{-1}(x)$ using the fact that

$$\tan^{-1}(x) = \int_0^x \frac{1}{1+t^2} dt$$

Solution: To begin with, if we replace x with $-t^2$ in the geometric series, we obtain

$$\frac{1}{1-(-t^2)} = 1 - t^2 + t^4 - \ldots + (-1)^n t^{2n} + \ldots, \qquad -1 < t < 1$$

If we compute the definite integral from 0 to x on each side, we obtain

$$\begin{aligned}\int_0^x \frac{1}{1+t^2} dx &= \int_0^x \left[1 - t^2 + t^4 - \ldots + (-1)^n t^{2n} + \ldots\right] dt \\ &= \int_0^x 1dt - \int_0^x t^2 dt + \int_0^x t^4 dt - \ldots + (-1)^n \int_0^x t^{2n} dt + \ldots\end{aligned}$$

As a result, we have

$$\begin{aligned}\tan^{-1}(x) &= t|_0^x - \left.\frac{t^3}{3}\right|_0^x + \left.\frac{t^5}{5}\right|_0^x - \ldots + (-1)^n \left.\frac{t^{2n+1}}{2n+1}\right|_0^x + \ldots \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} + \ldots\end{aligned}$$

Consequently, we have

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + \frac{(-1)^n x^{2n+1}}{2n+1} + \ldots, \qquad -1 < x < 1$$

EXAMPLE 4 Find an infinite series representation of the definite integral

$$\int_0^{\pi/4} \cos\left(x^2\right) dx$$

Solution: The Maclaurin's series of $\cos(x)$ implies that

$$\begin{aligned}\cos\left(x^2\right) &= 1 - \frac{\left(x^2\right)^2}{2!} + \frac{\left(x^2\right)^4}{4!} - \ldots + \frac{(-1)^n \left(x^2\right)^{2n}}{(2n)!} + \ldots \\ &= 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \ldots + \frac{(-1)^n x^{4n}}{(2n)!} + \ldots\end{aligned}$$

for all x. As a result, we have

$$\int_0^{\pi/4} \cos\left(x^2\right) dx = \int_0^{\pi/4} \left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \ldots + \frac{(-1)^n x^{4n}}{(2n)!} + \ldots\right) dx$$

Integrating term by term then expresses the definite integral as an infinite series:

$$\int_0^{\pi/4} \cos\left(x^2\right) dx = \frac{\pi}{4} - \frac{(\pi/4)^5}{5 \cdot 2!} + \frac{(\pi/4)^9}{9 \cdot 4!} - \ldots + \frac{(-1)^n (\pi/4)^{4n+1}}{(4n+1)(2n)!} + \ldots$$

Check your Reading *Evaluate* $\int_0^{\pi/4} \cos\left(x^2\right) dx$ *numerically, and then compare to the partial sum*

$$\frac{\pi}{4} - \frac{(\pi/4)^5}{5 \cdot 2!} + \frac{(\pi/4)^9}{9 \cdot 4!}$$

In many applications, new functions are defined using power series. Indeed, the vibration of an airplane wing or the top of a drum are not sinusoidal and must instead be modeled with *Bessel Functions,* where the 0^{th} Bessel function is defined

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4\,(2!)^2} - \frac{x^6}{2^6\,(3!)^2} + \ldots + \frac{(-1)^n\,x^{2n}}{2^{2n}\,(n!)^2} + \ldots \tag{9.57}$$

and the 1^{st} Bessel function is defined

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^3\,1!\,2!} + \frac{x^5}{2^5\,2!\,3!} - \ldots + \frac{(-1)^n\,x^{2n+1}}{2^{2n+1}\,n!\,(n+1)!} + \ldots \tag{9.58}$$

These series converge for all x, as will be shown in the exercises.

EXAMPLE 5 Use (9.57) to find a series representation for

$$\int_0^1 xJ_0(x)\,dx$$

Solution: To do so, we first multiply (9.57) by x to obtain

$$xJ_0(x) = x - \frac{x^3}{2^2} + \frac{x^5}{2^4\,(2!)^2} - \frac{x^7}{2^6\,(3!)^2} + \ldots + \frac{(-1)^n\,x^{2n+1}}{2^{2n}\,(n!)^2} + \ldots$$

Term by term integration then yields

$$\begin{aligned}\int_0^1 xJ_0(x)\,dx &= \int_0^1 \left(x - \frac{x^3}{2^2} + \frac{x^5}{2^4\,(2!)^2} - \ldots + \frac{(-1)^n\,x^{2n+1}}{2^{2n}\,(n!)^2} + \ldots\right)dx \\ &= \left.\frac{x^2}{2} - \frac{x^4}{4\cdot 4} + \frac{x^6}{6\cdot 16\cdot 4} - \ldots + \frac{(-1)^n\,x^{2n+2}}{(2n+2)\,2^{2n}\,(n!)^2} + \ldots\right|_0^1\end{aligned}$$

As a result, we have

$$\int_0^1 xJ_0(x)\,dx = \frac{1}{2} - \frac{1}{16} + \frac{1}{384} - \ldots + \frac{(-1)^n}{(2n+2)\,2^{2n}\,(n!)^2} + \ldots$$

EXAMPLE 6 Use (9.57) to find $J_0'(x)$.

Solution: Differentiating term by term yields

$$J_0'(x) = 0 - \frac{2x}{2^2} + \frac{4x^3}{2^4\,(2!)^2} - \frac{6x^5}{2^6\,(3!)^2} + \ldots + \frac{(-1)^n\,2n\,x^{2n-1}}{2^{2n}\,(n!)^2} + \ldots$$

which simplifies to

$$J_0'(x) = -\frac{x}{2} + \frac{2\cdot 2x^3}{2^4\,2!2!} - \frac{3\cdot 2x^5}{2^6\,3!3!} + \ldots + \frac{(-1)^n\,2n\,x^{2n-1}}{2^{2n}\,n!n!} + \ldots$$

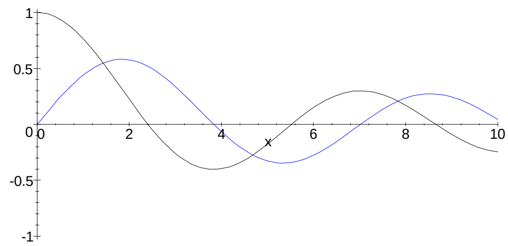
If we now factor out the negative and cancel, we obtain

$$J_0'(x) = -\left(\frac{x}{2} - \frac{x^3}{2^3\,1!\,2!} + \frac{x^5}{2^5\,2!\,3!} - \ldots + \frac{(-1)^n\,x^{2n-1}}{2^{2n-1}\,(n-1)!\,n!} + \ldots\right)$$

Comparison with (9.58) reveals that we have just shown that

$$J_0'(x) = -J_1(x).$$

Check your Reading *Use (9.57) and (9.58) to determine which of the graphs below is the graph of $J_0(x)$ and which is the graph of $J_1(x)$.*



7-1: Graphs of J_0 and J_1

Limit Evaluation with Power Series

Finally, let us notice that Taylor's and MacLaurin's series can also be used to evaluate limits. Indeed, power series are often used as an alternative to L'hopital's rule.

EXAMPLE 7 Use MacLaurin's series expansions to compute

$$\lim_{x\to 0}\frac{\sin\left(x^2\right)-\sin^2(x)}{x^4} \tag{9.59}$$

Solution: To begin with, substituting x^2 into the MacLaurin's series for $\sin(x)$ yields

$$\sin\left(x^2\right)=x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}+\dots$$

Moreover, $\sin^2(x)=\frac{1}{2}-\frac{1}{2}\cos(2x)$ and since

$$\begin{aligned}\frac{1}{2}\cos(2x) &= \frac{1}{2}\left(1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\dots\right)\\ &= \frac{1}{2}-\frac{4x^2}{2\cdot 2!}+\frac{16x^4}{2\cdot 4!}-\frac{64x^6}{2\cdot 6!}+\dots\end{aligned}$$

we have that

$$\begin{aligned}\sin^2(x) &= \frac{1}{2}-\left(\frac{1}{2}-\frac{4x^2}{2\cdot 2!}+\frac{16x^4}{2\cdot 4!}-\frac{64x^6}{2\cdot 6!}+\dots\right)\\ &= \frac{4x^2}{2\cdot 2}-\frac{16x^4}{2\cdot 24}+\frac{64x^6}{2\cdot 720}-\dots\\ &= x^2-\frac{x^3}{3}+\frac{2x^6}{45}-\dots\end{aligned}$$

Thus, the Maclaurin's series of the numerator of (9.59) is

$$\begin{aligned}\sin\left(x^2\right)-\sin^2(x) &= \left(x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}+\dots\right)-\left(x^2-\frac{x^4}{3}+\frac{2x^6}{45}-\dots\right)\\ &= \frac{x^4}{3}-\frac{11}{90}x^6+\dots\end{aligned}$$

Division by x^4 then yields

$$\frac{\sin\left(x^2\right) - \sin^2(x)}{x^4} = \frac{1}{3} - \frac{11}{90}x^2 + \dots$$

so that

$$\lim_{x\to 0} \frac{\sin\left(x^2\right) - \sin^2(x)}{x^4} = \lim_{x\to 0}\left(\frac{1}{3} - \frac{11}{90}x^2 + \dots\right) = \frac{1}{3}$$

Exercises:

Find the Taylor's series expansions of each of the following by differentiating or integrating a known series. State the interval of convergence of the result.

1. $\int \sin\left(x^2\right) dx$
2. $\int \cos\left(x^2\right) dx$
3. $\ln(1+x)$
4. $\ln(1-2x)$
5. $\dfrac{1}{(1+x)^2}$
6. $\dfrac{-2x}{\left(1+x^2\right)^2}$
7. $\dfrac{1}{(1-x)^3}$
8. $\dfrac{1}{(1+x)^3}$
9. $\ln\left(\dfrac{1-x}{1+x}\right)$
10. $\ln\left(1-x^2\right)$
11. $\ln\left(1+x^2\right)$
12. $\tan^{-1}\left(x^2\right)$
13. $\int J_0(x)\, dx$
14. $\int J_1(x)\, dx$
15. $J_1'(x)$
16. $\frac{d}{dx}\left(xJ_0(x)\right)$
17. $\int xJ_1(x)\, dx$
18. $\int x^2 J_0(x)\, dx$

Use a known Taylor's series (or the given Taylor's series) to find a series representation of the given definite integral.

19. $\int_0^2 e^{-x^2} dx$
20. $\int_0^{\sqrt{\pi}} \sin\left(x^2\right) dx$
21. $\displaystyle\int_0^1 \frac{dx}{1+x}$
22. $\displaystyle\int_{-1}^0 \frac{dx}{1-x}$
23. $\int_0^1 \frac{\sin(x)}{x} dx$
24. $\int_{-1}^1 \frac{e^x - 1}{x} dx$
25. $\int_1^2 \ln(x)\, dx$ using $\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$ for x in $(0,2)$
26. $\int_1^2 \ln(x)\, dx$ using $\ln(x) = \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}(x-2)^n \ln(2)$ for x in $(0,2)$

27. Use a MacLaurin's series expansion to evaluate the limit

$$\lim_{x\to 0} \frac{\sin\left(x^2\right) - x^2}{x^6}$$

28. Use a Maclaurin's series expansion to evaluate the limit

$$\lim_{x\to 0} \frac{2\cos(x) - 2 + x^2}{x^4}$$

29. In this exercise, we evaluate the limit

$$\lim_{x\to 0} \frac{\cos\left(x^2\right) - \cos^2(x)}{x^2}$$

(a) Find the Maclaurin's series expansions of $\sin\left(x^2\right)$ and $\sin^2(x)$.

(b) Use (a) and the Maclaurin's series for $\sin(x)$ to find the Maclaurin's series of

$$\sin\left(x^2\right) - \sin^2(x)$$

(c) Divide the resulting series in (b) by x^4 and then let x approach 0.

30. In this exercise, we evaluate the limit

$$\lim_{x\to 0} \frac{4\sin\left(x^3\right) - 3\sin(x) + \sin(3x)}{x^5}$$

(a) Find the Maclaurin's series expansion of $\sin\left(x^3\right)$ and $\sin(3x)$.

(b) Use (a) and the Maclaurin's series for $\sin(x)$ to find the Maclaurin's series of

$$4\sin\left(x^3\right) - 3\sin(x) + \sin(3x)$$

(c) Divide the resulting series in (b) by x^5 and then let x approach 0.

31. Sine Integral: The sine integral is defined to be

$$\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t}\,dt$$

Find the Maclaurin's series expansion of $\mathrm{Si}(x)$.

32. The *error function* is defined to be

$$\mathrm{erf}(x) = \int_0^x e^{-t^2}\,dt$$

(a) Find the Maclaurin's series expansion of the error function.

(b) Use $T_9(x)$ to approximate $\mathrm{erf}(0.5)$. How close is this to the approximation obtained by applying numerical integration to

$$\int_0^{0.5} e^{-x^2}\,dx$$

33. In this exercise, we explore the *Fresnel Cosine Integral*

$$C(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt$$

(a) Find the Maclaurin's series expansion of $C(x)$.

(b) Graph $T_{13}(x)$ over the interval $[-3, 3]$.

(c) Compute $T_{13}(1)$ to estimate the integral

$$\int_0^1 \cos\left(\frac{\pi t^2}{2}\right) dt$$

How close is this to the result obtained with numerical integration?

34. In this exercise, we explore the *Fresnel Sine Integral*

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$$

(a) Find the Maclaurin's series expansion of $C(x)$.
(b) Graph $T_{13}(x)$ over the interval $[-3, 3]$.
(c) Compute $T_{13}(1)$ to estimate the integral

$$\int_0^1 \sin\left(\frac{\pi t^2}{2}\right) dt$$

How close is this to the result obtained with numerical integration?

35. Show that the derivative of the Maclaurin's series of e^x is the Maclaurin's series of e^x

36. Show that the derivative of the Maclaurin's series of $\sin(x)$ is the Maclaurin's series of $\cos(x)$

37. Show that

$$J_1'(x) = J_0(x) - \frac{1}{x} J_1(x)$$

38. Show that

$$J_0''(x) = \frac{1}{x} J_1(x) - J_0(x)$$

39. **Write to Learn:** Suppose the power series $\sum_{n=0}^{\infty} a_n (x-p)^n$ converges on the closed interval $[p-r, p+r]$. In a short essay, explain why $|x-p| \leq r$ for all x in $[p-r, p+r]$ and thus why

$$\sum_{n=0}^{\infty} |a_n (x-p)^n| \leq \sum_{n=0}^{\infty} |a_n| r^n$$

implies that the power series $\sum_{n=0}^{\infty} a_n (x-p)^n$ converges absolutely and uniformly in x over $[p-r, p+r]$.

9.8 Differential Equations

First Order Linear Differential Equations

When a first order differential equation has only one solution for a given initial value, then its solution is an analytic function and a power series can be used to represent that solution. In this section, our initial values will be of the form $y(0) = a_0$, thus allowing us to solve a differential equation with a power series of the form

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots \quad (9.60)$$

We begin by writing the solution as

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots$$

where $a_0, a_1, a_2, \ldots$ are to be determined. Since the derivative y' is

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \ldots$$

we can substitute y and y' into the differential equation and then determine a function $f(n)$ such that

$$a_n = f(n)$$

That is, we seek the general term of the power series solution (9.60).

EXAMPLE 1 Find a power series solution to the initial value problem

$$y' = 2y, \qquad y(0) = 1$$

Solution: To obtain a solution of the form (9.60), we substitute for y and y' :

$$\begin{aligned} a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \ldots &= 2\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots\right) \\ a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \ldots &= 2a_0 + 2a_1x + 2a_2x^2 + 2a_3x^3 + 2a_4x^4 + \ldots \end{aligned}$$

In order to organize our work, we write the equation vertically so that we can match up the coefficients of corresponding powers of x:

	a_1	+	$2a_2\ x$	+	$3a_3\ x^2$	+	$4a_4\ x^3$	+	$5a_5\ x^4$	+	$\ldots$
=	$2a_0$	+	$2a_1\ x$	+	$2a_2\ x^2$	+	$2a_3\ x^3$	+	$2a_4\ x^4$	+	$\ldots$

Do not simplify calculations when looking for a pattern.

By equating the coefficient above the double line with the one below, we obtain

$$a_1 = 2a_0, \qquad 2a_2 = 2a_1, \qquad 3a_3 = 2a_2, \qquad 4a_4 = 2a_3, \qquad 5a_5 = 2a_4,$$

Since $a_0 = 1$, we must have $a_1 = 2a_0 = 2 \cdot 1$. The next equation implies that $2a_2 = 2a_1 = 2 \cdot 2$, which gives us $a_2 = 2(2)/2$. Continuing in this fashion gives us

$$\begin{aligned} a_1 &= 2a_0 = 2 \cdot 1, & a_1 &= 2 \\ 2a_2 &= 2a_1 = 2 \cdot 2, & a_2 &= \frac{2 \cdot 2}{2} = \frac{2^2}{2!} \\ 3a_3 &= 2a_2 = 2 \cdot \frac{2 \cdot 2}{2}, & a_3 &= \frac{2 \cdot 2 \cdot 2}{3 \cdot 2} = \frac{2^3}{3!} \\ 4a_4 &= 2a_3 = 2 \cdot \frac{2 \cdot 2 \cdot 2}{3 \cdot 2}, & a_4 &= \frac{2 \cdot 2 \cdot 2 \cdot 2}{4 \cdot 3 \cdot 2} = \frac{2^4}{4!} \\ 5a_5 &= 2a_4 = 2 \cdot \frac{2 \cdot 2 \cdot 2 \cdot 2}{4 \cdot 3 \cdot 2}, & a_5 &= \frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{5 \cdot 4 \cdot 3 \cdot 2} = \frac{2^5}{5!} \end{aligned}$$

so that clearly the general term is

$$a_n = \frac{2^n}{n!}$$

Substituting into (9.60) thus yield the power series of the solution

$$y = 1 + 2x + \frac{2^2}{2!}x^2 + \ldots + \frac{2^n}{n!}x^n + \ldots \tag{9.61}$$

Check your Reading *Explain why (9.61) is the same as $y = e^{2x}$.*

Patterns Requiring Several Terms

Often, the pattern may not hold for the first few coefficients, so that several coefficients are required before a pattern is observed.

EXAMPLE 2 Find a power series solution to

$$y' - y = x, \quad y(0) = 1 \tag{9.62}$$

Solution: To do so, we construct the combination $y' - y$ and set it equal to $x = 0 + 1 \cdot x + 0x^2 + 0x^3 + \ldots$, which we place in the last row below the double line:

$$\begin{array}{ccccccccccccc} y' & & a_1 & + & 2a_2\,x & + & 3a_3\,x^2 & + & 4a_4\,x^3 & + & 5a_5\,x^4 & + & \ldots \\ -y & & -a_0 & - & a_1\,x & - & a_2\,x^2 & - & a_3\,x^3 & - & a_4\,x^4 & - & \ldots \\ \hline\hline x & = & 0 & + & 1\,x & + & 0\,x^2 & + & 0\,x^3 & + & 0\,x^4 & + & \ldots \end{array}$$

Since $a_0 = y(0) = 1$, the first column after the equals sign thus implies that

$$a_1 - a_0 = 0, \qquad a_1 - 1 = 0, \qquad a_1 = 1$$

The column of coefficients of x implies that

$$2a_2 - a_1 = 1, \qquad 2a_2 - 1 = 1, \qquad a_2 = \frac{2}{2}$$

and the column of coefficients of x^2 implies that

$$3a_3 - a_2 = 0, \quad 3a_3 - \frac{2}{2} = 0, \qquad a_3 = \frac{2}{3 \cdot 2}$$

Moreover, the next two columns of coefficients imply that

$$\begin{array}{rcl} 4a_4 & = & a_3 = \dfrac{2}{3 \cdot 2}, \qquad a_4 = \dfrac{2}{4 \cdot 3 \cdot 2} = \dfrac{2}{4!} \\ 5a_5 & = & a_4 = \dfrac{2}{4 \cdot 3 \cdot 2}, \qquad a_5 = \dfrac{2}{5 \cdot 4 \cdot 3 \cdot 2} = \dfrac{2}{5!} \end{array}$$

Thus, the general term is $a_n = \frac{2}{n!}$ and the solution to (9.62) is

$$y = 1 + x + \frac{2}{2!}x^2 + \frac{2}{3!}x^3 + \ldots + \frac{2}{n!}x^n + \ldots \tag{9.63}$$

Notice however, that the first two terms do not fit the pattern.

When finding a power series solution to a second order equation, it is not at all uncommon for many terms to be required before the pattern becomes apparent.

EXAMPLE 3 Find the power series solution to

$$y'' - 2y' + y = 1, \quad y(0) = 2, \quad y'(0) = 1 \tag{9.64}$$

Solution: We begin by writing the solution as

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots$$

where $a_0, a_1, a_2, \ldots$ are to be determined. The first and second derivatives are thus

$$\begin{aligned} y' &= a_1 \quad + 2a_2 x \quad + 3a_3 x^2 \quad + 4a_4 x^3 \quad + 5a_5 x^4 \quad + 6a_6 x^5 + \ldots \\ y'' &= 2a_2 + 3\cdot 2a_3 x + 4\cdot 3a_4 x^2 + 5\cdot 4a_5 x^3 + 6\cdot 5a_6 x^4 + 7\cdot 6a_7 x^5 + \ldots \end{aligned}$$

We then construct the combination $y'' - 2y' + y$ and set it equal to $1 = 1 + 0x^2 + 0x^3 + \ldots$, which is in the last row of the table.

y''		$2a_2$	$+$	$3\cdot 2a_3\,x$	$+$	$4\cdot 3a_4\,x^2$	$+$	$5\cdot 4a_5\,x^3$	$+$	$\ldots$
$-2y'$		$-2a_1$	$-$	$2\cdot 2a_2\,x$	$-$	$2\cdot 3a_3\,x^2$	$-$	$2\cdot 4a_4\,x^3$	$-$	$\ldots$
y		a_0	$+$	$a_1\,x$	$+$	$a_2\,x^2$	$+$	$a_3\,x^3$	$+$	$\ldots$
1	$=$	1	$+$	$0\,x$	$+$	$0\,x^2$	$+$	$0\,x^3$	$+$	$\ldots$

The initial conditions $y(0) = 2$ and $y'(0) = 1$ imply that $a_0 = 2$ and $a_1 = 1$. The first column after the equals sign thus implies that

$$2a_2 - 2a_1 + a_0 = 1, \qquad 2a_2 - 2 + 2 = 1, \qquad a_2 = \frac{1}{2}$$

The column of x coefficients then implies that

$$3\cdot 2a_3 - 2\cdot 2a_2 + a_1 = 0, \qquad 3\cdot 2a_3 - 2\cdot 2\cdot \frac{1}{2} + 1 = 0, \qquad a_3 = \frac{1}{3\cdot 2}$$

and the column of x^2 coefficients implies that

$$4\cdot 3a_4 - 2\cdot 3a_3 + a_2 = 0, \qquad 4\cdot 3a_4 - 2\cdot 3\cdot \frac{1}{3\cdot 2} + \frac{1}{2} = 0, \qquad a_4 = \frac{1}{4\cdot 3\cdot 2}$$

It thus appears that $a_n = 1/n!$, which we can verify with the column of x^3 coefficients:

$$5\cdot 4a_5 - 2\cdot 4a_4 + a_3 = 0, \qquad 5\cdot 4a_5 - \frac{2\cdot 4}{4\cdot 3\cdot 2} + \frac{1}{3\cdot 2} = 0, \qquad a_5 = \frac{1}{5\cdot 4\cdot 3\cdot 2}$$

That is, we have $a_n = 1/n!$, so that the solution to (9.64) is

$$y = 2 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots \tag{9.65}$$

Notice, however, that $a_0 = 2$ does not fit our general pattern of $a_n = 1/n!$.

Check your Reading *Explain why (9.65) implies that $y = e^x + 1$ is a solution to (9.64).*

Binomial Expansions

If it is known that a certain function $y = f(x)$ is the only solution to a given initial value problem, then the power series solution of the initial value problem will be the MacLaurin's series of $y = f(x)$. To illustrate, consider that if $y = (1+x)^\alpha$, then

$$y' = \alpha (1+x)^{\alpha - 1}$$

If we now multiply both sides by $(1+x)$, we obtain

$$(1+x)\,y' = \alpha (1+x)^\alpha$$

and since $(1+x)^\alpha = y$, the Maclaurin's series expansion of $(1+x)^\alpha$ is the power series solution of

$$y' + xy' = \alpha y, \qquad y(0) = 1 \tag{9.66}$$

In the exercises it is shown that (9.66) has only one solution for each α. Consequently, the solution to (9.66) is the MacLaurin's series expansion of $(1+x)^\alpha$ and are called *Binomial series* expansions since they generalize the binomial expansion of $(1+x)^n$ when n is an integer.

EXAMPLE 4 Find the binomial expansion of $y = (1+x)^2$ by constructing the power series solution to the differential equation

$$y' + xy' = 2y, \quad y(0) = 1 \tag{9.67}$$

Solution: To do so, we construct the sum $y' + xy'$, being careful to place like powers in the same column, and then set it equal to $2y$:

$$\begin{array}{ccccccccccccc} y' & & a_1 & + & 2a_2\,x & + & 3a_3\,x^2 & + & 4a_4\,x^3 & + & 5a_5\,x^4 & + & \ldots \\ +xy' & & & & a_1\,x & + & 2a_2\,x^2 & + & 3a_3\,x^3 & + & 4a_4\,x^4 & + & \ldots \\ \hline 2y & = & 2a_0 & + & 2a_1\,x & + & 2a_2\,x^2 & + & 2a_3\,x^3 & + & 2a_4\,x^4 & + & \ldots \end{array}$$

The columns of coefficients then yield the following equations:

$$a_1 = 2a_0, \qquad 2a_2 + a_1 = 2a_1, \qquad 3a_3 + 2a_2 = 2a_2, \qquad 4a_4 + 3a_3 = 2a_3$$

Since $a_0 = 1$, the first equation tells us that $a_1 = 2a_0 = 2$. In fact, we have

$$\begin{array}{lll} a_1 = 2a_0 = 2, & & a_1 = 2 \\ 2a_2 + a_1 = 2a_1, & 2a_2 = a_1, & a_2 = 1 \\ 3a_3 + 2a_2 = 2a_2 & 3a_3 = 0 & a_3 = 0 \\ 4a_4 + 3a_3 = 2a_3 & 4a_4 = -a_3 & a_4 = 0 \end{array}$$

so that likewise, $a_5 = a_6 = \ldots = 0$. Thus, the solution to (9.67) is

$$y = 1 + 2x + 1x^2 + 0 + 0 + 0 + \ldots$$

which is the same as

$$y = x^2 + 2x + 1$$

Check your Reading *How is $y = (1+x)^2$ related to the solution to (9.67).*

The Binomial Series

Let's finish by developing the Binomial series for arbitrary α. To begin with, (9.66) says that $y = (1+x)^\alpha$ is the solution to

$$y' + xy' = \alpha y, \qquad y(0) = 1$$

To solve this equation, we set $y' + xy'$ equal to αy and match coefficients:

$$\begin{array}{ccccccccccccc} y' & & a_1 & + & 2a_2\,x & + & 3a_3\,x^2 & + & 4a_4\,x^3 & + & 5a_5\,x^4 & + & \ldots \\ +xy' & & & & a_1\,x & + & 2a_2\,x^2 & + & 3a_3\,x^3 & + & 4a_4\,x^4 & + & \ldots \\ \hline \alpha y & = & \alpha a_0 & + & \alpha a_1\,x & + & \alpha a_2\,x^2 & + & \alpha a_3\,x^3 & + & \alpha a_4\,x^4 & + & \ldots \end{array}$$

The columns of coefficients thus imply the following equations:

$$a_1 = \alpha a_0, \qquad 2a_2 + a_1 = \alpha a_1, \qquad 3a_3 + 2a_2 = \alpha a_2, \qquad 4a_4 + 3a_3 = \alpha a_3$$

Since $a_0 = 1$, the first equation tells us that $a_1 = \alpha a_0 = \alpha$. In fact, we have

$$\begin{array}{lll} a_1 = \alpha a_0 = \alpha, & & a_1 = \alpha \\ 2a_2 + a_1 = \alpha a_1, & 2a_2 = (\alpha - 1) a_1, & a_2 = \dfrac{\alpha(\alpha-1)}{2} \\ 3a_3 + 2a_2 = \alpha a_2 & 3a_3 = (\alpha - 2) a_2, & a_3 = \dfrac{\alpha(\alpha-1)(\alpha-2)}{3 \cdot 2} \end{array}$$

As a result, the binomial series expansion is given by

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \tag{9.68}$$

for *any* real number α and for all x in $(-1, 1)$. Moreover, if we let

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdot \ldots \cdot(\alpha-n+1)}{n!}$$

then the binomial series can be written in the form

$$(1+x)^{\alpha} = 1 + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \ldots + \binom{\alpha}{n}x^n + \ldots$$

EXAMPLE 5 Use (9.68) to expand $(1+x)^2$.

Solution: If $\alpha = 2$, then (9.68) becomes

$$\begin{aligned} (1+x)^2 &= 1 + 2x + \frac{2(2-1)}{2!}x^2 + \frac{2(2-1)(2-2)}{3!}x^3 + \ldots \\ &= 1 + 2x + x^2 + 0 + 0 + \ldots \end{aligned}$$

EXAMPLE 6 Use (9.68) to expand $(1+x)^{-1}$.

Solution: If $\alpha = -1$, then (9.68) becomes

$$\begin{aligned} (1+x)^{-1} &= 1 - x + \frac{-1(-1-1)}{2!}x^2 + \frac{-1(-1-1)(-1-2)}{3!}x^3 + \ldots \\ &= 1 - x + \frac{1 \cdot 2}{2!}x^2 - \frac{1 \cdot 2 \cdot 3}{3!}x^3 + \ldots \\ &= 1 - x + x^2 - x^3 + \ldots \end{aligned}$$

In fact, the binomial theorem (9.68) was discovered by Sir Isaac Newton in his quest to extend binomial expansions with Pascal's triangle to negative and fractional exponents. Indeed, his discovery of the binomial theorem was one of his breakthroughs on the way to developing the new field of calculus.

Exercises:

Find a power series solution of the given differential equation.

1. $y' = 3y, \quad y(0) = 2$
2. $y' = -y, \quad y(0) = 3$
3. $y' = y - 1, \quad y(0) = 0$
4. $y' = 2y + 2, \quad y(0) = 2$
5. $y' = x + y, \quad y(0) = 2$
6. $y' = x - y, \quad y(0) = 1$
7. $y' = -2xy, \quad y(0) = 3$
8. $y' = x^2 y, \quad y(0) = 2$
9. $y' = xy + 1, \quad y(0) = 2$
10. $y' = x^2 y - 1, \quad y(0) = 2$
11. $y' + xy = -y, \quad y(0) = 1$
12. $y' + xy = y, \quad y(0) = 1$
13. $y'' - y = 0, \quad y(0) = 1, \quad y'(0) = 1$
14. $y'' + y = 0, \quad y(0) = 1, \quad y'(0) = 0$
15. $y'' - y = x, \quad y(0) = 1, \quad y'(0) = 0$
16. $y'' + y = x, \quad y(0) = 1, \quad y'(0) = 0$
17. $y'' - x^2 y = 0, \quad y(0) = 1, \quad y'(0) = 0$
18. $y'' + x^2 y = 0, \quad y(0) = 1, \quad y'(0) = 0$

Find the binomial series expansion of $(1+x)^\alpha$ by solving the given differential equation: Then verify your answer using the expansion (9.68).

19. Of $(1+x)^3$ by solving $y' + xy' = 3y, \quad y(0) = 1$

20. Of $(1+x)^4$ by solving $y' + xy' = 4y, \quad y(0) = 1$

21. Of $(1+x)^{-1}$ by solving $y' + xy' = -y, \quad y(0) = 1$

22. Of $(1+x)^{-2}$ by solving $y' + xy' = -2y, \quad y(0) = 1$

23. Of $(1+x)^{1/2}$ by solving $y' + xy' = \frac{1}{2}y, \quad y(0) = 1$

24. Of $(1+x)^{-1/2}$ by solving $y' + xy' = \frac{-1}{2}y, \quad y(0) = 1$

25. Of $(1-x)^{-1}$ by solving $y' - xy' = -y, \quad y(0) = 1$

26. Of $(1-x)^2$ by solving $y' - xy' = 2y, \quad y(0) = 1$

27. Use the binomial series to find the first four terms of the MacLaurin's series expansion of

$$\frac{1}{\sqrt{1-x^2}}$$

and then integrate to find the first four terms of the MacLaurin's series expansion of

$$\sin^{-1}(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}}$$

28. Use a binomial series to find the first four terms of Maclaurin's series of

$$f(x) = \sqrt{R^2 - x^2}$$

(Hint: First show that

$$f(x) = R\left(1 - \left(\frac{x}{R}\right)^2\right)^{1/2}$$

29. Find the power series solution to

$$y' = y + x, \quad y(0) = 1$$

and then find its interval of convergence.

30. Find the power series solution to

$$y' = y + x, \quad y(0) = 1$$

and then find its interval of convergence.

31. Show that for every number k, the function $y = kx$ is a solution to

$$xy' = y, \quad y(0) = 0$$

Which solution does the power series solution represent?

32. Show that for every number k, the function $y = kx^2$ is a solution to

$$xy' = 2y, \quad y(0) = 0$$

Which solution does the power series solution represent?

33. Show that for every number k, the function $y = kx^2 + 1$ is a solution to

$$xy' = 2y - 2, \quad y(0) = 1$$

Which solution does the power series solution represent?

34. Show that for every number k, the function $y = x^3 + kx^2 + 1$ is a solution to

$$xy'' = y' + 3x^2, \quad y(0) = 1, \quad y'(0) = 0$$

Which solution does the power series solution represent?

35. * The binomial coefficients have many interesting properties. For example,

(a) Show that if α is a positive integer and $n > \alpha$, then

$$\binom{\alpha}{n} = 0$$

(b) In Pascal's triangle, a given binomial coefficient in a given row is the sum of two binomial coefficients in the previous row. Symbolically, this is equivalent to

$$\binom{\alpha+1}{n+1} = \binom{\alpha}{n} + \binom{\alpha}{n+1} \tag{9.69}$$

since α is the row index in Pascal's triangle if it is a positive integer. Show that (9.69) is true even if α is not a positive integer (Hint: compute $\binom{\alpha+1}{n+1} - \binom{\alpha}{n+1}$)

36. Apply the ratio test to (9.68) to show the following:

(a) If α is not a positive integer, then the interval of convergence for the binomial series is $(-1, 1)$.

(b) If α is a positive integer, then the binomial series converges for all x.

37. In this exercise, we show that if $g(x)$ is continuous at $x = 0$, then

$$y' = \alpha g(x)\, y, \quad y(0) = 1 \tag{9.70}$$

has only one solution:

(a) Suppose u, v are both solutions to (9.70) and let $h = \frac{u}{v}$. Compute h' using the quotient rule and then use (9.70) to show that $h' = 0$

(b) Explain why h is constant and then use the initial condition in (9.70) to determine the value of that constant. Why does this imply that $u = v$?

38. Why does the result in the previous exercise imply that $y = (1+x)^{\alpha}$ is the only solution to

$$xy' + y' = \alpha y, \quad y(0) = 1$$

Why does this imply that (9.68) is true?

Exercises 39-43 explore orthogonal polynomials and special functions

39. The *Hermite polynomials,* which are important in quantum mechanics, are solutions to

$$y'' - 2xy' + 2py = 0, \qquad y(0) = 0, \qquad y'(0) = 1 \tag{9.71}$$

when p is a non-negative integer.

(a) Solve (9.71) when $p = 2$.

(b) Solve (9.71) when $p = 5$

40. The *Chebyshev's polynomials,* which are important in numerical analysis, are solutions to

$$\left(1 - x^2\right) y'' - xy' + p^2 y = 0, \quad y(0) = 1, \quad y'(0) = 0 \tag{9.72}$$

when p is a non-negative integer.

(a) Solve (9.72) when $p = 2$?

(b) Solve (9.72) when $p = 3$?

41. The *Legendre polynomials,* which are important in spherical harmonics, are solutions to

$$\left(1 - x^2\right) y'' - 2xy' + p(p+1)y = 0, \quad y(0) = 1, \quad y'(0) = 0 \tag{9.73}$$

when p is a non-negative integer.

(a) Solve (9.73) when $p = 1$?

(b) Solve (9.73) when $p = 2$?

42. Use a power series to find a solution to *Airy's Equation*

$$y'' + xy = 0, \qquad y(0) = 1, \quad y'(0) = 0$$

43. The *confluent hypergeometric function* is the solution to

$$xy'' + (\gamma - x)y' - \alpha y = 0, \qquad y(0) = 1, \quad y'(0) = \frac{\alpha}{\gamma}$$

Find the Maclaurin's series expansion of the confluent hypergeometric function by solving the differential equation with a power series.

44. The oscillations of a vibrating membrane (such as the top of a drum) are not harmonic oscillations, but are instead oscillations which satisfy *Bessel's equation,*

$$x^2y'' + xy' + \left(x^2 - p^2\right) y = 0, \quad y\left(0\right) = 1, \quad y'\left(0\right) = 0$$

for a non-negative constant p.

(a) Show that the Bessel function $J_0\left(x\right)$ can be written as

$$J_0\left(x\right) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots + \frac{\left(-1\right)^n x^{2n}}{2^2 \cdot 4^2 \cdot \ldots \cdot \left(2n\right)^2} + \ldots$$

(b) Show that $J_0\left(x\right)$ as shown in (b) is a solution to Bessel's equation when $p = 0$.

Self Test

A variety of questions are asked in a variety of ways in the problems below. Answer as many of the questions below as possible before looking at the answers in the back of the book.

1. Answer each statement as true or false. If the statement is false, then supply an appropriate counterexample or explanation.

   (a) In general $T_n(x)$ for a specific function $f(x)$ differs according to the number p at which T_n is centered.

   (b) If $f(x) = ax^3 + bx + 1$, then $T_3(x) = f(x)$ regardless of the number p at which T_3 is centered.

   (c) The second Taylor approximation of $f(x) = 4x^5 + 3x^3 + 2x^2 - 1$ at $p = 0$ is $T_2(x) = 2x^2 - 1$.

   (d) The linearization $L_p(x)$ of a function at an input p is a Taylor polynomial approximation of that function at p.

   (e) If $T_n(x)$ is a family of Taylor approximations to $f(x)$ centered at $x = p$, then $f(0) = T_n(0)$ for all positive integers n.

   (f) Taylor's theorem allows one to approximate a bound for the error in approximating a function $f(x)$ by its Taylor polynomial $T_n(x)$ in an interval $[a, b]$.

   (g) An estimate derived from a Taylor expansion for $\int_0^1 e^{-x^2}dx$ is given by $1 - \frac{1}{3} + \frac{1}{12} - \frac{1}{42}$.

   (h) If a power series has a finite interval of convergence (a, b) then it will diverge for all $x > b$.

   (i) $1 + 2(x-1) + 3(x-1)^2 + ... + (n+1)(x-1)^n + ...$ is a geometric series.

   (j) If every derivative of $f(x)$ exists at a number p, then the power series expansion of $f(x)$ at p must the same as $f(x)$ on some neighborhood of p.

   (k) Considering the binomial series for $(1+x)^{3/2}$, the coefficient in the term for x^3 is $\frac{-1}{16}$.

   (l) Every power series converges for at least at one value of x.

2. If $T_n(x)$ denotes the n^{th} Taylor polynomial approximation of a function $f(x)$ at a number p and if

$$T_3(x) = T_4(x) = \ldots = T_n(x) = \ldots$$

   for all numbers x, then which of the following must be true:

   (a) $f(x)$ is a polynomial with degree no more than 3.

   (b) $f(x)$ is a polynomial with degree no less than 3.

   (c) $T'''(p) = 0$.

   (d) $f(x)$ is an exponential function.

3. Find the MacLaurin series for the function $f(x) = \cos(2x)$

   (a) $1 - 4x^2 + 16x^4 - 64x^6 + \ldots$ (b) $1 - x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots$

   (c) $2 - 2x^2 + \frac{2}{4!}x^4 - \frac{2}{6!}x^6 + \ldots$ (d) $1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \ldots$

4. For what values of x does $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$ converge?

(a)	x in $(-1, 1)$	(b)	x in $(-2, 2)$
(c)	x in $(-\infty, \infty)$	(d)	$x = 0$ only

5. Find the Taylor series generated by $f(x) = e^x$ at $x = -1$.

(a) $e^{-1} + e^{-1}(x-1) + \frac{e^{-1}}{2}(x-1)^2 + \ldots$ (b) $e^{-1} + e^{-1}(x+1) + \frac{e^{-1}}{2}(x+1)^2 + \ldots$

(c) $e + e(x+1) + \frac{e}{2}(x+1)^2 + \ldots$ (d) $e^{-1} + e^{-1}(x-1) + e^{-1}(x-1)^2 + \ldots$

6. Find $T_2(x)$, the quadratic approximation, for the solution to

$$y' = y^2 + e^x, \quad y(0) = 2$$

7. Find $T_3(x)$ for the function $f(x) = \ln(x)$ centered at $p = 1$

8. Estimate the error bound in approximating $f(x) = x^{-3/2}$ by $L_1(x) = \frac{-3}{2}x + \frac{5}{2}$ on $[0.8, 1.2]$.

9. Find the Maclaurin series and interval of convergence of the function

$$1 + \frac{x}{1-x}$$

10. Find the open interval of convergence for the series

$$\sum_{n=0}^{\infty} \frac{2n+1}{n!} x^n$$

11. Find the Taylor series expansion at $p = 1$ for the function

$$f(x) = \frac{1}{2-x}$$

What is the open interval of convergence of the series?

12. Find the power series representation for the solution to

$$y' = 2y + 1, y(0) = 1$$

13. Find the first five terms in the binomial expansion of $(1-x)^{5/3}$ using formula 8.55.

14. Find the Maclaurin series expansion of the function

$$f(x) = \frac{1}{1 + \frac{1}{1+x}}$$

15. Find the power series solution to

$$(1-x)^2 y'' = 2y, \; y(0) = 1, y'(0) = 1$$

and then rewrite the solution in closed form.

Would you believe that nearly every one of the Maclaurin's series introduced thus far can be considered a special case of a single MacLaurin's series? Our next step is to introduce a power series formula which can be used to study practically all the power series expansions of the elementary functions, and then some.

To begin with, however, let us introduce some new notation. Let us let $(a)_n$ denote

$$(a)_n = a(a+1)\cdot\ldots\cdot(a+n-1)$$

Clearly, $(a)_n$ is a generalization of the factorial, since

$$(1)_n = 1\cdot 2\cdot\ldots\cdot n = n!$$

For example, $(2)_n = 2\cdot 3\cdot\ldots\cdot(n+1) = (n+1)!$ and in general

$$(k)_n = (n+k)!$$

Since $(a)_n$ is a generalization of the factorial, the *hypergeometric series*

$$F(x;a,b,c) = 1 + \frac{ab}{c}x + \frac{a(a+1)\cdot b(b+1)}{c(c+1)}\frac{x^2}{2!} + \ldots + \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!} + \ldots \quad (9.74)$$

generalizes many of the series we have seen in this chapter (note: c cannot be a negative integer). For example, the geometric series is represented by $F(x;1,b,b)$ for any number b, since

$$\begin{aligned} F(x;1,b,b) &= 1 + \frac{b}{b}x + \frac{1(2)\cdot b(b+1)}{b(b+1)}\frac{x^2}{2!} + \ldots + \frac{(1)_n(b)_n}{(b)_n}\frac{x^n}{n!} + \ldots \\ &= 1 + x + \frac{2!}{1}\frac{x^2}{2!} + \ldots + \frac{n!}{1}\frac{x^n}{n!} + \ldots \\ &= 1 + x + x^2 + \ldots + x^n + \ldots \end{aligned}$$

Similarly, if $a = 2$, then for any number b we have

$$\begin{aligned} F(x;2,b,b) &= 1 + \frac{2b}{b}x + \frac{2\cdot 3}{1}\frac{x^2}{2!} + \frac{2\cdot 3\cdot 4}{1}\frac{x^3}{3!} + \frac{2\cdot 3\cdot 4\cdot 5}{1}\frac{x^4}{4!} + \ldots \\ &= 1 + \frac{2b}{b}x + \frac{2!\cdot 3}{1}\frac{x^2}{2!} + \frac{3!\cdot 4}{1}\frac{x^3}{3!} + \frac{4!\cdot 5}{1}\frac{x^4}{4!} + \ldots \\ &= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots \end{aligned}$$

Consequently, $F(x;2,b,b)$ converges to the derivative of the geometric series:

$$F(x;2,b,b) = \frac{d}{dx}\left(1 + x + x^2 + x^3 + x^4 + x^5 + \ldots\right) = (1-x)^{-2}$$

Indeed, it is easy to show that

$$(1+x)^\alpha = F(-x;-\alpha,b,b)$$

so that the hypergeometric series includes the binomial series.

In fact, it is straightforward to show that

$$\begin{aligned} \ln(1-x) &= -xF(x;1,1,2) \\ \sin^{-1}(x) &= xF\left(-x^2;\frac{1}{2},1,\frac{3}{2}\right) \\ \tan^{-1}(x) &= xF\left(-x^2;\frac{1}{2},1,\frac{3}{2}\right) \end{aligned}$$

Moreover, the *confluent hypergeometric series* is derived from the hypergeometric series and is of the form

$$\Phi(x;a,c) = 1 + \frac{a}{c}x + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2!} + \ldots + \frac{(a)_n}{(c)_n}\frac{x^n}{n!} + \ldots$$

If we let $a = c$ in $\Phi(x;a,c)$, then we obtain

$$\begin{aligned}\Phi(x;a,a) &= 1 + \frac{a}{a}x + \frac{a(a+1)}{a(a+1)}\frac{x^2}{2!} + \ldots + \frac{(a)_n}{(a)_n}\frac{x^n}{n!} + \ldots \\ &= 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots\end{aligned}$$

That is, for any nonzero a, we have $\Phi(x;a,a) = e^x$. Moreover, the trigonometric and hyperbolic functions can also be represented with confluent hypergeometric series.

Write to Learn Write a short essay in which you show that applying the ratio test to the hypergeometric series reduces to

$$\rho = \lim_{n\to\infty}\left|\frac{(a+n)(b+n)}{(c+n)(n+1)}x\right|$$

and then explains why the open interval of convergence of the hypergeometric series is $(-1,1)$.

Write to Learn Write a short essay in which you use the ratio test to show that $\Phi(x;a,c)$ converges for all x.

Write to Learn Write a short essay in which you show that

$$\frac{d}{dx}F(x;a,b,c) = \frac{ab}{c}\,F(x;a+1,b+1,c+1)$$

and then explain why

$$\frac{d^m}{dx^m}F(x;a,b,c) = \frac{(a)_m(b)_m}{(c)_m}\,F(x;a+m,b+m,c+m)$$

Write to Learn Search the library and/or the internet for information about the hypergeometric series. You might be especially interested in their use in probability and statistics. Write a formal report in which you report your results.

Group Learning The Hypergeometric and confluent hypergeometric functions not only generalize MacLaurin's series of elementary functions, but they also generalize Pascal's triangle. Have each member of the group determine the hypergeometric function $F(x;-n,1,1)$ for a different value of n (e.g., $n = 2,3,4,5$). Present these examples to your class along with the connection between hypergeometric functions and Pascal's triangle.

Advanced Contexts:

Finally, Taylor's series are not the only series used to represent functions. There are many other types of series which are important in mathematics and science. For example, recall that the Riemann zeta function is defined to be

$$\zeta(x) = 1 + \frac{1}{2^x} + \frac{1}{3^x} + \ldots + \frac{1}{n^x} + \ldots$$

and notice that we will study *Fourier series* in the next section.

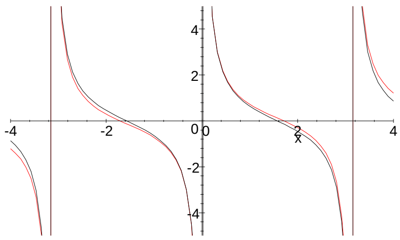
Indeed, functions with vertical asymptotes are often better represented by a series of rational functions. For example, it can be shown that

$$\cot(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2 - n^2\pi^2}$$

which is called the *partial fraction series* expansion of $\cot(x)$. The graph of $\cot(x)$ in black and

$$s_2(x) = \frac{1}{x} + \frac{2x}{x^2 - \pi^2} + \frac{2x}{x^2 - 4\pi^2}$$

in red are graphed in figure NS-1.



NS-1: Graph of function and a partial fraction approximation

1. Graph the function

$$s_5(x) = \frac{8x}{\pi^2 - 4x^2} + \frac{8x}{9\pi^2 - 4x^2} + \frac{8x}{25\pi^2 - 4x^2}$$

over the interval $[-5, 5]$. (You may want to restrict the y-range to $[-10, 10]$) What trigonometric function does it seem to be approximating?

2. * It can be shown that the sine function can be represented by the infinite product

$$\sin(x) = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\left(1 - \frac{x^2}{16\pi^2}\right)\cdots$$

(a) Show that the infinite product implies that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

(b) Find at least one other property of the infinite product that connects it to the sine function.

3. * Use the infinite product to find an infinite series representation of $\ln|\sin(x)|$. Then compute the derivative of the resulting series to recover the partial fraction series for $\cot(x)$.

10. FOURIER SERIES

This chapter is our *capstone chapter* for single variable calculus. We do not develop any new concepts in calculus, but instead, we develop and explore an application of calculus that is at the heart of modern mathematics and science—*Fourier Analysis* (pronounced "Four–ee–ay). In particular, we will explore how a function defined on an interval can be decomposed into an infinite series of sines and cosines, and then we will explore how such decompositions are used in applications such as signal processing, filter design, and optics.

That is, in this our capstone chapter, we will use our knowledge of calculus as a tool for discovery, synthesis, analysis, and design. In the process, our knowledge of calculus will be stengthened as we gain valuable practice and experiences with its concepts, precepts, and techniques.

10.1 Fourier Coefficients

Fourier Cosine Series

Wave motion can often be decomposed into a possibly infinite combination of harmonic oscillations. To illustrate, consider that the "A" below middle "C" on a piano corresponds to a frequency of 440 cycles per second. However, when we strike the "A" note, we hear not only the *fundamental* or *first harmonic* of 440 hz, but we also hear the *second harmonic* of 880 hz, the *third harmonic* of $3 \cdot 440 = 1320$ hz, the fourth harmonic of $4 \cdot 440 = 1760$ hz, and so on. That is, the sound wave is a combination or *superposition* of its individual harmonics. Similarly, white light is the superposition of several different colors of light, as is evidenced by a rainbow.

Mathematically, the decomposition of waveforms is known as *Fourier Analysis,* in which we desire to decompose $f(x)$ defined on $[-\pi, \pi]$ into a *Fourier Series,* which is an infinite series of the form

$$\begin{array}{rl} a_0 + & a_1 \cos(x) + a_2 \cos(2x) + \ldots + a_n \cos(nx) + \ldots \\ + & b_1 \sin(x) + b_2 \sin(2x) + \ldots + b_n \sin(nx) + \ldots \end{array}$$

However, for simplicity we begin with two special classes of Fourier series. If $f(x)$ is an even function which is integrable on $[-\pi, \pi]$, then its Fourier series reduces to a *Fourier cosine series*

$$a_0 + a_1 \cos(x) + a_2 \cos(2x) + \ldots + a_n \cos(nx) + \ldots \tag{10.1}$$

where a_0 is called the *average value* of $f(x)$ over $[-\pi, \pi]$, and is defined

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx \tag{10.2}$$

and where the numbers a_n are the *Fourier cosine coefficients of* $f(x)$ and are defined

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \tag{10.3}$$

EXAMPLE 1 Find the Fourier cosine coefficients of the even function $f(x) = x^2$.

Solution: To begin with, since $f(x)$ is even, we can use the identity

$$\int_{-p}^{p} f(x)\,dx = 2\int_{0}^{p} f(x)\,dx \tag{10.4}$$

(see page 328 for details). To compute a_0, we use (10.2) with $f(x) = x^2$:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2}{2\pi}\int_{0}^{\pi} x^2 dx = \frac{1}{\pi}\frac{x^3}{3}\bigg|_0^{\pi} = \frac{\pi^2}{3}$$

To compute a_n, we first recognize that its integrand is even:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos(nx)\,dx \tag{10.5}$$

Integration by parts is often used in determining Fourier coefficients.

Integration by parts in the form of tabular integration then yields

u		dv
x^2		$\cos(nx)$
	$\searrow^{+}$	
$2x$		$\frac{1}{n}\sin(nx)$
	$\searrow^{-}$	
2		$\frac{-1}{n^2}\cos(nx)$
	$\searrow^{+}$	
0		$\frac{-1}{n^3}\sin(nx)$

As a result, the evaluation of the definite integral yields

$$\begin{aligned}\frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx &= \frac{2}{\pi}\left(\frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)\right)\bigg|_0^{\pi} \\ &= \frac{2}{\pi}\left(\frac{\pi^2}{n}\sin(n\pi) + \frac{2\pi}{n^2}\cos(n\pi) - \frac{2}{n^3}\sin(n\pi) - (0+0-0)\right)\end{aligned}$$

Since $\sin(n\pi) = 0$ and since $\cos(n\pi) = (-1)^n$, the Fourier cosine coefficients are

$$a_n = \frac{2}{\pi}\left(0 + \frac{2\pi}{n^2}(-1)^n - 0\right) = (-1)^n \frac{4}{n^2}$$

That is, $a_1 = (-1)^1 \frac{4}{1^2}$, $a_2 = (-1)^2 \frac{4}{2^2}$, and consequently, the Fourier series generated by $f(x) = x^2$ is

$$\frac{\pi^2}{3} - \frac{4}{1^2}\cos(x) + \frac{4}{2^2}\cos(2x) + \ldots + \frac{4(-1)^n}{n^2}\cos(nx) + \ldots$$

Check your Reading *Show that the integrand of a_n in (10.5) is even.*

Fourier Sine Series

Similarly, if $f(x)$ is an odd function which is integrable on $[-\pi, \pi]$, then its Fourier series reduces to a *Fourier sine series*

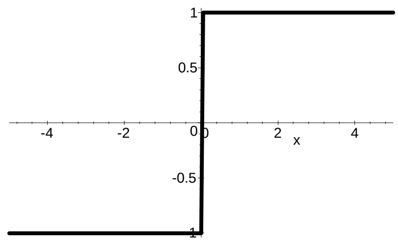
$$b_1 \sin(x) + b_2 \sin(2x) + \ldots + b_n \sin(nx) + \ldots \tag{10.6}$$

where the numbers b_n are called *Fourier sine coefficients* and are defined

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \tag{10.7}$$

EXAMPLE 2 Find the Fourier sine coefficients and the Fourier sine series of

$$f(x) = \begin{cases} -1 & if \quad x < 0 \\ 1 & if \quad x \geq 0 \end{cases}$$



Solution: If $f(x) = 1$, then $f(-x) = -1$ and vice versa, thus implying that $f(x)$ is an odd function. (Equivalently, $f(x)$ is odd because its graph is symmetric about the origin). As a result, its Fourier sine coefficients are

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

Since the product of two odd functions is even, the identity (10.4) implies that

The product of two odd functions is an even function.

$$b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin(nx)\, dx$$

Since $f(x) = 1$ if x is in $[0, \pi]$, we thus have

$$b_n = \frac{2}{\pi} \int_0^{\pi} \sin(nx)\, dx = \frac{-2}{\pi} \left(\left. \frac{\cos(nx)}{n} \right|_0^{\pi} \right) = \frac{2}{\pi} \left(\frac{-\cos(n\pi)}{n} - \frac{-\cos(0)}{n} \right)$$

Because $\cos(n\pi) = (-1)^n$ for every integer n, the Fourier sine coefficients are

$$b_n = \frac{2}{\pi} \frac{1 - (-1)^n}{n}$$

Thus, $b_1 = \frac{4}{\pi}$, $b_2 = 0$, $b_3 = \frac{4}{3\pi}$, etcetera, so that the Fourier sine series of $f(x)$ is

$$\frac{4}{\pi} \sin(x) + \frac{4}{3\pi} \sin(3x) + \ldots + \frac{2\left[1 - (-1)^n\right]}{n\pi} \sin(nx) + \ldots$$

EXAMPLE 3 Find the Fourier sine coefficients of $\sinh(x)$.

Solution: To begin with, $\sinh(x)$ and $\sin(nx)$ are both odd, so their product is even and thus

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sinh(x)\sin(nx)\,dx \tag{10.8}$$

If we use tabular integration, then we notice that the third row is also a $\sinh(x)$, $\sin(nx)$ combination.

u		dv	
$\sinh(x)$		$\sin(nx)$	
	$\searrow^{+}$		
$\cosh(x)$		$\frac{-1}{n}\cos(nx)$	
	$\searrow^{-}$		
$\sinh(x)$	$\xrightarrow{+}$	$\frac{-1}{n^2}\sin(nx)$	$\longleftarrow$same as $\frac{-1}{n^2}\int_0^{\pi}\sinh(x)\sin(nx)\,dx$

Fortunately, a product along a horizontal in tabular integration implies the integral of that product, so that

$$b_n = \frac{2}{\pi}\left(\frac{-1}{n}\sinh(x)\cos(nx)\big|_0^{\pi} + \frac{1}{n^2}\cosh(x)\sin(nx)\big|_0^{\pi} - \frac{1}{n^2}\int_0^{\pi}\sinh(x)\sin(nx)\,dx\right)$$

Since $\sinh(0) = 0$ and since $\sin(n\pi) = 0$ for all integers n, b_n simplifies to

$$b_n = \frac{-2}{n\pi}\sinh(\pi)\cos(n\pi) - \frac{1}{n^2}\frac{2}{\pi}\int_0^{\pi}\sinh(x)\sin(nx)\,dx$$

However, (10.8) implies that $b_n = \frac{2}{\pi}\int_0^{\pi}\sinh(x)\sin(nx)\,dx$, so that

$$b_n = \frac{-2}{n\pi}\sinh(\pi)\cos(n\pi) - \frac{1}{n^2}b_n$$

To solve for b_n, add the last term to both sides,

$$b_n + \frac{1}{n^2}b_n = \frac{-2}{n\pi}\sinh(\pi)\cos(n\pi)$$

factor out b_n on the left,

$$b_n\left(1 + \frac{1}{n^2}\right) = \frac{-2}{n\pi}\sinh(\pi)\cos(n\pi)$$

and then divide by the term in the parentheses:

$$b_n = \frac{1}{\left(1+\frac{1}{n^2}\right)}\frac{-2}{n\pi}\sinh(\pi)\cos(n\pi) = \frac{-2n\ (-1)^n}{\pi\left(n^2+1\right)}\sinh(\pi)$$

Check your Reading *Explain why we can also write b_n as*

$$b_n = \frac{2n\ (-1)^{n+1}}{\pi\left(n^2+1\right)}\sinh(\pi)$$

Fourier Series for Functions that are Neither Even nor Odd

If a function is neither even nor odd, then both the sine and cosine coefficients must be determined. Because of the amount of computation involved, a computer algebra system is often used when a functions is neither even nor odd.

EXAMPLE 4 Use a computer algebra system to determine the Fourier coefficients of $f(x) = e^x$.

Solution: The constant coefficient a_0 is straightforward:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x dx = \left.\frac{e^x}{2\pi}\right|_{-\pi}^{\pi} = \frac{e^{\pi} - e^{-\pi}}{2\pi}$$

The Fourier cosine coefficients are then given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos(nx)\, dx \tag{10.9}$$

Evaluating (10.9) with the computer algebra system Maple© yields

$$a_n = \frac{1}{\pi}\frac{e^{\pi}\cos n\pi + e^{\pi} n \sin n\pi - e^{-\pi}\cos n\pi + e^{-\pi} n \sin n\pi}{1+n^2}$$

We must now interpret the result in light of the fact that n is a positive integer.[1] In particular, $\cos(n\pi) = (-1)^n$ and $\sin(n\pi) = 0$ when n is a positive integer, so that

$$\begin{aligned} a_n &= \frac{1}{\pi}\frac{e^{\pi}(-1)^n + e^{\pi} n\,(\,0\,) - e^{-\pi}(-1)^n + e^{-\pi} n\,(\,0\,)}{1+n^2} \\ &= \frac{1}{\pi}\frac{e^{\pi}(-1)^n - e^{-\pi}(-1)^n}{1+n^2} \\ &= \frac{(-1)^n}{\pi}\frac{e^{\pi} - e^{-\pi}}{1+n^2} \end{aligned}$$

Check your Reading *Use a computer algebra system to show that the Fourier sine coefficients of* e^x *are*

$$b_n = \frac{(-1)^n}{\pi}\frac{n\left(e^{-\pi} - e^{\pi}\right)}{1+n^2}$$

Formal Derivation of the Coefficient Formulas

Finally, let us provide some justification for our definition of the Fourier sine and cosine coefficients. To do so, we will use the fact that

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\, dx = 0 \tag{10.10}$$

when $n \neq m$, but that

$$\int_{-\pi}^{\pi} \cos^2(nx)\, dx = \pi \quad \text{and} \quad \int_{-\pi}^{\pi} \sin^2(nx)\, dx = \pi \tag{10.11}$$

These follow immediately from the trig identities

$$2\cos(mx)\cos(nx) = \cos[(m-n)x] + \cos[(m+n)x] \tag{10.12}$$
$$2\sin(mx)\sin(nx) = \cos[(m-n)x] - \cos[(m+n)x] \tag{10.13}$$

[1] There are ways of telling Maple that n is a positive integer. See the manual for details.

To begin with, let us suppose that $f(x)$ is an even function which is integrable on $[-\pi, \pi]$ and which is actually equal to its Fourier cosine series:

$$f(x) = a_0 + a_1 \cos(x) + a_2 \cos(2x) + \ldots + a_n \cos(nx) + \ldots$$

To begin with, let us multiply $f(x)$ by $\cos(nx)$ for some fixed value of n:

$$f(x)\cos(nx) = a_0 \cos(nx) + a_1 \cos(x)\cos(nx) + \ldots + a_n \cos(nx)\cos(nx) + \ldots$$

If we now integrate from $-\pi$ to π, we obtain

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx &= a_0 \int_{-\pi}^{\pi} \cos(nx)\,dx + a_1 \int_{-\pi}^{\pi} \cos(x)\cos(nx)\,dx \\ &\quad + \ldots + a_n \int_{-\pi}^{\pi} \cos^2(nx)\ dx + \ldots\end{aligned}$$

Each of the integrals is 0 except for the one corresponding to a_n:

$$\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = 0 + 0 + \ldots + 0 + a_n \cdot \pi + 0 + \ldots$$

As a result, we have $a_n \cdot \pi = \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$, which yields the Fourier coefficient (10.3). Likewise, the identities (10.4) and (10.11) can be used to find a_0 and b_n. However, we leave these derivations to the exercises.

Exercises:

If the functions is even or odd, find its Fourier coefficients analytically without the aid of a computer algebra system. If the function is neither even nor odd, use a computer algebra system to determine its Fourier coefficients. Then write down the Fourier series generated by the coefficients.

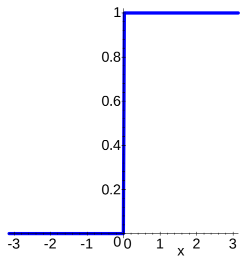
1. $f(x) = x$
2. $f(x) = \dfrac{|x|}{x}$
3. $f(x) = |x|$
4. $f(x) = x(\pi - |x|)$
5. $f(x) = 2$
6. $f(x) = 0$
7. $f(x) = x^2 + 1$
8. $f(x) = x^2 - 1$
9. $f(x) = \pi^2 - x^2$
10. $f(x) = \frac{\pi}{2} - |x|$
11. $f(x) = x^3$
12. $f(x) = x^4$
13. $f(x) = \begin{cases} -3 & if\ x < 0 \\ 3 & if\ x \geq 0 \end{cases}$
14. $f(x) = \begin{cases} 1 & if\ \frac{-\pi}{2} \leq x \leq \frac{\pi}{2} \\ 0 & otherwise \end{cases}$
15. $f(x) = \begin{cases} x & if\ \frac{-\pi}{2} \leq x \leq \frac{\pi}{2} \\ 0 & otherwise \end{cases}$
16. $f(x) = \begin{cases} 0 & if\ x < 0 \\ \sin(x) & if\ x \geq 0 \end{cases}$
17. $f(x) = e^{-x}$
18. $f(x) = e^{2x}$
19. $f(x) = |\sin(x)|$
20. $f(x) = |\cos(x)|$
21. $f(x) = \cosh(x)$
22. $f(x) = \sinh(2x)$
23. $f(x) = \cos\left(\frac{1}{2}x\right)$
24. $f(x) = \sin(\pi x)$
25. $f(x) = \begin{cases} 1 & if\ \frac{-\pi}{2} \leq x \leq \pi \\ 0 & otherwise \end{cases}$
26. $f(x) = \begin{cases} x + \pi & if\ x < 0 \\ x & if\ x \geq 0 \end{cases}$

27. Use numerical integration to estimate b_1, b_2 and b_3 for $f(x) = \tan(x/4)$.

28. Use numerical integration to estimate b_1, b_2 and b_3 for $f(x) = \tanh(x)$.

29. The *Heaviside step function* is defined

$$H(x) = \begin{cases} 0 & if \quad x \leq 0 \\ 1 & if \quad x > 0 \end{cases}$$



(a) Show that the Fourier cosine coefficients of $H(x)$ are

$$a_0 = \frac{1}{2}, \qquad a_n = 0$$

(b) Explain why the even sine coefficients vanish. That is, explain why $b_{2n} = 0$

(c) Explain why the odd sine coefficients are

$$b_{2n+1} = \frac{2}{(2n+1)\pi}$$

30. The *delta function* $\delta(x)$ is defined to be the "function" which is equal to 0 everywhere except at $x = 0$, and which satisfies

$$\int_a^b \delta(x) f(x)\, dx = f(0)$$

when 0 is in $[a, b]$. (Remark: $\delta(x)$ is not actually a function. It is a *distribution*, which is a generalization of the function concept.) Show that the Fourier coefficients of $\delta(x)$ are

$$a_0 = \frac{1}{2\pi}, \qquad a_n = \frac{1}{\pi}, \qquad b_n = 0$$

31. It can be shown that if $f(x)$ is continuous on $[-\pi, \pi]$ and if $f(-\pi) = f(\pi)$, then its Fourier coefficients vanish. Indeed, show that the amplitudes of $f(x) = |x|$ on $[-\pi, \pi]$ satisfy

$$\lim_{n\to\infty} a_n = 0$$

and explain why we would have expected this result in light of the comment above.

32. It can be shown that if $f(x)$ is differentiable on $[-\pi, \pi]$ and if $f(-\pi) = f(\pi)$ and $f'(-\pi) = f'(\pi)$, then its Fourier sine and cosine coefficients both satisfy

$$\lim_{n\to\infty} n\, a_n = 0 \qquad \lim_{n\to\infty} n\, b_n = 0$$

Show that this does not hold for the amplitudes of $f(x) = |x|$ and explain why not.

33. Show that if a is not an integer, then the Fourier series of $f(x) = \sin(ax)$ is

$$\frac{-2\sin(a\pi)}{\pi(a^2-1)}\sin(x) + \frac{4\sin(a\pi)}{\pi(a^2-4)}\sin(2x) - \frac{6\sin(a\pi)}{\pi(a^2-9)}\sin(3x) + \frac{8\sin(a\pi)}{\pi(a^2-16)}\sin(4x) - \ldots$$

34. **Computer algebra System:** In this exercise, we derive the Fourier coefficients of

$$f(x) = \sin^2(x)$$

(a) Show if $n \neq 0$ and $n \neq 2$, then $a_n = 0$.
(b) What is a_0?
(c) What is a_2?
(d) How is this related to a trig identity involving $\sin^2(x)$ and $\cos(2x)$?

35. **Computer algebra System:** In this exercise, we derive the Fourier coefficients of

$$f(x) = \cos^2(x)$$

(a) Show if $n \neq 0$ and $n \neq 2$, then $a_n = 0$.
(b) What is a_0?
(c) What is a_2?
(d) How is this related to a trig identity involving $\sin^2(x)$ and $\cos(2x)$?

36. **Computer algebra System:** In this exercise, we derive the Fourier coefficients of

$$f(x) = \sin^3(x)$$

(a) Show if $n \neq 1$ and $n \neq 3$, then $b_n = 0$.
(b) What is b_1?
(c) What is b_3?
(d) Use your computer algebra system to simplify $\sin^3(x)$ to an identity involving $\sin(x)$ and $\sin(3x)$? How is this related to (a)-(c)?

37. **Computer algebra System:** In this exercise, we derive the Fourier coefficients of

$$f(x) = \cos(x - \phi)$$

where ϕ is a constant.

(a) Show that $a_n = 0$ and $b_n = 0$ for all $n \neq 1$.
(b) What is a_1?
(c) What is b_1?
(d) How is (a)-(c) related to an identity involving $\cos(x - \phi)$?

38. Derive the identities in (10.10) using (10.12) and (10.13).

39. Derive the identities in (10.11) by letting $m = n$ in (10.12) and (10.13).

40. Assume that $f(x)$ is integrable on $[-\pi, \pi]$ and that

$$f(x) = a_0 + a_1\cos(x) + a_2\cos(2x) + \ldots + a_n\cos(nx) + \ldots$$

for all x in $[-\pi, \pi]$. Integrate both sides from $-\pi$ to π to derive (10.2).

41. Assume that $f(x)$ is integrable on $[-\pi, \pi]$ and that

$$f(x) = b_1\sin(x) + b_2\sin(2x) + \ldots + b_n\sin(nx) + \ldots$$

for all x in $[-\pi, \pi]$. Multiply both sides by $\sin(nx)$ for a fixed integer n and then integrate from $-\pi$ to π to derive (10.7).

10.2 Convergence of Fourier Series

Partial Sums of Fourier Series

Fourier series do not always converge, and even when they do converge, they do not always converge to the function that generated them. In this section, we explore the convergence of Fourier series so that we can better understand how and when these series converge.

To begin with, if $f(x)$ is an even function, then its *Fourier cosine series* over $[-\pi, \pi]$ is

$$a_0 + a_1 \cos(x) + a_2 \cos(2x) + \ldots + a_n \cos(nx) + \ldots \tag{10.14}$$

where a_n is the n^{th} Fourier cosine coefficient of $f(x)$. The n^{th} partial sum of the series is

$$s_n(x) = a_0 + a_1 \cos(x) + \ldots + a_n \cos(nx)$$

If the sequence of partial sums converges to $f(x)$ for some x in $[-\pi, \pi]$—that is, if

$$f(x) = \lim_{n\to\infty} [a_0 + a_1 \cos(x) + \ldots + a_n \cos(nx)]$$

then we say that the series *converges* to $f(x)$ at that value of x and we write

$$f(x) = a_0 + a_1 \cos(x) + \ldots + a_n \cos(nx) + \ldots \tag{10.15}$$

EXAMPLE 1 What are the partial sums of the Fourier series of $f(x) = x^2$ over $[-\pi, \pi]$?

Solution: In the last section, we showed that the Fourier series of the function $f(x) = x^2$ is

$$\frac{\pi^2}{3} - \frac{4}{1^2}\cos(x) + \frac{4}{2^2}\cos(2x) + \ldots + \frac{4(-1)^n}{n^2}\cos(nx) + \ldots \tag{10.16}$$

It follows that the partial sums are functions of x of the form

$$s_n(x) = \frac{\pi^2}{3} - \frac{4}{1^2}\cos(x) + \frac{4}{2^2}\cos(2x) + \ldots + \frac{4(-1)^n}{n^2}\cos(nx)$$

Likewise, the n^{th} partial sum of a *Fourier sine series* is

$$s_n(x) = b_1 \sin(x) + b_2 \sin(2x) + \ldots + b_n \sin(nx)$$

and if the sequence of partial sums converges to an odd function $f(x)$ for some x in $[-\pi, \pi]$—that is, if

$$f(x) = \lim_{n\to\infty} [b_1 \sin(x) + b_2 \sin(2x) + \ldots + b_n \sin(nx)]$$

then we say that the series *converges* to $f(x)$ at that value of x and we write

$$f(x) = b_1 \sin(x) + b_2 \sin(2x) + \ldots + b_n \sin(nx) + \ldots$$

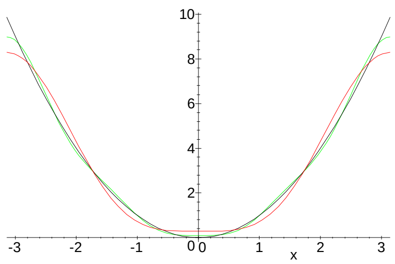
Moreover, by graphing the partial sums of a Fourier series, we can often determine visually which function the Fourier series is converging to.

EXAMPLE 2 Use graphs of the partial sums of $f(x) = x^2$ to determine what its Fourier series converges to over $[-\pi, \pi]$.

Solution: The second and fourth partial sums of $f(x) = x^2$ are

$$\begin{aligned} s_2(x) &= \frac{\pi^2}{3} - 4\cos(x) + \cos(2x) \\ s_4(x) &= \frac{\pi^2}{3} - 4\cos(x) + \cos(2x) - \frac{4}{9}\cos(3x) + \frac{1}{4}\cos(4x) \end{aligned}$$

They are shown versus $f(x) = x^2$ in black, with $s_4(x)$ being the better approximation.



Indeed, as n increases, the graph of $s_n(x)$ becomes more and more difficult to distinguish from the graph of $f(x) = x^2$. Thus, the Fourier series of $f(x) = x^2$ converges to $f(x) = x^2$ for all x in $[-\pi, \pi]$.

EXAMPLE 3 Use graphs of the partial sums of $f(x) = x^3 - \pi^2 x$ to determine what its Fourier series converges to over $[-\pi, \pi]$.

Solution: It can be shown that the Fourier series of $f(x) = x^3 - \pi^2 x$ is

$$-12\sin(x) + \frac{12}{2^3}\sin(2x) - \frac{12}{3^3}\sin(3x) + \frac{12}{4^3}\sin(4x) - \ldots$$

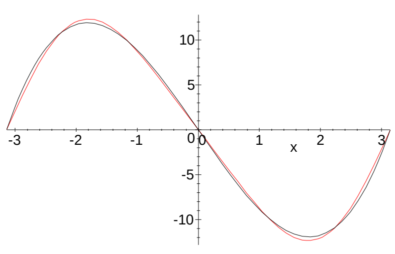
Thus, the second partial sum of $f(x) = x^3 - \pi^2 x$ is

$$s_2(x) = -12\sin(x) + \frac{12}{2^3}\sin(2x)$$

and the fourth partial sum is

$$s_4(x) = -12\sin(x) + \frac{12}{2^3}\sin(2x) - \frac{12}{3^3}\sin(3x) + \frac{12}{4^3}\sin(4x) \quad (10.17)$$

Shown below is the graph of $f(x)$ versus the graph of $s_2(x)$ in red.



and indeed, it can be shown that the Fourier series of $f(x) = x^3 - \pi^2 x$ does indeed converge to $f(x) = x^3 - \pi^2 x$ over $[-\pi, \pi]$.

Check your Reading *Graph $f(x) = x^3 - \pi^2 x$ and $s_4(x)$ as in (10.17) over $[-3.14, 3.14]$. Can you tell them apart?*

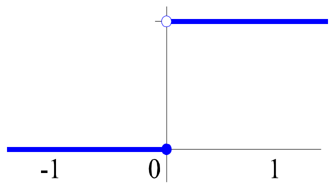
Statement of a Fourier Theorem

In order to understand when a Fourier series converges to the function that generated it, we need some additional definitions and notations. To begin with, we denote the limits from the left and right, respectively, by

$$f(p-) = \lim_{x \to p^-} f(x) \qquad f(p+) = \lim_{x \to p^+} f(x)$$

For example, the *Heaviside step function* is given by

$$H(x) = \begin{cases} 0 & if \quad x \leq 0 \\ 1 & if \quad x > 0 \end{cases}$$



The limit from the left at $p = 0$ is 0 and the limit from the right at $p = 0$ is 1. That is,

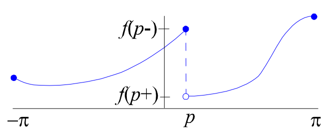
$$\lim_{x \to 0^-} H(x) = 0 \qquad \text{and} \qquad \lim_{x \to 0^+} H(x) = 1$$

In our new notation, these limits are $H(0-) = 0$ and $H(0+) = 1$, respectively.

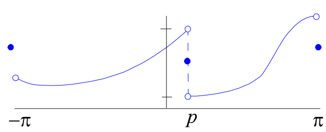
Moreover, a function is said to have a *jump discontinuity* at $x = p$ if $f(p-)$ and $f(p+)$ exist, but $f(p-) \neq f(p+)$. For example, $H(x)$ has a jump discontinuity at $p = 0$. Finally, a function $f(x)$ is said to be *piecewise continuous* on $[-\pi, \pi]$ if it is continuous at all but a finite number of points in $[-\pi, \pi]$ and has only jump discontinuities. This leads us to the following theorem:

Fourier Theorem: If $f(x)$ is piecewise continuous on $[-\pi, \pi]$ with a piecewise continuous derivative, then its Fourier series converges to $f(x)$ over $[-\pi, \pi]$ except at jump discontinuities, in which case it converges to the midpoint of the jump.

Moreover, if $f(-\pi) \neq f(\pi)$, then its Fourier series converges to the average of the endpoints at both ends.



Piecewise continuous Function



Fourier Series Converges to This

EXAMPLE 4 Use graphs of the partial sums of the Heaviside function $H(x)$ to determine what its Fourier series converges to over $[-\pi, \pi]$.

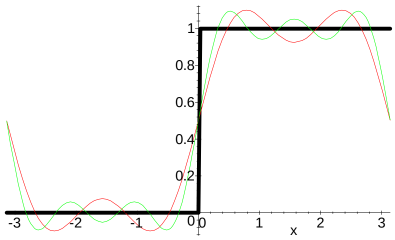
Solution: The Fourier series of the Heaviside step function is[2]

$$\frac{1}{2}+\frac{2}{\pi}\sin(x)+\frac{2}{3\pi}\sin(3x)+\frac{2}{5\pi}\sin(5x)+\frac{2}{7\pi}\sin(7x)+\ldots \quad (10.18)$$

The third and fifth partial sums of the Fourier series of $H(x)$ are

$$\begin{aligned} s_3(x) &= \frac{1}{2}+\frac{2}{\pi}\sin(x)+\frac{2}{3\pi}\sin(3x) \\ s_5(x) &= \frac{1}{2}+\frac{2}{\pi}\sin(x)+\frac{2}{3\pi}\sin(3x)+\frac{2}{5\pi}\sin(5x) \end{aligned}$$

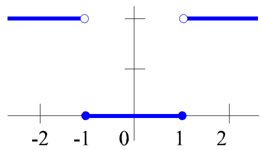
and are shown in red, green and blue respectively versus the graph of $H(x)$ below:



Both approximations have a value of $\frac{1}{2}$ when $x=0$. Indeed, if $x=0$ in (10.18), then clearly the Fourier series converges to $\frac{1}{2}$, which is the midpoint of the jump discontinuity at $p=0$.

EXAMPLE 5 Use the Fourier theorem to discuss what the Fourier series of the following function converges to

$$f(x)=\begin{cases} 0 & if \quad |x|\le 1 \\ 2 & if \quad |x|>1 \end{cases}$$



Solution: Notice that the function has jump discontinuities at $x=1$ and $x=-1$. The Fourier theorem tells us that the Fourier cosine series of $f(x)$ will converge to 1 when $x=1$ and to 1 when $x=-1$—i.e., to the midpoint of the jumps.

Check your Reading *Sketch the function that the Fourier series of $f(x)$ in the example above converges to.*

Periodicity of a Convergent Fourier Series

The Fourier series of $f(x)$ converges to the average of $f(\pi)$ and $f(-\pi)$ at the endpoints because a Fourier series converges to a periodic function with period 2π. That is, the function it represents on $[-\pi,\pi]$ is reproduced over every interval of length 2π on the real line.

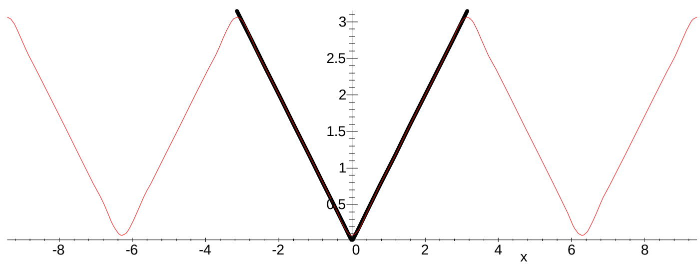
[2] The general term in $H(x)$ is omitted because the pattern is easily recognized.

EXAMPLE 6 Sketch the graph of the function the Fourier series of $f(x) = |x|$ converges to over $[-3\pi, 3\pi]$.

Solution: The Fourier series of $f(x) = |x|$ is

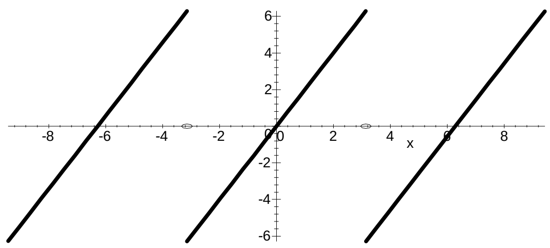
$$|x| = \frac{\pi}{2} - \frac{4\cos(x)}{\pi \cdot 1^2} - \frac{4\cos(3x)}{\pi \cdot 3^2} - \frac{4\cos(5x)}{\pi \cdot 5^2} - \frac{4\cos(7x)}{\pi \cdot 7^2} - \ldots$$

which converges quickly to $|x|$ on $[-\pi, \pi]$. However, outside of the interval $[-\pi, \pi]$, the series produces repeated copies of $|x|$ over $[-\pi, \pi]$.



EXAMPLE 7 Sketch the graph of the function that the Fourier series of $g(x) = 2x$ converges to over $[-3\pi, 3\pi]$.

Solution: If $g(x) = 2x$, then $g(-\pi) = -2\pi$ and $g(\pi) = 2\pi$. Thus, its Fourier series converges to 0 at $-\pi$ and π, which implies that the Fourier series of $g(x)$ converges to the function graphed below:

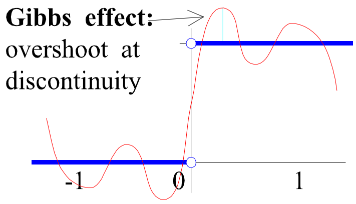


That is, the Fourier series of $g(x) = 2x$ is converging to a function which is only piecewise continuous over the whole x-axis.

Check your Reading *What does the Fourier series of $g(x) = 2x$ converge to at $x = 3\pi$?*

The Gibbs Effect

In the neighborhood of a jump discontinuity, the partial sums $s_n(x)$ converge to the function in a rather unique way. In particular, each approximation overshoots the jump, a phenomenon known as the *Gibb's effect.*



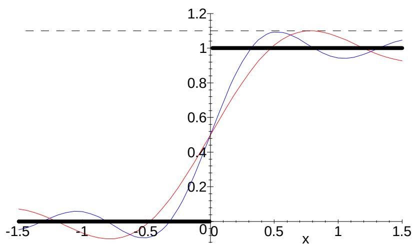
Remarkably, the height of the overshoot is about the same for every partial sum. That is, no matter how large n becomes, the n^{th} partial sum approximation overshoots the function by practically the same amount.

EXAMPLE 8 Estimate the Gibb's overshoot of the Fourier series of the Heaviside step function $H(x)$.

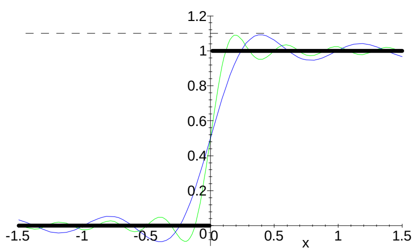
Solution: The third and fifth partial sums of the Fourier series of the Heaviside step function $H(x)$ are

$$\begin{aligned} s_3(x) &= \frac{1}{2} + \frac{2}{\pi}\sin(x) + \frac{2}{3\pi}\sin(3x) \\ s_5(x) &= \frac{1}{2} + \frac{2}{\pi}\sin(x) + \frac{2}{3\pi}\sin(3x) + \frac{2}{5\pi}\sin(5x) \end{aligned}$$

The graphs of $s_3(x)$, in red and $s_5(x)$, in blue, are shown below.



Clearly they both overshoot $H(x)$ by about 0.1 units. Moreover, the graphs of $s_7(x)$ and $s_{15}(x)$ also overshoot the jump of 1 by about 0.1 units:



Thus, the Gibb's overshoot of the Heaviside step function $H(x)$ is about 0.1 units.

Exercises:

Graph $s_3(x)$ and $s_7(x)$ along with the function itself on $[-\pi,\pi]$. Then graph $s_7(x)$ over $[-3\pi,3\pi]$. What function does the sequence of partial sums appear to be converging to?

1. $\frac{|x|}{x} = \frac{4}{\pi}\sin(x) + \frac{4}{3\pi}\sin(3x) + \frac{4}{5\pi}\sin(5x) + \frac{4}{7\pi}\sin(7x) + \dots$
2. $\pi - 2|x| = \frac{8\cos(x)}{\pi\cdot 1^2} + \frac{8\cos(3x)}{\pi\cdot 3^2} + \frac{8\cos(5x)}{\pi\cdot 5^2} + \frac{8\cos(7x)}{\pi\cdot 7^2} + \dots$
3. $x(\pi - |x|) = \frac{8\sin(x)}{\pi\cdot 1^3} + \frac{8\sin(3x)}{\pi\cdot 3^3} + \frac{8\sin(5x)}{\pi\cdot 5^3} + \frac{8\sin(7x)}{\pi\cdot 7^3} + \dots$
4. $\left(x^2 - \pi^2\right)^2 = \frac{8\pi^4}{15} + \frac{48\cos(x)}{1^4} + \frac{48\cos(2x)}{2^4} + \frac{48\cos(3x)}{3^4} + \dots$
5. $x^3 - \pi^2 x = -12\sin(x) + \frac{12}{2^3}\sin(2x) - \frac{12}{3^3}\sin(3x) + \frac{12}{4^3}\sin(4x) - \dots$
6. $\frac{\pi\cosh(x)}{2\sinh(\pi)} = \frac{1}{2} - \frac{1}{2}\cos(x) + \frac{1}{5}\cos(2x) - \frac{1}{10}\cos(3x) + \frac{1}{17}\cos(4x) + \frac{1}{26}\cos(5x) + \dots$
7. $\frac{\pi\sinh(x)}{2\sinh(\pi)} = \frac{1}{2}\sin(x) - \frac{2}{5}\sin(2x) + \frac{3}{10}\sin(3x) - \frac{4}{17}\sin(4x) + \frac{5}{26}\sin(5x) + \dots$
8. $\frac{1}{2}\tan^{-1}\left(\frac{4}{3}\sin(x)\right) = \frac{\sin(x)}{2^1\cdot 1} + \frac{\sin(3x)}{2^3\cdot 2} + \frac{\sin(5x)}{2^5\cdot 5} + \frac{\sin(7x)}{2^7\cdot 7} + \dots$
9. $\tan^{-1}\left(\sqrt{3}\cos(x)\right) = \frac{2\cos(x)}{\sqrt{3}\cdot 1} - \frac{2\cos(3x)}{\sqrt{3^3}\cdot 2} + \frac{2\cos(5x)}{\sqrt{3^5}\cdot 5} - \frac{2\cos(7x)}{\sqrt{3^7}\cdot 7} + \dots$
10. $\ln\left|\sec\left(\frac{x}{2}\right)\right| = \ln(2) - \cos(x) + \frac{1}{2}\cos(2x) - \frac{1}{3}\cos(3x) + \frac{1}{4}\cos(4x) - \frac{1}{5}\cos(5x)$
11. $\ln\left|\csc\left(\frac{x}{2}\right)\right| = \ln(2) + \cos(x) + \frac{1}{2}\cos(2x) + \frac{1}{3}\cos(3x) + \frac{1}{4}\cos(4x) + \frac{1}{5}\cos(5x)$
12. $\frac{\pi}{8}\sin\left(\frac{x}{2}\right) = \frac{\sin(x)}{4\cdot 1 - 1} - \frac{2\sin(2x)}{4\cdot 2^2 - 1} + \frac{3\sin(3x)}{4\cdot 3^2 - 1} - \frac{4\sin(4x)}{4\cdot 4^2 - 1} + \dots$

Sketch the graph of the function on $[-3\pi,3\pi]$ that the Fourier series of $f(x)$ over $[-\pi,\pi]$ converges to.

14. $f(x) = x$
15. $f(x) = \pi - x$
16. $f(x) = x^2$
17. $f(x) = e^x$
18. $f(x) = \begin{cases} -2 & if \quad x < 0 \\ 4 & if \quad x \ge 0 \end{cases}$
19. $f(x) = \begin{cases} 1 & if \quad x < 0 \\ -4 & if \quad x \ge 0 \end{cases}$
20. $f(x) = \begin{cases} 3 & if \quad x < 0 \\ 4 & if \quad x \ge 0 \end{cases}$
21. $f(x) = \begin{cases} 4 & if \quad |x| < 1 \\ 0 & if \quad |x| \ge 1 \end{cases}$
22. $f(x) = \begin{cases} x+\pi & if \quad x < 0 \\ x & if \quad x \ge 0 \end{cases}$
23. $f(x) = \begin{cases} x^2 & if \quad x < 0 \\ -x^2 & if \quad x \ge 0 \end{cases}$

24. The Fourier sine series of $|x|/x$ is

$$\frac{|x|}{x} = \frac{4\sin(x)}{\pi\cdot 1} + \frac{4\sin(3x)}{\pi\cdot 3} + \frac{4\sin(5x)}{\pi\cdot 5} + \frac{4\sin(7x)}{\pi\cdot 7} + \dots$$

Graph $s_3(x)$, $s_5(x)$ and $s_7(x)$ and estimate numerically the magnitude of the overshoot at $x = 0$ due to the Gibb's effect.

25. The Fourier sine series of $2|x|/x$ is

$$\frac{2|x|}{x} = \frac{8\sin(x)}{\pi \cdot 1} + \frac{8\sin(3x)}{\pi \cdot 3} + \frac{8\sin(5x)}{\pi \cdot 5} + \frac{8\sin(7x)}{\pi \cdot 7} + \ldots$$

Graph $s_3(x)$, $s_5(x)$ and $s_7(x)$ and estimate numerically the magnitude of the overshoot at $x = 0$ due to the Gibb's effect.

26. Let $f(x)$ be the function defined by

$$f(x) = \begin{cases} -1 & if \quad -\pi \le x < \frac{-\pi}{2} \\ 1 & if \quad \frac{-\pi}{2} \le x < \frac{\pi}{2} \\ -1 & if \quad \frac{\pi}{2} \le x \le \pi \end{cases}$$

(a) Graph $f(x)$ and determine if it is even or odd.

(b) Find the Fourier series of $f(x)$ by determining its Fourier coefficients. (Hint: use the fact that $\sin\left(\frac{n\pi}{2}\right) = 0$ if n is even).

(c) What does the Fourier Series of $f(x)$ converge to when $x = \frac{\pi}{2}$? Test your answer by substituting $\frac{\pi}{2}$ for x in the Fourier series of $f(x)$ and simplifying.

27. * Let $f(x)$ be the function defined by

$$f(x) = \begin{cases} -1 & if \quad -\pi \le x < \frac{-\pi}{2} \\ 0 & if \quad \frac{-\pi}{2} \le x < \frac{\pi}{2} \\ 1 & if \quad \frac{\pi}{2} \le x \le \pi \end{cases}$$

(a) Graph $f(x)$ and determine if it is even or odd.

(b) Find the Fourier series of $f(x)$ by determining its Fourier coefficients. (Hint: use the fact that $\cos\left(\frac{n\pi}{2}\right) = 0$ if n is odd).

(c) What does the Fourier Series of $f(x)$ converge to when $x = \frac{\pi}{2}$? Test your answer by substituting $\frac{\pi}{2}$ for x in the Fourier series of $f(x)$ and simplifying.

28. Can an even function have a jump discontinuity at the origin? Explain.

29. If a piecewise continuous odd function has a jump discontinuity at the origin, what does its Fourier series converge to when $x = 0$?

30. Suppose that $f(x)$ is even. What does it mean for

$$f(0) = a_0 + a_1 + a_2 + \ldots + a_n + \ldots$$

and what must we know about $f(x)$ in order for this to be true?

Fourier series of the form

$$\frac{q^{1/2}}{1-q}\sin(x) + \frac{q^{3/2}}{1-q^3}\sin(3x) + \frac{q^{5/2}}{1-q^5}\sin(5x) + \frac{q^{7/2}}{1-q^7}\sin(7x) + \ldots \quad (10.19)$$

are generated by elliptic functions. You will use these series in exercises 31-34.

31. Determine the Fourier series for the elliptic function for which $q = 0.25$ (i.e., $q = 1/4$). Then graph $s_5(x)$, $s_7(x)$, and $s_9(x)$ over $[-3\pi, 3\pi]$.

32. Determine the Fourier series for the elliptic function for which $q = 0.9801$. Then graph $s_5(x)$, $s_7(x)$, and $s_9(x)$ over $[-3\pi, 3\pi]$.

33. When q is close to 0, the elliptic function represented by (10.19) is practically a sine wave. Determine the Fourier series for the elliptic function for which $q = 0.01$. Then graph $s_7(x)$ over $[-\pi, \pi]$. About where does the first maximum of $s_7(x)$ occur? How high is that maximum? For what value of A does $s_7(x)$ most resemble the sine wave $f(x) = A\sin(x)$?

34. Repeat exercise 33 for $q = 0.0001$.

35. The Fourier series below converges to a function which is continuous everywhere but differentiable nowhere.

$$2\sin(x) + \frac{1}{2}\sin(4x) + \frac{1}{6}\sin(36x) + \frac{1}{24}\sin\left((24)^2 x\right) + \ldots + \frac{1}{n!}\sin\left((n!)^2 x\right) + \ldots$$

where $n!$ is the *factorial function,* defined $n! = n \cdot (n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$. Graph the partial sums of the series and explain why we might expect that it converges to a continuous-everywhere, differentiable nowhere function.

36. For a noninteger number a, the Fourier series of the function $f(x) = \sin(ax)$ is given by the expansion

$$\frac{-2\sin(a\pi)}{\pi(a^2-1)}\sin(x) + \frac{4\sin(a\pi)}{\pi(a^2-4)}\sin(2x) - \frac{6\sin(a\pi)}{\pi(a^2-9)}\sin(3x) + \frac{8\sin(a\pi)}{\pi(a^2-16)}\sin(4x) - \ldots$$

(a) Graph $s_4(x)$ and $s_5(x)$ for $a = 1.1$.
(b) Graph $s_4(x)$ and $s_5(x)$ for $a = 2.01$.
(c) Graph $s_4(x)$ and $s_5(x)$ for $a = 3.001$.

37. Write to Learn: In a short essay, explain why the Fourier theorem implies the following:

If $f(x)$ is piecewise continuous and periodic with a period 2π, then the Fourier series of $f(x)$ converges to

$$\frac{f(p-) + f(p+)}{2}$$

for all real numbers p.

In particular, discuss what happens at the endpoints of the interval $[-\pi, \pi]$.

38. Write to Learn: In a letter to the journal *Nature* in 1899, J.W. Gibbs showed that the n^{th} partial sum overshoots the jump by an amount equal to the magnitude of product of the jump and the integral

$$\frac{-1}{\pi}\int_{\pi}^{\infty}\frac{\sin(s)}{s}ds \tag{10.20}$$

However, it can be shown that the integral in (10.20) is the same as

$$\frac{1}{\pi}\int_{0}^{\pi}\frac{\sin(s)}{s}ds - \frac{1}{2} \tag{10.21}$$

Approximate the integral in (10.20) numerically, subtract it from $\frac{1}{2}$, and in a short essay explain how it relates to the Gibb's overshoot for the Heaviside Step function $H(x)$.

10.3 Fourier Series on Other Intervals

Fourier Series on intervals of the form $[-L, L]$

All of our work with Fourier series has to this point been with Fourier series over the interval $[-\pi, \pi]$. In this section, we extend our results for Fourier series over $[-\pi, \pi]$ to Fourier series to Fourier series of functions over an interval of the form $[-L, L]$.

To begin with, if $f(x)$ is integrable over an interval $[-L, L]$, then the Fourier series of $f(x)$ is of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n x}{L}\right) + b_n \sin\left(\frac{\pi n x}{L}\right)$$

where the Fourier cosine and sine coefficients, respectively, are given by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{\pi n x}{L}\right) dx, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{\pi n x}{L}\right) dx$$

Moreover, if $f(x)$ is an even function, then all the sine coefficients b_n are equal to zero so that we need only find the Fourier cosine coeffficients, a_n. Likewise, if $f(x)$ is odd, then we need only find the sine coefficients, b_n.

EXAMPLE 1 Find the Fourier Coefficients of $f(x) = x$ over $[-2, 2]$.

Solution: Since $f(x) = x$ is odd, we need only find the sine coefficients for $L = 2$:

$$b_n = \frac{1}{2}\int_{-2}^{2} x \sin\left(\frac{n\pi x}{2}\right) dx = \frac{2}{2}\int_{0}^{2} x \sin\left(\frac{n\pi x}{2}\right) dx$$

To do so, we use integration by parts with $u = x$ and $dv = \sin\left(\frac{n\pi x}{2}\right) dx$

u		dv
x		$\sin\left(\frac{n\pi x}{2}\right)$
	$\searrow^{+}$	
1		$\frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)$
	$\searrow^{-}$	
0		$-\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{2}\right)$

As a result, tabular integration implies that

$$\begin{aligned} b_n &= \left.\left(\frac{-2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{2}\right)\right)\right|_0^2 \\ &= \left(\frac{-2\cdot 2}{n\pi}\cos\left(\frac{n\pi 2}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi 2}{2}\right) - 0 - 0\right) \\ &= \frac{-4}{n\pi}(-1)^n \end{aligned}$$

Thus, the Fourier series of $f(x) = x$ over $[-2, 2]$ is

$$\sum_{n=1}^{\infty} \frac{-2(-1)^n}{n\pi} \sin\left(\frac{n\pi x}{2}\right) = \frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right) - \frac{2}{2\pi}\sin\left(\frac{2\pi x}{2}\right) + \frac{2}{3\pi}\sin\left(\frac{3\pi x}{2}\right) - \ldots$$

EXAMPLE 2 Find the Fourier Coefficients over $[-3, 3]$ of the function

$$f(x) = \begin{cases} 1 & if \quad |x| > 1 \\ 0 & if \quad |x| < 1 \end{cases}$$

Solution: Since $f(x)$ is an even function, we need only determine the Fourier cosine coefficients a_n.

$$a_n = \frac{1}{3}\int_{-3}^{3} f(x)\cos\left(\frac{n\pi x}{3}\right)dx$$

Since the integrand is an even function, we can simplify this to

$$a_n = \frac{2}{3}\int_0^3 f(x)\cos\left(\frac{n\pi x}{3}\right)dx = \frac{2}{3}\int_0^1 \cos\left(\frac{n\pi x}{3}\right)dx$$

since $f(x) = 0$ if $x > 1$. As a result, we have

$$\begin{aligned} a_n &= \frac{2}{3}\int_0^1 \cos\left(\frac{n\pi x}{3}\right)dx \\ &= \frac{2}{3}\frac{3}{n\pi}\sin\left(\frac{n\pi x}{3}\right)\Big|_0^1 \\ &= \frac{2}{n\pi}\sin\left(\frac{n\pi}{3}\right) \end{aligned}$$

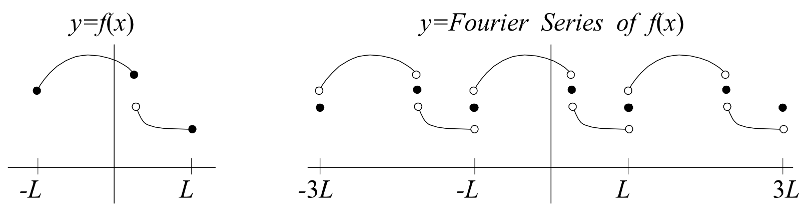
Check your Reading *What are a_1, a_2, and a_3 in example 2?*

Convergence of Fourier Series on Arbitrary Intervals

Convergence of a Fourier Series on $[-L, L]$ is similar to convergence of a Fourier series on $[-\pi, \pi]$. In particular, if $f(x)$ is piecewise continuous on $[-L, L]$ with a piecewise continuous derivative, then its Fourier series converges to a function with a period of $2L$ whose value at each input x is

$$\frac{f(x+) + f(x-)}{2}$$

In addition, the Fourier series converges to $f(x)$ if the function f is continuous at x.



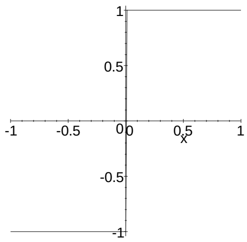
Moreover, there is a Gibb's effect in the convergence of the partial sums of the Fourier series at each jump discontinuity.

EXAMPLE 3 The Fourier series of $f(x) = \frac{|x|}{x}$ over $[-1, 1]$ is given by

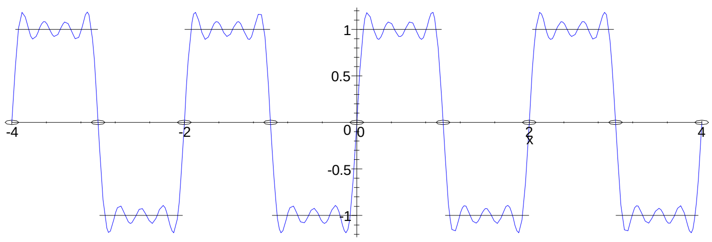
$$\frac{4}{\pi}\sin(\pi x) + \frac{4}{3\pi}\sin(3\pi x) + \frac{4}{5\pi}\sin(5\pi x) + \frac{4}{7\pi}\sin(7\pi x) + \ldots$$

What function does it converge to?

Solution: The graph of $f(x) = \frac{|x|}{x}$ over $[-1, 1]$ is given by



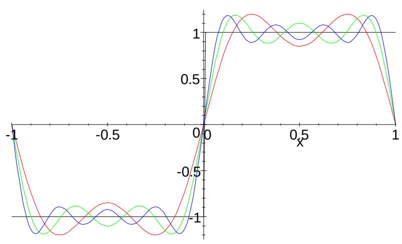
Thus, its Fourier series converges to -1 if x is in $(-1, 0)$, converges to 1 if x is in $(0, 1)$, and converges to 0 at $x = -1$, $x = 0$, and $x = 1$. Moreover, the Fourier series has a period of 2, and we have included the graph of $s_7(x)$ to illustrate this convergence:



Moreover, the first few partial sums of $f(x) = \frac{|x|}{x}$ over $[-1, 1]$ are given by

$$\begin{aligned} s_3(x) &= \frac{2}{\pi}\sin(\pi x) + \frac{2}{3\pi}\sin(3\pi x) \\ s_5(x) &= \frac{2}{\pi}\sin(\pi x) + \frac{2}{3\pi}\sin(3\pi x) + \frac{2}{5\pi}\sin(5\pi x) \\ s_7(x) &= \frac{2}{\pi}\sin(\pi x) + \frac{2}{3\pi}\sin(3\pi x) + \frac{2}{5\pi}\sin(5\pi x) + \frac{2}{7\pi}\sin(7\pi x) \end{aligned}$$

and when graphed versus the function itself, it is clear that there is a Gibb's effect at the jump discontinuities.



Check your Reading *What is the Gibb's effect illustrated in the figure above?*

Parsevals' Theorem

If $f(x)$ is an odd function which is equal to its Fourier series, then we can write

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

As a result, squaring the absolute value of $f(x)$ yields

$$|f(x)|^2 = f(x) \cdot f(x) = \left(\sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right)\right)\left(\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

If we assume that the Fourier series converges absolutely, the product can be rearranged into

$$|f(x)|^2 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_m b_n \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

If we allow that the series can be integrated term by term, then we have

$$\int_{-L}^{L} |f(x)|^2\, dx = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_n b_m \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx$$

If $m \neq n$, then the integral is zero, so the only terms that remain are those where $n = m$:

$$\int_{-L}^{L} |f(x)|^2\, dx = \sum_{n=1}^{\infty} b_n^2 \int_{-L}^{L} \sin^2\left(\frac{n\pi x}{L}\right) dx$$

However, $\int_{-L}^{L} \sin^2\left(\frac{n\pi x}{L}\right) dx = L$, which implies that

$$\int_{-L}^{L} |f(x)|^2\, dx = \sum_{n=1}^{\infty} b_n^2\, L \qquad (10.22)$$

Advanced techniques can be used to show that (10.22) holds in general, and likewise, that a similar identity holds for the Fourier cosine coefficients. Moreover, a little rearranging yields the following

Parseval's Formula: If $f(x)$ is odd, then

$$\frac{1}{L}\int_{-L}^{L} |f(x)|^2\, dx = \sum_{n=1}^{\infty} b_n^2 \qquad (10.23)$$

Likewise, if $f(x)$ is even, then

$$\frac{1}{L}\int_{-L}^{L} |f(x)|^2\, dx = \frac{a_0^2}{2} + \sum_{n=0}^{\infty} a_n^2 \qquad (10.24)$$

Moreover, Parseval's formula can be used to determine the sum of many different series.

EXAMPLE 4 Apply Parseval's formula to the Fourier series of $f(x) = x$ over $[-2, 2]$, which is given by

$$\sum_{n=1}^{\infty} \frac{-4(-1)^n}{n\pi} \sin\left(\frac{n\pi x}{2}\right)$$

Solution: As was shown in example 1, the Fourier sine coefficients of $f(x)$ are $b_n = \frac{-2(-1)^n}{n\pi}$, so that Parseval's formula implies that

$$\sum_{n=1}^{\infty} \left[\frac{-4(-1)^n}{n\pi}\right]^2 = \frac{1}{2}\int_{-2}^{2} |x|^2 \, dx$$

Since $|x|^2$ is an even function, we can simplify this to the following:

$$\sum_{n=1}^{\infty} \frac{16}{n^2\pi^2} = \frac{2}{2}\int_0^2 x^2 dx = \frac{8}{3}$$

Thus, Parseval's formula yields

$$\frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{8}{3}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{16}\frac{8}{3} = \frac{\pi^2}{6}$$

That is, the p-series with $p = 2$ has a sum of

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Indeed, we can use Parseval's formula to compute the sum of all of the p-series in which p is even. However, the p-series with p odd are much more complicated. Indeed, no one has yet been able to find a closed form representation of the p-series in which $p = 3$.

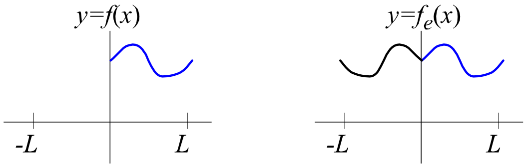
Check your Reading *Use a calculator to compute the decimal representation of the sum of* $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Fourier Series on Intervals of the form $[0, L]$

In many applications, we are given $f(x)$ on an interval of the form $[0, L]$ and asked to find its Fourier Series representation. We can do so in one of two equivalent ways. First, if we define a new function $f_e(x)$ by

$$f_e(x) = \begin{cases} f(x) & if \quad 0 \le x \le L \\ f(-x) & if \quad -L \le x < 0 \end{cases}$$

then $f_e(x)$ is called the *even extension* of $f(x)$ to $[-L, L]$.



Indeed, $f_e(x)$ is an even function, so that its Fourier series is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \qquad \text{where} \qquad a_n = \frac{1}{L}\int_{-L}^{L} f_e(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

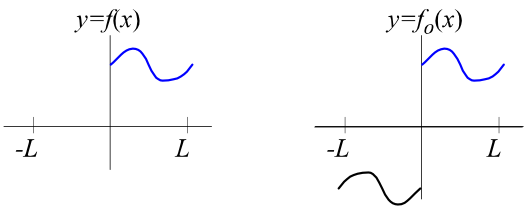
Applying symmetry to the integral then gives us that the Fourier series of $f(x)$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \qquad \text{where} \qquad a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

Likewise, if we define a function $f_o(x)$ by

$$f_o(x) = \begin{cases} f(x) & if \quad 0 \le x \le L \\ -f(-x) & if \quad -L \le x < 0 \end{cases}$$

then $f_o(x)$ is called the *odd extension* of $f(x)$ to $[-L, L]$.



Indeed, $f_o(x)$ is an even function, so that its Fourier series is given by

$$\sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right) \qquad \text{where} \qquad b_n = \frac{1}{L}\int_{-L}^{L} f_o(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Applying symmetry to the integral then gives us that the Fourier series of $f(x)$ is

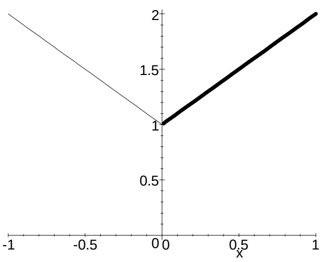
$$\sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right) \qquad \text{where} \qquad b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

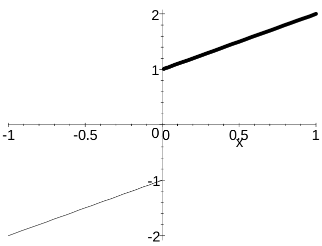
EXAMPLE 5 Find the even and odd extensions of $f(x) = 1 + x$ over $[0, 1]$.

Solution: By definition, the even and odd extensions are

$$f_e(x) = \begin{cases} 1+x & if \quad 0 \le x \le 1 \\ 1-x & if \quad -1 \le x < 0 \end{cases} \qquad \textit{and} \qquad f_o(x) = \begin{cases} 1+x & if \quad 0 \le x \le 1 \\ -(1-x) & if \quad -1 \le x < 0 \end{cases}$$

which are shown below:





Exercises:

Find the Fourier sine and cosine coefficients of the given function over the given interval.

1. $f(x) = x$ over $[-1,1]$
2. $f(x) = x$ over $[-3,3]$
3. $f(x) = |x|$ over $[-1,1]$
4. $f(x) = |x|$ over $[-2,2]$
5. $f(x) = x^2$ over $[-1,1]$
6. $f(x) = x^2$ over $[-2,2]$
7. $f(x) = x^2 - 1$ over $[-1,1]$
8. $f(x) = x^2 - 4$ over $[-2,2]$
9. $f(x) = \dfrac{|x|}{x}$ over $[-1,1]$
10. $f(x) = \dfrac{|x|}{x}$ over $[-2,2]$
11. $f(x) = \begin{cases} 1 & if \quad |x| \le 1 \\ 1 & if \quad |x| > 1 \end{cases}$ over $[-2,2]$
12. $f(x) = \begin{cases} 1 & if \quad |x| \le 1 \\ 1 & if \quad |x| > 1 \end{cases}$ over $[-3,3]$

Apply Parseval's formula to each of the following Fourier series over $[-\pi,\pi]$.

13. $x^3 = \sum_{n=1}^{\infty} \frac{2n^2\pi^2 + 12}{n^3} \sin(nx)$
14. $\frac{\pi^2}{3} - x^2 = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n^2} \cos(nx)$
15. $x^3 - \pi^2 x = \sum_{n=1}^{\infty} \frac{12(-1)^n}{n^3} \sin(nx)$
16. $\pi - 2|x| = \sum_{n=1}^{\infty} \frac{8\cos((2n-1)x)}{\pi \cdot (2n-1)^2}$
17. $\frac{|x|}{x} = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin[(2n-1)x]$
18. $\frac{\pi}{8} \sin\left(\frac{x}{2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n \sin(nx)}{4n^2 - 1}$
19. $\left(x^2 - \pi^2\right)^2 = \frac{8\pi^4}{15} + \sum_{n=1}^{\infty} \frac{48\cos(nx)}{n^4}$
20.* $\frac{\pi \sinh(x)}{2\sinh(\pi)} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+1} \sin(nx)$

Find the Fourier series for the even extension of the given function. Then find the Fourier series for the odd extension of the given function.

21. $f(x) = x$ on $[0,\pi]$
22. $f(x) = x$ on $[0,1]$
23. $f(x) = 1$ on $[0,1]$
24. $f(x) = x^2$ on $[0,1]$

25. It can be shown that if $|a| < 1$, then

$$\frac{1 - a\cos(x)}{1 - 2a\cos(x) + a^2} = \sum_{n=0}^{\infty} a^n \cos(nx)$$

Show that the Fourier series reduces to the geometric series formula when $x = 0$.

26. It can be shown that if $|a| < 1$, then

$$\frac{a\sin(x)}{1 - 2a\cos(x) + a^2} = \sum_{n=1}^{\infty} a^n \sin(nx)$$

Use Parseval's formula to relate the integral

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{a\sin(x)}{1 - 2a\cos(x) + a^2} \right)^2 dx$$

to the geometric series $a^2 + a^4 + \ldots + a^{2n} + \ldots$. Then use the geometric series formula to evaluate the integral.

27. It can be shown that

$$\left(x^2-\pi^2\right)^2=\frac{8\pi^4}{15}+\frac{48\cos(x)}{1^4}+\frac{48\cos(2x)}{2^4}+\ldots+\frac{48\cos(nx)}{n^4}+\ldots$$

Let $x=0$ and then simplify the result to find the sum of the p-series when $p=4$.

28. Earlier in this chapter, we showed that

$$\frac{|x|}{x}=\frac{4}{\pi}\sin(x)+\frac{4}{3\pi}\sin(3x)+\frac{4}{5\pi}\sin(5x)+\frac{4}{7\pi}\sin(7x)+\ldots$$

Let $x=\pi/2$ to find the sum of the series

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$

29. The derivative of a Fourier series need not converge to the Fourier series of the function that generated it. We consider that in this problem by noticing that

$$2x=4\sin(x)-\frac{4}{2}\sin(2x)+\frac{4}{3}\sin(3x)-\frac{4}{4}\sin(4x)+\ldots \tag{10.25}$$

for all x in $(-\pi,\pi)$.

(a) Application of the derivative to both sides of (**??**) yields

$$2\overset{?}{=}4\cos(x)-4\cos(2x)+4\cos(3x)-4\cos(4x)+\ldots \tag{10.26}$$

(b) Let $x=0$ in (10.26) and explain why (10.26) cannot possibly be true.

(c) Graph $s_3(x)$, $s_4(x)$, and $s_5(x)$ for (10.26). Do the partial sums appear to be converging to 2?

(d) Given the information in (b) and (c), would you conclude that the derivative of the Fourier series of $2x$ converges to the derivative of $2x$?

30. **Write to Learn:** The Fourier series of the function $f(x)=\sinh(x)$ is

$$\sinh(x)=\frac{2\sinh(\pi)}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}\sin(nx)$$

The second derivative of $\sinh(x)$ is $\sinh(x)$, but is the second derivative of the Fourier series of $\sinh(x)$ equal to the Fourier series of $\sinh(x)$? Why or why not? Present your conclusions in a short essay.

31. Show that if $m\neq n$, then $\int_{-L}^{L}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx=0$.

32. Show that if $m\neq n$, then $\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx=0$

33. Show that $\int_{-L}^{L}\sin^2\left(\frac{m\pi x}{L}\right)dx=L$.

34. Here we prove Parseval's formula for the cosine coefficients for the special case of $a_0=0$. Show that if $\sum_{n=1}^{\infty}|a_n|$ converges and if $g(x)=\sum_{n=1}^{\infty}a_n\cos(nx)$, then

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|g(x)|^2\,dx=a_1^2+a_2^2+\ldots+a_n^2+\ldots$$

The Next Step... Naive Signal Processing

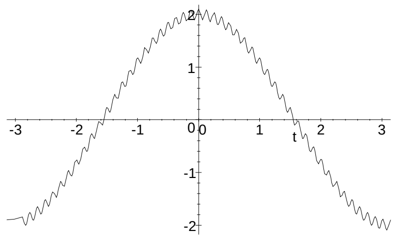
Why are Fourier Series so important to twentieth century science? What do Fourier series allow us to do that cannot be done otherwise? Our next step is to demonstrate the importance of Fourier series by illustrating their use in *signal processing.*

To begin with, a function $s(t)$ is called a *signal* if s is the amplitude of a wave measured at a certain point in space and t is time. Signal processing means modifying a wave so that it has certain desirable properties.

Indeed, the removal of noise from a signal is an example of a type of signal processing known as *filtering*. To do this type of filtering, *noise* $n(t)$ is defined to be a high frequency, low amplitude signal, and a real signal $s(t)$ is then assumed to be the superposition of an ideal, noiseless signal $f(t)$ and the noise $n(t)$. That is,

$$s(t) = f(t) + n(t)$$

For example, $s(t)$ might represent the "noisy" $\cos(2t)$ function shown below:



The goal is to remove the noise by *filtering* out the higher frequencies.

To do so, we might simply write $s(t)$ as a Fourier Series over $[-L, L]$ as

$$s(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

Removing the higher frequencies means truncating the series for $s(t)$ at a finite value N:

$$s(t) \approx \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

However, we must choose N so that not too much of $s(t)$ has been removed. In order to do so, we use Parseval's formula, which says that

$$\frac{1}{L}\int_{-L}^{L} |s(t)|^2\, dt = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2$$

In particular, we choose N so that Parseval's formula is approximately true—that is, we choose N such that

$$\frac{1}{L}\int_{-L}^{L} |s(t)|^2\, dt \approx \frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 + b_n^2$$

This process for filtering the noise out of a signal is usually called *Naive Signal Processing* (Most signal processing algorithms employed by scientists and engineers involve the *Fourier Transform* and a process called *convolution*—see the *Advanced contexts* below).

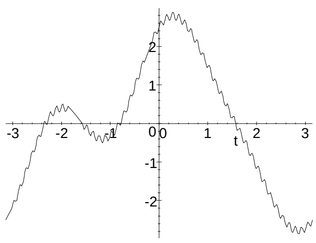
For example, consider the "signal"

$$s(t) = 2\cos(t) + \sin(2t) + 0.5\cos(3t) + 0.1\sin(50t)$$

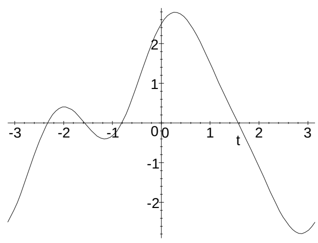
If we consider noise to be any part of $s(t)$ with a frequency greater than $N = 40$, then the "filtered" cosine is given by

$$f(t) = 2\cos(t) + \sin(2t) + 0.5\cos(3t)$$

The raw signal $s(t)$ and the filtered signal $f(t)$ are shown below:



$s(t)$



$f(t)$

Moreover, Parseval's formula says that the integral

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |s(t)|^2 \, dt$$

should be approximately the same as the sum of the squares of the Fourier coefficients of the filtered wave, $f(t)$. That is,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |s(t)|^2 \, dt \approx 2^2 + 1^2 + (0.5)^2$$

However, $2^2 + 1^2 + (0.5)^2 = 5.25$ while the integral is

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |s(t)|^2 \, dt = 5.26$$

Write to Learn Use a computer algebra system to evaluate

$$\frac{1}{\pi}\int_{-\pi}^{\pi} (2\cos(t) + \sin(2t) + 0.5\cos(3t) + 0.1\sin(50t))^2 \, dt$$

and then in a short essay, explain how it is related to the filtering of $s(t)$ above into the function

$$f(t) = 2\cos(t) + \sin(2t) + 0.5\cos(3t)$$

Write to Learn

Write to Learn Go to the library and/or search the internet to learn more about filtering and signal processing. Then write a short paper explaining how Fourier series can be used in signal processing.

Group Learning Using a computer algebra system such as Maple or Mathematica, a mathematical environment such as Matlab or MathCad, or a computer programming language such as C++ or Java, design and develop an algorithm which accomplishes Naive signal processing. In particular, the algorithm will consist of 3 steps given an arbitrary signal $s(t)$ on $[-L, L]$ and a "maximum frequency N".

- Use numerical integration to compute the first N Fourier Coefficients of $s(t)$.
- Use Fourier series to construct the filtered form of $s(t)$
- Compare $\frac{1}{L}\int_{-L}^{L}|s(t)|^2\,dt$ to the sum of the squares of the numerically-computed coefficients.

Advanced Contexts

Naive signal processing is so called because so many of the parameters involved are chosen arbitrarily. For example, in naive signal processing, the maximum frequency N is chosen almost at random, as is the interval $[-L, L]$. As a result, the signal processing techniques used in practice are not based on Fourier series per se, but are instead based on the closely related concept of the *Fourier transform* of a signal.

In particular, if $f(x)$ is a function defined on $(-\infty,\infty)$, then its *Fourier cosine Transform* is denoted by $A(\omega)$ and is defined

$$A(\omega) = \int_{-\infty}^{\infty} f(x)\cos(\omega x)\,dx$$

while its *Fourier Sine transform* is denoted by $B(\omega)$ and is defined

$$B(\omega) = \int_{-\infty}^{\infty} f(x)\sin(\omega x)\,dx$$

Moreover, these transforms are important because they are closed related to the *convolution* of two functions f and g, which is defined

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-t)\,g(t)\,dt$$

when the integral exists.

Exercise 1 *Find the Fourier cosine transform of*

$$f(x) = \begin{cases} 1 & if \quad |x| \le 1 \\ 0 & if \quad |x| > 1 \end{cases}$$

Exercise 2 *Explain why if $f(x)$ is odd, then its Fourier cosine transform is 0.*

Exercise 3 ** Show that $f*g = g*f$.*