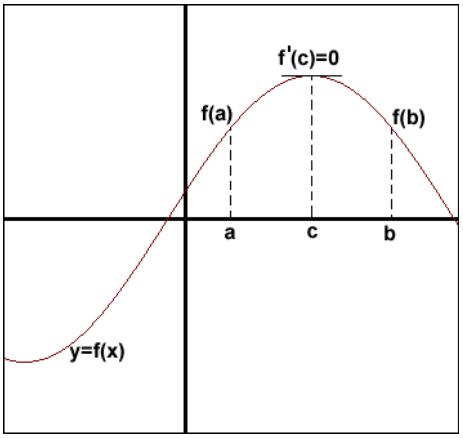
4.2 The Mean Value Theorem

Consider the following graph. If a graph goes through $f(a)$ and $f(b)$ then it must change directions. If it changes directions then the derivative must be zero since this is a maximum. This idea has a name:

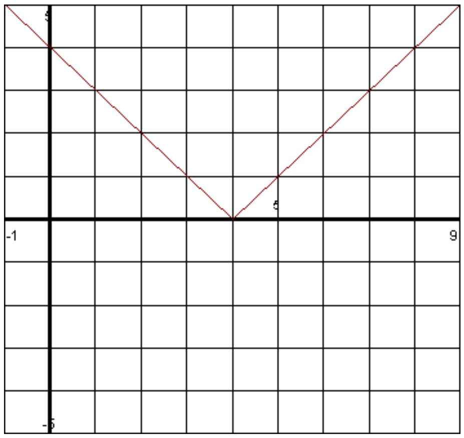


Rolle's Theorem

Let f be continuous on the closed interval [a, b] and differentiable on the open interval (a, b). If $f(a) = f(b)$ then there is at least one number c in (a, b) such that $f'(c) = 0$.

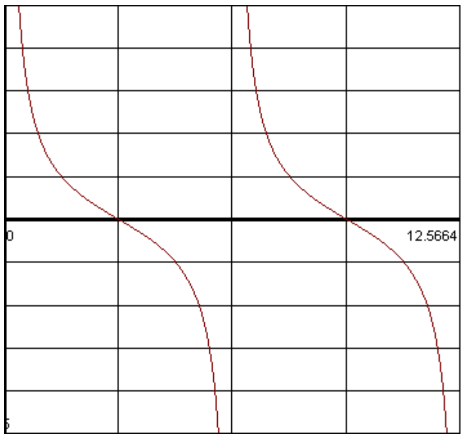
EXAMPLE: Explain why Rolle's Theorem does not apply for $f(x) = |x - 4|$ on [2, 5].

In order for Rolle's Theorem to be applied f needs to be differentiable on the open interval (2, 5). At x = 4 this absolute value function is not differentiable since there is corner, or cusp, in the graph.



EXAMPLE: Explain why Rolle's Theorem does not apply for $f(x) = \cot\left(\frac{x}{2}\right)$ on $[\pi,\ 3\pi]$.

There is a vertical asymptote at $x = 2\pi$ and therefore it is not differentiable at this point. Therefore Rolle's Theorem does not apply.



EXAMPLE: Determine whether Rolle's Theorem can be applied to $f(x) = x^2 - 3x$ on [0, 3]. If Rolle's Theorem can be applied, find all values of c in the open interval (0, -1) such that $f^{-1}(c) = 0$. If Rolle's Theorem can not be applied, explain why.

We first notice that this is continous on [0, 3]. The derivative is $f'(x) = 2x - 3$. This is defined for all numbers on (0, 3). Therefore f is differentiable on this interval. We need to make sure $f(a) = f(b)$. In other words we need to make sure $f(0) = f(3)$. We see that $f(0) = f(3) = 0$. Since all of these conditions have been met we can now apply Rolle's Theorem. We need to set the derivative equal to zero and solve for x: $0 = 2x - 3$. Solving this we get $x = \frac{3}{2}$, which is in our closed interval, [0, 3].

EXAMPLE: Determine whether Rolle's Theorem can be applied to $f(x) = x^4 - 2x^2$ on [2, 3]. If Rolle's Theorem can be applied, find all values of c in the open interval (2, 3) such that $f'(c) = 0$. If Rolle's Theorem can not be applied, explain why.

This is continuous on [2, 3]. The derivative would be $f'(x) = 4x^3 - 4x$. This is defined for all values on (2, 3) and therefore it is differentiable on this interval. However we notice that $f(2) = 8$ and $f(3) = 63$. Since these are not the same then Rolle's Theorem can not be applied, which means if we set the derivative equal to zero we are not guaranteed to find a number in our interval. Let's try this: $0 = 4x^3 - 4x$. Factoring we will get $0 = 4x(x^2 - 1)$, or $0 = 4x(x+1)(x-1)$. Solving this we get $x = 0, \pm 1$, which are all not in [2, 3].

EXAMPLE: Determine whether Rolle's Theorem can be applied to $f(x) = (x-3)(x+1)^2$ on [-1, 3]. If Rolle's Theorem can be applied, find all values of c in the open interval (-1, 3) such that $f'(c) = 0$. If Rolle's Theorem can not be applied, explain why.

This is continuous on [-1, 3]. You will need the product rule to find the derivative:

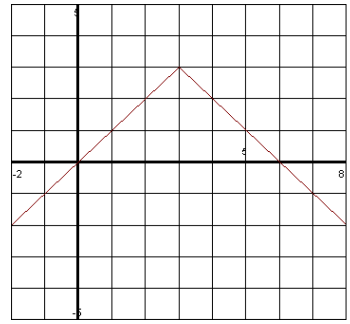
$$\underbrace{(x+3)}_{f}\underbrace{2(x+1)}_{g'} + \underbrace{(x+1)^2}_{g}\underbrace{(1)}_{f'}$$

$f'(x) = (x+3)2(x+1) + (x+1)^2(1)$ Which simplifies to $f'(x) = 3x^2 - 2x - 5$. This is defined for all values on (1, 3). Also we notice that $f(-1) = f(3) = 0$. Therefore Rolle's Theorem can be applied. Set the derivative equal to zero: $0 = 3x^2 - 2x - 5$. Factoring we get $0 = (x+1)(3x-5)$. Solving this we get $x = -1$, $x = \frac{5}{3}$. Both of these are in are interval, so Rolle's Theorem is satisfied.

EXAMPLE: Determine whether Rolle's Theorem can be applied to $f(x) = \frac{x^2 - 1}{x}$ on [-1, 1]. If Rolle's Theorem can be applied, find all values of c in the open interval (-1, 1) such that $f'(c) = 0$. If Rolle's Theorem can not be applied, explain why.

This is not continuous at x = 0. Therefore Rolle's Theorem can not be applied. Suppose we did take the derivative anyway. We would rewrite f as $f(x) = x - x^{-1}$ by dividing everything on the top by x. The derivative would be $f'(x) = 1 + x^{-2}$, or $f(x) = 1 + \frac{1}{x^2}$. This is undefined at x = 0 and is therefore not differentiable at this point. If we try to set the derivative equal to zero we would not be able to get any solutions, so this verifies that Rolle's Theorem can not be applied.

EXAMPLE: Determine whether Rolle's Theorem can be applied to $f(x) = 3 - |x - 3|$ on [0, 6]. If Rolle's Theorem can be applied, find all values of c in the open interval (0, 6) such that $f'(c) = 0$. If Rolle's Theorem can not be applied, explain why.



We first notice that this is continuous on [0, 6]. Next we notice that there is a corner in the graph at x = 3. Therefore it is not differentiable at x = 3 and so Rolle's Theorem can not be applied.

EXAMPLE: Determine whether Rolle's Theorem can be applied to $f(x) = -3x\sqrt{x+1}$ on [-1, 0]. If Rolle's Theorem can be applied, find all values of c in the open interval (0, -1) such that $f'(c) = 0$. If Rolle's Theorem can not be applied, explain why.

First we note that f is continuous on [-1, 0]. Next we need to take the derivative and make sure this is defined for all values on (-1, 0). Remember that for continuity we always look at the **closed** interval and for differentiability we look at the **open** interval. We will first rewrite our problem as $f(x) = -3x(x+1)^{\frac{1}{2}}$.

$$\overset{f}{}\qquad \overset{g'}{}\qquad \overset{g}{}\qquad \overset{f'}{}$$

$$f'(x) = -3x \cdot \frac{1}{2}(x+1)^{-\frac{1}{2}}(1) + (x+1)^{\frac{1}{2}}(-3)$$ We needed to use the product rule. Now simplify.

$$f'(x) = \frac{-3x}{2\sqrt{x+1}} - 3\sqrt{x+1}$$

In the derivative above, we need to see if any numbers make this function undefined on the open interval, (0, -1). Even though -1 makes the function undefined this is not included on the open interval, so we know that this is differentiable on (0, -1). Now we need to see if $f(a) = f(b)$. In other words we need to see if $f(0) = f(-1)$. Remember that the 0 and -1 need to be put into the original equation. We find that $f(0) = f(-1) = 0$. All of the conditions are right for us to apply Rolle's Theorem. We know that we can set the derivative equal to zero and this should be a number in our interval.

$$0 = \frac{-3x}{2\sqrt{x+1}} - 3\sqrt{x+1}$$ We need to solve for x.

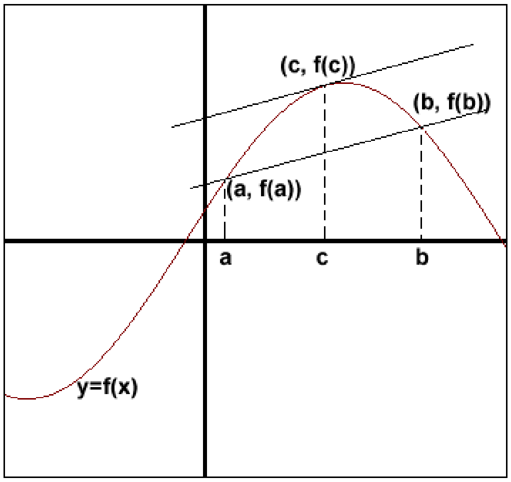
$$\frac{-3x}{2\sqrt{x+1}} = \frac{3\sqrt{x+1}}{1}$$ Cross multiply.

$$-3x = 6(x+1)$$
$$-3x = 6x + 6$$
$$-9x = 6$$
$$x = -\frac{2}{3}.$$ This is in our closed interval, [-1, 0], so it does satisfy Rolle's Theorem.

Looking at the picture to the right I can find two points such that the slope of the line going through these two points is the same as the slope of a line going through point x. This is called the

Mean Value Theorem.

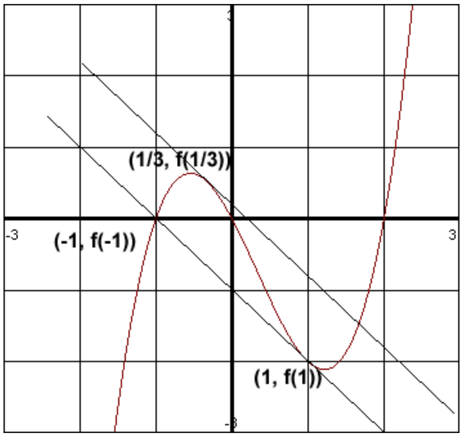
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



In order for the Mean Value Theorem to be applied, f must be continuous on [a, b] and differentiable on (a, b). Then c must be on (a, b).

EXAMPLE: Determine whether the Mean Value Theorem can be applied to $f(x) = x(x^2 - x - 2)$ on [-1, 1]. If yes, then find all values of c on (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

We know f is continuous on [-1, 1]. To find the derivative, we rewrite f as $f(x) = x^3 - x^2 - 2x$. Then the derivative is $f'(c) = 3c^2 - 2c - 2$. Now let's find the other side of the equation, $\frac{f(b) - f(a)}{b - a}$. Plugging in -1 for a and 1 for b we get: $\frac{f(1) - f(-1)}{1 - (-1)} = \frac{-2 - 0}{2} = -1$. So now we set $f'(c) = -1$. We will get: $-1 = 3c^2 - 2c - 2$. Setting it equal to zero we get $0 = 3c^2 - 2c - 1$. Factoring will give us $0 = (3c + 1)(c - 1)$. Solving for c we get $c = -\frac{1}{3}, 1$. Both of these would be our answer. In the graph to the right you can see what we just did. The line that connects the points at x = -1 and x = 1 has the same slope as at the point $x = \frac{1}{3}$.



EXAMPLE: Determine whether the Mean Value Theorem can be applied to $f(x) = \frac{x + 1}{x}$ on $\left[\frac{1}{2}, 2\right]$. If yes, then find all values of c on (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Even though f is not continuous at x = 0 it doesn't matter because we only need to make sure it is continuous between ½ and 2, which it is. To find the derivative we can rewrite the problem as $f(x) = 1 + x^{-1}$. The derivative is $f'(c) = -c^{-2}$ which is $f'(c) = \frac{-1}{c^2}$. We know that f is differentiable on $\left(\frac{1}{2}, 2\right)$. Now let's look at our Mean Value Theorem formula. Now let's look at the right side of the formula, which is $\frac{f(b)-f(a)}{b-a}$. This is $\frac{f\left(\frac{1}{2}\right)-f(2)}{\frac{1}{2}-2}=\frac{\frac{3}{2}-3}{-\frac{3}{2}}=\frac{-\frac{3}{2}}{\frac{3}{2}}=-1$. So now $f'(c)=-1$, or $\frac{-1}{c^2}=-1$. We need to solve this for c by cross multiplying: $-c^2=-1$. Solving for c we get $c=\pm 1$. Our answer is just c = 1 since this is the only answer in our interval.

EXAMPLE: Determine whether the Mean Value Theorem can be applied to $f(x)=2\sin x+\sin 2x$ on $[0,\ \pi]$. If yes, then find all values of c on (a, b) such that $f'(c)=\frac{f(b)-f(a)}{b-a}$.

We know that f is continuous on our interval. The next thing to do is find the derivative: $f'(c)=2\cos(c)+2\cos(2c)$ which is defined for all x on $(0,\ \pi)$ and therefore differentiable. Now let's look at the right side of the formula, which is $\frac{f(b)-f(a)}{b-a}$, which is $\frac{f(\pi)-f(0)}{\pi-0}=\frac{0-0}{\pi}=0$. So now $f'(c)=0$. We will now solve $2\cos(c)+2\cos(2c)=0$. I will using the identity $\cos(2c)=2\cos^2 c-1$ which is a double angle formula. You will get: $2\cos(c)+2(2\cos^2 c-1)=0$. After distributing we will get: $2\cos(c)+4\cos^2 c-2=0$, or $4\cos^2(c)+2\cos(c)-2=0$. We can divide both sides by 2: $2\cos^2(c)+\cos(c)-1=0$. After factoring we get: $(\cos(c)+1)(2\cos(c)-1)=0$. Solving this we get $\cos(c)=-1$, in which $c=\pi$. The other equation is $\cos(c)=\frac{1}{2}$. Solving this you get $c=\frac{\pi}{3}$. Both of these are in our interval, so it satisfies the Mean Value Theorem.

EXAMPLE: Determine whether the Mean Value Theorem can be applied to $f(x)=\sqrt{x(1-x)}$ on [0, 1]. If yes, then find all values of c on (a, b) such that $f'(c)=\frac{f(b)-f(a)}{b-a}$.

Since we are only using numbers between 0 and 1 we know this is continuous. It is okay to take the square root of zero. To see if it is differentiable, we need to take the derivative. It would be easier to first distribute the x inside the square root so the product rule is not necessary. I will also change the square root into an exponent: $f(x)=\left(x-x^2\right)^{\frac{1}{2}}$. Now we take the derivative using the chain rule: $f(x)=\frac{1}{2}\left(x-x^2\right)^{-\frac{1}{2}}(1-2x)$. This can be rewritten as: $f(x)=\frac{1-2x}{2\sqrt{x(1-x)}}$. We see that the derivative will be undefined at x = 0 and x = 1, however the Mean Value Theorem works on the open interval, which is (0, 1). Therefore 0 and 1 are not concluded so we know we can apply the Mean Value Theorem. So now we need to use the formula:

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$\frac{1-2c}{2\sqrt{c(1-c)}} = \frac{f(1)-f(0)}{1-0}$$

We see that $f(1)=0$ and $f(0)=0$. So we can simplify.

$$\frac{1-2c}{2\sqrt{c(1-c)}} = \frac{0-0}{1}$$

Now we cross-multiply.

$$0 = 1-2c$$

Solve for c.

$$c = \frac{1}{2}$$

This number is in our interval, so it satisfies the Mean Value Theorem.