4.2 The Mean Value Theorem and Its Consequences

If you averaged 30 miles per hour during a trip, then at some instant during the trip you were traveling exactly 30 miles per hour.

That relatively obvious statement is the Mean Value Theorem as it applies to a particular trip. It may seem strange that such a simple statement would be important or useful to anyone, but the Mean Value Theorem is important and some of its consequences are very useful for people in a variety of areas. Many of the results in the rest of this chapter depend on the Mean Value Theorem, and one of the corollaries of the Mean Value Theorem will be used every time we calculate an "integral" in later chapters. A truly delightful aspect of mathematics is that an idea as simple and obvious as the Mean Value Theorem can be so powerful.

Before we state and prove the Mean Value Theorem and examine some of its consequences, we will consider a simplified version called Rolle's Theorem.

Rolle's Theorem

Suppose we pick any two points on the x–axis and think about all of the differentiable functions which go through those two points (Fig. 1). Since our functions are differentiable, they must be continuous and their graphs can not have any holes or breaks. Also, since these functions are differentiable, their derivatives are defined everywhere between our two points and their graphs can not have any "corners" or vertical tangents. The graphs of the functions in Fig. 1 can still have all sorts of shapes, and it may seem unlikely that they have any common properties other than the ones we have stated, but Michel Rolle (1652–1719) found one. He noticed that every one of these functions has one or more points where the tangent line is horizontal (Fig. 2), and this result is named after him.

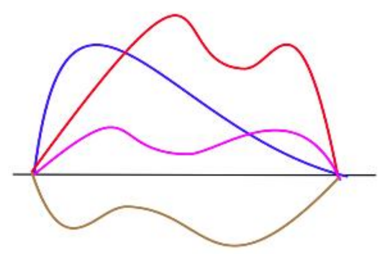

Fig. 1

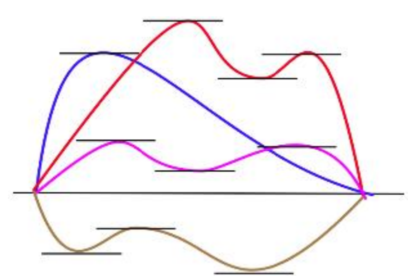

Fig. 2

Rolle's Theorem:

If $f(a) = f(b)$, and $f(x)$ is continuous for $a \le x \le b$ and differentiable for $a < x < b$,

then there is at least one number c, between a and b, so that $f'(c) = 0$.

Proof: We consider two cases: when $f(x) = f(a)$ for all x in (a,b) and when $f(x) \ne f(a)$ for some x in (a,b).

Case I, $f(x) = f(a)$ for all x in (a,b) : If $f(x) = f(a)$ for all x between a and b, then f is a horizontal line segment and $f'(c) = 0$ for all values of c strictly between a and b.

Case II, $f(x) \ne f(a)$ for some x in (a,b): Since f is continuous on the closed interval [a,b], we know from the Extreme Value Theorem that f must have a maximum value in the closed interval [a,b] and a minimum value in the interval.

If f(x) > f(a) for some value of x in [a,b], then the maximum of f must occur at some value c strictly between a and b, a < c < b. (Why can't the maximum be at a or b?) Since f(c) is a local maximum of f, then c is a critical number of f and f '(c) = 0 or f '(c) is undefined. But f is differentiable at all x between a and b, so the only possibility left is that f '(c) = 0.

If f(x) < f(a) for some value of x in [a,b], then f has a minimum at some value x = c strictly between a and b, and f '(c) = 0.

In either case, there is at least one value of c between a and b so that f '(c) = 0.

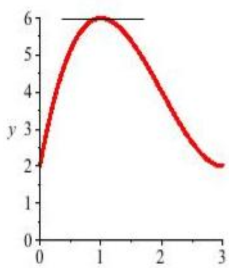

Fig. 3

Example 1: Show that $f(x) = x^3 - 6x^2 + 9x + 2$ satisfies the hypotheses of Rolle's Theorem on the interval [0, 3] and find the value of c which the theorem says exists.

Solution: f is a polynomial so it is continuous and differentiable everywhere. f(0) = 2 and f(3) = 2. $f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$ so f '(x) = 0 at 1 and 3. The value c = 1 is between 0 and 3. Fig. 3 shows the graph of f.

Practice 1: Find the value(s) of c for Rolle's Theorem for the functions in Fig. 4.

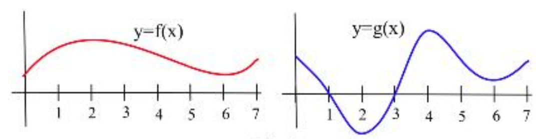

Fig. 4

The Mean Value Theorem

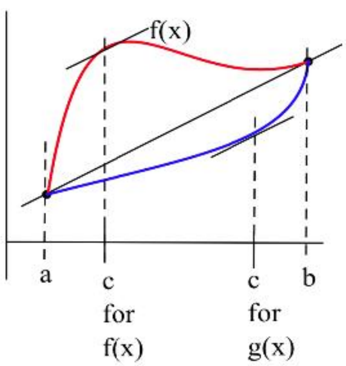

Fig. 5

Geometrically, the Mean Value Theorem is a "tilted" version of Rolle's Theorem (Fig. 5). In each theorem we conclude that there is a number c so that the slope of the tangent line to f at x = c is the same as the slope of the line connecting the two ends of the graph of f on the interval [a,b]. In Rolle's Theorem, the two ends of the graph of f are at the same height, f(a) = f(b), so the slope of the line connecting the ends is zero. In the Mean Value Theorem, the two ends of the graph of f do not have to be at the same height so the line through the two ends does not have to have a slope of zero.

Mean Value Theorem:

If f(x) is continuous for $a \le x \le b$ and differentiable for $a < x < b$,

then there is at least one number c, between a and b, so the tangent line at c is parallel to the secant line through the points (a, f(a)) and (b,f(b)): $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Proof: The proof of the Mean Value Theorem uses a tactic common in mathematics: introduce a new function which satisfies the hypotheses of some theorem we already know and then use the conclusion of that previously proven theorem. For the Mean Value Theorem we introduce a new function, h(x), which satisfies the hypotheses of Rolle's Theorem. Then we can be certain that the conclusion of Rolle's Theorem is true for h(x), and the Mean Value Theorem for f follows from the conclusion of Rolle's Theorem for h.

First, let g(x) be the straight line through the ends (a, f(a)) and (b, f(b)) of the graph of f. The function g goes through the point (a, f(a)) so g(a) = f(a). Similarly, g(b) = f(b). The slope of the linear function g is $\frac{f(b) - f(a)}{b - a}$ so $g'(x) = \frac{f(b) - f(a)}{b - a}$ for all x between a and b, and g is continuous and differentiable. (The formula for g is $g(x) = f(a) + m(x - a)$ with $m = (f(b) - f(a))/(b - a)$.)

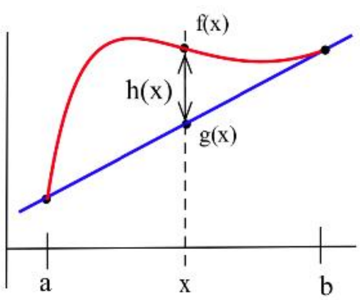

Fig. 6

Define $h(x) = f(x) - g(x)$ for $a \le x \le b$ (Fig. 6). The function h satisfies the hypotheses of Rolle's theorem:

$h(a) = f(a) - g(a) = 0$ and $h(b) = f(b) - g(b) = 0$,

h(x) is continuous for $a \le x \le b$ since both f and g are continuous there, and

h(x) is differentiable for $a < x < b$ since both f and g are differentiable there,

so the conclusion of Rolle's Theorem applies to h:

there is a c, between a and b, so that $h'(c) = 0$.

The derivative of $h(x) = f(x) - g(x)$ is $h'(x) = f'(x) - g'(x)$ so we know that there is a number c, between a and b, with $h'(c) = 0$. But $0 = h'(c) = f'(c) - g'(c)$ so $f'(c) = g'(c) = \frac{f(b) - f(a)}{b - a}$.

Graphically, the Mean Value Theorem says that there is at least one point c where the slope of the tangent line, f '(c), equals the slope of the line through the end points of the graph segment, (a, f(a)) and (b, f(b)). Fig. 7 shows the locations of the parallel tangent lines for several functions and intervals.

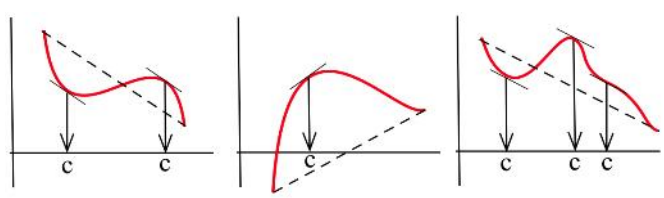

Fig. 7

The Mean Value Theorem also has a very natural interpretation if f(x) represents the position of an object at time x: f '(x) represents the velocity of the object at the **instant** x, and

$\frac{f(b) - f(a)}{b - a} = \frac{\text{change in position}}{\text{change in time}}$ represents the **average** (mean) velocity of the object during the time interval from time a to time b. The Mean Value Theorem says that there is a time c, between a and b, when the instantaneous velocity, f '(c), is equal to the average velocity for the entire trip,

$\frac{f(b) - f(a)}{b - a}$. If your average velocity during a trip is 30 miles per hour, then at some instant during the trip you were traveling exactly 30 miles per hour.

Practice 2: For $f(x) = 5x^2 - 4x + 3$ on the interval $[1,3]$, calculate $m = \frac{f(b) - f(a)}{b - a}$ and find the value of c so that $f'(c) = m$.

Some Consequences of the Mean Value Theorem

If the Mean Value Theorem was just an isolated result about the existence of a particular point c, it would not be very important or useful. However, the Mean Value Theorem is the basis of several results about the behavior of functions over entire intervals, and it is these consequences which give it an important place in calculus for both theoretical and applied uses.

The next two corollaries are just the first of many results which follow from the Mean Value Theorem.

We already know, from the Main Differentiation Theorem, that the derivative of a constant function $f(x) = k$ is always 0, but can a nonconstant function have a derivative which is always 0? The first corollary says no.

Corollary 1: If $f'(x) = 0$ for all x in an interval I, then $f(x) = K$, a constant, for all x in I.

Proof: Assume $f'(x) = 0$ for all x in an interval I, and pick any two points a and b ($a \neq b$) in the interval. Then, by the Mean Value Theorem, there is a number c between a and b so that $f'(c) = \frac{f(b) - f(a)}{b - a}$. By our assumption, $f'(x) = 0$ for all x in I so we know that $0 = f'(c) = \frac{f(b) - f(a)}{b - a}$ and we can conclude that $f(b) - f(a) = 0$ and $f(b) = f(a)$. But a and b were any two points in I, so the value of $f(x)$ is the same for any two values of x in I, and f is a constant function on the interval I.

We already know that if two functions are parallel (differ by a constant), then their derivatives are equal, but can two nonparallel functions have the same derivative? The second corollary says no.

Corollary 2: If $f'(x) = g'(x)$ for all x in an interval I,
then $f(x) - g(x) = K$, a constant, for all x in I,
so the graphs of f and g are "parallel" on the interval I.

Proof: This corollary involves two functions instead of just one, but we can imitate the proof of the Mean Value Theorem and introduce a new function $h(x) = f(x) - g(x)$. The function h is differentiable, and $h'(x) = f'(x) - g'(x) = 0$ for all x in I, so, by Corollary 1, $h(x)$ is a constant function and $K = h(x) = f(x) - g(x)$ for all x in the interval. Then $f(x) = g(x) + K$.

We will use Corollary 2 hundreds of times in Sections 4 and 5 when we work with "integrals". Typically you will be given the derivative of a function, f '(x) , and asked to find **all** functions f which have that derivative. Corollary 2 tells us that if we can find **one** function f which has the derivative we want, then the only other functions which have the same derivative are $f(x) + K$: once you find one function with the right derivative, you have essentially found all of them.

Example 2: (a) Find **all** functions whose derivatives equal 2x .

(b) Find a function g(x) with $g'(x) = 2x$ and $g(3) = 5$.

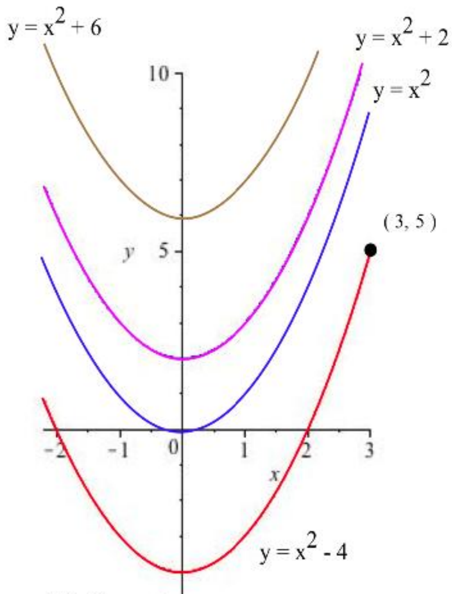

Fig. 8

Solution: (a) We can recognize that if $f(x) = x^2$ then $f'(x) = 2x$ so one function with the derivative we want is $f(x) = x^2$. Corollary 2 guarantees that every function g whose derivative is 2x has the form $g(x) = f(x) + K = x^2 + K$. The only functions with derivative 2x have the form $x^2 + K$.

(b) Since $g'(x) = 2x$, we know that g must have the form $g(x) = x^2 + K$, but this is a whole "family" of functions (Fig. 8), and we want to find one member of the family . We know that $g(3) = 5$ so we want to find the member of the family which goes through the point (3,5). All we need to do is replace the g(x) with 5 and the x with 3 in the formula $g(x) = x^2 + K$, and then solve for the value of K: $5 = g(3) = (3)^2 + K$ so $K = -4$. The function we want is $g(x) = x^2 - 4$.

Practice 3: Restate Corollary 2 as a statement about the positions and velocities of two cars.

PROBLEMS

1. In Fig. 9, find the location of the number(s) "c" which Rolle's Theorem promises (guarantees).

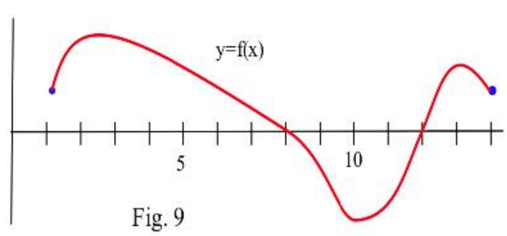

Fig. 9

For problems 2 – 4, verify that the hypotheses of Rolle's Theorem are satisfied for each of the functions on the given intervals, and find the value of the number(s) "c" which Rolle's Theorem promises exists.

2. (a) $f(x) = x^2$ on $[-2, 2]$ (b) $f(x) = x^2 - 5x + 8$ on $[0, 5]$
3. (a) $f(x) = \sin(x)$ on $[0, \pi]$ (b) $f(x) = \sin(x)$ on $[\pi, 5\pi]$
4. (a) $f(x) = x^3 - x + 3$ on $[-1, 1]$ (b) $f(x) = x \cdot \cos(x)$ on $[0, \pi/2]$

5. Suppose you toss a ball straight up and catch it when it comes down. If $h(t)$ is the height of the ball at time t, then what does Rolle's Theorem say about the velocity of the ball? Why is it easier to catch a ball which someone on the ground tosses up to you on a balcony, than for you to be on the ground and catch a ball which someone on a balcony tosses down to you?

6. If $f(x) = 1/x^2$, then $f(-1) = 1$ and $f(1) = 1$ but $f'(x) = -2/x^3$ is never equal to 0. Why doesn't this function violate Rolle's Theorem?

7. If $f(x) = |x|$, then $f(-1) = 1$ and $f(1) = 1$ but $f'(x)$ is never equal to 0. Why doesn't this function violate Rolle's Theorem?

8. If $f(x) = x^2$, then $f'(x) = 2x$ is never 0 on the interval $[1, 3]$. Why doesn't this function violate Rolle's Theorem?

9. If I take off in an airplane, fly around for awhile and land at the same place I took off from, then my starting and stopping heights are the same but the airplane is always moving. Doesn't this violate Rolle's theorem which says there is an instant when my velocity is 0?

10. Prove the following corollary of Rolle's Theorem: If $P(x)$ is a polynomial, then between any two roots of P there is a root of P'.

11. Use the corollary in problem 10 to justify the conclusion that the **only** root of $f(x) = x^3 + 5x - 18$ is 2. (Suggestion: What could you conclude about f' if f had another root?)

12. In Fig. 10, find the location(s) of the "c" which the Mean Value Theorem promises (guarantees).

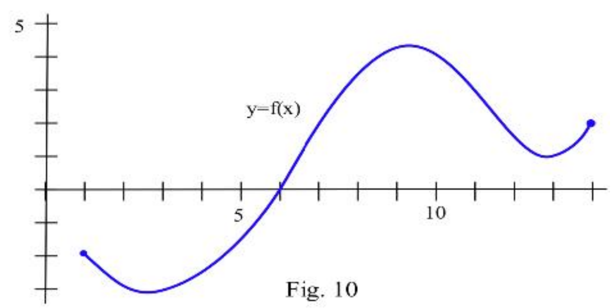

Fig. 10

In problems 13–15, verify that the hypotheses of the Mean Value Theorem are satisfied for each of the functions on the given intervals, and find the value of a number(s) "c" which Mean Value Theorem guarantees.

13. (a) $f(x) = x^2$ on $[0, 2]$ (b) $f(x) = x^2 - 5x + 8$ on $[1, 5]$
14. (a) $f(x) = \sin(x)$ on $[0, \pi/2]$ (b) $f(x) = x^3$ on $[-1, 3]$
15. (a) $f(x) = 5 - \sqrt{x}$ on $[1, 9]$ (b) $f(x) = 2x + 1$ on $[1, 7]$

16. For the quadratic functions in parts (a) and (b) of problem 13, the number c turned out to be the midpoint of the interval, $c = (a + b)/2$.

(a) For $f(x) = 3x^2 + x - 7$ on $[1, 3]$, show that $f'(2) = \dfrac{f(3) - f(1)}{3 - 1}$.

(b) For $f(x) = x^2 - 5x + 3$ on $[2, 5]$, show that $f'(7/2) = \dfrac{f(5) - f(2)}{5 - 2}$.

(c) For $f(x) = Ax^2 + Bx + C$ on $[a, b]$, show that $f'\left(\dfrac{a+b}{2}\right) = \dfrac{f(b) - f(a)}{b - a}$.

17. If $f(x) = |x|$, then $f(-1) = 1$ and $f(3) = 3$ but $f'(x)$ is never equal to $\frac{f(3) - f(-1)}{3 - (-1)} = \frac{1}{2}$. Why doesn't this function violate the Mean Value Theorem?

In problems 18 and 19, you are a traffic court judge. In each case, a speeding ticket has been given and you need to decide if the ticket is appropriate.

18. The toll taker says, "Your Honor, based on the elapsed time from when the car entered the toll road until the car stopped at my booth, I know the average speed of the car was 83 miles per hour. I did not actually see the car speeding, but I know it was and I gave the driver a speeding ticket."

19. The driver in the next case heard the toll taker and says, "Your Honor, my average velocity on that portion of the toll road was only 17 miles per hour, so I could not have been speeding. I don't deserve a ticket."

20. Find three different functions f, g and h so that $f'(x) = g'(x) = h'(x) = \cos(x)$.

21. Find a function f so that $f'(x) = 3x^2 + 2x + 5$ and $f(1) = 10$.

22. Find a function g so that $g'(x) = x^2 + 3$ and $g(0) = 2$.

23. Find values for A and B so that the graph of the parabola $f(x) = Ax^2 + B$ is
    (a) tangent to the line $y = 4x + 5$ at the point (1,9)
    (b) tangent to the line $y = 7 - 2x$ at the point (2,3)
    (c) tangent to the parabola $y = x^2 + 3x - 2$ at the point (0,2)

24. Sketch the graphs of several members of the "family" of functions whose derivatives always equal 3. Give a formula which defines every function in this family.

25. Sketch the graphs of several members of the "family" of functions whose derivatives always equal $3x^2$. Give a formula which defines every function in this family.

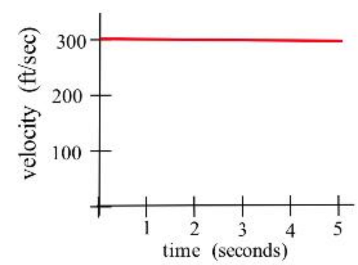

Fig. 11

26. At t seconds after takeoff, the upward velocity of a helicopter was $v(t) = 2t - 7$ feet/second. Two seconds after takeoff, the helicopter was 8 feet above sea level. Find a formula for the height of the helicopter at every time t.

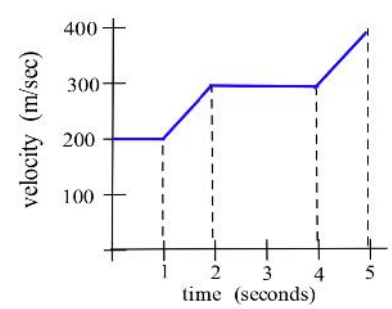

Fig. 12

27. Assume that a rocket is fired from the ground and has the upward velocity shown in Fig. 11. Estimate the height of the rocket when t = 1, 2, and 5 seconds.

28. Fig. 12 shows the upward velocity of a rocket. Use the information in the graph to estimate the height of the rocket when t = 1, 2, and 5 seconds.

29. Use the following information to determine an equation for $f(x)$: $f''(x) = 6$, $f'(0) = 4$, and $f(0) = -5$.

30. Use the following information to determine an equation for $g(x)$: $g''(x) = 12x$, $g'(1) = 9$, and $g(2) = 30$.

31. Define $A(x)$ to be the **area** bounded by the x–axis, the line $y = 3$, and a vertical line at x (Fig. 13).

(a) Find a formula for $A(x)$? (b) Determine $A'(x)$

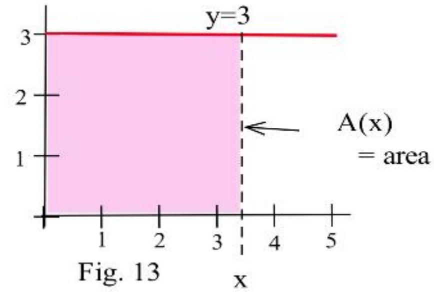

Fig. 13

32. Define $A(x)$ to be the **area** bounded by the x–axis, the line $y = 2x$, and a vertical line at x (Fig. 14).

(a) Find a formula for $A(x)$? (b) Determine $A'(x)$

33. Define $A(x)$ to be the **area** bounded by the x–axis, the line $y = 2x + 1$, and a vertical line at x (Fig. 15).

(a) Find a formula for $A(x)$? (b) Determine $A'(x)$

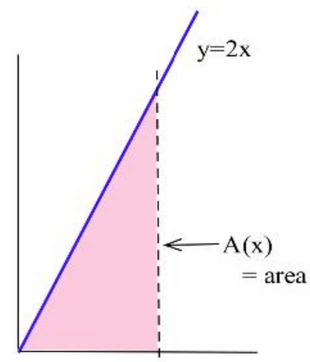

Fig. 14

In problems 34 – 36, we have a list of numbers $a_1, a_2, a_3, a_4, \ldots$, and the consecutive differences between numbers in the list are $a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots$

34. If $a_1 = 5$ and the difference between consecutive numbers in the list is always 0, what can you conclude about the numbers in the list?

35. If $a_1 = 5$ and the difference between consecutive numbers in the list is always 3, find a formula for a_n?

36. Suppose the "a" list starts 3, 4, 7, 8, 6, 10, 13, . . ., and there is a "b" list which has the same differences between consecutive numbers as the "a" list.

(a) If $b_1 = 5$, find the next six numbers in the "b" list. How is b_n related to a_n?

(b) If $b_1 = 2$, find the next six numbers in the "b" list. How is b_n related to a_n?

(c) If $b_1 = B$, find the next six numbers in the "b" list. How is b_n related to a_n?

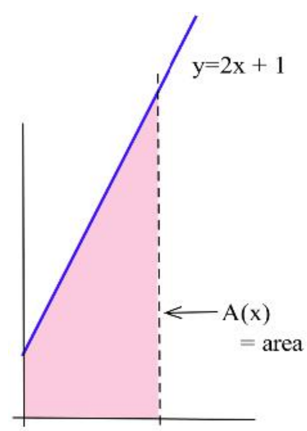

Fig. 15

Section 4.2 **PRACTICE Answers**

Practice 1: f '(x) = 0 when x = 2 and 6 so c = 2 and c= 6.

g '(x) = 0 when x = 2, 4, and 6 so c = 2, c = 4, and c = 6.

Practice 2: $f(x) = 5x^2 - 4x + 3$ on [1, 3]. f(1) = 4 and f(3) = 36 so

$$m = \frac{f(b) - f(a)}{b - a} = \frac{36 - 4}{3 - 1} = 16.$$

f '(x) = 10x – 4 so f '(c) = 10c – 4 = 16 if 10c = 20 and **c = 2**.

The graph of f and the location of c are shown in Fig. 16.

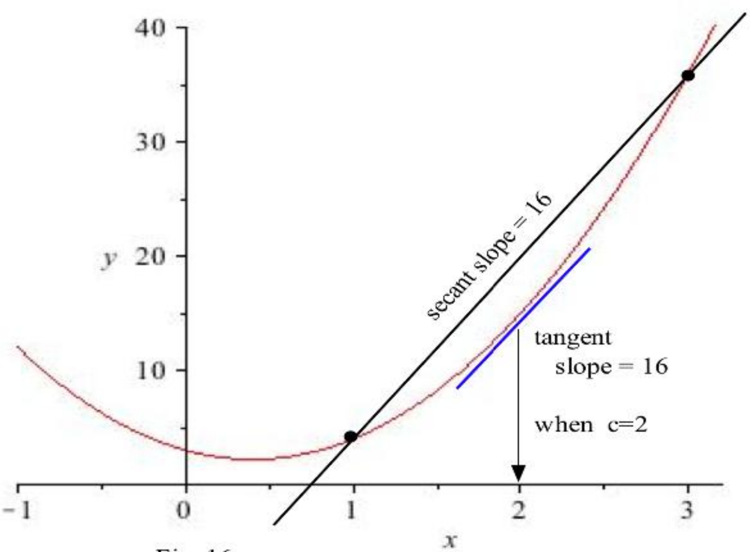

Fig. 16

Practice 3:

If two cars have the same velocities during an interval of time (f '(t) = g '(t) for t in I)

then the cars are always a constant distance apart during that time interval.

(Note: The "same velocity" means **same speed** and **same direction**. If two cars are traveling at the same speed but in different directions, then the distance between them changes and is not constant.)